Lecture One: *Vernacular of the S-Matrix*

Problem 1: Consistency Conditions from Quantum Mechanics

As we discussed at the end of the lecture, the uniqueness of the (analytic continuation of the) three-particle S-matrix for massless particles places some surprisingly strong conditions on viable theories in four dimensions. In this problem, I'd like you to derive a few famous results from this perspective.

To complete the following, it suffices to consider the forms of various factorization-channels of particular four-point amplitudes involving some sets of external states. For the sake of the problems below, you may assume that $\sigma\in\mathbb{Z}_+$.

a. Weinberg's theorem

In lecture, we discussed how four-particle amplitudes must factorize, depending on the spins of the particles involved.

Consider an interacting theory of spin-σ particles. Provided the theory is local and unitarity, the factorization-structure of any four-particle amplitude is uniquely determined by three-particle amplitudes.

Show that if $\sigma>2$, the theory cannot consistently factorize.

Solution: Consider for example the helicity amplitude $\mathcal{A}(1^{-\sigma},2^{-\sigma},3^{+\sigma},4^{+\sigma})$. On general grounds, the little-group scaling part of the amplitude can be factored out, allowing us to write

$$\mathcal{A}(1^{-\sigma},2^{-\sigma},3^{+\sigma},4^{+\sigma})\eqqcolon(\langle12\rangle[34])^{2\sigma}\mathscr{F}(s,t,u)\,, \tag{1.1}$$

where $s\coloneqq\langle12\rangle[12], t\coloneqq\langle23\rangle[23], u\coloneqq\langle13\rangle[13]$ are the usual Mandelstam invariants (which satisfy $s{+}t{+}u=0$). From dimensional analysis, we know that $\mathscr{F}$ must be rational function with fixed scaling dimension $-(\sigma+1)$ in momentum-squared; in particular, it must be homogeneous.

In lecture we saw how (for any local, unitary quantum theory) any factorization channel associated with internal particle exchange (long range mediation) resulted in a pole whose residue took a very precise form:

$$\operatorname*{Res}_{s=0}\big(\mathscr{F}\big)\propto\frac{1}{u^{\sigma}}\,, \tag{1.2}$$

and similarly for t and u-channels. (Recall that, when $s\to0$, $t{+}u=0$; so t and u are interchangeable in the above.) When $\sigma\geq3$, $\mathscr{F}$ would need to be of degree -4 in momenta-squared, and thus would necessarily have degree ≥2 in at least one Mandelstam. Put another way, it is not possible for $\mathscr{F}$ to have (*merely*) simple poles in all three channels if its total degree where ≤-4. Thus, $\mathscr{F}$ cannot support residues on all factorization channels.

As factorization channels are the signal of long-range interactions, it is clear that this must break down when $\sigma\geq3$.

$*$ b. Uniqueness of Yang-Mills theory

For interacting particles of spin $\sigma=1$, show that the three-point coupling constants *must* satisfy a *Jacobi* identity:

$$\sum_{\star}\Big(f^{c_1,c_2,\star}f^{\star,c_3,c_4}+f^{c_2,c_3,\star}f^{\star,c_1,c_4}+f^{c_3,c_1,\star}f^{\star,c_2,c_4}\Big)=0\,. \tag{1.3}$$

Thus, any 'charges' (distinguishing quantum number-labels) they may have *must* transform according to (the *adjoint* representation of) some Lie algebra—and consequently, the only (local, long-range, unitary) theory of interacting spin-one particles is Yang-Mills theory.

Solution: Let us denote the three terms appearing in (1.3) c_s, c_t, c_u, respectively—so that (1.3) reads $c_s{+}c_t{+}c_u=0$.

Consider the four-particle amplitude involving particles labelled by some distinguishing (non-kinematic) quantum numbers $c_1,\ldots,c_4$. Peeling-off the helicity-dependent part as above, and this time taking into account the coupling-constants appearing in three-point amplitudes, and being careful about signs, we can easily show that

$$\underset{s=0}{\mathrm{Res}}\big(\mathscr{F}\big)=\Big(\sum_{\star}f^{c_1,c_2,\star}f^{\star,c_3,c_4}\Big)\frac{1}{u}=+c_s\frac{1}{u} \tag{1.4}$$

$$\underset{t=0}{\mathrm{Res}}\big(\mathscr{F}\big)=\Big(\sum_{\star}f^{c_1,c_4,\star}f^{\star,c_2,c_3}\Big)\frac{1}{u}=-c_t\frac{1}{u} \tag{1.5}$$

$$\underset{u=0}{\mathrm{Res}}\big(\mathscr{F}\big)=\Big(\sum_{\star}f^{c_1,c_3,\star}f^{\star,c_2,c_4}\Big)\frac{1}{s}=-c_u\frac{1}{s} \tag{1.6}$$

where we have used the fact that Bose symmetry dictates the complete antisymmetry of the coupling constants $f^{a,b,c}$.

Now, on general grounds, we can deduce that $\mathscr{F}$ may be expanded into a basis of degree-(-2) functions $\{\frac{1}{st},\frac{1}{tu},\frac{1}{su}\}$; but it is easy to see that momentum conservation—$s{+}t{+}u=0$—implies that only two of these are independent. Thus, we may use any pair we'd like to represent the complete function.

Choosing to express $\mathscr{F}$ in terms of $\{\frac{1}{su},\frac{1}{tu}\}$, we see that the coefficients of these two terms are uniquely determined by the first two residue conditions above, (1.4) and (1.5). Specifically, we see that

$$\mathscr{F}=c_s\frac{1}{s\,u}-c_t\frac{1}{t\,u}\,. \tag{1.7}$$

However, comparing the u-residue of this expression with the result in (1.6) gives us an interesting constraint:

$$\underset{u=0}{\mathrm{Res}}\big(\mathscr{F}\big)=-c_u\frac{1}{s}=c_s\frac{1}{s}-c_t\frac{1}{t}=\frac{1}{s}\big(c_s+c_t\big) \tag{1.8}$$

[*] extra credit

—where, in the last equality, we have used the fact that $t = -s$ on the support of $u = 0$. Thus, we conclude that $c_s + c_t + c_u = 0$, which was what we were asked to show.

*** c. Equivalence Principle**

Suppose that there exists a spin-2 particle with some self-coupling constant κ.

Show that its coupling to *any* other field must also have the strength κ.

Solution: We can merely sketch the argument—as the details are not difficult to work out (and are not especially illuminating). The basic argument is to consider any four-particle amplitude involving a spin-2 particle with another state of arbitrary spin. For example, let us consider the amplitude $\mathcal{A}(1^{-\sigma}, 2^{+\sigma}, 3^{-2}, 4^{+2})$. Again, on general grounds we may factor-out the helicity-dependent part of the amplitude to consider

$$\mathcal{A}(1^{-\sigma}, 2^{+\sigma}, 3^{-2}, 4^{+2}) \eqqcolon \big(\langle 13\rangle[41]\big)^4 \left(\frac{\langle 13\rangle[24]}{\langle 32\rangle[41]}\right)^{\sigma} \mathscr{F}(s,t,u)\,, \tag{1.9}$$

where, as before, $\mathscr{F}$ must be a homogeneous function of degree -3. Thus, it must be proportional to $\frac{1}{stu}$—and any particular factorization channel will test the same constant of proportionality.

Let us use κ to denote the coupling constant for the three-graviton amplitude and $\widetilde{\kappa}$ to denote that for its interactions with the spin-σ particles.

The key insight is that different factorization channels will depend on different combinations of κ and $\widetilde{\kappa}$—and consistency between them will require that they are identical.

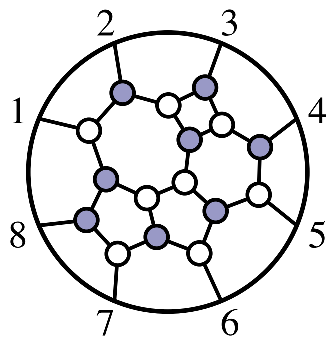
Consider for example the s-channel factorization; this involves the exchange of gravitons, and therefore will be proportional to $\kappa\widetilde{\kappa}$. In contrast, the t-channel factorization involves only spin-σ particle exchange. As such, it will be proportional to $\widetilde{\kappa}^2$. Comparing these two requires that $\kappa = \widetilde{\kappa}$.

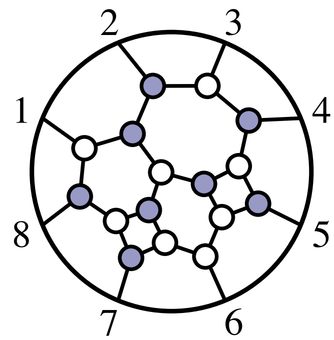
*extra credit

Lecture Two: *Combinatorics of On-Shell Diagrams*

Problem 1: Basic Combinatorics of On-Shell Diagrams

In this problem, I'd like you to analyze the combinatorics associated with the following on-shell diagrams:



and


(1.1)

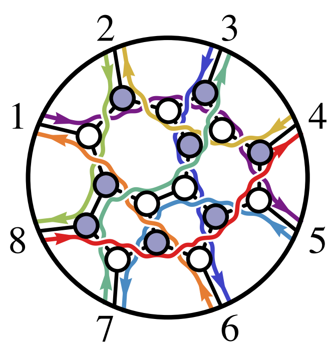
a. What are the *decorated*[3] permutation-labels of the on-shell diagrams above?[4]

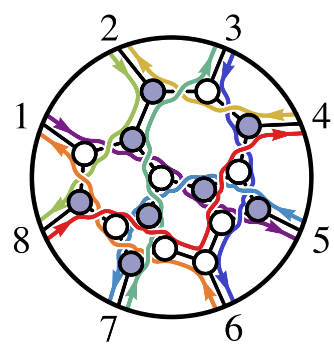
Do these diagrams represent to the same on-shell *function*?

Solution: Both graphs have the same permutation labels—namely,

$$\sigma \coloneqq \left(\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 5 & 8 & 6 & 10 & 7 & 9 & 11 & 12 \end{array}\right). \tag{1.2}$$

We can see this directly by drawing the left-right paths according to the rules discussed in lecture:



and


(1.3)

b. Find a sequence of square/merge-moves which transforms the first of these diagrams into the second.

Solution: Left to the reader ;) .

[3]Recall that when $\sigma\!:\!a\mapsto b$ with $b\!<\!a$, we *conventionally* define its image to be $\sigma(a)\!=\!b\!+\!n$.

[4]Recall that the map $a\mapsto\sigma(a)$ is defined by starting at leg a and turning left/right at every white/blue vertex, respectively; $\sigma(a)$ labels the leg where the path terminates—*decorated* so that $\sigma(a)\!\geq\!a$ for all a.

Problem 2: Building with BCF(W) Bridges

Consider the following decorated permutation:

$$\sigma \coloneqq \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 5 & 7 & 6 & 8 & 9 \end{pmatrix}. \tag{2.1}$$

a. This decorated permutation labels a cell in some Grassmannian $Gr(k,n)$. What are k and n?

Solution: It is clear that the permutation σ labels a cell in $G(k,6)$—as there are 6 pre-images. To determine k, we merely note that 3 of the images *exceed* n—namely, 7,8,9. Thus, $k=3$.

b. Use the lexicographic BCFW-bridge decomposition to construct a *reduced* on-shell diagram which will be labeled by the permutation given above: How many bridges, d, are required to reach the identity?

Solution: Recall that the lexicographic bridge decomposition expresses $\sigma \eqqcolon (a\,b)\circ\sigma'$ where '$(a b)$' represents a transposition of the images of a,b, and this pair is chosen to be the *first* pair of consecutive—*but skipping over self-identified legs*—legs whose images are *ordered*—that is, such that $\sigma(a)<\sigma(b)$. Thus, we see that

$$\sigma = (1\,2)\circ(2\,3)\circ(1\,2)\circ(3\,4)\circ(2\,3)\circ(3\,5)\circ(2\,3)\circ(3\,6)\circ\sigma_0 \tag{2.2}$$

where σ_0 is the decoration of the identity,

$$\sigma_0 \coloneqq \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 8 & 9 & 4 & 5 & 6 \end{pmatrix}. \tag{2.3}$$

Thus, the number of bridges to connect σ to the identity is $d=8$.

c. As discussed in lecture, any on-shell function of planar, maximally supersymmetric Yang-Mills theory can be represented in the form[5]

$$f_\sigma = \int \frac{d\alpha_1}{\alpha_1}\wedge\cdots\wedge\frac{d\alpha_d}{\alpha_d}\,\delta^{k\times 4}\big(C_\sigma(\vec{\alpha})\cdot\widetilde{\eta}\big)\delta^{k\times 2}\big(C_\sigma(\vec{\alpha})\cdot\widetilde{\lambda}\big)\delta^{2\times(n-k)}\big(\lambda\cdot C_\sigma^\perp(\vec{\alpha})\big). \tag{2.4}$$

For the same permutation σ, and using the bridge decomposition obtained above, determine the form of the matrix $C(\vec{\alpha})$ (representing a configuration in $Gr(k,n)$). To be clear: let α_1 denote the *last* transposition to the identity in the decomposition—the one parameterizing a one-dimensional configuration in $Gr(k,n)$, and let α_d denote the 'uppermost' bridge.

Solution: Recall that the action of adding a bridge '$(a b)$' (white-to-blue from a to b) between legs a,b translates into a shift of the $C\mapsto\widehat{C}$ where $\widehat{c}_b = c_b + \alpha\, c_a$; that is, it shifts column b by some new parameter times column a.

Applying the sequence of shifts given in (2.2) results in the following C-matrix representative of the on-shell function:

[5] Recall that, as appearing in the δ-functions here, '$A\cdot B$' means $A.B^T$—or, a sum over the index $a\in[n]$.

$$C(\vec{\alpha})\coloneqq\begin{pmatrix}1&(\alpha_6{+}\alpha_8)&\alpha_6\,\alpha_7&0&0&0\\0&1&(\alpha_2{+}\alpha_4{+}\alpha_7)&(\alpha_2{+}\alpha_4)\alpha_5&\alpha_2\,\alpha_3&0\\0&0&1&\alpha_5&\alpha_3&\alpha_1\end{pmatrix}. \tag{2.5}$$

* **d.** Assume that you have a momentum-conserving set of momenta ($\lambda\!\cdot\!\widetilde{\lambda}\!=\!0$); use whatever remains of the δ-functions in (2.4) to localize as many of the bridge coordinates $\vec{\alpha}$ as possible; find an analytic solution for each α in terms of Lorentz-invariant spinor products.

Solution: Because any computer algebra package can easily solve the requisite equations, the difficult part of this problem lies almost entirely in doing this in terms of Lorentz-invariant spinor products. Sparing the details, the (unique) solution to these equations is:

$$\begin{array}{lll}\alpha_1\mapsto\alpha_1^*\coloneqq-\dfrac{\langle6|(1{+}2)|3]}{s_{123}} & \alpha_4\mapsto\alpha_4^*\coloneqq\dfrac{s_{123}\langle45\rangle[23]}{\langle4|(1{+}2)|3]\langle5|(1{+}2|3]} & \alpha_7\mapsto\alpha_7^*\coloneqq-\dfrac{\langle4|(1{+}3)2]}{\langle4|(1{+}2)|3]}\\ \alpha_2\mapsto\alpha_2^*\coloneqq\dfrac{\langle56\rangle[23]s_{123}}{\langle5|(1{+}2)3]\langle6|(1{+}2)3]} & \alpha_5\mapsto\alpha_5^*\coloneqq-\dfrac{\langle4|(1{+}2)3]}{s_{123}} & \alpha_8\mapsto\alpha_8^*\coloneqq-\dfrac{\langle4|(2{+}3)|1]}{\langle4|(1{+}3)|2]}\\ \alpha_3\mapsto\alpha_3^*\coloneqq-\dfrac{\langle5|(1{+}2)|3]}{s_{123}} & \alpha_6\mapsto\alpha_6^*\coloneqq-\dfrac{[12]\langle4|(1{+}2)|3]}{[23]\langle4|(1{+}3)|2]} & \end{array} \tag{2.6}$$

and the Jacobian associated with integration against the δ-function constraints turns out to be:

$$\mathfrak{J}=\frac{s_{123}^2}{\langle4|(1{+}3)|2]\langle5|(1{+}2)|3]\langle6|(1{+}2)3]}. \tag{2.7}$$

Thus, upon using the δ-functions to localize all of the α_i variables, the analytic form of this particular on-shell function would be:

$$\begin{aligned}f_\sigma&=\frac{\mathfrak{J}}{\alpha_1^*\,\alpha_2^*\,\alpha_3^*\,\alpha_4^*\,\alpha_5^*\,\alpha_6^*\,\alpha_7^*\,\alpha_8^*}\delta^{3\times4}\big(C(\vec{\alpha}^*)\!\cdot\!\widetilde{\eta}\big)\delta^{2\times2}\big(\lambda\!\cdot\!\widetilde{\lambda}\big)\\&=\frac{s_{123}^3}{\langle4|(2{+}3)|1][12]\langle45\rangle\langle56\rangle[23]\langle6|(1{+}2)|3]}\delta^{3\times4}\big(C(\vec{\alpha}^*)\!\cdot\!\widetilde{\eta}\big)\delta^{2\times2}\big(\lambda\!\cdot\!\widetilde{\lambda}\big).\end{aligned} \tag{2.8}$$

where the matrix of $\widetilde{\eta}$-coefficients is given by

$$C(\vec{\alpha}^*)\coloneqq\begin{pmatrix}1&(\alpha_6^*{+}\alpha_8^*)&\alpha_6^*\,\alpha_7^*&0&0&0\\0&1&(\alpha_2^*{+}\alpha_4^*{+}\alpha_7^*)&(\alpha_2^*{+}\alpha_4^*)\alpha_5^*&\alpha_2^*\,\alpha_3^*&0\\0&0&1&\alpha_5^*&\alpha_3^*&\alpha_1^*\end{pmatrix}. \tag{2.9}$$

e. Consider the (on-shell) superfunction[7]

$$f(\lambda,\widetilde{\lambda},\widetilde{\eta})=\frac{\delta^{3\times4}\big(C\!\cdot\!\widetilde{\eta}\big)\delta^{2\times2}\big(\lambda\!\cdot\!\widetilde{\lambda}\big)}{s_{123}\langle4|(2{+}3)|1][12]\langle45\rangle\langle56\rangle[23]\langle6|(1{+}2)|3]} \tag{2.10}$$

where the C-matrix (merely of $\widetilde{\eta}$ coefficients this time) is given by

*extra credit

[7]Recall that '$\langle a|(b{+}c)|d]$' $\coloneqq\langle a|(p_b{+}p_c)|d]=\langle a|(b\rangle[b{+}c\rangle[c)|d]=\langle ab\rangle[bd]+\langle ac\rangle[cd]$.

$$C \coloneqq \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ [23] & [31] & [12] & 0 & 0 & 0 \end{pmatrix}. \tag{2.11}$$

Representing a particular configuration in $Gr(3,6)$—and for generic (momentum-conserving) spinors $\lambda, \widetilde{\lambda}$—what permutation would label this (on-shell) function's C-matrix, *as a configuration* in the Grassmannian $Gr(3,6)$?

Solution: The *geometric* definition of the permutation associated with a configuration in the (positive) Grassmannian $C \eqqcolon (c_1\, c_2 \cdots c_n)$ is simply that $a \mapsto \sigma(a)$ if $\sigma(a)$ represents the *nearest* column-vector $c_{\sigma(a)}$ such that $c_a \in \text{span}\{c_{a+1}, \ldots, c_{\sigma(a)}\}$—where cyclic labeling is understood. (If we ever have $c_a = \vec{0}$, then $\sigma(a) = a$ by definition.)

Consider first a completely generic 3×6 matrix. As each column represents a generic three-vector, a generic three-dimensional subspace is required to represent any particular column. Thus, $c_a \in \text{span}\{c_{a+1}, \ldots, c_{a+3}\}$ in general, and $\sigma(a) = a+3$.

For the matrix given above, it is not hard to see that the first three columns are generic 3-vectors, while the last three span only a 2-dimensional subspace. Moreover, it is fairly easy to see that every consecutive set of columns is full-rank, with the exception of the set $(c_4\, c_5\, c_6)$. In particular, this means that $c_3 \notin \text{span}\{c_4, \ldots, c_6\}$, but $c_3 \in \text{span}\{c_4, \ldots, c_7\}$; as such, $\sigma(3) = 7$. Similarly, it is easy to see that $c_4 \in \text{span}\{c_5, c_6\}$; as such, $\sigma(4) = 6$. All other sets of columns behave similarly to the generic case. Thus, the permutation encoding this configuration in $Gr(3,6)$ would be given precisely by the one given in (2.1) above.

f. It is not hard to see that the *matrices* C generated by the two constructions are fairly different; and yet, each of these two are supposed to represent the 'same' on-shell function. Explain how this gets resolved.

Solution: This is not in fact hard to understand. By inspection, the two expressions differ by a factor of s_{123}^4 in their prefactors, and involve different C-matrices. Let's denote the matrix $C(\vec{\alpha}^*)$ appearing in problems **c**,**d** by C^A and that appearing in problem **e** as C^B.

Before we make the critical (mathematical) argument, we can easily see that the super-functions have *identical* component amplitudes. Consider for example extracting component function involving particles $1, \ldots, 6$ all being gluonic states with helicities $(-,-,-,+,+,+)$. This component would be extracted by multiplying the relevant bosonic functions by the fourth-power of the determinant of the corresponding matrix of $\widetilde{\eta}$-coefficients. For the first expression, this corresponds simply to the function

$$\int \prod_{I=1}^{4} \left(d\widetilde{\eta}_1^I\, d\widetilde{\eta}_2^I\, d\widetilde{\eta}_3^I \right) f_\sigma^A = \frac{s_{123}^3}{\langle 4|(2{+}3)|1][12]\langle 45\rangle\langle 56\rangle[23]\langle 6|(1{+}2)|3]} \delta^{2\times 2}\big(\lambda \cdot \widetilde{\lambda}\big), \tag{2.12}$$

as the determinant $\det\{c_1^A, c_2^A, c_3^A\} = 1$; while for the second, it would correspond to

$$\int \prod_{I=1}^{4} \left(d\widetilde{\eta}_1^I\, d\widetilde{\eta}_2^I\, d\widetilde{\eta}_3^I \right) f_{(2.10)}^B = \frac{s_{123}^4}{s_{123}\langle 4|(2{+}3)|1][12]\langle 45\rangle\langle 56\rangle[23]\langle 6|(1{+}2)|3]} \delta^{2\times 2}\big(\lambda \cdot \widetilde{\lambda}\big), \tag{2.13}$$

as $\det\{c_1^B, c_2^B, c_3^B\} = s_{123}$. Thus, the two superfunctions appear to match on bosonic components.

The logic behind this (and the way to see that this will always work) is to notice that, for fermionic δ-functions,

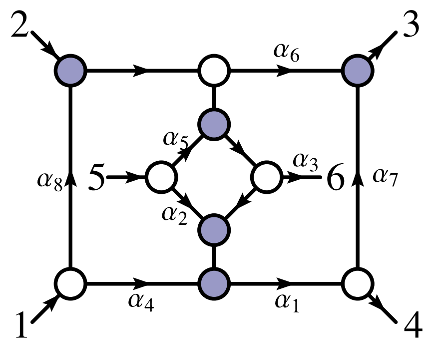
$$\delta^{k\times 4}\big(D\cdot\widetilde{\eta}\big) = \det(M)^4\delta^{k\times 4}\big(C\cdot\widetilde{\eta}\big) \quad \text{for} \quad D = M.C\,. \tag{2.14}$$

By noticing that $C^A \sim C^B \in Gr(3,6)$—that is, there exists some M (namely, $(c_1^A c_2^A c_3^A).(c_1^B c_2^B c_3^B)^{-1}$) —it is easy to use the rule above to show that they are in fact identical as superfunctions.

Lecture Three: *Non-Planar On-Shell Varieties*

Problem 1: Non-Planar On-Shell Functions

In this problem, we'd like to analyze the following non-planar on-shell diagram:



(1.1)

a. Determine the boundary-measurements matrix $C(\vec{\alpha})$ for the graph above (assuming all unlabeled edges have weight 1).

Solution: Following the instructions discussed in lecture, we see that the 'sources' $\{1,2,5\}$ should correspond to the columns set to the identity matrix; the other entries can be read off directly from the sum over paths, resulting in:

$$C(\vec{\alpha}) = \begin{pmatrix} 1 & 0 & \alpha_6\alpha_8+\alpha_1\alpha_7(\alpha_4+\alpha_8) & \alpha_1(\alpha_4+\alpha_8) & 0 & \alpha_3\alpha_8 \\ 0 & 1 & \alpha_6+\alpha_1\alpha_7 & \alpha_1 & 0 & \alpha_3 \\ 0 & 0 & \alpha_1\alpha_7(\alpha_2+\alpha_5) & \alpha_1(\alpha_2+\alpha_5) & 1 & \alpha_3\alpha_5 \end{pmatrix} \tag{1.2}$$

b. Find the number of solutions that exist to the (universal) 'constraints' on the α's—namely, $\delta^{3\times 2}\big(C(\vec{\alpha})\cdot\widetilde{\lambda}\big)$ and $\delta^{2\times 3}\big(\lambda\cdot C^{\perp}(\vec{\alpha})\big)$—assuming momentum-conserving spinors.

(Hint: you can solve the equations directly—in MATHEMATICA, say—for some particular (generic, momentum conserving) spinors.)

Solution: Using MATHEMATICA and a generic set of (momentum-conserving) spinors, we find that the number of solutions to the constraints is **2**; in particular, this 'leading singularity' involves square-root

$$\sqrt{(s_{13}+s_{14}+s_{23}+s_{24})^2 - 4s_{12}s_{34}}\,. \tag{1.3}$$

Lecture Four: *Tree-Level Recursion in* Mathematica

Problem 1: Implementing Recursion Yourself

The goal of this problem is to have you do for yourself what we did 'in real time' during lecture: to build *your own* implementation of tree-level recursion in Mathematica, say, and to improve upon what was done in several important ways.

a. Write your own implementation of the tree-level BCFW recursion relations in momentum-twistor variables.

* **b.** Implement this more efficiently than done during the lecture. (This is not very hard: there are many obvious places for improvement.)

c. Any superfunction expressed in momentum-twistor space according to

$$f(Z)\times\delta^{k\times n}\big(C\cdot\eta\big)=\frac{1}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}f(\lambda,\widetilde{\lambda})\delta^{(k+2)\times n}\big(\widehat{C}\cdot\widetilde{\eta}\big) \tag{1.1}$$

Construct the general map between superfunctions expressed in terms of momentum-twistor variables (Z,η) into those expressed in terms of spinor variables $(\lambda,\widetilde{\lambda},\widetilde{\eta})$. That is, given the equality

$$\begin{aligned}f(Z)\times\delta^{k\times n}\big(C\cdot\eta\big)&=\frac{\delta^{2\times4}\big(\lambda\cdot\widetilde{\eta}\big)\delta^{2\times2}\big(\lambda\cdot\widetilde{\lambda}\big)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}f(Z)\delta^{k\times n}\big(C(Z)\cdot\eta\big)\\&=:\widehat{f}(\lambda,\widetilde{\lambda})\delta^{(k+2)\times n}\big(\widehat{C}(\lambda,\widetilde{\lambda})\cdot\eta\big)\delta^{2\times2}\big(\lambda\cdot\widetilde{\lambda}\big)\end{aligned} \tag{1.2}$$

determine the explicit form of $\widehat{C}$ of η-coefficients.

(*Hint*: As the bosonic part of any superfunction is near-trivially translated—as any momentum twistor four-bracket can be directly expanded as a function of $(\lambda,\widetilde{\lambda})$—the only non-trivial part of this problem is the translation between η-coefficients $C(Z)$ and $\widetilde{\eta}$-coefficients $\widehat{C}(\lambda,\widetilde{\lambda})$.)

d. Implement this transformation in your code, to directly represent the results of BCFW recursion in momentum-space variables. This translation is needed when discussion ordinary component amplitudes in sYM, as momentum-twistor super-states are not identical to momentum-space super-states.

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*extra credit