Polylogs, elliptic polylogs, etc

The Amplitude Games

MITP School 2021

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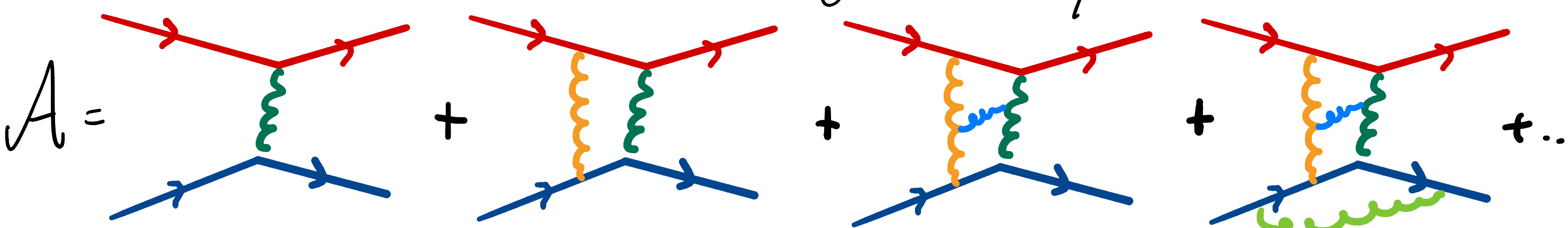
$\Rightarrow$ We want to compute scattering amplitudes with external gluon and quark states!

$\Rightarrow$ We only know how to do that in Perturbation Theory (PT):

$\Rightarrow$ E.g. 2-to-2 scattering of quarks:

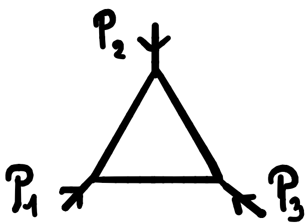


Tree-Level	1-Loop	2-Loops	3-Loops
EASY	NOT EASY	HARD	VERY HARD

2. Feynman Integrals

- Beyond tree-level, we need to compute integrals.

- <u>Example</u>



$$\sim \int \frac{d^4k}{k^2 (k+p_1)^2 (k+p_1+p_2)^2} = \,?$$

$p_1 + p_2 + p_3 = 0$

N.B.: In general, such integrals diverge and need to be regularised.

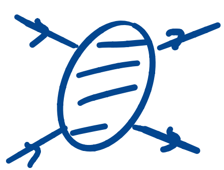
⇒ What does this integral evaluate to?

⇒ Lorentz invariance: Function of ratios:

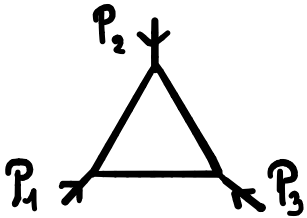
$$u = p_1^2 / p_3^2 \qquad v = p_2^2 / p_3^2$$

⇒ Is it a simple rational funtion of u and v?

⇒ No! ⟶ Unitarity:

Disc  $\sim 2\pi i$ 

Intermediate Particles on-shell



$$\sim \frac{2}{z-\bar{z}}\left[\mathrm{Li}_2(z)-\mathrm{Li}_2(\bar{z})+\log(z\bar{z})\log\frac{1-z}{1-\bar{z}}\right]$$

$$u = z\bar{z} \qquad v = (1-z)(1-\bar{z})$$

Log(arithm)

$$\log x = \int_1^x \frac{dt}{t} \qquad \log(1-x) = \int_0^x \frac{dt}{t-1}$$

Recursive structure!

Dilog(arithm)

$$\mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t}\log(1-t) = \int_0^x \frac{dt}{t}\int_0^t \frac{dt'}{1-t'}$$

Polylog(arithm):

$$\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t}\, \mathrm{Li}_{n-1}(t) \qquad \mathrm{Li}_1(x) = -\log(1-x)$$

These are all examples of

$$\boxed{\text{Iterated integrals}}$$

Goal of this lecture:

Review / define / study iterated integrals that show up in multiloop computations:

- Multiple polylogs
- elliptic multiple polylogs
- iterated integrals of modular forms

N.B. Strong overlap/connections with lectures by Ruth Britto & Erik Panzer.

Multiple polylogarithms

$$G(a_1,\dots,a_n;x) = \int_0^x \frac{dt}{t-a_1}\, G(a_2,\dots,a_n;t) \quad [n = \text{weight}]$$

N.B: If $a_n = 0$, the integral diverges

⇝ tangential base point [See Erik's lecture]

⇝ This leads to the "rule":

$$G(\underbrace{0,\dots,0}_{n \text{ times}};x) = \frac{1}{n!}\log^n x$$

- MPLs contain log and Li_n as special cases:

$$G(a;z) = \log\left(1-\frac{z}{a}\right) \qquad a \neq 0$$

$$G(\underbrace{0,\dots,0}_{n-1},1;z) = -\mathrm{Li}_n(z)$$

$$G(\underbrace{0,\dots,0}_{p},\underbrace{1,\dots,1}_{q};z) = (-1)^q S_{p,q}(z)$$

- Up to weight 3, all MPLs can be expressed in terms of classical polylogs and log

$\rightsquigarrow$ Fails from weight 4.

• MPLs are examples of iterated integrals

⇝ They have all properties of iterated integrals

⇝ In particular they form a shuffle algebra

Example:

$$G(a;z)\,G(b;z) = G(a,b;z) + G(b,a;z)$$

$$G(a;z)\,G(b,c;z) = G(a,b,c;z) + G(b,a,c;z) + G(b,c,a;z)$$

$$G(a,b;z)\,G(c,d;z) = G(a,b,c,d;z) + G(a,c,b,d;z) + G(a,c,d,b;z) + G(c,a,b,d;z) + G(c,a,d,b;z) + G(c,d,a,b;z)$$

<u>Property</u>: The multiplication of MPLs preserves the weight.

$\mathcal{A}_n$ = "$\mathbb{Q}$-vector space spanned by all MPLs of weight n"

$\mathcal{A}_0 = \mathbb{Q}$

$\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n$ = "$\mathbb{Q}$-vector space of all MPLs"

$\mathcal{A}$ is graded algebra!

$\mathcal{A}_m \cdot \mathcal{A}_n \subseteq \mathcal{A}_{m+n}$

3.2. The Coaction

- A is an algebra: vect. sp. with multipl. $\mu: A\otimes A \to A$

Associativity

$$\begin{tikzcd} A\otimes A\otimes A \arrow[r, "\mu\otimes id"'] \arrow[d, "id\otimes\mu"] & A\otimes A \arrow[d, "\mu"] \\ A\otimes A \arrow[r, "\mu"] & A \end{tikzcd}$$

$a\otimes b\otimes c$, $(ab)\otimes c$, $a\otimes(bc)$, $(ab)c = a(bc)$

Co-Associativity

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A\otimes A \arrow[d, "\Delta\otimes id"] \\ A\otimes A \arrow[r, "id\otimes\Delta"] & A\otimes A\otimes A \end{tikzcd}$$

- A is a coalgebra: vect. sp. with coproduct $\Delta: A \to A\otimes A$
- Bialgebra / Hopf algebra: $\Delta(a\cdot b) = \Delta(a)\cdot\Delta(b)$

- MPLs form a shuffle algebra.
- MPLs form a Hopf algebra [GONCHAROV]

<u>Example:</u> $\Delta(\log x) = 1 \otimes \log x + \log x \otimes 1$

$$\Delta(\log x \log y) = 1 \otimes (\log x \log y) + \log x \otimes \log y + \log y \otimes \log x + (\log x \log y) \otimes 1$$

$$\Delta(\mathrm{Li}_2(x)) = 1 \otimes \mathrm{Li}_2(x) - \log(1-x) \otimes \log x + \mathrm{Li}_2(x) \otimes 1$$

$$\Delta(\mathrm{Li}_n(x)) = 1 \otimes \mathrm{Li}_n(x) + \mathrm{Li}_n(x) \otimes 1 + \sum_{k=1}^{n-1} \mathrm{Li}_{n-k}(x) \otimes \frac{\log^k x}{k!}$$

$$\Delta\left(\mathrm{Li}_n(x)\right) = \underbrace{1 \otimes \mathrm{Li}_n(x)}_{1 \otimes \zeta_n} + \underbrace{\mathrm{Li}_n(x) \otimes 1}_{\zeta_n \otimes 1} + \underbrace{\sum_{k=1}^{n-1} \mathrm{Li}_{n-k}(x) \otimes \frac{\log^k x}{k!}}_{\cdots \otimes 0 = 0}$$

$$\mathrm{Li}_n(1) = \zeta_n = \sum_{k=1}^{\infty} \frac{1}{k^n}$$

$$\Delta(\zeta_n) = 1 \otimes \zeta_n + \zeta_n \otimes 1$$

$$\zeta_2 = \frac{\pi^2}{6} \qquad \zeta_4 = \frac{\pi^4}{90} \rightsquigarrow \zeta_4 = \frac{2}{5}\zeta_2^2$$

$$\rightsquigarrow \Delta(\zeta_4) = \frac{2}{5}\Delta(\zeta_2)^2$$

$$= 1 \otimes \zeta_4 + \zeta_4 \otimes 1 + \frac{4}{5}\,\zeta_2 \otimes \zeta_2$$

???

⚠ We need $\zeta_2 \otimes \zeta_2 = 0$

1) Work "mod π": $\zeta_2 = \frac{\pi^2}{6} = 0 \mod \pi$

2) Work "mod π" only in second entry [Brown]

This gives:

$$\Delta(\zeta_2) = \zeta_2 \otimes 1, \quad \Delta(\zeta_4) = \zeta_4 \otimes 1, \ldots$$

$$\Delta(\underbrace{i\pi}_{= \log(-1+i0)}) = i\pi \otimes 1$$

Technical interlude

$$\Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes \underbrace{\mathcal{H}}_{= \text{"}\mathcal{A} \bmod \pi\text{"}} \qquad \text{[Brown]}$$

Δ : Coaction $\qquad$ $\mathcal{A}$: comodule

$$\Delta : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$$

Δ : Coproduct $\qquad$ $\mathcal{H}$: Hopf algebra

Symbols

Recap: Coaction: Decompose an MPL into MPLs of lower weights.

Example:

$$\underset{2}{\Delta(\mathrm{Li}_2(x))} = \underset{(0,2)}{1 \otimes \mathrm{Li}_2(x)} + \underset{(2,0)}{\mathrm{Li}_2(x) \otimes 1} - \underset{(1,1)}{\log(1-x) \otimes \log x}$$

$$\underset{3}{\Delta(\mathrm{Li}_3(x))} = \underset{(3,0)}{\mathrm{Li}_3(x) \otimes 1} + \underset{(0,3)}{1 \otimes \mathrm{Li}_3(x)} + \underset{(2,1)}{\mathrm{Li}_2(x) \otimes \log x} - \underset{(1,2)}{\frac{1}{2}\log(1-x) \otimes \log^2 x}$$

Symbol $=$ $(1,1,\dots,1)$ part of the coaction

$\rightsquigarrow$ "Invariant" attached to a polylog

Examples:

$$\mathcal{S}(\log x) = \log x \to x$$

$$\mathcal{S}(\mathrm{Li}_2(x)) = -\log(1-x) \otimes \log x \to -(1-x) \otimes x$$

$$\mathcal{S}(\mathrm{Li}_n(x)) = -\log(1-x) \otimes \underbrace{\log x \otimes \dots \otimes \log x}_{n-1 \text{ times}}$$

$$\to -(1-x) \otimes x \otimes \dots \otimes x$$

- Properties of $\mathcal{S}$ are inherited from coaction:

$$\ldots \otimes (a \cdot b) \otimes \ldots = \ldots \otimes a \otimes \ldots + \ldots \otimes b \otimes \ldots$$

$[\log(a \cdot b) = \log a + \log b]$

$$\ldots \otimes (\pm 1) \otimes \ldots = 0$$

$[\log(\pm 1) = 0 \mod \pi]$

$$\mathcal{S}(\zeta_n) = 0$$

$[\Delta(\zeta_n) = 1 \otimes \zeta_n + \zeta_n \otimes 1]$

↳ no non-trivial decomposition

$\mathcal{S}(F \cdot G) =$ Shuffle of $\mathcal{S}(F)$ and $\mathcal{S}(G)$

E.g: $\mathcal{S}(\log x \log y) = x \otimes y + y \otimes x$

Consider a Rational function with poles at most at $x = a_1, \dots, a \in \mathbb{C}$.

$$R(x) = \frac{P(x)}{(x-a_1)^{n_1} \cdots (x-a_m)^{n_m}}$$

$n_i \in \mathbb{N}$,

P = polynomial

How can we compute $\int R(x)\,dx$?

- Step 1: Partial fractioning

$\rightsquigarrow$ only consider x^k, $\dfrac{1}{(x-a_i)^k}$, $k \geq 0$

- Step 2: Compute primitives

$$\int dx\, x^k = \frac{1}{k+1} x^{k+1}$$

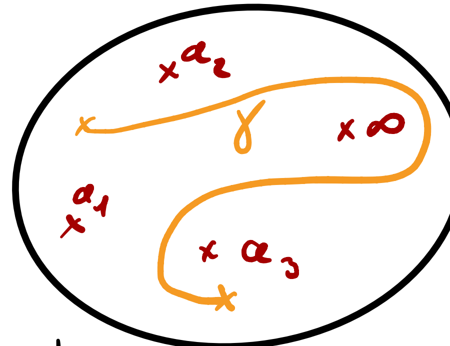
$$\int \frac{dx}{(x-a_i)^k} = \begin{cases} \frac{1}{1-k} \frac{1}{(x-a_i)^{k-1}} , & k \neq 1 \\ \log(x-a_i) , & k = 1 \end{cases}$$

Conclusion:

* $\int R(x)\,dx$ only involves rational functions and logarithms.
* If you iterate this (e.g. by integrating over α_1), you get rational functions, logarithms, and dilogarithms.
* ...

Geometrical picture:

* Consider Riemann sphere with the points $a_1, \ldots, a_m, \infty$ removed

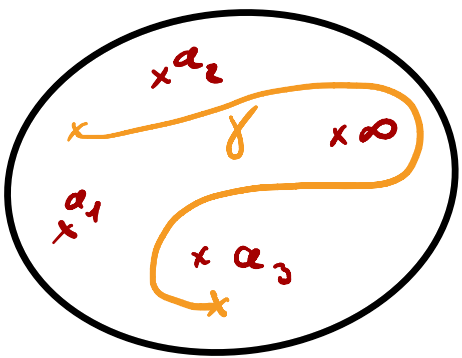


* Meromorphic functions on $\mathbb{CP}^1 / \{a_1, \ldots, a_m, \infty\}$

  = Rational functions with poles at most at $a_1, \ldots, a_m$

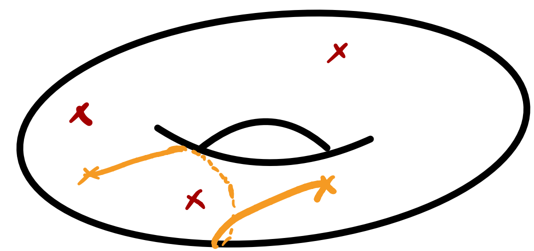
* MPLs: Arise from iterated integrations on $\mathbb{CP}^1 / \{a_1, \ldots, a_m, \infty\}$

We can repeat this construction with other "geometric spaces", e.g:



Punctured sphere

$\rightsquigarrow$



Punctured torus

torus = Elliptic curve

Elliptic curves:

Consider 2-dimensional complex space with coordinates (x,y).

$\rightsquigarrow$ Consider hypersurface

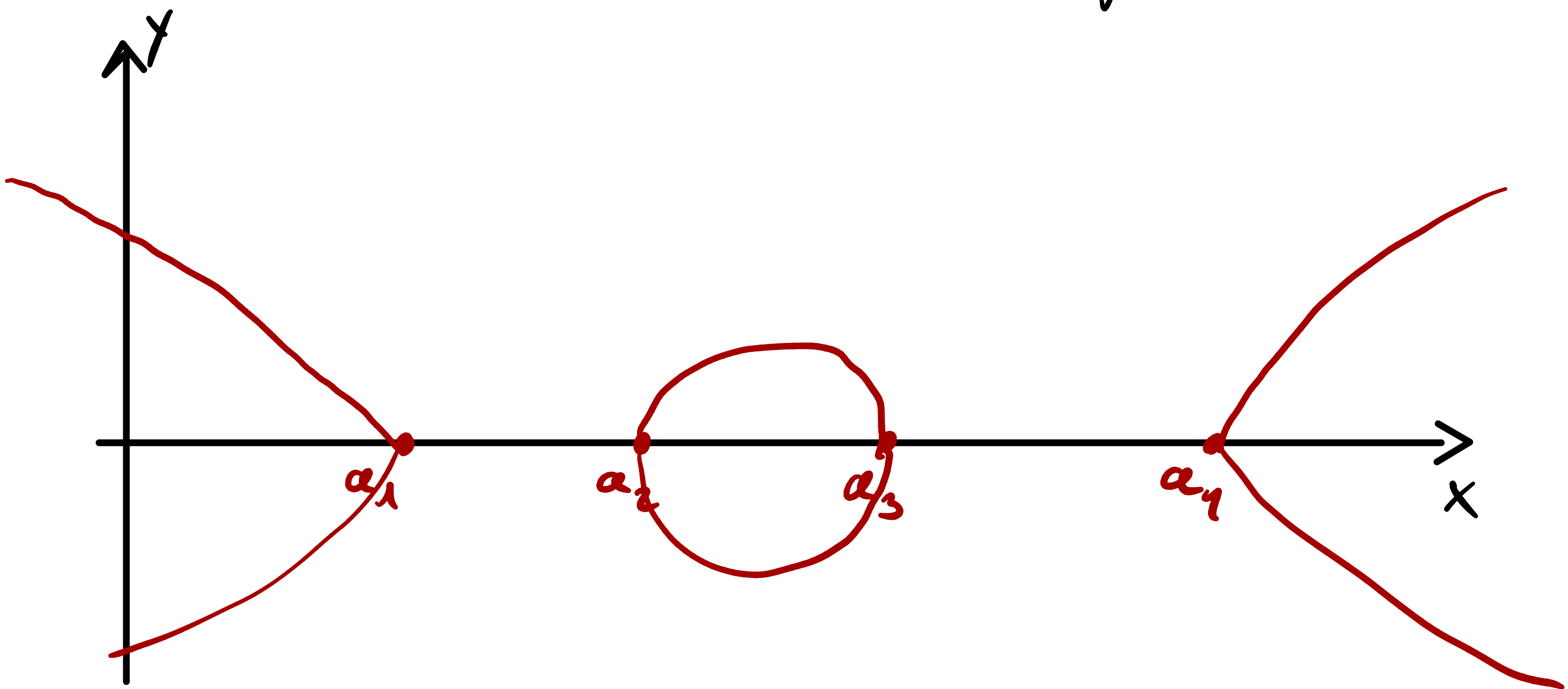
$$y^2 = P_4(x) = (x-a_1)\dots(x-a_4)$$

N.B: Can always change coordinates s.t. $a_4 = \infty$.

$\rightsquigarrow$ Everything also applies to cubics.

Example: Assume $a_1 < a_2 < a_3 < a_4$ real.

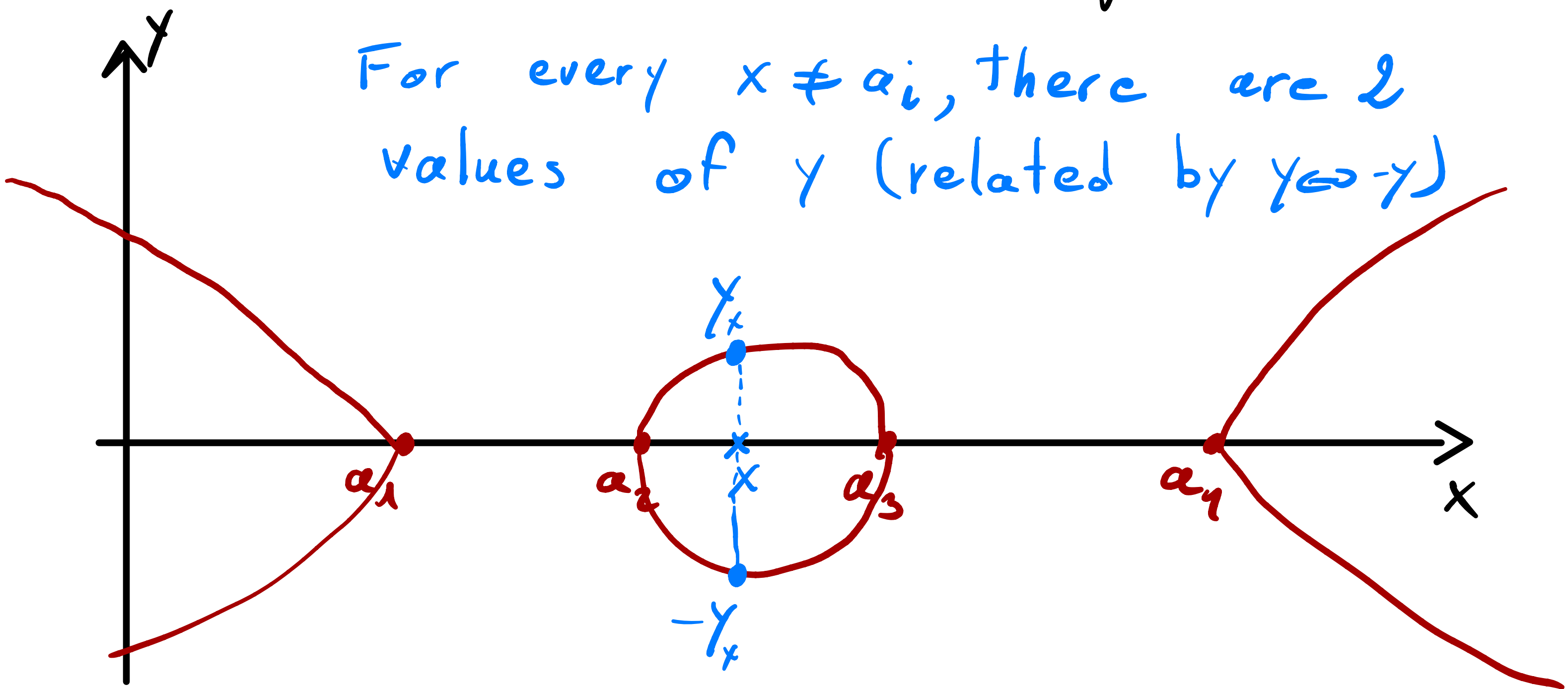
$\rightsquigarrow$ We can draw the real points



Example: Assume $a_1 < a_2 < a_3 < a_4$ real.

$\rightsquigarrow$ We can draw the real points

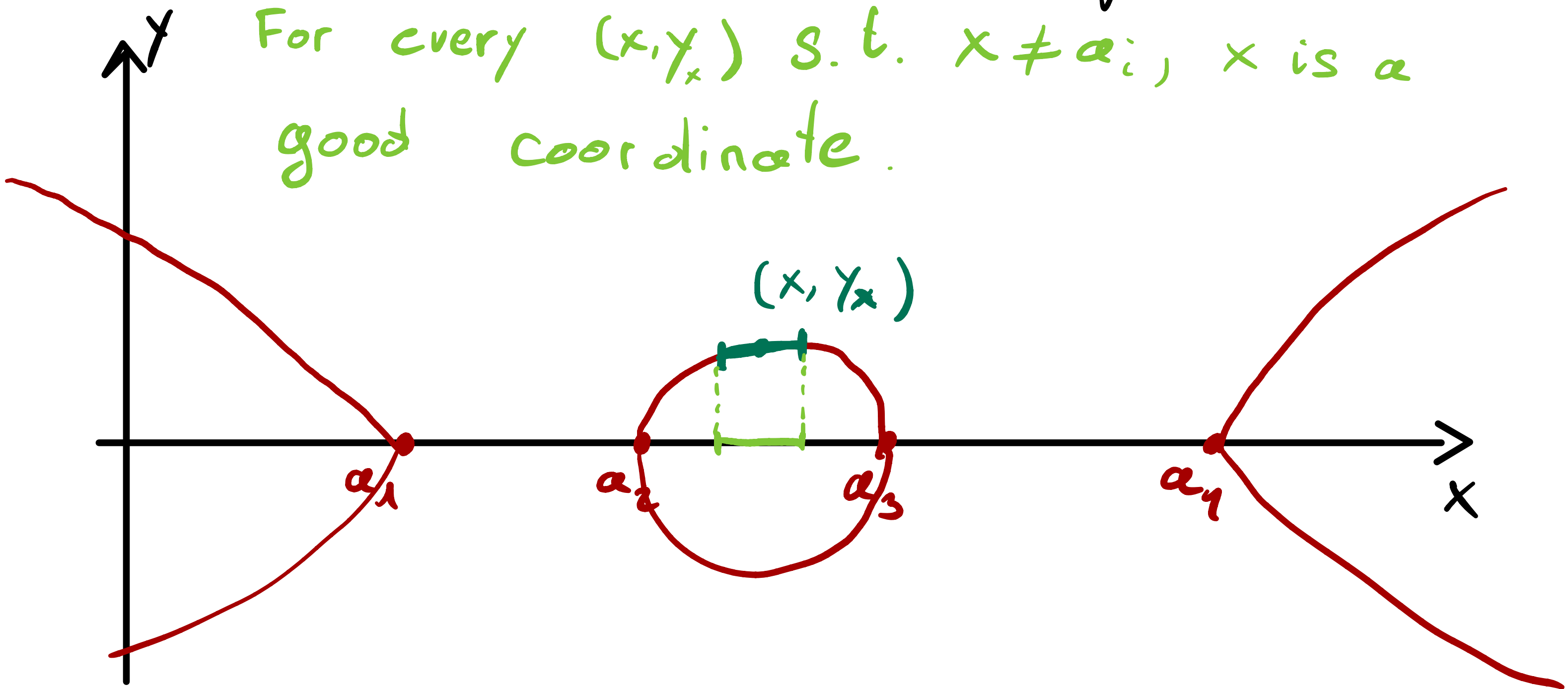
For every $x \neq a_i$, there are 2 values of y (related by $y \leftrightarrow -y$)



Example: Assume $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ real.

$\rightsquigarrow$ We can draw the real points

For every (x, y_x) s.t. $x \neq \alpha_i$, x is a good coordinate.

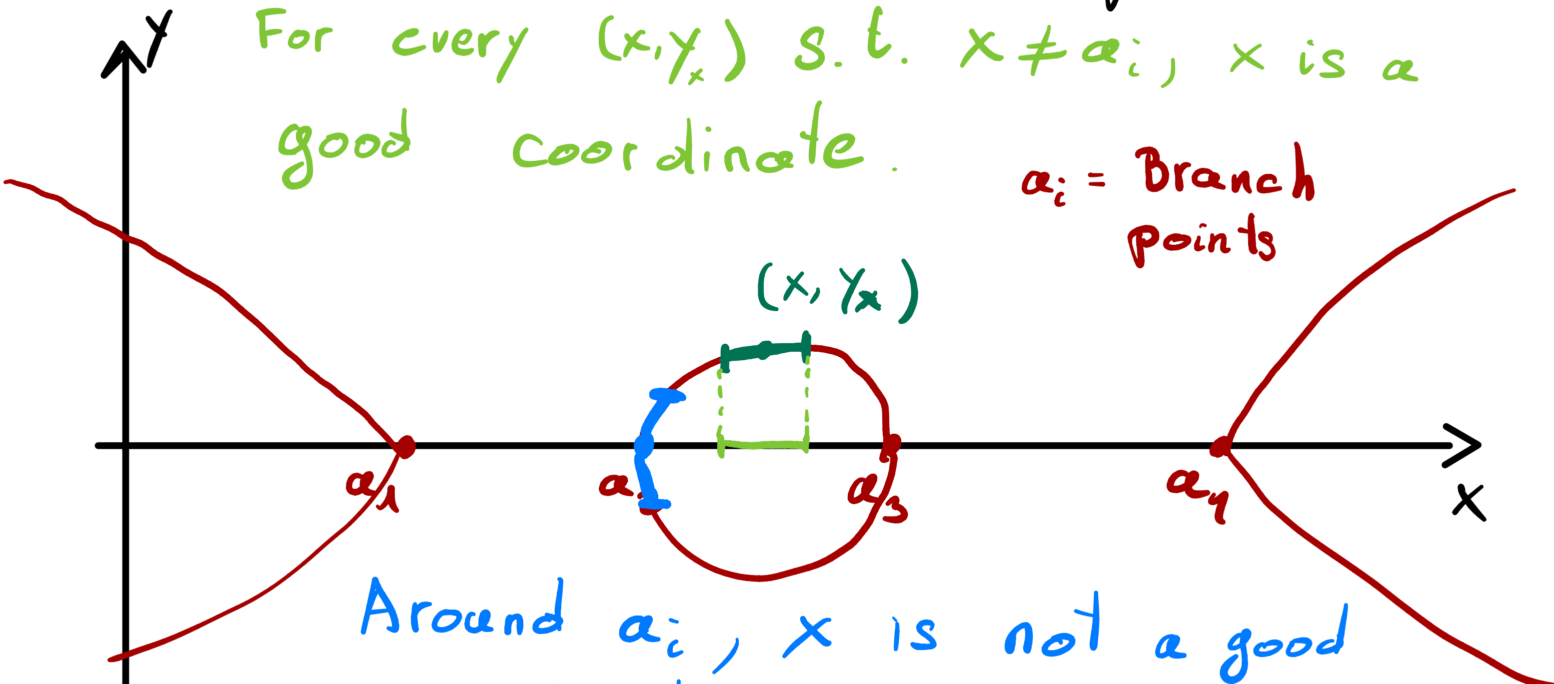


Example: Assume $a_1 < a_2 < a_3 < a_4$ real.

$\rightsquigarrow$ We can draw the real points

For every (x, y_x) s.t. $x \neq a_i$, x is a good coordinate.

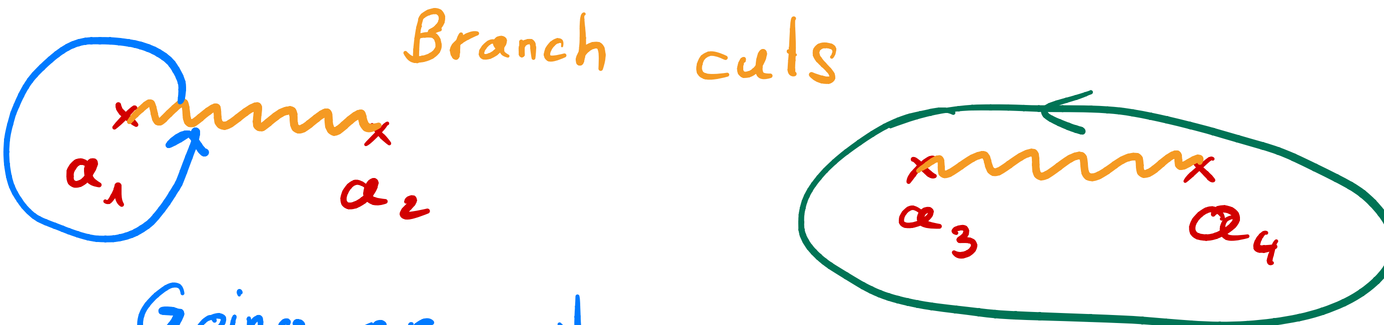
a_i = Branch points



Around a_i, x is not a good coordinate

Picture in the complex x-plane:

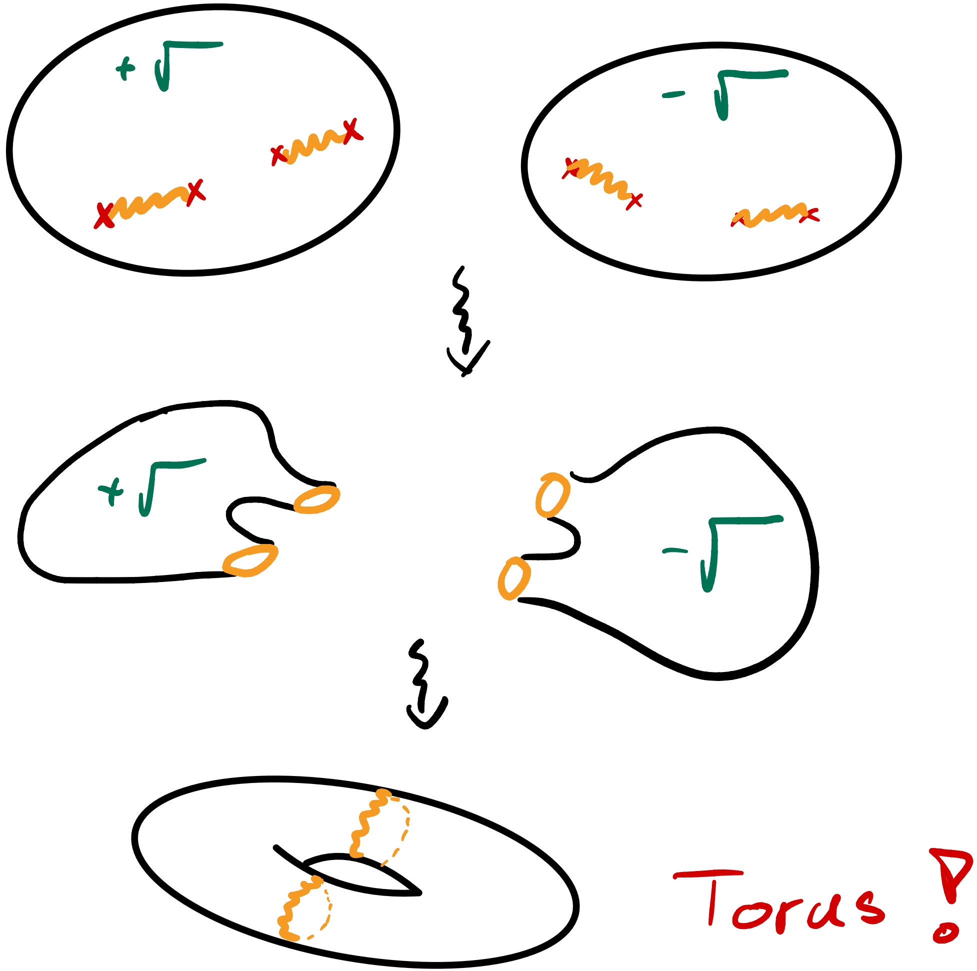
$$y_x = \sqrt{(x-a_1)(x-a_2)(x-a_3)(x-a_4)}$$



Going around a single a_i changes sign of $\sqrt{}$

$$y_x \longleftrightarrow -y_x$$

Going arond 2 a_i's does not change sign


+√
−√
+√
−√
Torus ?

* Meromorphic functions on an elliptic curve:

⇝ Rational functions in (x,y)

⇝ Subject to the constraint $y^2 = P_4(x)$.

⇝ $$R(x) = \frac{P_1(x) + P_2(x)\sqrt{P_4(x)}}{Q_1(x) + Q_2(x)\sqrt{P_4(x)}} = R_1(x) + \frac{1}{y_x} R_2(x)$$

⇝ $$\int dx\, R(x) = \underbrace{\int dx\, R_1(x)}_{\text{As before: rational + Log's}} + \underbrace{\int \frac{dx}{y} R_2(x)}_{\text{New piece}}$$

Partial fractioning $\rightsquigarrow$ only need to consider

$$\frac{x^m}{y} \qquad \frac{1}{y(x-c)^m}$$

* Here: Cubic case (easier to write formulas):

$$y^2 = (x-a_1)(x-a_2)(x-a_3)$$

$$\partial_x\left(y\, x^{m-2}\right) = \frac{1}{2}\sum_{\ell=0}^{3}(-1)^\ell \underbrace{S_\ell(a_1,a_2,a_3)}_{\text{Elem. Symmetric polynomial}}(2m-\ell-1)\frac{x^{m-\ell}}{y}$$

$S_0 = 1$, $S_1 = a_1+a_2+a_3$, $S_2 = a_1a_2+a_1a_3+a_2a_3$ $S_3 = a_1a_2a_3$

$$\partial_x \left(y\, x^{m-2} \right) = \frac{1}{2} \sum_{\ell=0}^{3} (-1)^{\ell}\, S_{\ell}(a_1, a_2, a_3)\,(2m-\ell-1)\, \frac{x^{m-\ell}}{y}$$

$\rightsquigarrow$ Acting with $\int dx$ gives recursion of depth 3 for $\int \frac{dx}{y} x^m$

$\rightsquigarrow$ Can express all such integrals in terms of 3 of them:

$$\frac{dx}{y} \qquad \frac{x\,dx}{y} \qquad \frac{dx}{x\,y}$$

* Similar reasoning can be applied to $\frac{1}{Y(x-a)^m}$.
* All integrals can be reduced to

$$\frac{dx}{Y} \qquad \frac{x\,dx}{Y} \qquad \frac{dx}{Y(x-c)} \qquad \frac{dx}{x-c}$$

Differential of 1st kind (no pole)

2nd kind (pole with vanishing Res)

$c \neq a_i$

3rd kind (Pole with Residue)

$$K(\lambda) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\lambda x^2)}}$$

$$E(\lambda) = \int_0^1 dx \sqrt{\frac{1-\lambda x^2}{1-x^2}}$$

$\Pi(n|\lambda)$ $\qquad$ $\log(x-c)$

Summary:

* Riemann sphere $\xrightarrow{\int dx}$ rational, log

  $\xrightarrow{\text{Iterated}}$ rational, MPLs

* Elliptic curve

  $\xrightarrow{\int dx}$ rational $R(x,y)$, log, K, E, Π

  $\xrightarrow{\text{Iterated}}$ rational $R(x,y)$, MPLs, elliptic MPLs

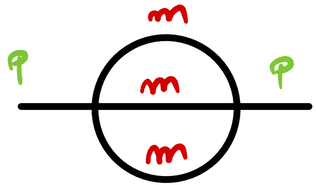
Corollary:

* MPLs are a subset of eMPLs

* $G(a_1(x,y),\dots a_n(x,y);z(x,y))$ are eMPLs

Proof: Differentiate and integrate back.

Example : The 2-loop equal-mass sunrise :



$$= \int \frac{d^2k\, d^2\ell}{(k^2-m^2)(\ell^2-m^2)((k+\ell-p)^2-m^2)}$$

$$= -\int_0^\infty \frac{dx_1\, dx_2\, dx_3\ \delta(1-x_3)}{\mathcal{F}(x_1,x_2,x_3)}$$

$$\mathcal{F} = (-p^2)\, x_1 x_2 x_3 + m^2 (x_1+x_2+x_3)(x_1x_2+x_1x_3+x_2x_3)$$

With $x_1 = x/(1-x)$, we get

$$\frac{1}{m^2-p^2}\int_0^1 \frac{dx}{y}\, \log\frac{x(1-x)t+t+(t+1)y}{x(1-x)t+t-(t+1)y}$$

$$t = m^2/(-p^2)$$

$$y^2 = P_4(x)$$

$$\frac{1}{m^2-p^2}\int_0^1 \frac{dx}{y}\,\log\frac{x(1-x)t+t+(t+1)y}{x(1-x)t+t-(t+1)y}$$

$$t = m^2/(-p^2)$$

$$y^2 = P_4(x)$$

$$a_1(t) = \frac{1}{2}\left(1-\sqrt{1+\rho}\right) \qquad a_2(t) = \frac{1}{2}\left(1+\sqrt{1+\rho}\right)$$

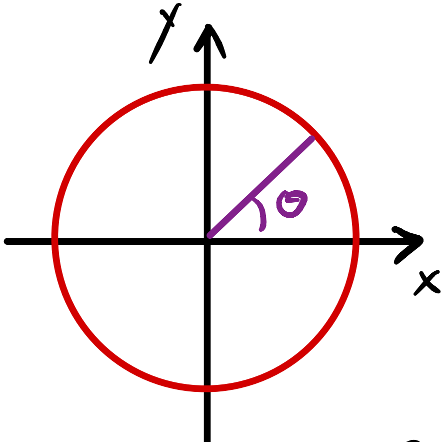
$$a_3(t) = \frac{1}{2}\left(1-\sqrt{1+\bar{\rho}}\right) \qquad a_4(t) = \frac{1}{2}\left(1+\sqrt{1+\bar{\rho}}\right)$$

$$\rho = \frac{-4t}{(1+\sqrt{t})^2} \qquad \bar{\rho} = \frac{-4t}{(1-\sqrt{t})^2}$$

$\rightsquigarrow$ Integral can be done in terms of eMPLs!

N.B.: Kinematics t determines "shape" of elliptic curve.

Interlude: Consider the circle $y^2 = 1 - x^2$



Local coordinate = angle Θ

Map: $[0,2\pi[\longrightarrow$ circle

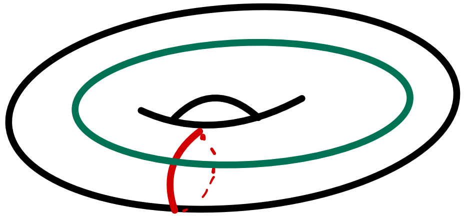
$$\Theta \longmapsto \begin{aligned} x &= \sin\Theta \\ y &= (\sin\Theta)' = \cos\Theta \end{aligned}$$

$\rightsquigarrow$ Constraint $y^2 = 1 - x^2 \longleftrightarrow$ DEQ for sin

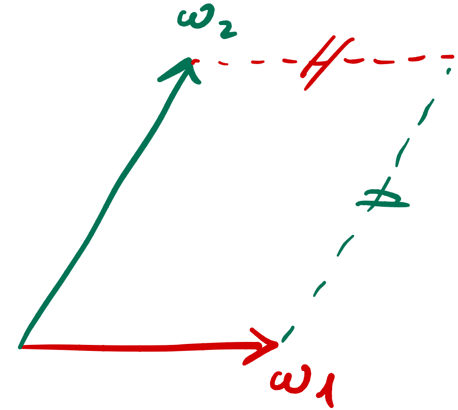
$\rightsquigarrow$ inverse function

$$\Theta = \int \frac{dx}{\sqrt{1-x^2}} = \int \frac{dx}{y}$$

The torus



$\rightsquigarrow$



ω_i = periods.

$\rightsquigarrow$ we can rescale $(\omega_1, \omega_2) \longrightarrow (1, \tau)$

$\rightsquigarrow$ Consider the lattice : $\Lambda = \mathbb{Z} \oplus \tau \mathbb{Z}$

Torus = $\mathbb{C}/\Lambda$

Meromorphic functions on a torus

Def: An *elliptic function* is a meromorphic function that is doubly periodic:

$$f(z) = f(z+1) = f(z+\tau)$$

Q: Example of such a function?

Does it exist at all?

Assume for now that it exists...

Claim 1: $\#(\text{poles in } \square) > 1$

(counted with multiplicities), or it is constant.

Claim 2: $\#(\text{poles in } \square) = \#(\text{zeroes in } \square)$

Proof: Homework

Hint: Consider $\int_{\square} f(z)\,dz$ and $\int_{\square} \frac{f'(z)}{f(z)}\,dz$

Example of an elliptic function:

Weierstrass $\wp$-function

$$\wp(z;\tau) = \underbrace{\frac{1}{z^2}}_{\text{2 poles in } \square} + \underbrace{\sum_{(m,n)\in\mathbb{Z}^2/(0,0)} \left(\frac{1}{(z+m+n\tau)^2}\right.}_{\text{periodicity}} - \left.\underbrace{\frac{1}{(m+n\tau)^2}}_{\text{required for convergence}} \right)$$

$$\wp(-z,\tau) = \wp(z,\tau)$$

Other examples: $\wp'$, $\wp''$, $\wp'''$, ...

Theorem: There are $g_2, g_3 \in \mathbb{C}$ s.t.

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3$$

Theorem: Every elliptic function is a rational function in $\wp$ and $\wp'$.

* Consider the map:

$$\mathbb{C}/\Lambda \longrightarrow \mathbb{C}^2$$
$$z \longmapsto (x,y) = (\wp(z), \wp'(z))$$

The image of $\mathbb{C}/\Lambda$ in $\mathbb{C}^2$ satisfies

$$y^2 = \wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 = 4x^3 - g_2x - g_3$$

$\rightsquigarrow$ elliptic curve!

N.B: For every elliptic curve we can change coordinates s.t. $y^2 = 4x^3 - g_2 x - g_3$.

N.B.: $z \in \Lambda \longrightarrow x = \infty$, because $\wp(z)$ has a pole at $z \in \Lambda$!

Differentials:

$$\frac{dx}{y} = \frac{d\wp(z)}{\wp'(z)} = \frac{\wp'(z)\,dz}{\wp'(z)} = dz\,!$$

$$\frac{x\,dx}{y} = \wp(z)\,dz = dz\Big(\underbrace{\frac{1}{z^2} + \frac{0}{z}}_{\text{double pole without residue}} + O(z^0)\Big)$$

* Inverse map

$$(x_0, y_0) \longmapsto z_0 = \int_{\infty}^{x_0} \frac{dx}{y} \bmod \Lambda$$

[cf. definition of arcsin!]

Indeed, with $x_0 = \wp(z_0, \tau)$,

$$\int_{\infty}^{x_0} \frac{dx}{y} = \int_0^{z_0} dz = z_0 - 0 = z_0 \,!$$

The Kronnecker function

Def:

$$F(z,\alpha,\tau) = \frac{\theta_1'(0,\tau)\,\theta_1(z+\alpha,\tau)}{\theta_1(z,\tau)\,\theta_1(\alpha,\tau)}$$

$$= \frac{1}{\alpha} \sum_{n \geq 0} g^{(n)}(z,\tau)$$

$$\theta_1(z,\tau) = 2iq^{1/8} \sin(\pi z) \prod_{j=1}^{\infty} (1-q^j)(1-e^{2\pi i z} q^j) \times (1-e^{-2\pi i z} q^j)$$

Properties

1) $g^{(0)}(z,\tau) = 1$ & $g^{(n)}(-z;\tau) = (-1)^n g^{(n)}(z,\tau)$

2) $\partial_z g^{(1)}(z,\tau) = -\wp(z,\tau) + \text{constant}$

3) $g^{(n)}(z,\tau)$ = polynomial in $g^{(1)}, \wp, \wp'$

e.g.: $$g^{(2)}(z,\tau) = \frac{1}{2}\underbrace{g^{(1)}(z,\tau)^2}_{\sim\left(\frac{1}{z}\right)^2} - \frac{1}{2}\underbrace{\wp(z,\tau)}_{\sim\frac{1}{z^2}}$$

4) $g^{(1)}(z,\tau)$ has a simple pole for $z \in \Lambda$

$g^{(n)}(z,\tau)$ has a simple pole for $z \in \Lambda - \mathbb{R}$

5) Mixed-Heat equation

$$2\pi i\, \partial_\tau g^{(n)}(z,\tau) = n \cdot \partial_z g^{(n)}(z,\tau)$$

6) Fay identity ($\sim$ partial fractioning)

$$g^{(n_1)}(z_1,\tau)\, g^{(n_2)}(z_2,\tau) = -(-1)^{n_2} g^{(n_1+n_2)}(z_1-z_2,\tau)$$
$$+ \sum_{r=0}^{n_2} \binom{n_1+r-1}{n_1-1} g^{(n_2-r)}(z_2-z_1,\tau)\, g^{(n_1+r)}(z_1,\tau)$$
$$+ \sum_{r=0}^{n_1} \binom{n_2+r-1}{n_2-1} g^{(n_1-r)}(z_1-z_2;\tau)\, g^{(n_2+r)}(z_2;\tau)$$

cf $\frac{1}{(x-a)(x-b)} = \frac{1}{a-b}\frac{1}{x-a} + \frac{1}{b-a}\frac{1}{x-b}$ $\quad z_2 = x-a$, $z_1 = x-b$

Elliptic MPLs: $n_i \in \mathbb{N}$, $z_i \in \mathbb{C}$, $z \in \mathbb{C}$

$$\tilde{\Gamma}\begin{pmatrix} n_1 & \cdots & n_k \\ z_1 & \cdots & z_k \end{pmatrix}; z, \tau\Big)$$

$$= \int_0^z dz' \, g^{(n_1)}(z' - z_1, \tau) \, \tilde{\Gamma}\begin{pmatrix} n_2 & \cdots & n_k \\ z_2 & \cdots & z_k \end{pmatrix}; z', \tau\Big)$$

N.B.: * All properties of iterated integrals

Hold

* singularity at $z=0 \rightsquigarrow$ tangential base-point

(see Erik's lecture)

* What has this to do with $\frac{dx}{x-c}$, $\frac{dx}{y(x-c)}$?

* Let z_c be s.t. $c = \wp(z_c;\tau)$.

unit residue!

$$\frac{dx\, y_c}{y(x-c)} = dz \underbrace{\frac{\wp'(z_c;\tau)}{\wp(z,\tau) - \wp(z_c;\tau)}}_{= f(z)}$$

- Zeroes of $f(z)$: $f(z) = 0 \iff \wp(z,\tau) = \infty$
  $\iff z = 0 \bmod \Lambda$

  $\rightsquigarrow \exists$ 2 zeroes, and so 2 poles!

- Poles of f: $z = \pm z_c \bmod \Lambda$

We have

$$\wp(z;\tau) = \wp(\pm z_c;\tau) + \wp'(\pm z_c;\tau)(z \mp z_c) + \dots$$

$$= \wp(z_c;\tau) \pm \wp'(z_c,\tau)(z \mp z_c) + \dots$$

$$\rightsquigarrow \frac{dx\, y_c}{y(x-c)} = \frac{\pm dz}{z \mp z_c} + \dots$$

$\rightsquigarrow$ Simple pole with residue ± 1 at $z = \pm z_c$

Consider

$$\tilde{f}(z) = f(z) - \Big(\underbrace{g^{(1)}(z - z_c, \tau)}_{= \frac{+1}{z - z_c} + \dots} \underbrace{- g^{(1)}(z + z_c, \tau)}_{- \frac{1}{z + z_c} + \dots} \Big)$$

$\rightsquigarrow$ $\tilde{f}(z)$ is elliptic and has no poles.

$\rightsquigarrow$ $\tilde{f}(z) = \text{constant}$

$$\lim_{z \to 0} \tilde{f}(z) = 2 g^{(1)}(z_c, \tau) + \underbrace{\lim_{z \to 0} f(z)}_{= 0} = 2 g^{(1)}(z_c, \tau)$$

$$\frac{dx \, Y_c}{Y(x - c)} = dz \left(g^{(1)}(z - z_c, \tau) - g^{(1)}(z + z_c, \tau) - 2 g^{(1)}(z_c, \tau) \right)$$

Dictionary:

$$\frac{dx\, y_c}{y(x-c)} = dz\left(g^{(1)}(z-z_c,\tau) - g^{(1)}(z+z_c,\tau) - 2g^{(1)}(z_c,\tau)\right)$$

$$\frac{dx}{x-c} = dz\left(g^{(1)}(z-z_c,\tau) + g^{(1)}(z+z_c,\tau) - 2g^{(1)}(z,\tau)\right)$$

$$\frac{dx}{y} = dz = dz\, g^{(0)}(z,\tau)$$

$$\frac{x\,dx}{y} = \wp(z,\tau)\,dz = -dz\,\partial_z g^{(1)}(z,\tau)$$

Summary:

- Iterated integrals on Riemann sphere:

  MPLs

- Iterated integrals on elliptic curve:

  eMPLs

- ???

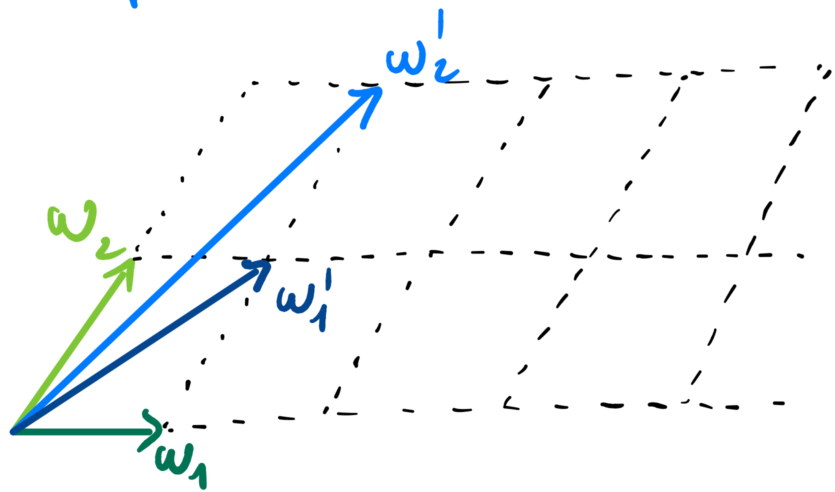
Kinematics t $\to$ shape of curve $\to$ τ

$\rightsquigarrow$ Iterated integrals in τ?

The moduli space of elliptic curves

Q: How can we "classify" elliptic curves with different "shapes"?

Elliptic curve $\mathbb{C}/\Lambda$ $\longleftrightarrow$ Lattice $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$



$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}$$

When do (ω_2, ω_1) and (ω_2', ω_1') generate the same lattice Λ?

$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} \qquad a, b, c, d \in \mathbb{Z}$$

and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ invertible $\Rightarrow \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm 1$!

$\rightsquigarrow \det = -1$ changes orientation, so we can restrict to $\det = +1$

$$\rightsquigarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

$\tau = \frac{\omega_2}{\omega_1}$ and $\tau' = \frac{\omega_2'}{\omega_1'}$ define the same elliptic curve iff $\tau' = \frac{a\tau + b}{c\tau + d}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$!

$\rightsquigarrow$ Moduli space of elliptic curves

$$= SL(2,\mathbb{Z}) \backslash \mathbb{H}$$

$$\mathbb{H} = \{\tau \in \mathbb{C} : \operatorname{Im} \tau > 0\}$$

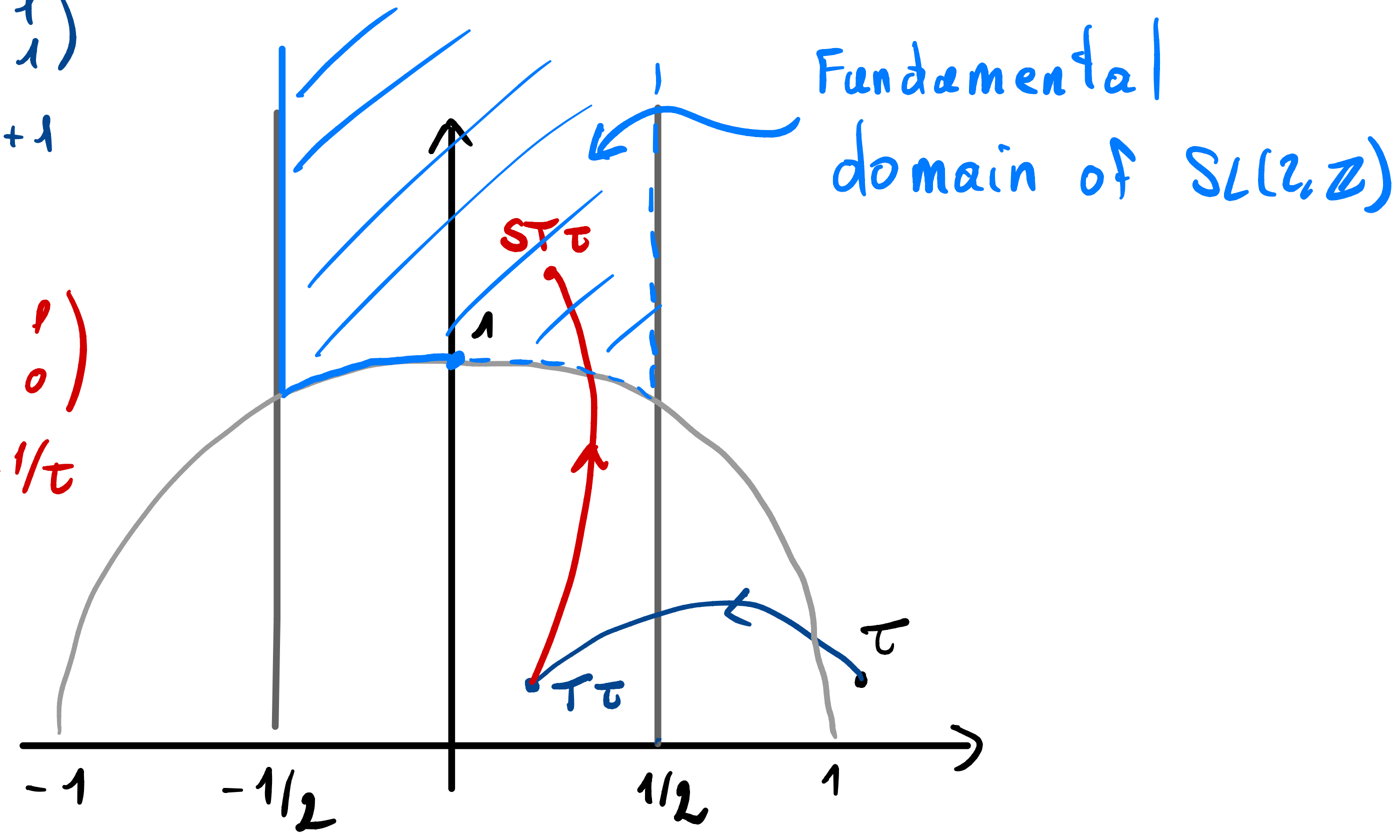
What does $SL(2,\mathbb{Z})\backslash\mathbb{H}$ "look like"?

$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$T\tau = \tau + 1$

$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$S\tau = -1/\tau$



In applications, one encounters subgroups Γ of $SL(2,\mathbb{Z})$ [of finite index].

$\rightsquigarrow$ Particularly important:

Γ is <u>Congruence subgroups of level N</u> if

$$\Gamma \supseteq \Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}$$

<u>Examples</u>: $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : c = 0 \bmod N \right\}$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : c = 0 \bmod N, a, d = 1 \bmod N \right\}$$

N.B: $SL(2,\mathbb{Z}) = \Gamma(1)$, sunrise $\leftarrow \Gamma_1(6)$.

General fact: If $\Gamma \subseteq SL(2,\mathbb{Z})$ is a subgroup [of finite index], then $Y_\Gamma := \Gamma \backslash \mathbb{H}$ is a [non-compact] Riemann surface.

<u>Example:</u> $Y_{\Gamma(1)} = \mathbb{CP}^1 / \{\infty\} = \mathbb{C}$

$$Y_{\Gamma_1(6)} = \mathbb{CP}^1 / \underbrace{\{0, 1, 3, \infty\}}_{\text{"cusps"}}$$

"Cusp of Γ": Equivalence classes of action of Γ on $\mathbb{Q} \cup \{\infty\}$.

Meromorphic functions on Y_Γ?

$\rightsquigarrow$ Must be invariant under Γ.

Modular function: Meromorphic function on $\mathbb{H}$ invariant under Γ:

$$f(\gamma\cdot\tau)=f(\tau)\ ,\ \forall\ \gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma\qquad\left[\gamma\cdot\tau=\frac{a\tau+b}{c\tau+d}\right]$$

Modular functions are not enough for Feynman integrals!

Modular forms: A modular form of weight k for Γ is a function $f: \mathbb{H} \to \mathbb{C}$ s.t.

1) f is holomorphic on $\mathbb{H}$
2) f is holomorphic at the cusps [see later]
3) $f(\gamma \cdot \tau) = (c\tau + d)^k f(\tau)$, $\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

Variants:

- Meromorphic modular form: only 3).
- Weakly-holomorphic MF: 1) & 3), but with poles at the cusps.

Let $\mathcal{M}_k(\Gamma)$ be the $\mathbb{C}$-vector space of modular forms of weight k for Γ, and

$$\mathcal{M}_\bullet(\Gamma) := \bigoplus_k \mathcal{M}_k(\Gamma)$$

$\mathcal{M}_\bullet(\Gamma)$ is an algebra.

Properties:

1) $\dim \mathcal{M}_k(\Gamma) < \infty$ [we can construct an explicit basis!]

2) $\dim \mathcal{M}_k(\Gamma) = 0$ for $k < 0$ and $\mathcal{M}_0(\Gamma) = \mathbb{C}$.

3) If $-\mathbb{1} \in \Gamma$, $\dim \mathcal{M}_k(\Gamma) = 0$ for odd k.

Example: Modular form for $\Gamma = SL(2,\mathbb{Z})$:

Define: $$G_k(\tau) := \sum_{(m,n)\in\mathbb{Z}^2/\{0,0\}} (m\tau+n)^{-k} \qquad k \geq 1.$$

Homework:
- Show that 3) holds.
- Deduce $G_k(\tau) = 0$ for k odd.

Proposition: For k even and $k \geq 4$, $G_k(\tau)$ is a modular form of weight k for $SL(2,\mathbb{Z})$.

k	$\dim \mathcal{M}_k(\Gamma)$	basis
2	0	—
4	1	G_4
6	1	G_6
8	1	$G_8 = \frac{3}{7} G_4^2$
10	1	$G_{10} = \frac{5}{11} G_4 G_6$
12	2	$G_{12} = \frac{18}{143} G_4^3 + \frac{25}{143} G_6^2$

$$\Delta := \frac{1}{1728}\left(\frac{1}{(2\zeta_4)^3} G_4^3 - \frac{1}{(2\zeta_6)^2} G_6^2 \right)$$

<u>Proposition:</u> $\mathcal{M}_{\bullet}(SL(2,\mathbb{Z})) = \mathbb{C}[G_4, G_6]$

There are similar descriptions for other Γ (such that Y_Γ has genus 0)

Fourier ("q"-) expansions

Let Γ be a congruence subgroup of level N.

Check that $T^N = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \Gamma$!

Let $f \in \mathcal{M}_k(\Gamma)$.

$$f(T^N \cdot \tau) = f(\tau + N) = (0 \cdot \tau + 1)^k f(\tau) = f(\tau)$$

$\rightsquigarrow$ f is periodic with period N

$\rightsquigarrow$ Fourier expansion !

Fourier / q-expansion:

$$f(\tau) = \sum_n \underbrace{a_n}_{\in \mathbb{C}} e^{n \cdot 2\pi i \tau / N} = \sum_n a_n q_N^n$$

- Holomorphicity: $a_n = 0$ for $n < 0$.
- Notation: $q = e^{2\pi i \tau}$ $\quad q_N \doteq q^{1/N}$

q-expansions allow us to evaluate modular forms numerically?

Convergence? [We only disuss $\Gamma = SL(2, \mathbb{Z})$]

Convergence? Let $\tau = x + iy$, $y > 0$.

$$q = e^{2\pi i \tau} = e^{-2\pi y} e^{2\pi i x}$$

- If $y = \mathrm{Im}\,\tau \gg 1$, $|q| \ll 1 \rightsquigarrow$ fast convergence
- If $y = \mathrm{Im}\,\tau \ll 1$, $|q| \simeq 1 \rightsquigarrow$ slow convergence

But: For every τ, there is $\tau' \in$ fundamental domain and $\gamma \in SL(2,\mathbb{Z})$ s.t. $\tau = \gamma \cdot \tau'$, and $\mathrm{Im}\,\tau' > \sqrt{3}/2$!

$\rightsquigarrow$ Transform: $f(\tau) = (c\tau' + d)^k \underbrace{f(\tau')}_{\text{fast convergence}}$

Iterated integrals of modular forms:

Let $f_1, \dots, f_p$ be modular forms of weights $k_1, \dots, k_p$ for Γ.

$$I(f_1, \dots, f_p; \tau) := \int_{i\infty}^{\tau} d\tau' \, f_1(\tau') \, I(f_2, \dots, f_p; \tau')$$

N.B: If $a_0(f_p) \neq 0$, then the integral diverges at $i\infty$

$\rightsquigarrow$ tangential base-point!

<u>Example</u>: $G_k(\tau) = 2\zeta_k + \sum_{n=1}^{\infty} a_{k,n} q^n$

$$q = e^{2\pi i \tau} \qquad \tau = \frac{1}{2\pi i} \log q \qquad d\tau = \frac{dq}{2\pi i q}$$

$$I(G_k;\tau) = \int_{\vec{1}_0}^{q} \frac{dq'}{2\pi i q'} G_k(q)$$

$$= \int_{\vec{1}_0}^{q} \frac{dq'}{2\pi i q'} 2\zeta_k + \int_0^q \frac{dq'}{2\pi i q'} \sum_{n=1}^{\infty} a_{k,n} q^n$$

$$= 2\zeta_k \frac{1}{2\pi i} \log q + \sum_{n=1}^{\infty} \frac{a_{k,n}}{2\pi i n} q^n$$

$\rightsquigarrow$ q-series + log's !

• Relation to MPLs & eMPLs?

Assume all f_i have weight 2 ($n_i = 2$).

- If Y_Γ has genus 0, then $I(f_1, \dots, f_p; \tau)$ can be expressed in terms of MPLs
- If Y_Γ has genus 1, then $I(f_1, \dots, f_p; \tau)$ can be expressed in terms of eMPLs

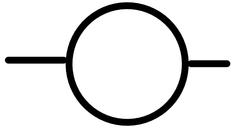
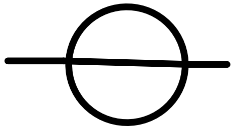
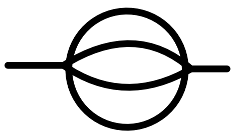
- Consider $g^{(n)}\left(\frac{r}{N}\tau+\frac{s}{N};\tau\right)$, $r,s\in\mathbb{Z}$, $N\in\mathbb{N}$.

This is a modular form for $\Gamma(N)$!

- A point $z=\frac{r}{N}\tau+\frac{s}{N}$ is called a torsion point of order N.

- One can show:

If all z_i are torsion points of order N, then $\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ z_1 & \dots & z_k \end{smallmatrix}; z_{k+1}, \tau\right)$ can be expressed in terms of iterated integrals of modular forms for $\Gamma(N)$.

	MPLs (all orders in ε)
	• At least 1 zero mass MPLs (all ε) • 3 non-zero masses : eMPLs (all ε) • 3 equal masses : I. I mod. form $\Gamma_1(6)$ (all ε)
	• At least 2 zero masses : MPLs (all ε) • 1 zero mass : eMPLs (all ε ?) equal masses : IIMF for $\Gamma_1(6)$ • 4 non-zero masses $\to$ Calabi-Yau 2-fold equal masses : $D=2$ IIMF for $\Gamma_1(6)$ eMPLs $D=2-2\varepsilon$ MEROMORPHIC IIMF