Lecture One: *Vernacular of the S-Matrix*

Problem 1: Consistency Conditions from Quantum Mechanics

As we discussed at the end of the lecture, the uniqueness of the (analytic continuation of the) three-particle S-matrix for massless particles places some surprisingly strong conditions on viable theories in four dimensions. In this problem, I'd like you to derive a few famous results from this perspective.

To complete the following, it suffices to consider the forms of various factorization-channels of particular four-point amplitudes involving some sets of external states. For the sake of the problems below, you may assume that $\sigma \in \mathbb{Z}_+$.

a. Weinberg's theorem

In lecture, we discussed how four-particle amplitudes must factorize, depending on the spins of the particles involved.

Consider an interacting theory of spin-σ particles. Provided the theory is local and unitarity, the factorization-structure of any four-particle amplitude is uniquely determined by three-particle amplitudes.

Show that if $\sigma > 2$, the theory cannot consistently factorize.

* **b. Uniqueness of Yang-Mills theory**

For interacting particles of spin $\sigma = 1$, show that the three-point coupling constants *must* satisfy a *Jacobi* identity:

$$\sum_{\star}\Big(f^{c_1,c_2,\star}f^{\star,c_3,c_4}+f^{c_2,c_3,\star}f^{\star,c_1,c_4}+f^{c_3,c_1,\star}f^{\star,c_2,c_4}\Big)=0\,. \tag{1.1}$$

Thus, any 'charges' (distinguishing quantum number-labels) they may have *must* transform according to (the *adjoint* representation of) some Lie algebra—and consequently, the only (local, long-range, unitary) theory of interacting spin-one particles is Yang-Mills theory.

* **c. Equivalence Principle**

Suppose that there exists a spin-2 particle with some self-coupling constant κ. Show that its coupling to *any* other field must also have the strength κ.

---

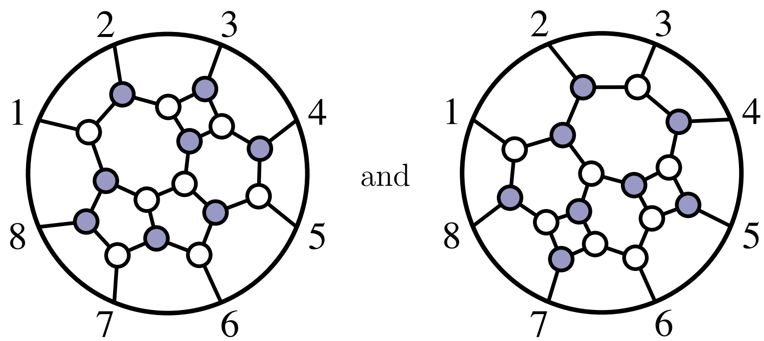
* extra credit

* extra credit

Lecture Two: *Combinatorics of On-Shell Diagrams*

Problem 1: Basic Combinatorics of On-Shell Diagrams

In this problem, I'd like you to analyze the combinatorics associated with the following on-shell diagrams:



(1.1)

a. What are the *decorated*[3] permutation-labels of the on-shell diagrams above?[4]
Do these diagrams represent to the same on-shell *function*?

b. Find a sequence of square/merge-moves which transforms the first of these diagrams into the second.

Problem 2: Building with BCF(W) Bridges

Consider the following decorated permutation:

$$\sigma \coloneqq \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 5 & 7 & 6 & 8 & 9 \end{pmatrix}. \tag{2.1}$$

a. This decorated permutation labels a cell in some Grassmannian $Gr(k,n)$. What are k and n?

b. Use the lexicographic BCFW-bridge decomposition to construct a *reduced* on-shell diagram which will be labeled by the permutation given above:
How many bridges, d, are required to reach the identity?

c. As discussed in lecture, any on-shell function of planar, maximally supersymmetric Yang-Mills theory can be represented in the form[5]

$$f_\sigma = \int \frac{d\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{d\alpha_d}{\alpha_d}\, \delta^{k\times4}\big(C_\sigma(\vec{\alpha})\cdot\widetilde{\eta}\big)\delta^{k\times2}\big(C_\sigma(\vec{\alpha})\cdot\widetilde{\lambda}\big)\delta^{2\times(n-k)}\big(\lambda\cdot C_\sigma^\perp(\vec{\alpha})\big)\,. \tag{2.2}$$

For the same permutation σ, and using the bridge decomposition obtained above, determine the form of the matrix $C(\vec{\alpha})$ (representing a configuration

[3] Recall that when $\sigma\!:\!a\mapsto b$ with $b<a$, we *conventionally* define its image to be $\sigma(a)=b+n$.

[4] Recall that the map $a\mapsto\sigma(a)$ is defined by starting at leg a and turning left/right at every white/blue vertex, respectively; $\sigma(a)$ labels the leg where the path terminates—*decorated* so that $\sigma(a)\geq a$ for all a.

[5] Recall that, as appearing in the δ-functions here, '$A\cdot B$' means $A.B^T$—or, a sum over the index $a\in[n]$.

in $Gr(k,n)$). To be clear: let α_1 denote the *last* transposition to the identity in the decomposition—the one parameterizing a one-dimensional configuration in $Gr(k,n)$, and let α_d denote the 'uppermost' bridge.

* **d.** Assume that you have a momentum-conserving set of momenta ($\lambda\cdot\widetilde{\lambda}=0$); use whatever remains of the δ-functions in (2.2) to localize as many of the bridge coordinates $\vec{\alpha}$ as possible; find an analytic solution for each α in terms of Lorentz-invariant spinor products.

e. Consider the (on-shell) superfunction[7]

$$f(\lambda,\widetilde{\lambda},\widetilde{\eta})=\frac{\delta^{3\times4}\big(C\cdot\widetilde{\eta}\big)\delta^{2\times2}\big(\lambda\cdot\widetilde{\lambda}\big)}{s_{123}\langle4|(2{+}3)|1][12]\langle45\rangle\langle56\rangle[23]\langle6|(1{+}2)|3]} \tag{2.3}$$

where the C-matrix (merely of $\widetilde{\eta}$ coefficients this time) is given by

$$C\coloneqq\left(\begin{array}{cccccc}\lambda_1^1&\lambda_2^1&\lambda_3^1&\lambda_4^1&\lambda_5^1&\lambda_6^1\\\lambda_1^2&\lambda_2^2&\lambda_3^2&\lambda_4^2&\lambda_5^2&\lambda_6^2\\{[23]}&[31]&[12]&0&0&0\end{array}\right). \tag{2.4}$$

Representing a particular configuration in $Gr(3,6)$—and for generic (momentum-conserving) spinors $\lambda,\widetilde{\lambda}$—what permutation would label this (on-shell) function's C-matrix, *as a configuration* in the Grassmannian $Gr(3,6)$?

f. It is not hard to see that the *matrices* C generated by the two constructions are fairly different; and yet, each of these two are supposed to represent the 'same' on-shell function. Explain how this gets resolved.

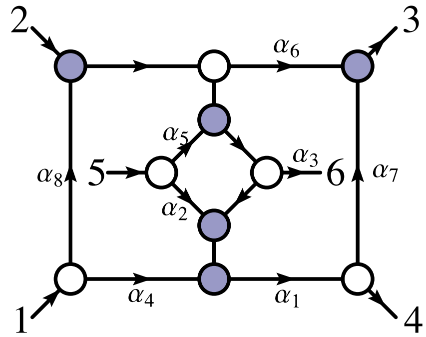
*extra credit

[7]Recall that '$\langle a|(b{+}c)|d]$' $\eqqcolon \langle a|(p_b{+}p_c)|d]=\langle a|(b\rangle[b{+}c\rangle[c)|d]=\langle ab\rangle[bd]+\langle ac\rangle[cd]$.

Lecture Three: *Non-Planar On-Shell Varieties*

Problem 1: Non-Planar On-Shell Functions

In this problem, we'd like to analyze the following non-planar on-shell diagram:



(1.1)

a. Determine the boundary-measurements matrix $C(\vec{\alpha})$ for the graph above (assuming all unlabeled edges have weight 1).

b. Find the number of solutions that exist to the (universal) 'constraints' on the α's—namely, $\delta^{3\times2}\big(C(\vec{\alpha})\cdot\widetilde{\lambda}\big)$ and $\delta^{2\times3}\big(\lambda\cdot C^{\perp}(\vec{\alpha})\big)$—assuming momentum-conserving spinors.

(Hint: you can solve the equations directly—in Mathematica, say—for some particular (generic, momentum conserving) spinors.)

Lecture Four: *Tree-Level Recursion in* Mathematica

Problem 1: Implementing Recursion Yourself

The goal of this problem is to have you do for yourself what we did 'in real time' during lecture: to build *your own* implementation of tree-level recursion in Mathematica, say, and to improve upon what was done in several important ways.

a. Write your own implementation of the tree-level BCFW recursion relations in momentum-twistor variables.

∗ **b.** Implement this more efficiently than done during the lecture.
(This is not very hard: there are many obvious places for improvement.)

c. Any superfunction expressed in momentum-twistor space according to

$$f(Z)\times\delta^{k\times n}\big(C\cdot\eta\big)=\frac{1}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}f(\lambda,\widetilde{\lambda})\delta^{(k+2)\times n}\big(\widehat{C}\cdot\widetilde{\eta}\big)\tag{1.1}$$

Construct the general map between superfunctions expressed in terms of momentum-twistor variables (Z,η) into those expressed in terms of spinor variables $(\lambda,\widetilde{\lambda},\widetilde{\eta})$. That is, given the equality

$$\begin{aligned}f(Z)\times\delta^{k\times n}\big(C\cdot\eta\big)&=\frac{\delta^{2\times 4}\big(\lambda\cdot\widetilde{\eta}\big)\delta^{2\times 2}\big(\lambda\cdot\widetilde{\lambda}\big)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}f(Z)\delta^{k\times n}\big(C(Z)\cdot\eta\big)\\&=:\widehat{f}(\lambda,\widetilde{\lambda})\delta^{(k+2)\times n}\big(\widehat{C}(\lambda,\widetilde{\lambda})\cdot\eta\big)\delta^{2\times 2}\big(\lambda\cdot\widetilde{\lambda}\big)\end{aligned}\tag{1.2}$$

determine the explicit form of $\widehat{C}$ of η-coefficients.

(*Hint*: As the bosonic part of any superfunction is near-trivially translated—as any momentum twistor four-bracket can be directly expanded as a function of $(\lambda,\widetilde{\lambda})$—the only non-trivial part of this problem is the translation between η-coefficients $C(Z)$ and $\widetilde{\eta}$-coefficients $\widehat{C}(\lambda,\widetilde{\lambda})$.)

d. Implement this transformation in your code, to directly represent the results of BCFW recursion in momentum-space variables. This translation is needed when discussion ordinary component amplitudes in sYM, as momentum-twistor super-states are not identical to momentum-space super-states.

∗extra credit