Jet Physics

Exercise 1 *Hadron collider kinematics*

Consider a particle with energy E and three momentum $\vec{p} = (p_x, p_y, p_z)^{\mathrm{T}}$. At hadron colliders, particles are commonly parametrised in terms of the kinematic variables

$$p_{\mathrm{T}} = \sqrt{p_x^2 + p_y^2} \qquad \text{(transverse momentum)}, \tag{1}$$

$$y = \frac{1}{2}\ln\left(\frac{E+p_z}{E-p_z}\right) \qquad \text{(rapidity)}, \tag{2}$$

$$\varphi = \arctan\frac{p_y}{p_x} \qquad \text{(azimuth)}. \tag{3}$$

(a) Show that for massless particles, y is identical to the pseudorapidity $\eta = -\ln\tan(\theta/2)$ with θ the angle w.r.t. the $+z$-direction.

(b) The LHC experiments have extensive detector coverage up to $y \lesssim 2.5$ followed by more coarse-grained instrumentation up to $y \lesssim 5$. What scattering angles do these values correspond to?

At the LHC, the two incoming partons that initiate the hard scattering reaction carry the momentum fractions $x_{1,2}$ of the proton momenta $P^\mu_{1,2} = \frac{\sqrt{s}}{2}(1,0,0,\pm1)^{\mathrm{T}}$. Consider the scattering kinematics associated with di-jet production, $p_a + p_b \to p_1 + p_2$, where $p^\mu_{a,b} = x_{1,2}P^\mu_{1,2}$ and $p_1^2 = p_2^2 = 0$.

(c) Show that the momentum fractions $x_{1,2}$ can be expressed in terms of

$$x_{1,2} = \frac{2\,p_{\mathrm{T,avg}}}{\sqrt{s}}\,\mathrm{e}^{\pm y_b}\,\cosh(y^*), \tag{4}$$

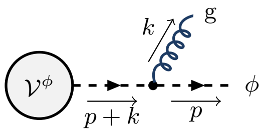
with the kinematic variables

$$p_{\mathrm{T,avg}} = \tfrac{1}{2}(p_{\mathrm{T},1} + p_{\mathrm{T},2}), \qquad y_b = \tfrac{1}{2}|y_1 + y_2|, \qquad y^* = \tfrac{1}{2}|y_1 - y_2|. \tag{5}$$

(d) What is the physical meaning of the variables y_b and y^*?

Exercise 2 *Soft and collinear limits with scalar particles*

Consider the emission of a gluon from an outgoing scalar particle ϕ:



. (6)

The scalar shall transform under the fundamental representation of $SU(N_\mathrm{c})$ and the Feynman rules are given by (momenta $p^\mu_\pm$ *ingoing*):

$$G^{\phi\phi^\dagger}(k) = \frac{\mathrm{i}}{k^2 - m_\phi^2 + \mathrm{i}\epsilon} \qquad \text{(scalar propagator)}, \tag{7}$$

$$V[\phi^\dagger(p_+), \phi(p_-), A^a_\mu] = \mathrm{i}g_\mathrm{s}t^a\,[p^\mu_+ - p^\mu_-] \qquad \text{(gluon–scalar vertex)}. \tag{8}$$

(a) Derive an expression for the eikonal current by considering the limit where the gluon becomes soft ($k^\mu \to 0$). How does it compare to the emission from a quark line?

(b) Consider the square of the diagram in Eq. (6) for a massless scalar ($m_\phi = 0$) and use the Sudakov parametrisation ($\tilde{p}^2 = n^2 = 0$, $(k_\perp \cdot \tilde{p}) = (k_\perp \cdot n) = 0$, $k_\perp^2 < 0$)

$$p^\mu = z\tilde{p}^\mu + k_\perp^\mu - \frac{k_\perp^2}{z}\,\frac{n^\mu}{2(\tilde{p}\cdot n)}, \tag{9}$$

$$k^\mu = (1-z)\tilde{p}^\mu - k_\perp^\mu - \frac{k_\perp^2}{(1-z)}\,\frac{n^\mu}{2(\tilde{p}\cdot n)}, \tag{10}$$

to obtain the behaviour in the limit $k_\perp^\mu \to 0$. Identify the Splitting function $\hat{P}_{\phi\phi}(z)$ for the scalar carrying the momentum fraction z (gluon carrying $(1-z)$).
Hint: Use $\sum_{\text{pol.}} \varepsilon_\mu^*(k)\varepsilon_\nu(k) = -g_{\mu\nu} + (k_\mu n_\nu + k_\nu n_\mu)/(k\cdot n)$ [$n_\mu \equiv$ gauge vector].

(c) Note that $\hat{P}_{\phi\phi}(z)$ computed in part (b) contains a soft divergence when the gluon becomes soft ($z \to 1$). We have so far only considered the real-emission corrections; convince yourself that the missing virtual corrections must be of the form $P_{\phi\phi}^{\text{virt}}(z) \sim \delta(1-z)$. Determine the *regularised* splitting function that includes both real and virtual corrections by using ϕ-number conservation:

$$\int_0^1 P_{\phi\phi}(z)\,\mathrm{d}z = 0. \tag{11}$$

Hint: In a first step introduce the *plus prescription* to regulate the soft divergence, $\int_0^1 \mathrm{d}z \frac{f(z)}{(1-z)_+} \equiv \int_0^1 \mathrm{d}z \frac{f(z)-f(1)}{(1-z)}$.

Exercise 3 *Infrared-safe observables*

Recall the conditions that an infrared-safe observable must fulfil. Explicitly check the criteria for the following observables and determine if they are infrared safe:

(a) The number of particles inside a jet.

(b) The number n_j of jets in an event.

(c) The sum of the (squared) energies

$$E_{\text{tot}} = \sum_{i=1}^{n} E_i, \qquad [E^2]_{\text{tot}} = \sum_{i=1}^{n} E_i^2, \tag{12}$$

where the sum runs over all final-state partons.

(d) The energy–energy correlation (EEC):

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\chi} = \int \mathrm{d}\sigma \sum_{i,j=1}^{n} \frac{E_i E_j}{(\sum_k E_k)^2}\,\delta(\chi - \theta_{ij}), \tag{13}$$

where the sums run over all final-state partons and θ_{ij} denotes the angle between parton i and j.

(e) The angularity

$$\tau_\beta = \sum_{i \in \text{jet}} z_i \, \theta_i^\beta , \tag{14}$$

where z_i denotes the energy fraction of particle i, θ_i its angle w.r.t. the jet axis, and $\beta \in \mathbb{R}$ in the exponent. Remind yourself that the emission probability has a divergence $\sim \mathrm{d}\theta_i/\theta_i$; what are the allowed values for β?

Exercise 4 *quark–gluon discrimination*

(a) Consider the angularity from Eq. (14) for a quark emitting a gluon in the soft *and* collinear limit. Show that up to $\mathcal{O}(\alpha_\mathrm{s})$, the cumulant for the τ_β distribution reads

$$\Sigma_q(\tau_\beta) = 1 - \frac{\alpha_\mathrm{s}}{\pi} \frac{C_\mathrm{F}}{\beta} \ln^2(R^\beta/\tau_\beta). \tag{15}$$

(b) The all-order resummed expression in the double-logarithmic approximation is then given by

$$\Sigma_q(\tau_\beta) = \exp\left[-\frac{\alpha_\mathrm{s}}{\pi} \frac{C_\mathrm{F}}{\beta} \ln^2(R^\beta/\tau_\beta) \right]. \tag{16}$$

How does the analogous expression look like for a gluon $\Sigma_\mathrm{g}(\tau_\beta)$?

(c) To separate quarks from gluon jets, we can place a cut τ_cut and only retain events with $\tau_\beta < \tau_\mathrm{cut}$; the fraction of quark (gluon) jets retained by such a cut is precisely the cumulant $\Sigma_{q(\mathrm{g})}(\tau_\mathrm{cut})$. Display the fraction of retained gluon jets v.s. quark jets in a receiver operating characteristic curve (ROC). The area under the ROC curve (AUC) is a discriminating metric often used in machine-learning applications. Compute the AUC.