- Schwinger parameters
- analytic regularization
- Landau varities
- monodromy

Scalar L-loop Feynman integral family:

$$I(D, n, z) = \left(\prod_{k=1}^{L} \int_{\mathbb{R}^{1,D-1}} \frac{d^D l_k}{i\pi^{D/2}} \right) \prod_{e=1}^{N} \frac{1}{Q_e^{n_e}}$$

D dimension

indices

m_i^2, $p_i \cdot p_j$

quadratic in $\{l_i, p_j, m_k\}$

Schwinger Parameters:

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \implies \frac{1}{Q^n} = \frac{1}{\Gamma(n)} \int_0^\infty x^{n-1} e^{-xQ} dx$$

Combined quadrics in exponent:

$$\sum_{e=1}^{N} x_e Q_e = \sum_{i,j=1}^{L} (-l_i \cdot l_j) A_{ij} + 2\sum_{i=1}^{L} B_i \cdot l_i + C$$

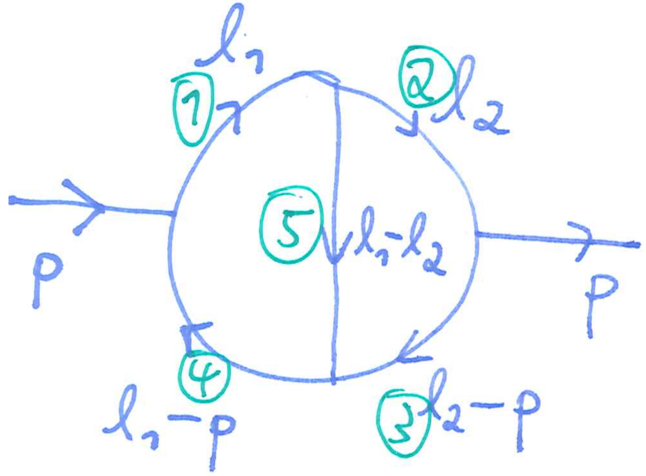
$$= \sum_{i,j=1}^{L} (-l_i' \cdot l_j') A_{ij} + \Delta$$

← independent of l's

by completing the square:

$$l_i' = l_i - \sum_{j=1}^{L} A^{-1}_{ij} B_j \, , \quad \Delta = C + \sum_{i,j=1}^{L} A^{-1}_{ij} (B_i \cdot B_j)$$

Example: $\quad Q_e = m_e^2 - k_e^2 - i\varepsilon$



$$\Rightarrow x_1\left(m_1^2 - l_1^2\right) + x_2\left(m_2^2 - l_2^2\right) + x_3\left(m_3^2 - (l_2-p)^2\right) + x_4\left(m_4^2 - (l_1-p)^2\right) + x_5\left(m_5^2 - (l_1-l_2)^2\right)$$

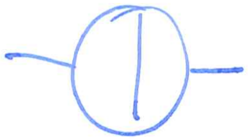
$$= -(l_1, l_2)\underbrace{\begin{pmatrix} x_1+x_4+x_5 & -x_5 \\ -x_5 & x_2+x_3+x_5 \end{pmatrix}}_{A}\begin{pmatrix} l_1 \\ l_2 \end{pmatrix}$$

$$+ 2\,(l_1, l_2)\underbrace{\begin{pmatrix} x_4 p \\ x_3 p \end{pmatrix}}_{B} + \underbrace{(-p^2)(x_3+x_4) + \sum_{i=1}^{5} m_i^2 x_i}_{C}$$

Def. $\mathcal{U} := \det A \in \mathbb{Z}[x_1,\dots,x_N]$ (homogeneous of degree L)

$\mathcal{F} := \mathcal{U}\cdot\Delta \in \mathbb{Z}[x_1,\dots,x_N, z]$ (—"— in x_i of degree $L+1$)

$m_i^2,\ p_i\cdot p_j$

Ex 

$$\mathcal{U} = \begin{vmatrix} x_1+x_4+x_5 & -x_5 \\ -x_5 & x_2+x_3+x_5 \end{vmatrix}$$

$$= (x_1+x_4)(x_2+x_3) + x_5(x_1+x_2+x_3+x_4)$$

[linear in each x_i]

Notice: Integral $\left(\prod_{k=1}^{L} \int d^D l_k'\right)\cdot e^{\sum_{i,j}(l_i'\cdot l_j')A_{ij}}$ is Gaußian

$$\Rightarrow \quad I(D, n, z) = \underbrace{\left(\prod_{e=1}^{N} \int_0^\infty \frac{x_e^{n_e - 1}\, dx_e}{\Gamma(n_e)} \right)}_{} \frac{e^{-F/U + i\varepsilon}}{U^{D/2}}$$

Lee-Pomeransky form:

$$I(D, n, z) = \frac{\Gamma(D/2)}{\Gamma(D/2 - \omega)} \left(\prod_{e=1}^{N} \int_0^\infty \cdots \right) (U + F)^{-D/2}$$

where $\omega = \sum_{e=1}^{N} n_e - L \cdot D/2$ ("superficial degree of convergence")

Projective form: Let $h : \mathbb{R}_+^N \to \mathbb{R}_+$ homogeneous of degree 1 $(h(\lambda x) = \lambda h(x))$

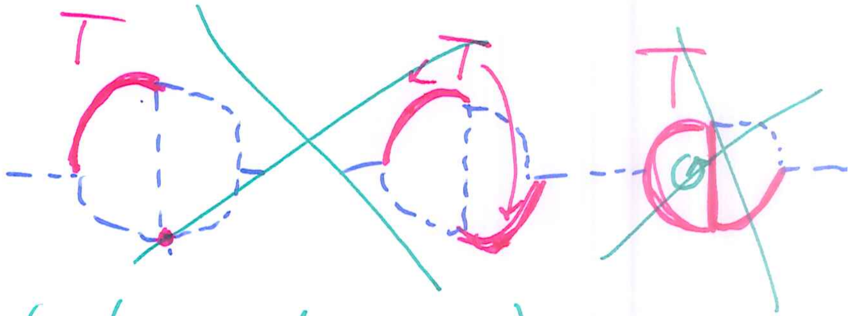
$$I(D, n, z) = \Gamma(\omega) \left(\prod_{e=1}^{N} \int_0^\infty \cdots \right) \frac{\delta(1 - h(x))}{U^{D/2 - \omega} F^{\omega}}$$

Lemma. (graph polynomials)

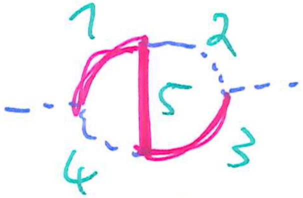
k_e = momentum flowing through edge e

If $Q_e = m_e^2 - k_e^2$ is given by a graph, then

- $$\mathcal{U} = \sum_{T:\ \text{spanning tree}} \ \prod_{e \notin T} x_e$$



(not connected/spanning) (has a loop)

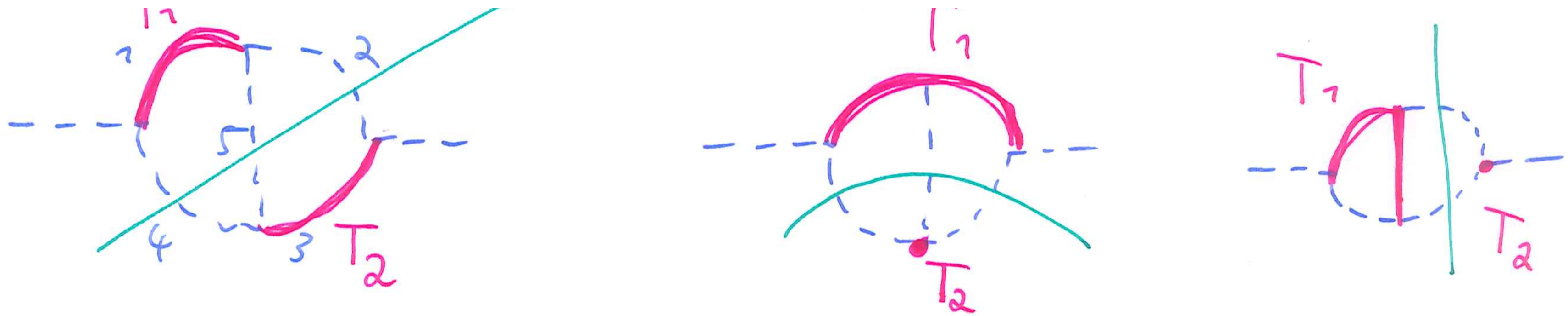
Ex: 

Tree ✓

$$\mathcal{U} = x_2 x_4 + \dots$$

(exercise: compare with det A)

- $$\mathcal{F} = \sum_{\substack{F:\ \text{spannig 2-forest} \\ = T_1 \sqcup T_2}} (-p_F)^2 \prod_{x \notin F} x_e \ + \ \mathcal{U} \cdot \sum_{e=1}^{N} x_e m_e^2$$

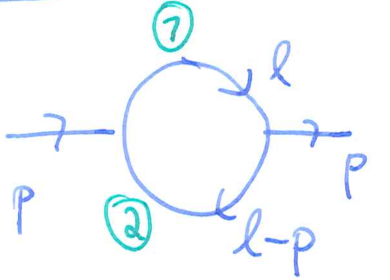


$$F = (-p^2)\, x_2 x_4 x_5 + \dots + 0 \cdot x_3 x_4 x_5 + (-p)^2 x_2 x_3 x_4 + \dots$$

Questions: 1) When does $I(n, D, z)$ converge?

2) Where are its singularities?

2') What type are the singularities?



$(m_1 = m_2 = 0)$

$$I(n, D, p^2) = \int_{\mathbb{R}^{1,D-1}} \frac{d^D l}{i\pi^{D/2}} \frac{1}{(-l^2 - i\varepsilon)^{n_1} \, (-(l-p)^2 - i\varepsilon)^{n_2}}$$

$$\underset{\substack{\uparrow \\ \text{proj. rep.} \\ h(x) = x_2}}{=} \frac{\Gamma(\overset{n_1+n_2-D/2}{\overset{\parallel}{\omega}})}{\Gamma(n_1)\,\Gamma(n_2)} \int_0^\infty \frac{x_1^{n_1-\omega-1} \, dx_1}{(x_1+1)^{D/2-\omega} \, (-p^2)^{\omega}}$$

$$\begin{pmatrix} \mathcal{U} = x_1 + x_2 \\ F = (-p)^2 x_1 x_2 \end{pmatrix}$$

This integral converges only when

- $\mathrm{Re}\,(n_1 - \omega) = \mathrm{Re}\,(D/2 - n_2) > 0$ $\quad (x_1 \to 0)$

- $\mathrm{Re}\,(n_1 - D/2) < 0$ $\quad (x_1 \to \infty)$

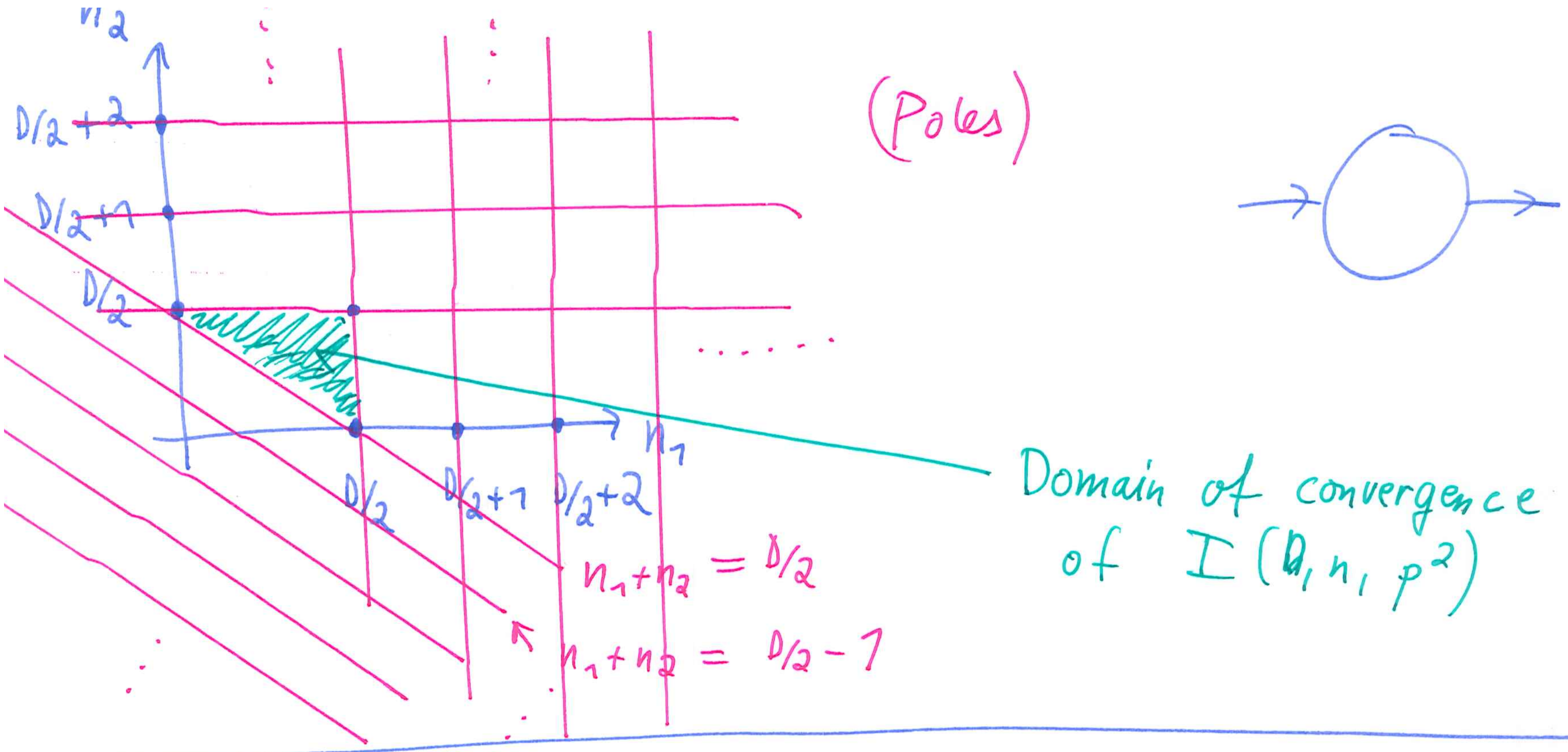
$$I(n, D, p^2) = (-p^2)^{-\omega} \frac{\Gamma(D/2 - n_1)\,\Gamma(D/2 - n_2)\,\Gamma(n_1 + n_2 - D/2)}{\Gamma(n_1)\,\Gamma(n_2)\,\Gamma(D - n_1 - n_2)}$$

$\Rightarrow$ • meromorphic continuation to all $\mathbb{C}^3 \ni (D, n_1, n_2)$

($\Rightarrow$ single-valued)

• simple poles on $D/2 - n_1,\ D/2 - n_2,\ n_1 + n_2 - D/2 = 0, -1, -2, \ldots$

(Poles)



Questions Given any Feynman integral,

① What is its domain of convergence? ⇒ power counting

② What is its analytic continuation? → integration by parts

[How to get expressions for residues / Laurent expansion]