## Exercise 1:

(a) massless $\Rightarrow$ $E = |\vec{p}|$ ; $p_z = |\vec{p}| \cdot \cos\theta$

$$\Rightarrow y = \frac{1}{2} \ln\left(\frac{|\vec{p}| + p_z}{|\vec{p}| - p_z}\right) = \frac{1}{2} \ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right)$$

$$1 - \cos\theta = 1 - \left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) = 2\sin^2\frac{\theta}{2}$$

$$1 + \cos\theta = 1 + (\text{——} \;{}'' \;\text{——}) = 2\cos^2\frac{\theta}{2}$$

$$= \frac{1}{2} \ln\left(\frac{\cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}\right) = -\ln\tan\frac{\theta}{2}$$

(b) $y = 2.5 \;\hat{=}\; \theta = 9.4°$ ; $\theta = 2\arctan(e^{-y})$

$y = 5 \;\hat{=}\; \theta = 0.77°$

(c) $p^\mu_{a,b} = x_{1/2} \frac{\sqrt{s}}{2} (1, 0, 0, \pm 1)^T$ ; $p^\mu_{1/2} = p_{T,1} (\cosh y_1, \hat{e}_{\varphi_1}, \sinh y_1)$

back-to-back in transverse plane $\Rightarrow$ $p_{T,1} = p_{T,2} = p_{T,avg}$ ; $\hat{e}_{\varphi_1} = -\hat{e}_{\varphi_2}$

$$* \quad (p_a + p_b)^2 = 2p_a p_b = x_1 x_2 s \overset{!}{=} (p_1 + p_2)^2 = 2p_1 p_2 = 2p_T^2 \underbrace{[\underbrace{ch y_1 \, ch y_2 - sh y_1 \, sh y_2}_{\cosh(y_1 - y_2) \,=\, \cosh 2y^*} + 1]}_{2\cosh^2 y^*}$$

$$\Rightarrow x_1 x_2 = \frac{4 p_{T,avg}^2}{s} \cosh^2 y^* \qquad (1)$$

$$* \quad \text{rapidity} \quad y = \frac{1}{2} \ln\left[\frac{(x_1 + x_2) + (x_1 - x_2)}{(x_1 + x_2) - (x_1 - x_2)}\right] = \frac{1}{2} \ln\left(\frac{x_1}{x_2}\right)$$

$$\overset{!}{=} \frac{1}{2} \ln\left[\frac{(ch y_1 + ch y_2) + (sh y_1 + sh y_2)}{(ch y_1 + ch y_2) - (sh y_1 + sh y_2)}\right] = \frac{1}{2} \ln\left[\frac{\overbrace{e^{+y_1} + e^{+y_2}}^{e^{(y_1+y_2)}}}{e^{-y_1} + e^{-y_2}}\right]$$

$$\Rightarrow \frac{x_1}{x_2} = e^{(y_1 + y_2)} \qquad (2)$$

$$* \quad (1) \cdot (2) \Rightarrow \quad x_1 = \frac{2 p_{T,avg}}{\sqrt{s}} e^{\pm y_b} \cosh y^*$$

$$(1) / (2) \Rightarrow \quad x_2 = \frac{2 p_{T,avg}}{\sqrt{s}} e^{\mp y_b} \cosh y^*$$

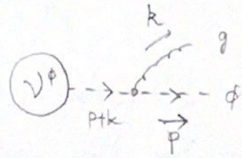
(d) $y_b = \left|\frac{1}{2} \ln\left(\frac{x_1}{x_2}\right)\right|$

boost from part. CoM $\Rightarrow$ Lab frame

$y^* = |\text{rapidity in part. CoM}| = |\hat{y}|$

$y_{1/2} = \pm \hat{y} + \frac{1}{2} \ln\left(\frac{x_1}{x_2}\right)$

Exercise 2:




$$= i g_s \left(t^a\right)_{ii'} \left(-p-(p+k)\right)^\mu \frac{i}{(p+k)^2 - m_\phi^2} \, V^{\phi^{i'}}(p+k) \; \varepsilon^a_\mu(k)^*$$

$$= g_s \left(t^a\right)_{ii'} \frac{(2p+k)^\mu}{2p\cdot k} \, V^{\phi^{i'}}(p+k) \; \varepsilon^a_\mu(k)^* \qquad (1)$$

(a) soft limit $k \to 0$

$$(1) \to \underbrace{g_s \left(t^a\right)_{ii'} \frac{p^\mu \varepsilon^a_\mu(k)^*}{(p\cdot k)}}_{(\text{eikonal current})\cdot \varepsilon_\mu} \; \underbrace{V^{\phi^{i'}}(p)}_{\substack{\text{process w/o} \\ \text{gluon emission}}}$$

same as in the quark case (eikonal does not "see" spin)

(b) collinear limit $(k \parallel p)$ using Sudakov parametrisation

* $2(p\cdot k) = -\dfrac{k_\perp^2}{z(1-z)}$
* $(n\cdot k) = (1-z)\,(n\cdot \tilde{p})$
* $(n\cdot p) = z\,(n\cdot\tilde{p})$

$$\Rightarrow \sum_{\text{d.o.f}} |(1)|^2 = g_s^2 C_F \frac{|V^\phi(p+k)|^2}{4(p\cdot k)^2} \underbrace{(2p_\mu + k_\mu)(2p_\nu + k_\nu)\left[-g^{\mu\nu} + \frac{k^\mu n^\nu + k^\nu n^\mu}{(k\cdot n)}\right]}_{4(p\cdot k)\frac{2z}{1-z}}$$

$$\xrightarrow{k_\perp^2 \to 0} g_s^2 \frac{1}{(p\cdot k)} \cdot \underbrace{C_F \frac{2z}{1-z}} \cdot |V^\phi(\tilde{p})|^2$$

splitting function $\hat{P}_{\phi\phi}(z) = C_F \dfrac{2z}{1-z}$

(c)

$\phi$-number conservation

$$\Rightarrow \int_0^1 dz \left[C_F \frac{2z}{(1-z)_+} + A\,\delta(1-z)\right] \overset{!}{=} 0 \Rightarrow A = 2C_F$$

$$\Rightarrow P_{\phi\phi}(z) = C_F\left[\frac{2z}{(1-z)_+} + 2\,\delta(1-z)\right]$$

"regularized" splitting function (including virtual corrections)

Exercise 3:

(a) particle multiplicity in general not IR safe (adding soft part. or splitting one $\Rightarrow$ n++) NO

(b) if the jets are defined in an IR-safe way, their # is IR safe

Depends on IR-safety of the jet def$^n$

(c)

* $E_{tot} = \sum_{i=1}^{n} E_i$

  $\hookrightarrow$ add a soft particle $(p_{n+1} \to \emptyset) \to E_{tot} = \sum_{i=1}^{n} E_i + \cancel{E_{n+1}}^{\to \emptyset} = \sum_{i=1}^{n} E_i$

  $\hookrightarrow$ split a particle $p_n \longmapsto \underbrace{z p_n}_{p_n'} + \underbrace{(1-z) p_n}_{p_{n+1}'} \Rightarrow E_{tot} = \sum_{i=1}^{n-1} E_i + \underbrace{E_n' + E_{n+1}'}_{zE_n + (1-z)E_n} = \sum_{i=1}^{n} E_i$ YES

* $[E^2]_{tot} = \sum_{i=1}^{n} E_i^2$

  $\hookrightarrow$ soft: ok (same as above)

  $\hookrightarrow$ collinear: $[E^2]_{tot} = \sum_{i=1}^{n-1} E_i^2 + \underbrace{(E_n')^2 + (E_{n+1}')^2}_{z^2 E_n^2 + (1-z)^2 E_n^2} \neq \sum_{i=1}^{n} E_i^2$ NO

(d) soft: $\sim E_{n+1} \to \emptyset$ does not contribute to the sums ✓

coll: $p_n \to \underbrace{z p_n}_{p_n'} + \underbrace{(1-z) p_n}_{p_{n+1}'}$

$\hookrightarrow$ denominator: $\sum_k E_k = E_{tot}$ ✓ (part (c))

$\hookrightarrow$ numerator: $\sum_{i,j=1}^{n} E_i E_j \, \delta(\chi - \cos\theta_{ij})$

$$\to \sum_{i,j=1}^{n-1} E_i E_j \, \delta(\chi - \cos\theta_{ij}) + \underbrace{2 \sum_{i=1}^{n-1} \sum_{j=n}^{n+1} (\dots)}_{(1)} + \underbrace{\sum_{i,j=n}^{n+1} (\dots)}_{(2)}$$

(1) $2 \sum_{i=1}^{n-1} E_i \underbrace{(zE_n + (1-z)E_n)}_{E_n} \delta(\chi - \cos\theta_{in})$ coll. split does not change direction!

(2) $\sum_{i,j=n}^{n+1} E_i' E_j' \, \delta(\chi - \cos\theta_{ij}) = E_n E_n \, \delta(\chi - 1)$

$= \sum_{i,j=1}^{n} E_i E_j \, \delta(\chi - \cos\theta_{ij})$ ✓

YES

(e) soft $z_i \to \emptyset$ is ok

coll : under splitting $\theta_i$ angle remains the same ✓

However, for $\frac{d\theta_i}{\theta_i}$ divergence, we need $\beta > \emptyset$

# Exercise 4:

(a) $\tau_\beta = \sum_{i \in jet} z_i \theta_i^\beta$ ; work in S&C limit $\Rightarrow d\omega^{S\&C}_{q\to qg} = \frac{2\alpha_s}{\pi} C_F \underbrace{\frac{dE}{E}}_{\frac{dz}{z}} \frac{d\theta}{\theta}$

cumulant @ $\mathcal{O}(\alpha_s)$

$$\alpha_s \Sigma_q^{(1)}(\tau_\beta) = \frac{2\alpha_s}{\pi} C_F \int \frac{dz}{z} \frac{d\theta}{\theta}$$

$$\cdot \{ \Theta[z\theta^\beta < \tau_\beta]\ \Theta(\theta < R) \quad \leftarrow \text{real emission insite jet}$$

$$+ \Theta[\emptyset < \tau_\beta]\ \Theta(\theta > R) \quad \leftarrow \text{outside jet}$$

$$- \Theta[\emptyset < \tau_\beta] \} \leftarrow \text{virtual via unitarity}$$

$$= -\frac{2\alpha_s}{\pi} C_F \int \frac{dz}{z} \int \frac{d\theta}{\theta}\ \Theta(\theta < R)\ \Theta[z\theta^\beta < \tau_\beta]$$

$$\Rightarrow \theta < R \ ; \ \theta > \left(\frac{\tau_\beta}{z}\right)^{1/\beta}$$

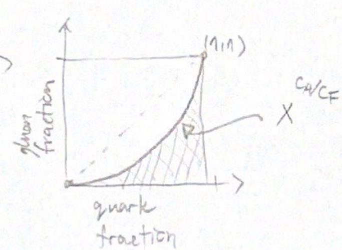
$$\Rightarrow z > \frac{\tau_\beta}{R^\beta}$$

$$= -\frac{2\alpha_s}{\pi} C_F \int_{\tau_\beta/R^\beta}^{1} \frac{dz}{z} \underbrace{\int_{(\tau_\beta/z)^{1/\beta}}^{R} \frac{d\theta}{\theta}}_{\frac{1}{\beta}\ln\left(\frac{R^\beta z}{\tau_\beta}\right)} = -\frac{2\alpha_s}{\pi} \frac{C_F}{\beta} \frac{1}{2} \ln^2\left(R^\beta/\tau_\beta\right)$$

(b) gluon case in S&C limit by $C_F \to C_A$ replacement

(c) ROC curve: $\quad ROC(x) = \Sigma_g\left(\Sigma_q^{-1}(x)\right) = x^{C_A/C_F}$

$\Rightarrow$




$$AUC = \int_0^1 dx\ x^{C_A/C_F} = \frac{C_F}{C_A + C_F}$$