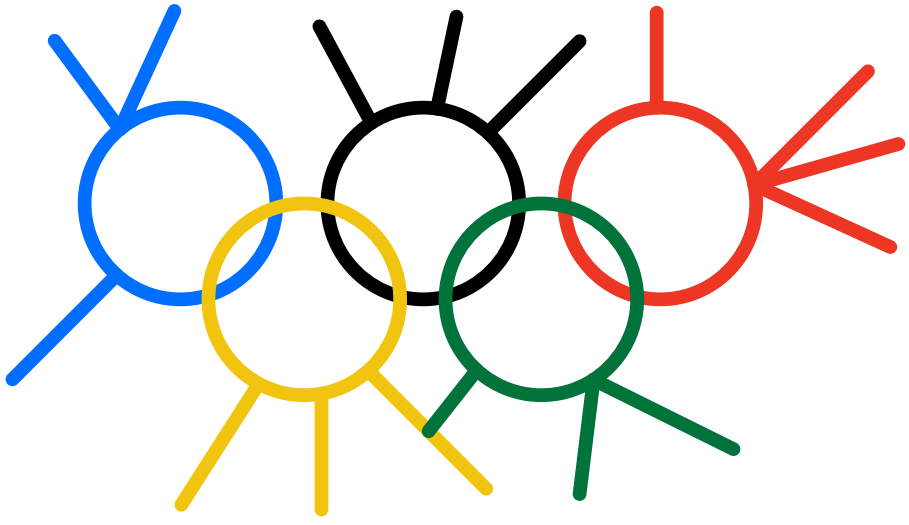
The Amplitudes Games



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Basics of Scattering Amplitudes

Lecture 3 Part 3

Soft Theorems

Henriette Elvang (LCTP & Univ. of Michigan)

LECTURE 3 PART 3

Apply the Master Formula:

$$\hat{A}_{n+1}(z,\varepsilon) = \sum_{k=1}^{n} g_k \frac{[sk]^{2h_s-\partial} \langle Xs \rangle^{1-\partial} \hat{A}_n^{(k)}(z)}{\varepsilon\, z^{\partial-1} \langle sk \rangle^{2-\partial} \left(1 - \frac{z}{\varepsilon}\frac{\langle Xk \rangle}{\langle sk \rangle}\right)} + O(\varepsilon^0)$$

w/ $\partial = [g_k] + 2h_s$ $\qquad z < \varepsilon << 1$

Photon $h_s = +1$ $\qquad [g_k] = 0$

$\partial = 2$

$$\hat{A}_{n+1}(z,\varepsilon) = \sum_{k=1}^{n} g_k \frac{[sk]^{\cancel{2h_s-\partial}\,0} \langle Xs \rangle^{(1-\partial)\,-1} \hat{A}_n^{(k)}(z)}{\varepsilon\, z^{\partial-1\,1} \langle sk \rangle^{\cancel{2-\partial}\,0} \left(1 - \frac{z}{\varepsilon}\frac{\langle Xk \rangle}{\langle sk \rangle}\right)} + O(\varepsilon^0)$$

Then

$$\hat{A}_{n+1}(z,\varepsilon) = \sum_{k=1}^{n} g_k \frac{\hat{A}_n(z)}{\varepsilon\, z \langle Xs \rangle \left(1 - \frac{z}{\varepsilon}\frac{\langle Xk \rangle}{\langle sk \rangle}\right)}$$

pole $\frac{1}{z}$! ξ finite, $z \to 0$ pole !

Amplitude can only have poles where physical particles are exchanged, ie. at

$$\xi = \xi_k$$

So $\frac{1}{z}$ pole is UNPHYSICAL !

Hence the residue at $z=0$ must vanish when summed over k.

$\frac{1}{z}$ - residue:

$$\sum_{k=1}^{n} g_k \frac{A_n(0)}{\langle X_S \rangle} \overset{!}{=} 0$$

$$\Rightarrow \boxed{\sum_{k=1}^{n} g_k = 0} \leftarrow$$ that's just charge conservation.

$$\frac{1}{1-\frac{z}{\varepsilon}\frac{\langle Xk\rangle}{\langle sk\rangle}} \simeq \left(1+\frac{z}{\varepsilon}\frac{\langle Xk\rangle}{\langle sk\rangle}+\dots\right)$$

$\frac{1}{z}$ pole $\Rightarrow$ zero by charge conservation

So as $\varepsilon \to 0$

$$\hat{A}_{n+1}(z,\varepsilon) \to \frac{1}{\varepsilon^2}\underbrace{\frac{\langle Xk\rangle}{\langle sk\rangle\langle Xs\rangle}}_{S^{(0)}} A_n$$

$S^{(0)}$ leading soft photon factor.

$$\frac{\langle Xk\rangle[ks]}{\langle Xs\rangle\langle sk\rangle[ks]} = \frac{\langle X|k|s]}{\langle Xs\rangle 2p_s\cdot k}$$

$\varepsilon_s \cdot k$

ref spinor of ε^+_μ polarization

Subleading soft photon theorem $S^{(1)}$

arises $\hat{A}_n^{(n)}(z) \simeq \hat{A}_n^{(k)}(z=0) + z\,\partial_z \hat{A}^{(k)}(z)\Big|_{z=0}$

$\hat{A}_n^{(k)}(z=0)$: leading $S^{(0)}$ $+\dots$

Gravitons $h_s = 2$

$[g_k] = -1 \qquad d = [g_k] + 2h_s = 3$

$$\hat{A}_{n+1}(z,\varepsilon) = \sum_{k=1}^{n} g_k \frac{[sk]^{2h_s-d}\,\langle Xs\rangle^{1-d}\,\hat{A}_n^{(k)}(z)}{\varepsilon\, z^{d-1}\langle sk\rangle^{2-d}\left(1-\frac{z}{\varepsilon}\frac{\langle Xk\rangle}{\langle sk\rangle}\right)} + O(\varepsilon^0)$$

($2h_s-d = 1$, $1-d = -2$, $d-1 = 2$, $2-d = -1$)

$$\hat{A}_{n+1}(z,\varepsilon) = \sum_{k=1}^{n} g_k \frac{[sk]\langle sk\rangle\,\hat{A}_n^{(k)}(z)}{\varepsilon\, z^2 \langle Xs\rangle^2\left(1-\frac{z}{\varepsilon}\frac{\langle Xk\rangle}{\langle sk\rangle}\right)} + O(\varepsilon^0)$$

$\frac{1}{z^2}$ pole? Unphysical? Residue must vanish.

$\frac{1}{z^2}$ residue:

$$0 \overset{!}{=} \sum_{k=1}^{n} g_k \frac{[s_k]\langle s_k\rangle \hat{A}_n(0)}{\varepsilon \langle X_s\rangle^2}$$

$$\Rightarrow \quad 0 = \sum_{k=1}^{n} \underbrace{[s_k]\langle k s\rangle}_{P_k} \underbrace{g_k}_{\text{same!}}$$

$\Downarrow$

this vanishes by mom. conservation

$g_k \equiv k \Leftarrow$ graviton must couple to every other particle with the SAME coupling!

$\Leftarrow$ **equivalence principle.**

What about $\frac{1}{z}$ pole?

$$\sum_{k=1}^{n} k\,[ks]\langle kX\rangle = 0 \qquad \checkmark$$

mom. consv.

$\frac{1}{z}$ OK.

$O(z^0) \Rightarrow$ soft factors $\underset{\frac{1}{\varepsilon^3}}{\mathcal{S}^{(0)}}, \underset{\frac{1}{\varepsilon^2}}{\mathcal{S}^{(1)}}, \underset{\frac{1}{\varepsilon}}{\mathcal{S}^{(2)}}$

$\mathcal{S}^{(0)}$:

$$\hat{A}_{n+1}(z_i,\varepsilon) = \sum_{k=1}^{n} g_k \frac{[sk]\langle sk\rangle \hat{A}_n^{(k)}(z)}{\varepsilon z^2 \langle Xs\rangle^2 \left(1 - \frac{z}{\varepsilon}\frac{\langle Xk\rangle}{\langle sk\rangle}\right)} + O(\varepsilon^0)$$

$z=0$

k

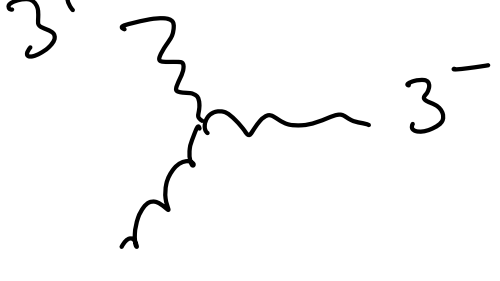
$$\frac{1}{\left(1-\frac{z}{\xi}\frac{\langle Xk\rangle}{\langle sk\rangle}\right)} = 1 + \underbrace{\frac{z}{\xi}\frac{\langle Xk\rangle}{\langle sk\rangle}}_{\frac{1}{z}\text{ vanishing residue at } z=0} + \frac{z^2}{\xi^2}\frac{\langle Xk\rangle^2}{\langle sk\rangle^2} + \cdots \quad \subset \frac{1}{\xi^3}$$

$$\hat{A}_{n+1}(z,\xi) = \frac{1}{\xi^3}\sum_{k=1}^{n} k\,\underbrace{[sk]\langle sk\rangle}_{\text{energy}}\,\underbrace{\frac{\langle Xk\rangle^2\,\hat{A}_n(0)}{\langle sk\rangle^2\langle Xs\rangle^2}}_{\left(\text{"}s^{(0)}\text{"}\right)^2}$$

$$= \frac{1}{\xi^3}\,\mathcal{J}^{(0)} A_n$$

Higher spin ?

Suppose spin 3 massless $h_s = 3$



$$[g_k] = 1 - 3 + 3 - 2 = -1$$

$$d = -1 + 2\cdot 3 = 5$$

$$\hat{A}_{n+1}(z,\varepsilon) = \sum_{k=1}^{n} g_k \frac{[sk]^{\overbrace{2h_s-d}^{1}} \langle Xs\rangle \hat{A}_n^{(k)}(z)}{\varepsilon\, z^{\underbrace{d-1}_{4}} \langle sk\rangle^{\overbrace{2-d}^{-3}} \left(1 - \frac{z}{\varepsilon}\frac{\langle Xk\rangle}{\langle sk\rangle}\right)} + O(\varepsilon^0)$$

$\overbrace{}^{-4}$ over $1-d$

$\frac{1}{z^4}$ -pole must have vanishing residue.

↳ $$0 \overset{!}{=} \sum_{k=1}^{n} g_k [sk]\langle sk\rangle^{3} \hat{A}_n^{(k)}(z)$$

$\neq 0$

massless

Therefore V spin 3 cannot couple to the graviton

Can prove more generally that massless spin $s > 2$ cannot couple consistently

to the graviton.

Very fundamental results from soft limits.

The recorded lecture ends here.

Below I show how a gravitino must couple supersymmetrically to gravitons

GRAVITINOS

- Suppose we have a gravitino $h = \frac{3}{2}$
- Know now (equivalence principle) that gravitons must couple to all other states w/ coupling $[\kappa]$:

$$A_3(1^{+2}\ 2^{+\frac{3}{2}}\ 3^{h_3})$$

graviton gravitino

dimensional analysis $\Rightarrow -1 = [\kappa] = 1 - h_1 - h_2 - h_3$

$$\Rightarrow -1 = 1 - 2 - \frac{3}{2} - h_3 \Rightarrow h_3 = -\frac{3}{2}$$

So we must have

$$A_3(1^{+2}\ 2^{+\frac{3}{2}}\ 3^{-\frac{3}{2}})$$

Consider now

$$A_{n+1}\left(s^{\frac{3}{2}}\, 1^{-\frac{3}{2}}\, 2^{+2}\, 3^{+2}\, \underbrace{4^{-2} \cdots n^{-2}}_{\text{all neg hel. gravitons}}\right)$$

has $\quad \alpha = [k] + 2h_s = -1 + 2\cdot\frac{3}{2} = 2$

Master formula

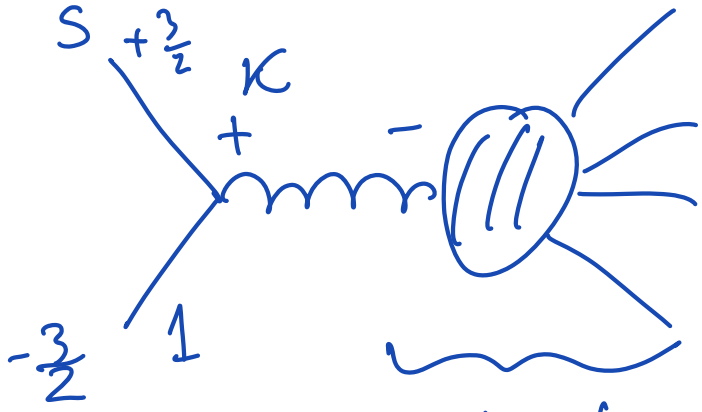
$$\hat{A}_{n+1}(z,\xi) = \sum_{k=1}^{n} g_k \frac{[sk]^{2h_s-\alpha}\, \langle Xs\rangle^{1-\alpha}\, \hat{A}_n^{(k)}(z)}{\xi\, z^{\alpha-1}\, \langle sk\rangle^{2-\alpha}\, \underbrace{\left(1-\frac{z}{\xi}\frac{\langle Xk\rangle}{\langle sk\rangle}\right)}} + O(\xi^0)$$

has $\frac{1}{z}$ pole $\therefore$ residue must be zero

$$\rightarrow \quad 0 \overset{!}{=} \sum_{k=1}^{n} g_k\, [sk]\, \langle Xs\rangle^{-1}\, \hat{A}^{(k)}(z=0)$$

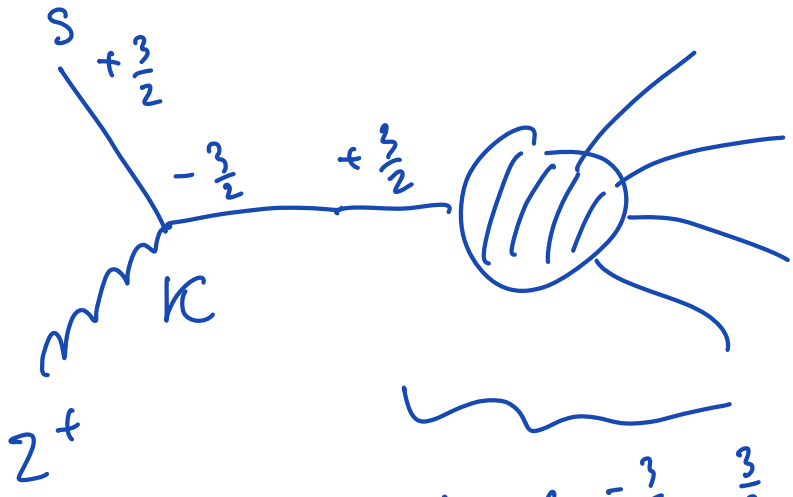
What do the terms look like:

$k=1$



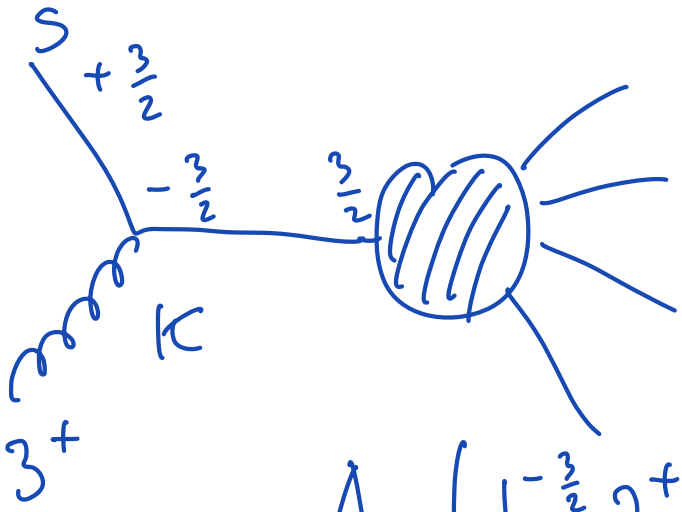
$$A_n(1^- 2^+ 3^+ 4^- \dots n^-)$$

$k=2$

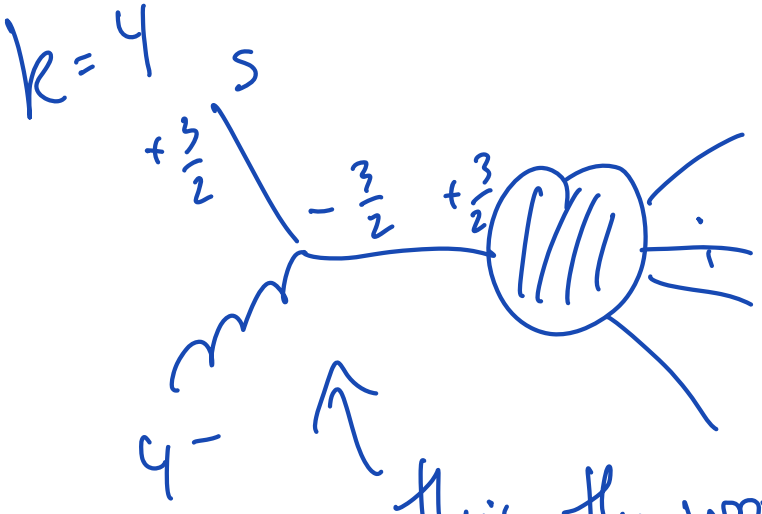


$$A_n(1^{-\frac{3}{2}} 2^{\frac{3}{2}} 3^+ 4^- \dots n^-)$$

$k=3$



$$A_n(1^{-\frac{3}{2}} 2^+ 3^{\frac{3}{2}} 4^- \dots n^-)$$



this the wrong kinematics!

$\sum h_i < 0$

$\Rightarrow$ does not contribute.

Same for $k = 4, 5, \ldots, n$.

So 3 terms contribute and the vanishing residue condition is

$$0 = [s1]\, A_n(1^- 2^+ 3^+ 4^- \cdots n^-)$$
$$+ [s2]\, A_n(1^{-\frac{3}{2}} 2^{+\frac{3}{2}} 3^+ 4^- \cdots n^-)$$
$$+ [s3]\, A_n(1^{-\frac{3}{2}} 2^+ 3^{+\frac{3}{2}} 4^- \cdots n^-)$$

This is a SUSY Ward identity!

If we choose $s=3$, it says

$$A_n(1^{-\frac{3}{2}}\, 2^{+\frac{3}{2}}\, 3^+ 4^- \dots n^-)$$

$$= -\frac{[13]}{[23]} A_n(1^- 2^+ 3^+ 4^- \dots n^-)$$

↖ An example of a SUSY MHV Ward identity!

Gravitinos must couple supersymmetrically to gravitons!