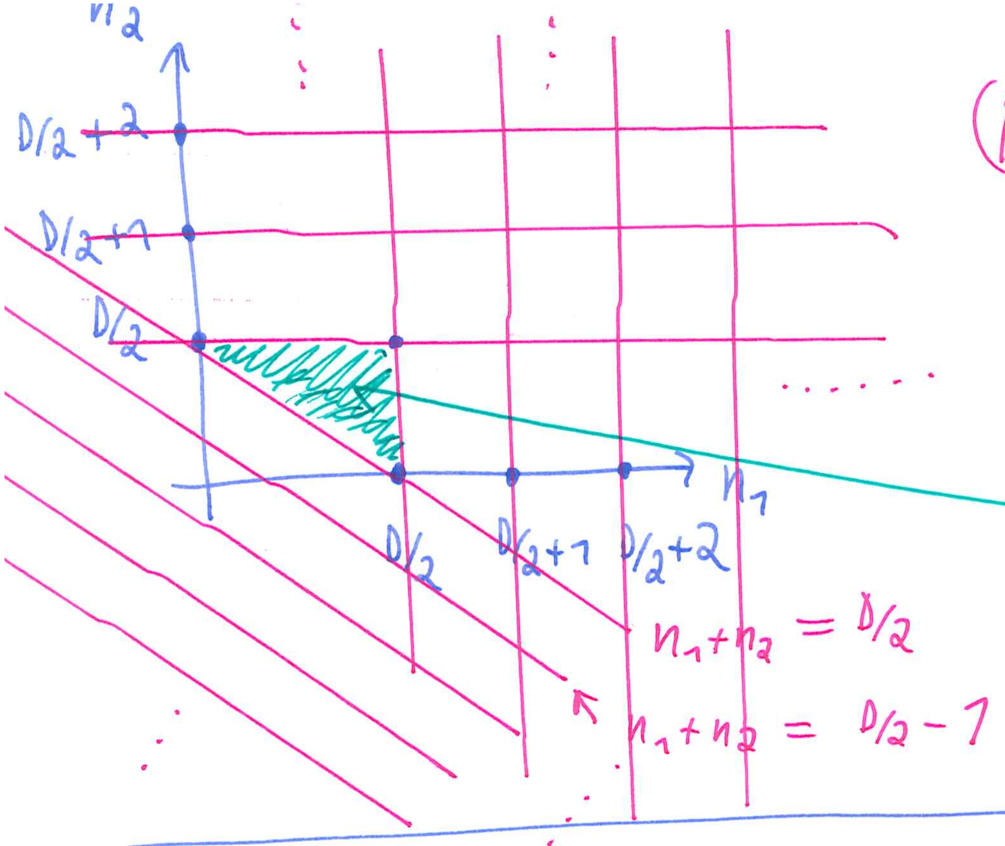
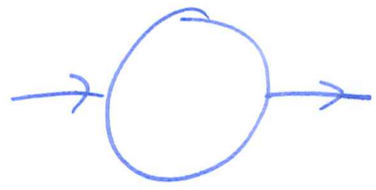
(Poles)



Domain of convergence of $I(D, n, p^2)$



Questions

Given any Feynman integral,

(1) What is its domain of convergence? $\Rightarrow$ power counting

(2) What is its analytic continuation? $\rightarrow$ integration by parts

[How to get expressions for residues / Laurent expansion]

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx$$

power counting @ $x \to 0$: converges only for $\mathrm{Re}\, n > 0$

integrate by parts

$$= \underbrace{\frac{x^n}{n} e^{-x} \Big|_0^\infty}_{0} + \frac{1}{n} \int_0^\infty x^n \left(-\frac{\partial}{\partial x}\right) e^{-x}\, dx$$

$$\vdots$$

$$= \underbrace{\frac{1}{n(n+1)\cdots(n+k)}}_{\text{poles @ } n = 0, -1, -2, \dots} \underbrace{\int_0^\infty x^{n+k} \left(-\frac{\partial}{\partial x}\right)^{k+1} e^{-x}\, dx}_{\text{holomorphic \& convergent for } \mathrm{Re}\, n > -k-1}$$

Corollary. $\Gamma(n)$ extends to a unique meromorphic function on $\mathbb{C} \ni n$ with simple poles @ $\mathbb{Z}_{\leq 0} = \{0, -1, -2, \dots\}$.

$\Rightarrow$ Generalize to several variables & different integrand.

Note: $\int_0^\infty x^{n-1} f(x)\, dx = \frac{1}{n(n+1)\dots(n+k)} \int_0^\infty x^{n+k} \left(-\frac{\partial}{\partial x}\right)^{k+1} f(x)\, dx$

$$\Rightarrow \operatorname*{Res}_{n=-k} \left(\int_0^\infty x^{n-1} f(x)\, dx \right) = \left(\frac{\partial}{\partial x}\right)^k \Big|_{x=0} f(x)$$

$(k \in \mathbb{Z}_{\geq 0})$

[→ reduces integrals with numerators in momenta $\frac{(l_i \cdot l_j)}{\prod Q_e^{n_e}}$]

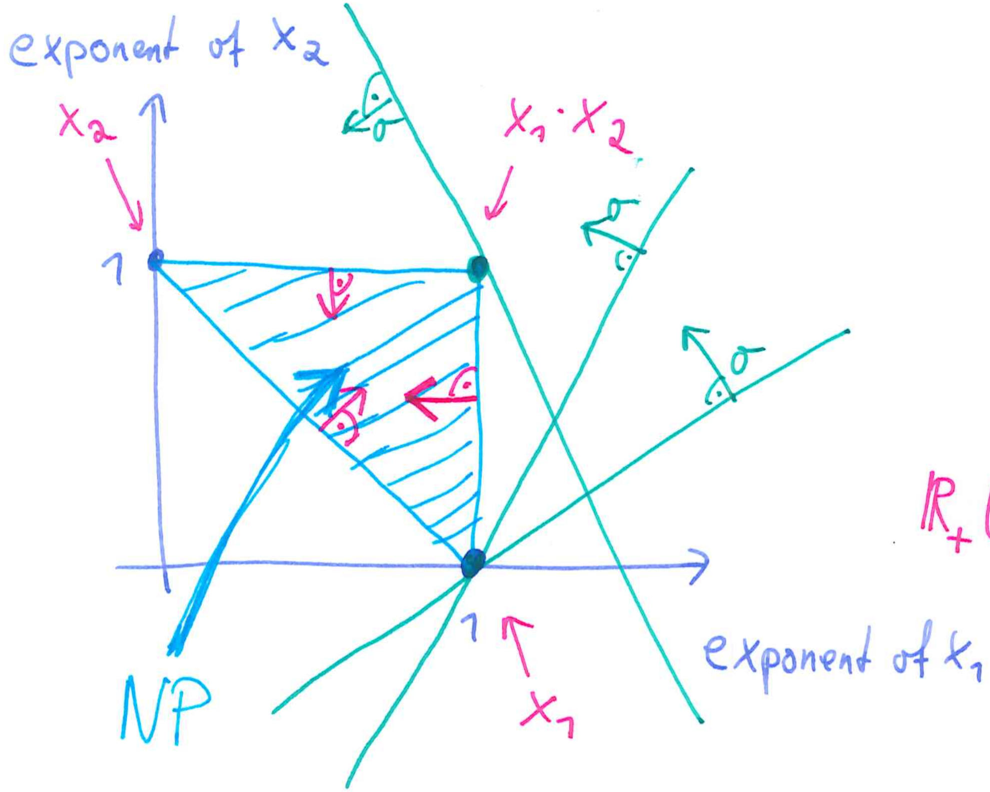
(no integral anymore!)

Def. The power counting degree $\omega(\sigma)$ of $f(x)$ in the direction $\sigma \in \mathbb{R}^N$ is the exponent such that (for $\rho \searrow 0$)

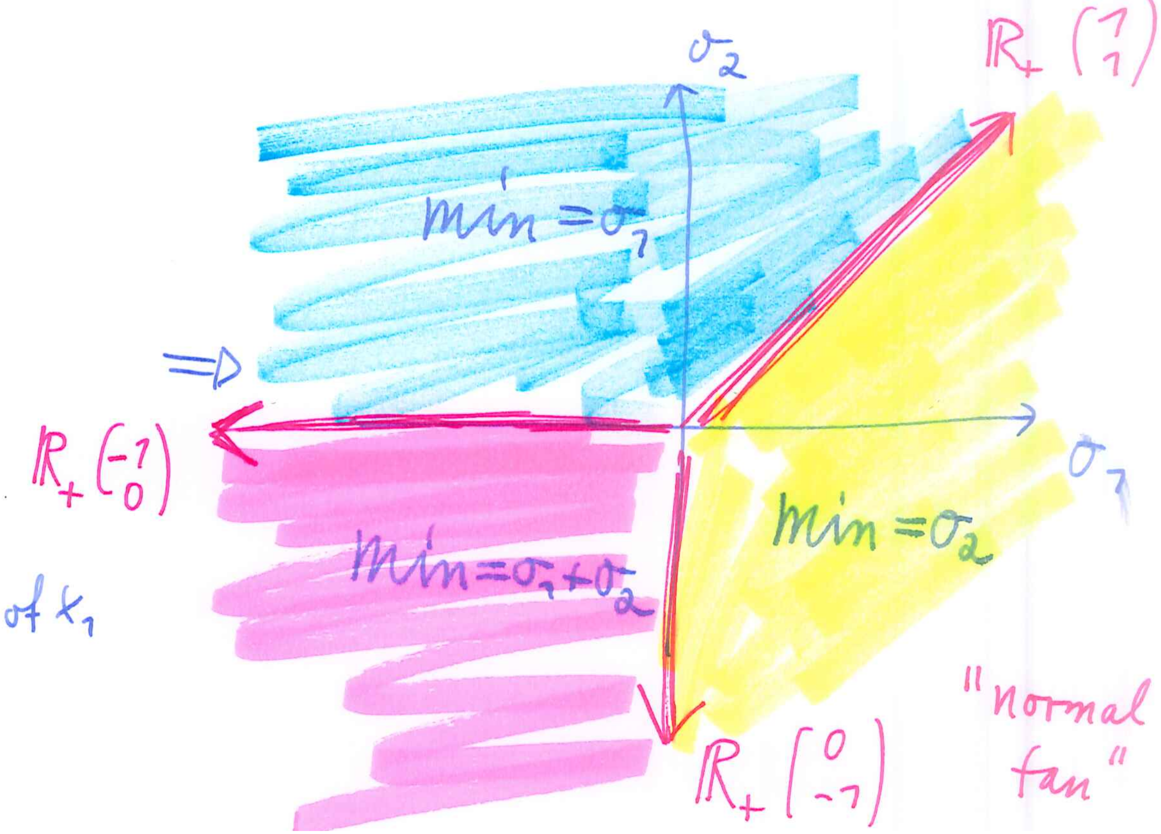
$$f(x_1 \rho^{\sigma_1}, \dots, x_N \rho^{\sigma_N}) \propto \rho^{\omega(\sigma)} \cdot (1 + \mathcal{O}(\rho^{>0}))$$

Ex. $\quad f = \underbrace{x_1^{n_1-1}\,dx_1}_{\propto \rho^{n_1\cdot\sigma_1}} \; \underbrace{x_2^{n_2-1}\,dx_2}_{\propto \rho^{n_2\cdot\sigma_2}} \, \big(\underset{\downarrow \, \rho^{\sigma_1}}{x_1} + \underset{\downarrow \, \rho^{\sigma_2}}{x_2} - \rho^2 \underset{\downarrow \, \rho^{\sigma_1+\sigma_2}}{x_1 x_2}\big)^{-D/2}$

$$\Rightarrow \quad \omega(\sigma) = n_1\cdot\sigma_1 + n_2\cdot\sigma_2 - D/2 \cdot \min\{\underset{\sigma\cdot\binom{1}{0}}{\underset{\parallel}{\sigma_1}},\ \underset{\sigma\cdot\binom{0}{1}}{\underset{\parallel}{\sigma_2}},\ \underset{\sigma\cdot\binom{1}{1}}{\underset{\parallel}{\sigma_1+\sigma_2}}\}$$



$\Rightarrow$



Note: The power counting function switches precisely along the inward normal directions to the Newton polytope

$$NP = \operatorname{conv}\{ v : c_v \neq 0 \}$$

where $\mathcal{U}+\mathcal{F} = \sum_{v \in \mathbb{Z}^N} c_v \cdot x^v$, $x^v = x_1^{v_1} \cdots x_N^{v_N}$

Thm. The following are equivalent:

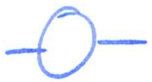
1) $\left(\prod_{e=1}^{N} \int_0^\infty x_e^{n_e - 1} dx_e \right) (\mathcal{U}+\mathcal{F})^{-D/2}$ converges

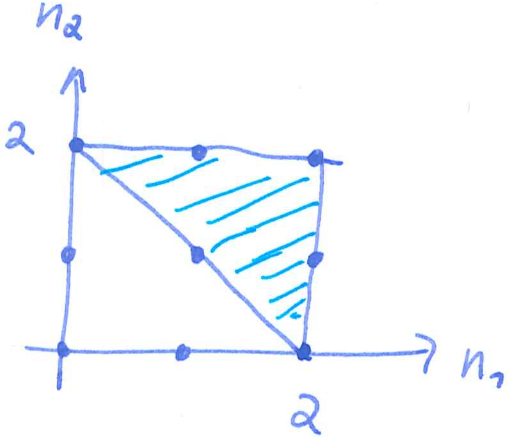
2) $\operatorname{Re} \omega(\sigma) > 0$ for all $\sigma \in \mathbb{R}^N \setminus \{0\}$

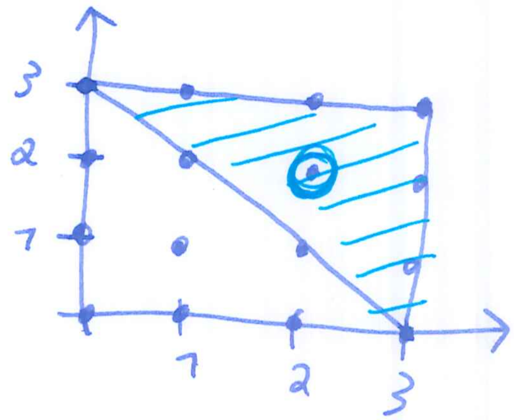
3) $\operatorname{Re} n \in \operatorname{interior}(D/2 \cdot NP)$

4) $\mathrm{Re}\ \omega(\sigma) > 0$ for all inward normals of NP

[only finitely many!]

Example  Convergent in $D=4$: Convergent in $D=6$:





$\Rightarrow$ in large dimensions, there are lots of convergent integrals

$\Rightarrow$ no convergent integral with $n_1, n_2 \in \mathbb{Z}$

$\Rightarrow I(\underset{D}{6}, \underbrace{2,2}_{n}, p^2)$ is convergent

($\Rightarrow$ Exists bases of finite integrals)

Thm 2. $I(D, n, z)$ have unique meromorphic analytic continuation, with simple poles @ hyperplanes

$$\{\omega(\sigma) = 0, -1, -2, \dots\}$$

for the normals σ of NP.

Special case: Ultraviolet divergences (UV) in graphs $(Q_e = m_e^2 - k_e^2)$

Lemma. Let $\gamma \subset \{1, \dots, N\}$ be a subgraph. For $x_e \to x_e \cdot \rho \;\; \forall e \in \gamma$,

1. $\mathcal{U}_G \to \rho^{\text{loops}(\gamma)} \left(\mathcal{U}_\gamma \cdot \mathcal{U}_{G/\gamma} + \mathcal{O}(\rho) \right)$

2. $F_G \to \rho^{\text{loops}(\gamma)} \left(\mathcal{U}_\gamma \cdot F_{G/\gamma} + \mathcal{O}(\rho) \right)$

$\sigma = (1, 0 \dots, 1, \dots 0)$

$$\Rightarrow \omega(\sigma) = \sum_{e\in\gamma} n_e - D/2 \cdot \mathrm{loops}(\gamma) = \omega(\gamma)$$

<u>Lemma.</u> Suppose that γ contains all massive edges and connects all external legs. Then $F_{G/\gamma} = 0$, and

(Infrared) "soft"

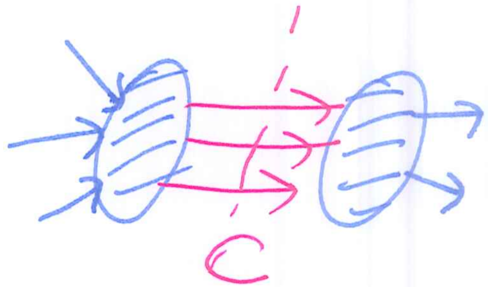
$$F_G \to \rho^{\mathrm{loops}(\gamma)+1} \left(\mathcal{U}_{G/\gamma} \cdot F_\gamma + \mathcal{O}(\rho) \right)$$

$$\Rightarrow \omega(-\sigma) = -\omega(G/\gamma)$$

[provided $F_\gamma \neq 0$]

<u>Thm.</u> Suppose that a graph G has generic momenta:

$$\left(\sum_{e\in C} k_e \right)^2 \neq \sum_{e\in C} m_e^2$$



[e.g. holds for Euclidean external momenta]

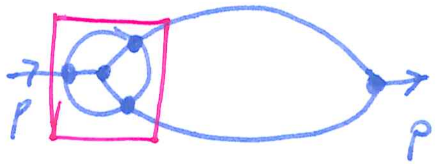
Then the UV- and IR divergences from above are <u>all</u> divergences.

[$\Rightarrow$ only need to check subgraphs]

$\boxed{\text{Ex}}$ $(n_e = 1, \; D = 4 - 2\varepsilon)$

$$\omega = 8 - 4\cdot(2-\varepsilon) = 4\varepsilon$$

$\Rightarrow$ overall logarithmic UV divergence

$G =$ 

γ

$$\hookrightarrow \omega(\gamma) = 6 - 3(2-\varepsilon) = 3\varepsilon$$

$\Rightarrow$ logarithmic subdivergence
UV

$\rightarrow$ Convergence for small $\varepsilon > 0$

"period"

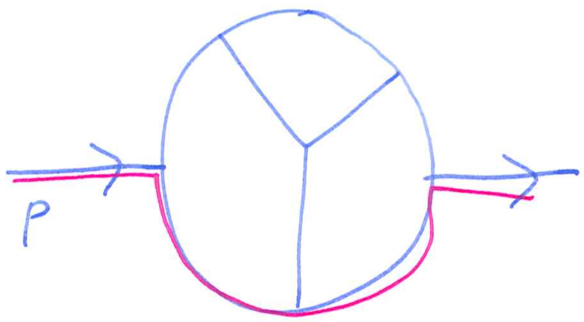
$$\operatorname*{Res}_{\omega(\gamma)\to 0} I_G = P_\gamma \cdot I_{G/\gamma} = P$$  $$\cdot \, I$$ 

$$\operatorname*{Res}_{\omega=0} I_\gamma = \left(\prod_{e=1}^{N} \int_0^\infty \ldots\right) \frac{\delta(1-h(x))}{\mathcal{U}^{D/2}}$$

$$\Rightarrow I = \frac{1}{3\varepsilon}\cdot\frac{1}{4\varepsilon}\cdot P \cdot P + O\left(\frac{1}{\varepsilon}\right)$$

$$P = 6\zeta(3), \quad P = 1$$

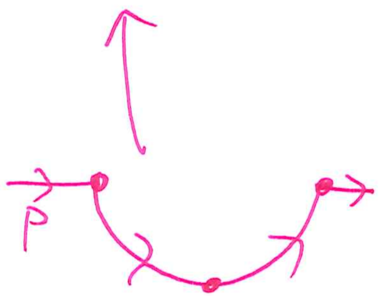
Infrared example $(n_e = 1,\ D = 4-2\varepsilon,\ m_e = 0)$



$$\omega = 8 - 3(2-\varepsilon) = 2 + 3\varepsilon$$

$\Rightarrow$ overall quadratically convergent

converges for small $\varepsilon < 0$

$\gamma =$ 

$$-\omega(G/\gamma) = -\omega(\,) = -3\varepsilon$$

$\Rightarrow$ logarithmic (soft) infrared subdivergence

$$\Rightarrow \operatorname*{Res}_{-\omega(G/\gamma)=0} I = \underbrace{P}_{=\,6\,\zeta(3)} \cdot \underbrace{I}_{\frac{1}{(-p^2)^2}}$$

$$\Rightarrow I = \frac{6\,\zeta(3)}{-3\varepsilon} \cdot \frac{1}{p^4} + \mathcal{O}(\varepsilon^0)$$