Feynman Integrals

$$I(D, n, z) = \left(\prod_{k=1}^{L} \int_{\mathbb{R}^{1,D-1}} \frac{d^D l_k}{i \pi^{D/2}} \right) \frac{1}{Q_e^{n_e}}$$

e.g. $m_e^2 - k_e^2$

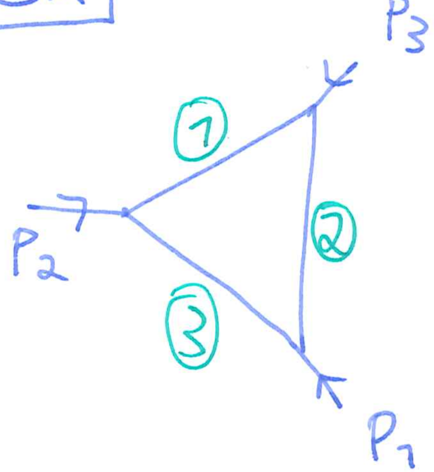
Schwinger parameters:

$$= \Gamma(\omega) \left(\prod_{e=1}^{N} \int_0^{\infty} \frac{x_e^{n_e - 1} \, dx_e}{\Gamma(n_e)} \right) \frac{\delta(1 - h(x))}{\mathcal{U}^{D/2 - \omega} F^{\omega}}$$

$\sum$ spanning trees

$\sum$ spanning 2-forests

Ex



$D = 4 - 2\varepsilon$, $n_e = 1$, $m_e^2 = 0$

(trees)

$$\mathcal{U} = x_1 + x_2 + x_3$$

(2-forests)

$$F = p_1^2 x_2 x_3 + p_2^2 x_1 x_3 + p_3^2 x_1 x_2$$

$$\omega = \underbrace{1+1+1}_{n_e} - 1 \cdot \underbrace{D/2}_{2-\varepsilon} = 1+\varepsilon$$

$$I \underset{\substack{\uparrow \\ h(x)=x_3}}{=} \Gamma(1+\varepsilon) \int_0^\infty \int_0^\infty \frac{dx_1\, dx_2}{\mathcal{U} F} \left(\frac{\mathcal{U}^2}{F} \right)^{\varepsilon} \Bigg|_{x_3=1}$$

(convergent @ $\varepsilon \to 0$)

$$\left(\frac{\mathcal{U}^2}{F} \right)^{\varepsilon} = e^{\varepsilon \ln \mathcal{U}^2/F}$$

$\Rightarrow$ ε – expansion:

$$= \underbrace{e^{-\gamma_E \varepsilon + \sum_{n \geq 2} \frac{\zeta(n)}{n} (-\varepsilon)^n}}_{\Gamma(1+\varepsilon)} \cdot \sum_{m \geq 0} \frac{\varepsilon^m}{m!} \underbrace{\int_0^\infty \int_0^\infty \frac{dx_1\, dx_2}{\mathcal{U} F} \left(\ln \frac{\mathcal{U}^2}{F} \right)^m}_{\text{convergent integral, independent of } \varepsilon}$$

- Divergences on hyperplanes $\omega(\sigma) = 0, -1, -2, \dots$

  power counting

(sector decomposition, expansion by regions)

$$\text{Integrand}(x_e \rho^{\sigma_e}) \propto \rho^{\omega(\sigma)} + \text{higher orders}$$

- analytic continuation: integrate by parts

  $\Rightarrow$ increase domain of convergence

$$\int_0^\infty \rho^{\omega(\sigma)-1} d\rho \, f(\rho) = \frac{1}{\omega(\sigma) \cdots [\omega(\sigma)+K]} \int_0^\infty \rho^{\omega(\sigma)+K} d\rho \left(-\frac{\partial}{\partial \rho}\right)^{K+1} f(\rho)$$

$\Rightarrow$ meromorphic continuation (with simple poles)

$\Rightarrow$ ε-expansion also for integrals on a pole

- factorization of graph polynomials:

$$\mathcal{U} \to \mathcal{U}_\gamma \, \mathcal{U}_{G/\gamma} \, \rho^{\mathrm{loops}(\gamma)} + \text{higher orders}$$

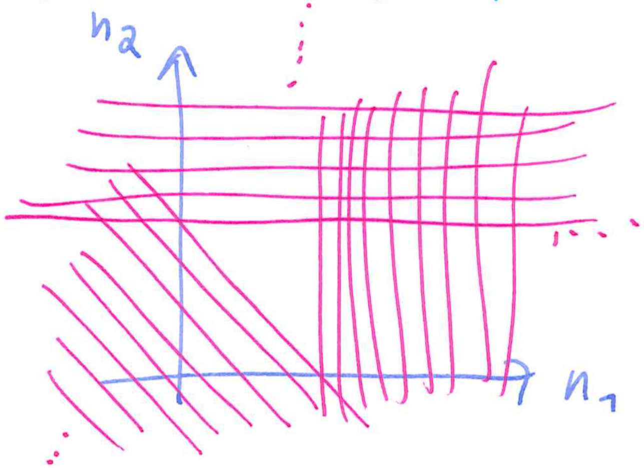
⇒ residues factor into sub- and quotient graph integrals

[renormalizability, "graph-by-graph" renormalization group,

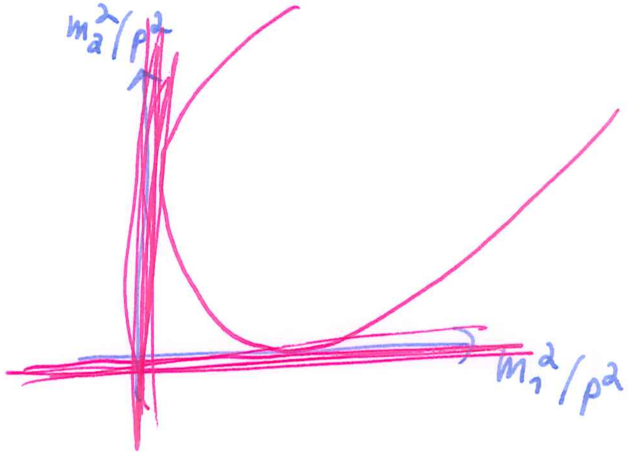
⇝ Hopf algebra of Feynman graphs]

$$I(D, n, \underbrace{z}_{m_i^2,\ p_i \cdot p_j})$$

- single-valued (meromorphic)
- poles on hyperplanes



- multivalued ($\log$, $\pm\sqrt{\ }$, ...)
- Landau varieties



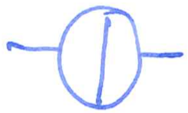
(can be non-linear)

(don't repeat, e.g. finitely many)

- power counting (easy)
  (newton polytope)
- Landau analysis (complicated)
- residues
- monodromy ("discontinuities across a "cut"")
- difference equations
- differential equations

$[\Gamma(n+1) = n \cdot \Gamma(n)]$

$$\frac{\partial}{\partial m_i^2} I = \sum_{j=1}^{r} c_r \cdot I(\ldots)$$

e.g.  $I(n_1+k_1, n_2+k_2, n_3+k_3, n_4+k_4, n_5+k_5)$ $\quad k \in \mathbb{Z}^5$

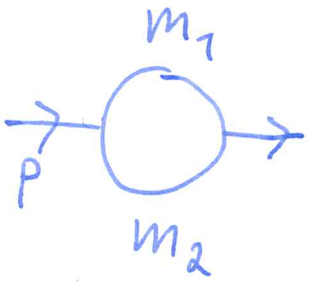
$(m_e = 0)$

$$= c_1(n,k) \cdot I(n_1, n_2, n_3, n_4, n_5)$$
$$+ c_2(n,k) \cdot I(n_1+1, n_2, n_3, n_4, n_5)$$
$$+ c_3(n,k) \cdot I(n_1, n_2, n_3, n_4, n_5+1)$$

rational fcts of n for each K

("3 master integrals")

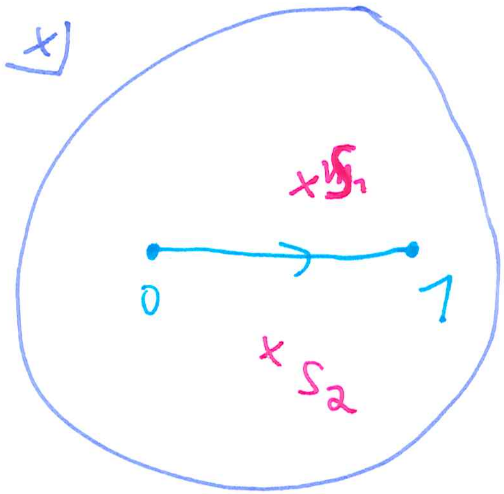
Dependence on masses and momenta



$D = 2 \qquad n_1 = n_2 = 1 \qquad (\Rightarrow \omega = 1)$

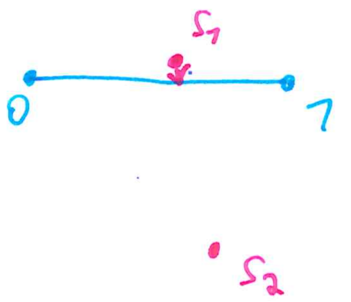
$$\underset{\substack{\parallel \\ (m_1^2,\, m_2^2,\, p^2)}}{I(z)} \overset{h(x)=x_1+x_2}{\overset{\downarrow}{=}} \int_0^1 \frac{dx}{F}$$

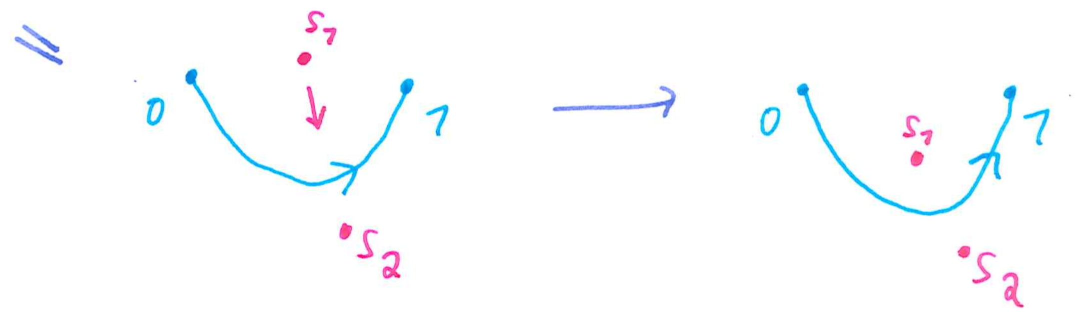
$$F = x \cdot m_1^2 + (1-x)\, m_2^2 - p^2 x \cdot (1-x)$$
$$= p^2 (x - s_1)(x - s_2)$$



- Vary $z \longrightarrow s_1, s_2$ move in the x-plane
- $I(z)$ is clearly smooth & holomorphic as long as $s_1, s_2 \notin [0,1]$

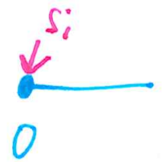
- $\frac{dx}{F}$ is holomorphic, so by Cauchy, we can deform the contour without changing the integral





$\Longrightarrow$ $I(z)$ has an analytic continuation even when $s_{1/2}$ cross the line $(0,1)$.

Contour deformation fails only when:

1) $s_i \to 0$  $\Longleftrightarrow$ $0 = F(0) = m_2^2$

2) $s_i \to 1$  $\Longleftrightarrow$ $0 = F(1) = m_1^2$

} endpoint singularities

3) $s_1 \to s_2$

"pinch singularity"

$$\Leftrightarrow \quad 0 = DF = (m_1^2 - m_2^2 - p^2)^2 - 4 m_2^2 p^2$$

"normal threshold"

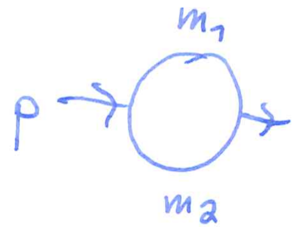
$$= [p^2 - (m_1+m_2)^2] \cdot [p^2 - (m_1 - m_2)^2]$$

"anomalous thresh"

discriminant $= B^2 - 4AC$

$$F = \underbrace{p^2}_{A} x^2 + \underbrace{(m_1^2 - m_2^2 - p^2)}_{B} x + \underbrace{m_2^2}_{C} \quad \Rightarrow \quad s_{1/2} = -\frac{B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A}$$

__Def.__ The Landau variety L of an integral $I(z)$ is the smallest subset ~~def~~ ~~a~~ in kinematic space (z-space), defined by polynomials $\{f_i = 0\}$, such that $I(z)$ has a (multivalued) analytic continuation in the entire domain away from L.

Ex 

$$L = \{m_1^2 = 0\} \cup \{m_2^2 = 0\} \cup \{DF = 0\}$$

endpoints

pinch