MITP Summer School 2021: The Amplitude Games

Lectures: Basics of Scattering Amplitudes

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Introduction. We learned in the lectures that the KLT-form of the double copy is

$$M_n^{\mathrm{L}\otimes\mathrm{R}} = \sum_{\alpha,\beta} A_n^{\mathrm{L}}[\alpha]\, S_n[\alpha|\beta]\, A_n^{\mathrm{R}}[\beta]\,, \tag{1}$$

where α and β each index a set of R_n color-orderings and $S_n[\alpha|\beta]$ is the *double-copy kernel.* We also learned that the field theory kernel S_n is the inverse of a rank $R_n = (n-3)!$ submatrix of bi-adjoint scalar (BAS) tree amplitudes m_n which are calculated from the Lagrangian (The Lagrangian is not needed for the problem, so don't worry about it if you are not familiar with it)

$$\mathcal{L}_{\mathrm{BAS}} = -\frac{1}{2}\left(\partial_\mu \phi^{aa'}\right)^2 - \frac{g}{6} f^{abc}\tilde{f}^{a'b'c'}\phi^{aa'}\phi^{bb'}\phi^{cc'}\,. \tag{2}$$

In this problem, we explore a generalization of the double-copy kernel that arises from adding terms of the schematic form ϕ^4 to $\mathcal{L}_{\mathrm{BAS}}$. The purpose of the problem is to allow you to get some hands-on experience with the double-copy in the KLT-form and it will prepare you to do your own future explorations of the double-copy. Note: Mathematica (or similar) may be helpful.

Background

We start with the usual field theory KLT double-copy as a warm-up. The tree amplitudes of a bi-adjoint scalar can be doubly color-decomposed to give partial color-ordered amplitudes $m_n[\beta|\alpha]$ where β and α run over $(n-1)!$ independent color-orderings. We focus on 4-point so we have 6 color-orderings:

$$1234, 1243, 1324, 1342, 1423, 1432\,. \tag{3}$$

To simplify our analysis, we use trace-reversal symmetry (the three Kleiss-Kuijf (KK) relations discussed in Lecture 4) together with cyclic symmetry to reduce the number of independent color-orderings to 3 . For example, $1432 = 4321 = 1234$. We let the 3 orderings label the rows and columns of the the 3×3 matrix $\mathbf{m}_4$ of tree amplitudes $m_n[\beta|\alpha]$:

$$\mathbf{m}_4 = \begin{pmatrix} m_4[1234|1234] & m_4[1234|1243] & m_4[1234|1324] \\ m_4[1243|1234] & m_4[1243|1243] & m_4[1243|1324] \\ m_4[1324|1234] & m_4[1324|1243] & m_4[1324|1324] \end{pmatrix}. \tag{4}$$

These 9 matrix elements are not independent: they are related by cyclicity and momentum relabelings and there are only 2 independent functions, namely

$$\begin{aligned} m_4[1234|1234] &= f_1(s,t) \quad \text{with} \quad f_1(s,t) = f_1(u,t) \text{ by cyclicity}\,, \\ m_4[1234|1243] &= f_2(s,t)\,, \\ m_4[1234|1324] &= f_2(u,t)\,. \end{aligned} \tag{5}$$

where $s = s_{12}$, $t = s_{13}$, and $u = u_{14}$ satisfy $s + t + u = 0$. The last line follows from

$$m_4[1234|1324] = m_4[4123|4132] = m_4[1234|1243]\Big|_{1\to4\to3\to2\to1} = f_2(u,t)\,. \tag{6}$$

The rest of the elements in the matrix (4) can likewise be expressed in terms of f_1 and f_2, for example

$$m_4[1243|1234] = m_4[1234|1243]\Big|_{3\leftrightarrow4} = f_2(s,u)\,. \tag{7}$$

Now it is time for you to calculate!

Problem 0: Express the 3×3 matrix (4) in terms of f_1 and f_2.

You'll be using this result in Problems 1 and 2 below. In Problem 1, we consider the usual double-copy with a kernel based on the the BAS model, as described in the lecture. In Problem 2, you'll explore a new form of the double-copy based on a modification of the BAS model.

Problem 1: warm-up

(a) For the BAS model (2), we have

$$f_1(s,t) = \frac{g^2}{s} + \frac{g^2}{u} \quad \text{and} \quad f_2(s,t) = -\frac{g^2}{s}\,. \tag{8}$$

Show that with these functions, the matrix $\mathbf{m_4}$ in (4) has rank 1.

(b) When a 3×3 matrix has rank 1, it must have two null vectors. Show that

$$\mathbf{n}_1 = \big(1,1,1\big) \quad \text{and} \quad \mathbf{n}_2 = \big(1, -\tfrac{t}{u}, 0\big) \tag{9}$$

are null vectors of $\mathbf{m_4}$.

(c) Consider now the color-ordered 4-point amplitudes A_4 in a single-color theory and construct a 3-vector

$$\mathbf{A}_4 = \big(A_4[1234], A_4[1243], A_4[1324]\big)\,. \tag{10}$$

One can show in general (but you are not expected to do so) that the null vectors can be used to encode the KK + BCJ relations. In particular, as you can quickly see yourself,

$$\mathbf{n}_1 \cdot \mathbf{A}_4 = 0 \quad \text{and} \quad \mathbf{n}_2 \cdot \mathbf{A}_4 = 0 \tag{11}$$

are precisely the $U(1)$ decoupling identity and the BCJ relation described in Lecture 4. As an example, consider χPT (chiral perturbation theory) which has

$$A_4[1234] = c_0\, t\,. \tag{12}$$

(Often the coupling is written in terms of the pion decay constant as $c_0 = 1/f_\pi^2$.) Show that χPT satisfies the 4-point KK + BCJ relations (11).

(d) Double-copy χPT with itself and show that the result is the 4-point amplitude of the special Galileon. (Bonus for showing that your result is basis-independent.)

(e) Let us now consider higher-derivative corrections to χPT. As we learned in Lecture 1, it is a lot easier to enumerate local higher-derivative operators in terms of their matrix elements and in this case we can write

$$A_4[1234] = c_0 t + c_1 t^2 + c_2 su + b_1 t^3 + b_2 stu + \dots \tag{13}$$

Note that we are including only terms compatible with cyclic symmetry and trace-reversal. Subject this ansatz to the KKBCJ conditions to find out which constant Wilson coefficients c_1, c_2, b_1, b_2 are allowed as input for the double-copy. Then double-copy the answer and identify the local operator (in the schematic form $\partial^{2k}\phi^4$, don't worry about the color-contractions) that each term corresponds to.

The purpose of this exercise was to get you familiar with the KLT formulation of the double-copy and produce a result that you can compare your new answers to in the next problem with. Now the real work begins!

Problem 2: modifying the double-copy Consider the following modification of the BAS model with ϕ^4 operators:

$$f_1(s,t) = \frac{g^2}{s} + \frac{g^2}{u} + a_1 \quad \text{and} \quad f_2(s,t) = -\frac{g^2}{s} + a_2\,. \tag{14}$$

That there are two constants signify that there are two independent contractions of the bi-adjoint indices in ϕ^4. For generic values of a_1 and a_2, the 3×3 matrix $\mathbf{m}_3$ has full rank. But there are two choices that give rank 2. One of those choices is

$$a_1 = a_2\,. \tag{15}$$

This is the solution we'll study first. When $\mathbf{m}_3$ has rank 2, the KLT algebra dictates that the 4-point KLT sum must be taken over 2 color-orderings since S_4 is now the inverse of a 2×2 submatrix of $\mathbf{m}_3$. The generalized KKBCJ relations (that follow from the null vector of $\mathbf{m}_3$) will ensure basis independence of which 2×2 submatrix is chosen for the kernel.

There is however an apparent problem with this new 2×2 double-copy kernel. Compute the determinant of some of the 2×2 submatrices. You'll see that they have zeroes that, when the submatrix is inverted, correspond to unphysical poles! It seems very dangerous to have unphysical poles in the double-copy kernel because of the risk that the result of the double copy, $M_n^{\mathrm{L}\otimes\mathrm{R}}$, on the RHS of (1), would inherit these spurious poles and therefore not correspond to a tree amplitude of a local field theory $\mathrm{L} \otimes \mathrm{R}$. Yet, there is a chance that spurious poles might cancel in the double-copy sum and that is exactly what you are now going to explore.

(a) For the choice (15), go through the analysis parallel to what you did in Problem 1. Show that χPT (12) can be double-copied with this rank 2 kernel: is the result local? Does it make sense?? What is it???

Compute the leading non-vanishing higher-derivative correction to χPT which can be double-copied with the rank 2 kernel. Double-copy it. Compare your answer to the result for the rank 1 double-copy in Problem 1(e). Comment on the result.

(b) Find the other choice of a_1 and a_2 that gives rank 2. Analyze it the same way as in part (a) above. What do you find??? Can you make $M_n^{\mathrm{L}\otimes\mathrm{R}}$ local? Tell me more!

(c) Say something qualitatively about what kind of group-theory structures enter into the two ϕ^4-operators the two types of rank 2 deformations correspond to.

If you are energetic, try to

(d) do the analysis of the two rank 2 double-copies for YM theory: do both double-copies work? What about YM + higher derivatives?

There is no guarantee that the modified double-copy kernels we have studied here is going to be sensible for higher point amplitudes (i.e. avoid spurious poles). To examine this at 5- and 6-point is basically a research-project. Will some of you take it on? If so, keep me posted on your findings!!

Epilogue. I hope this problem raises some curiousity — and hopefully also questions. There are a few things I'd like to comment on here:

The addition of ϕ^4 moves the true vacuum away from the origin; however, we continue here to do perturbation theory around the origin for the purpose of computing the tree amplitudes. In a sense this is not worse than doing perturbation theory around $\phi = 0$ in a ϕ^3 theory, and by doing this we avoid issues of giving a vev to a bi-adjoint field. We are in no way saying that this model is a sensible *quantum* field theory; it is not, but neither is BAS. Rather, it is a tool for exploring the double-copy in a systematic manner.

Perhaps you wonder why even try to modify the double-copy? After all, the standard double-copy based on the BAS amplitudes works just fine, right? Or the strings KLT double-copy kernel I mentioned in the lectures, that one is great too... except that its α'-dependence is delicately tuned to the α'-dependence of the open string amplitudes. It is not entirely obvious that one should use the field theory kernel or the α'-expanded string kernel to double-copy an effective field theory, such as YM + higher-derivative operators, with general higher-derivative corrections. Hence, it is of interest to explore what higher-derivative corrections are generically allowed in the double-copy kernel and how that affects which EFTs can be double-copied. And what EFTs can arise as the double-copy of others.

Moreover, exploring the inner workings of the double-copy is a way to examine the intricate nature of the map and learn more about how it works. Modifying it is kind of like bombarding gold-foil with alpha-particles just to see what happens. We poke at the double-copy and try to break it to figure out what the rules are and learn more about what models can be double-copied and which ones cannot. In the case here, the modification to BAS appears to immediately break it by causing the double-copy kernel to have spurious poles. Yet, can it prevail? How and why and when? Does it continue to hold at higher-point or must one always work with kernels of "minimal" rank $(n-3)!$? Is there some BCJ or CHY formulation of other double-copies? In the future, you guys may tell me the answers to these questions.