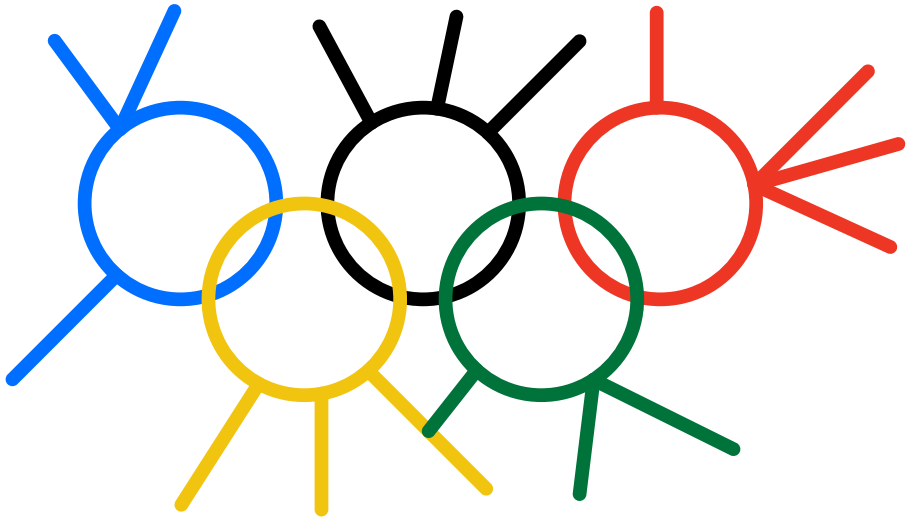
The Amplitudes Games



MITP School 2021

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Basics of Scattering Amplitudes

Lecture 1

Scalar Theories & The Soft Bootstrap

Henriette Elvang (LCTP & Univ. of Michigan)

LECTURE 1

Traditional QFT class

$\mathcal{L}$ → Feynman Rules → Amplitudes

Symmetries ⟶ Ward identities

$\delta\mathcal{L}$ = total derivative

you learn about trees, loops, renormalization, and much more.

All very good and useful stuff!

But: typically it isn't taught how to backwards ?

WHAT ???

Backwards?

- construct the physical observables without direct reference to a Lagrangian or Feynman rules, but exploit their physical properties:

  poles, cuts, scaling, dimensional analysis, ...

- use amplitudes to explore the space of field theories.

Goal of these lectures

Provide an introduction to a selection of modern amplitudes methods and their applications

Plan:

L1: scalar theories: foundations & soft bootstrap.

L2: spinor-helicity formalism in 4d

L3: soft theorems

L4: double-copy

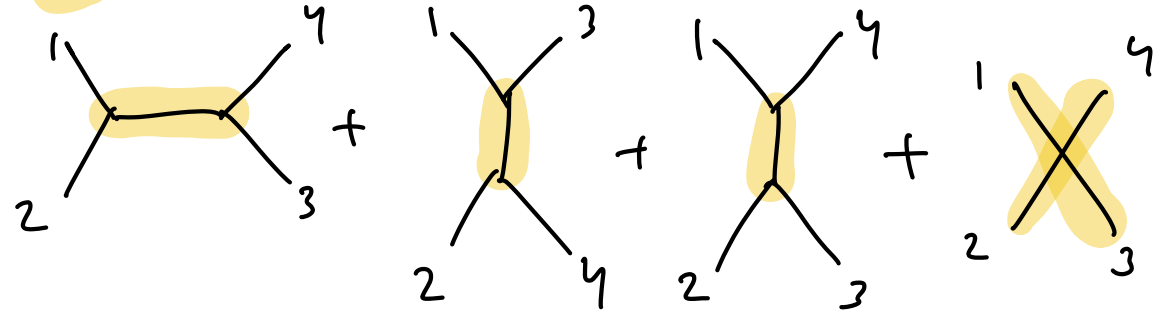
Let's start with something familiar. A model with a massless real scalar φ:

$$\mathcal{L} = -\frac{1}{2}(\partial\varphi)^2 + \frac{\lambda_3}{3!}\varphi^3 + \frac{\lambda_4}{4!}\varphi^4 + \frac{\lambda_5}{5!}\varphi^5 + \dots$$

Calculate the tree amplitudes:

$$\mathcal{A}_3 = \lambda_3$$

$$\mathcal{A}_4 =$$



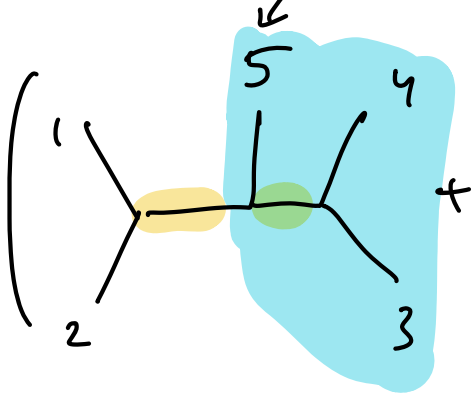
contact term

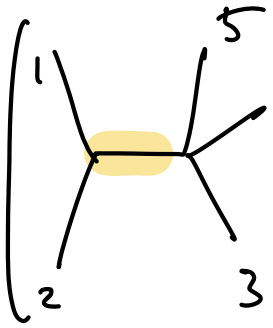
$$= \lambda_3^2\left(\frac{1}{s_{12}} + \frac{1}{s_{13}} + \frac{1}{s_{14}}\right) + \lambda_4$$

where $\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_4 = 0$

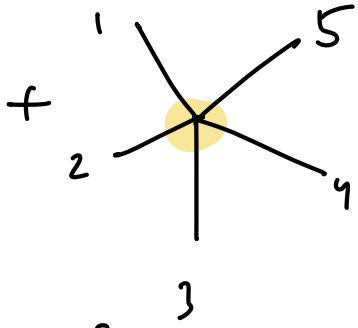
$s_{ij} \equiv (p_i + p_j)^2$ Mandelstam variables.

$s_{12} = s_{34}$
$s_{13} = s_{24}$
$s_{14} = s_{23}$

$$A_5 =$$ 

$+ \text{perms}) +$ 

$+ \text{perms})$

$+$ 

$$= \left(\frac{\lambda_3^3}{s_{12}\, s_{34}} + \text{perms} \right) + \left(\frac{\lambda_3 \lambda_4}{s_{12}} + \text{perms} \right) + \lambda_5$$

and so on.

Pole structure and factorization

- Tree amplitudes are rational functions with simple poles located precisely where physical particles are exchanged.

Residues

$$s_{12} A_4 \Big|_{s_{12}=0} = \lambda_3 \times \lambda_3 = A_3 \times A_3$$

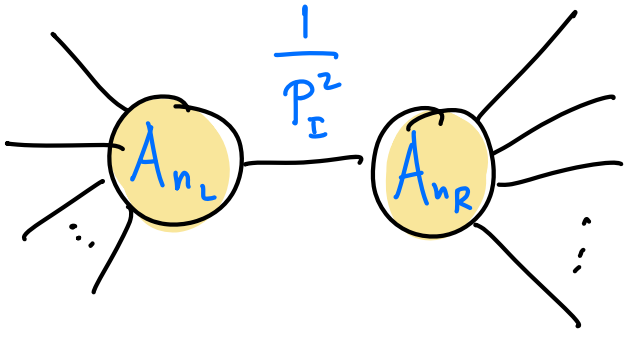
$$s_{12} A_5 \Big|_{s_{12}=0} = \lambda_3^3 \left(\frac{1}{s_{34}} + \frac{1}{s_{35}} + \frac{1}{s_{45}} \right) + \lambda_3 \lambda_4$$

$$= \underbrace{A_3(1,2,-P_{12})}_{\lambda_3} \underbrace{A_4(3,4,5,P_{12})}_{\lambda_3^2 (\text{poles}) + \lambda_4}$$

Lesson:

- On each simple pole, the tree amplitude factorizes into on-shell lower-point amplitudes

$$A_n \sim \frac{1}{P_I^2} A_{n_L} A_{n_R} \qquad P_I^2 = 0$$



$$n_L + n_R = n + 2$$

Upshot:

- We can construct tree amplitudes from their poles & local terms

  polynomial (no poles)

The understanding of this analytic structure is the basis of on-shell RECURSION RELATIONS

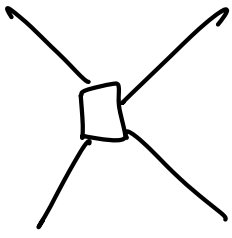
[BCFW, all-line shifts, soft recursion, ...]

↖ not covered in these lectures, but you can learn about them in various reviews & amplitude textbooks.

Scalar EFTs (Effective Field Theories)

$$\mathcal{L} = -\frac{1}{2}(\partial\varphi)^2 + \underbrace{\frac{g_4}{4}\varphi^2(\partial\varphi)^2} + \dots$$

$$\partial_\mu \to i p_\mu$$



$$= -g_4 \left(p_1 \cdot p_2 + \dots \right)$$

permutations

on-shell amplitude $p_i^2 = 0$

$$A_4 = -g_4 \left(\frac{1}{2}(p_1+p_2)^2 - \frac{1}{2}\cancel{p_1^2}^{\,0} - \frac{1}{2}\cancel{p_2^2}^{\,0} + \text{perms} \right)$$

$$= -g_4 \underbrace{\left(s_{12} + s_{23} + s_{13} \right)}_{s+t+u \;=\; 0}$$

$$\Rightarrow \boxed{A_4 = 0}$$

Why $A_4 = 0$?

(i) $\varphi^2 (\partial\varphi)^2 = \frac{1}{3}(\partial_\mu \varphi^3)\partial^\mu \varphi$

$= \underbrace{\frac{1}{3}\partial_\mu(\varphi^3 \partial^\mu \varphi)}_{\text{total derivative}} - \underbrace{\frac{1}{3}\varphi^3 \Box\varphi}_{\text{vanish by EOM}}$

$\Box\varphi = 0$

$p_1^\mu + p_2^\mu + p_3^\mu + p_4^\mu = 0$

$p_i^2 = 0$

(ii) Field redefinitions

$$\varphi \to \varphi + \alpha \varphi^3$$

$$\partial_\mu \varphi \to \partial \varphi + 3\alpha \varphi^2 \partial_\mu \varphi$$

$$(\partial_\mu \varphi)^2 = (\partial_\mu \varphi)^2 + 6\alpha \varphi^2 (\partial \varphi)^2 + O(\varphi^6)$$

$$-\frac{1}{2} 6\alpha = \frac{g_4}{4} \Rightarrow \alpha = \frac{-g_4}{12}$$

$\Rightarrow$ no 4-pt interaction term!

Lessons: Amplitudes are invariant under field redefinitions

Lagrangians more complicated than amplitudes.

On-shell methods avoids Lagrangian ambiguities related to

- field redef.
- partial integration
- EOM
- gauge invariance.

Example:

What can $A_n(\phi\phi\dots\phi)$ possibly be in a generic EFT for a massless scalar.

$\mathcal{L}$ $\longleftrightarrow$ scalar Ampl.

1) Lorentz inv. $\longleftrightarrow$ depends on $s_{ij} = (p_i + p_j)^2$.

mom. conservation.

2) Locality $\longleftrightarrow$

- simple poles w/ residues that are products of lower-pt on-shell amplitudes
- local operator n fields $\Leftrightarrow$ n-pt local matrix element.

polynomial in s_{ij}.

3) Dim'l analysis $\longleftrightarrow$ $[A_n] = 4 - n$

mass-dim.

local operator.

$\partial^{2n} \phi^k$ $\longleftrightarrow$ degree n polynomial in s_{ij}.

4) Bose sym $\longleftrightarrow$ symmetric in interchanges of momenta associated w/ identical particles.

Let's now see how this works:

3pt

$A_3(\phi\phi\phi) = \lambda_3$ nothing else!

b/c $s_{12} = (p_1+p_2)^2 = p_3^2 = 0$

$\phi^2\partial^2\phi \to 0$ by eom

4pt

$$A_4(\phi\phi\phi\phi) = (\text{pole terms}) + \lambda_4$$

$$+ g_4 \cancel{(s+t+u)} \to 0 \qquad \phi^2(\partial\phi)^2$$

$$+ g_4' (s^2+t^2+u^2)$$

$$+ g_4'' (s^3+t^3+u^3) = -3stu \leftarrow$$

$$+ g_4''' (s^4+t^4+u^4)$$

$$+ g_4'''' (s^5+t^5+u^5)$$

$$+ g_4''''' (s^6+t^6+u^6) \Big\} O(p^{12})$$

$$+ g_4'''''' (s^2t^2u^2)$$

$$+ \dots$$

$$\phi^2 (\underbrace{\partial \dots \partial}_{6} \phi)^2 \qquad \overbrace{(\partial\partial\dots\partial\varphi)}^{6}(\partial\varphi)(\partial\varphi)(\partial\varphi)(\partial\varphi)$$

↖ two indep local ↗ operators

$O(p^{24})$ → 3 indep ...

$O(p^{36})$ → 4 indep ...

Lesson Very easy to deal w/ polynomials
Very messy to deal w/ integration by parts & EOM.

Applications (w/ or w/o masses & spins)

- SMEFT
- characterizes higher-derivative operators in (S)YM or (super)gravity

(eg. α'-corrections in low-E string theory or first UV divergence).

Symmetries

- Unbroken symmetries $\hookrightarrow$ Ward identities

Ex. $\mathcal{L}$ has a global U(1)

$\Updownarrow$

charge conservation.

$$\left.\begin{aligned} z &\to e^{i\alpha} z \\ \bar{z} &\to e^{-i\alpha} \bar{z} \end{aligned}\right\} \to A_n(\underbrace{z \cdots z}_{n_z}\underbrace{\bar{z} \cdots \bar{z}}_{n_{\bar{z}}}) = 0$$

unless $n_z = n_{\bar{z}}$.

Duality symmetries (in this context)

$\mathcal{L}$ does not have symmetry but

EOM do have the symmetry.

$\Rightarrow$ tree amplitudes have an associated Ward identity.

ex. EM duality $\longleftrightarrow$ optical helicity conservation.

Spontaneously broken symmetry

Symmetry of $\mathcal{L}$, but not of the vacuum.

global sym. spontaneously broken

$\Rightarrow$ Ward id: soft theorems.

$G \longrightarrow H$

Goldstone modes = # broken generators

$= \dim G - \dim H = \dim(G/H)$

Goldstones $\in$ G/H

scalars

ϕ

$\hookrightarrow$ coset

$A_n(\phi \cdots \phi) \xrightarrow[\varepsilon \to 0]{} O(\varepsilon^\sigma)$

degree of softness

1 more soft: $p_\phi \to \varepsilon p_\phi$

- internal sym. broken $\sigma = 0$ or 1
- translational sym. $\sigma = 2$

$\sigma > 0$ amplitude vanishes in the soft limit :. Adler zero.

What scalar EFTs exist with vanishing soft theorems $\sigma > 0$?

- 3pt: $A_3 = \lambda_3$

$$\Rightarrow A_4 = \lambda_3^2 \left(\frac{1}{s} + \frac{1}{t} + \frac{1}{u} \right) \longrightarrow \infty$$

$p_1 \to 0$

"$\sigma = -1$"

So if we want $\sigma > 0$, then

$$\lambda_3 = 0 \quad \Rightarrow \quad A_3 = 0.$$

$\sigma = 0$

• 4pt

$$A_4(\phi\phi\phi\phi) = \cancel{\lambda_4}^{\,0} + g_4\,(\underbrace{s^2+t^2+u^2}_{\sigma = 2}) + \dots$$

b/c $\sigma > 0$ wanted.

$$s_{12}^2 + s_{13}^2 + s_{14}^2 \to \varepsilon^2$$

$$p_1 \to \varepsilon p_1$$
$$\varepsilon \to 0.$$

$\sigma = 0$

• 5pt

$$A_5(\phi\phi\phi\phi\phi) = \cancel{\lambda_5}^{\,0} + \#\,(\underbrace{s_{12} + s_{13} + \dots}_{=\,0})$$

$$+ g_5(\underbrace{s_{12}^2 + \text{perms}})$$

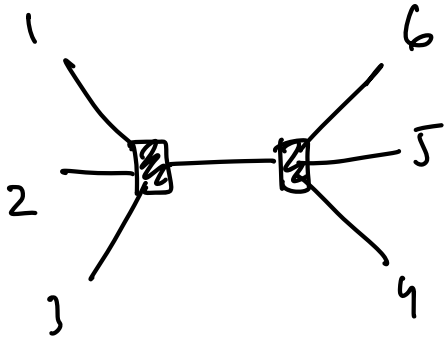
$$p_1 \to 0 \quad s_{23}^2 + s_{24}^2 + s_{25}^2 + s_{34}^2 + s_{35}^2 + s_{45}^2 \neq 0$$

$\sigma = 0$

+ higher-derivative order.
to get $\sigma > 0$.

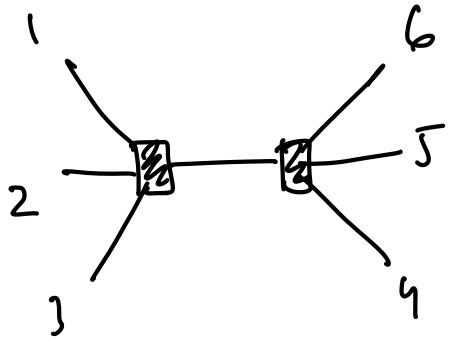
- 6pt

$$A_6(\phi\phi\phi\phi\phi\phi) =$$



+ perm.

+ polynomial terms

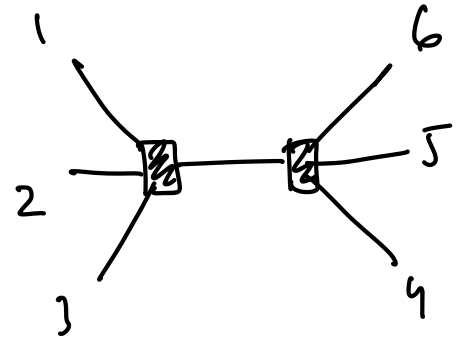


$$\xrightarrow{\text{at pole}} = \frac{A_4(1,2,3,-P_{123})\,A_4(4,5,6,P_{123})}{P_{123}^2}$$

$$= g_4^2 \frac{(s_{12}^2 + s_{13}^2 + s_{23}^2)(s_{45}^2 + s_{46}^2 + s_{56}^2)}{s_{456}}$$

$s_{46}^2 \to 0$, $s_{56}^2 \to 0$, $s_{456} \to 0$

$$\xrightarrow[P_6 \to 0]{} g_4^2 (s_{12}^2 + s_{13}^2 + s_{23}^2)\, s_{45} \neq 0$$



$+ \text{perms} \xrightarrow[p_6 \to 0]{} g_4^2 \left(S_{12}^3 + \text{perms}_{12345} \right)$

$\neq 0$

$\uparrow$

$O(p^6)$

$O(p^6)$

$$\text{poly}_6 = \lambda_6 \left(S_{12}^3 + \text{perms}_{123456} \right) \xrightarrow[p_6 \to 0]{} \lambda_6 \left(S_{12}^3 + \text{perms}_{12345} \right)$$

$$+ \tilde{\lambda}_6 \left(S_{12} S_{34} S_{56} + \text{perms}_{123456} \right) \xrightarrow[p_6 \to 0]{} \sigma = 1$$

I can tune $\lambda_6 = -g_4^2$ in order to get $\sigma = 1$.

What about $\tilde{\lambda}_6$?

$\Rightarrow$ $\sigma = 1$ model is NOT unique

But we can choose $\tilde{\lambda}_6$ to get

$$A_6(\phi\phi\cdots\phi) \longrightarrow O(\varepsilon^2) \qquad \sigma = 2.$$
$$p_6 \to 0$$

$$\tilde{\lambda}_6 \propto \# g_4^2$$

In fact the model w/ a single real scalar and $\sigma = 2$ is unique!

I can construct all A_n systematically

$\Downarrow$

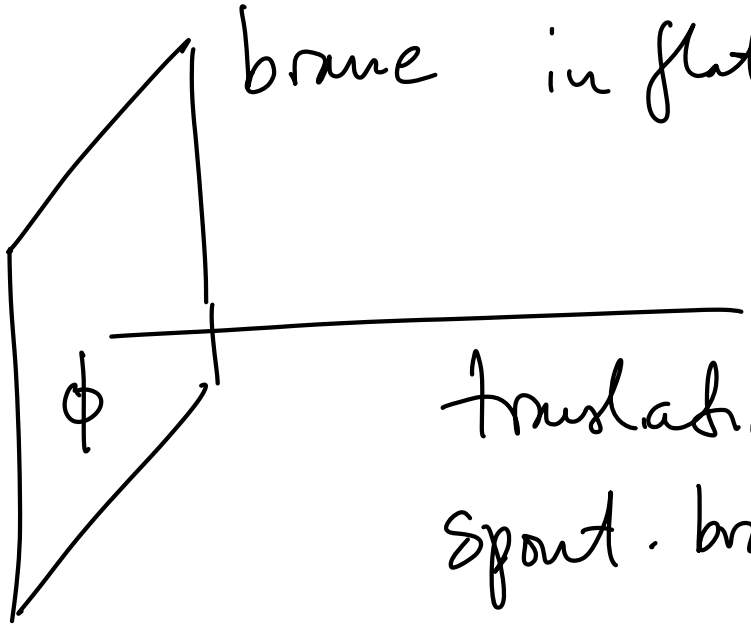
reconstruct the Lagrangian:

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \frac{1}{8}(\partial\phi)^4 - \frac{1}{16}(\partial\phi)^6 + \frac{5}{128}(\partial\phi)^8 + \cdots$$

$\Rightarrow$

$$\mathcal{L} = -\sqrt{1+(\partial\phi)^2} + 1$$

This is the $\mathcal{L}$ of Dirac-Born-Infeld model (DBI)



ϕ is the associated Goldstone mode!

Why $\sigma = 2$?

$\phi \to \phi + c$ (c: const.) $\Rightarrow \sigma = 1$

$\phi \to \phi + c + v_\mu x^\mu$ + field redef. $\Rightarrow \sigma = 2$.

↖ rotations / boosts.

- What about $\sigma = 3$?

$$A_4(\phi\phi\phi\phi) = \tilde{\lambda}_4\, stu \to \varepsilon^3$$

$p \to \varepsilon p$, $\varepsilon \to 0$.

$\mathcal{L} \supset (\partial\varphi)^2(\partial\partial\varphi)^2$ special Galileon ← $\sigma = 3$

- What about $\sigma \geq 4$? NO!

Scalar ϕ in adjoint repr. $\phi = \phi^a T^a$

color or flavor group.

$$A_4 = \sum_{\substack{\sigma \\ \text{perms} \\ 2,3,4}} \underbrace{A_4[\phi_1\ \phi_{\sigma(2}\ \phi_3\ \phi_{4)}]}_{\text{partial ampl. color-ordered.}} \times \mathrm{Tr}\left(T^{a_1} T^{\sigma(a_2} T^{a_3} T^{a_4)}\right)$$

$$A_4[\phi_1\phi_2\phi_3\phi_4] = A_4[\phi_4\phi_1\phi_2\phi_3]$$

$$\Downarrow$$

sym in $S \leftrightarrow u$

$$A_4[\phi\phi\phi\phi] = \cancel{\chi_4}^{\sigma=0} + g_4 t + \cdots$$

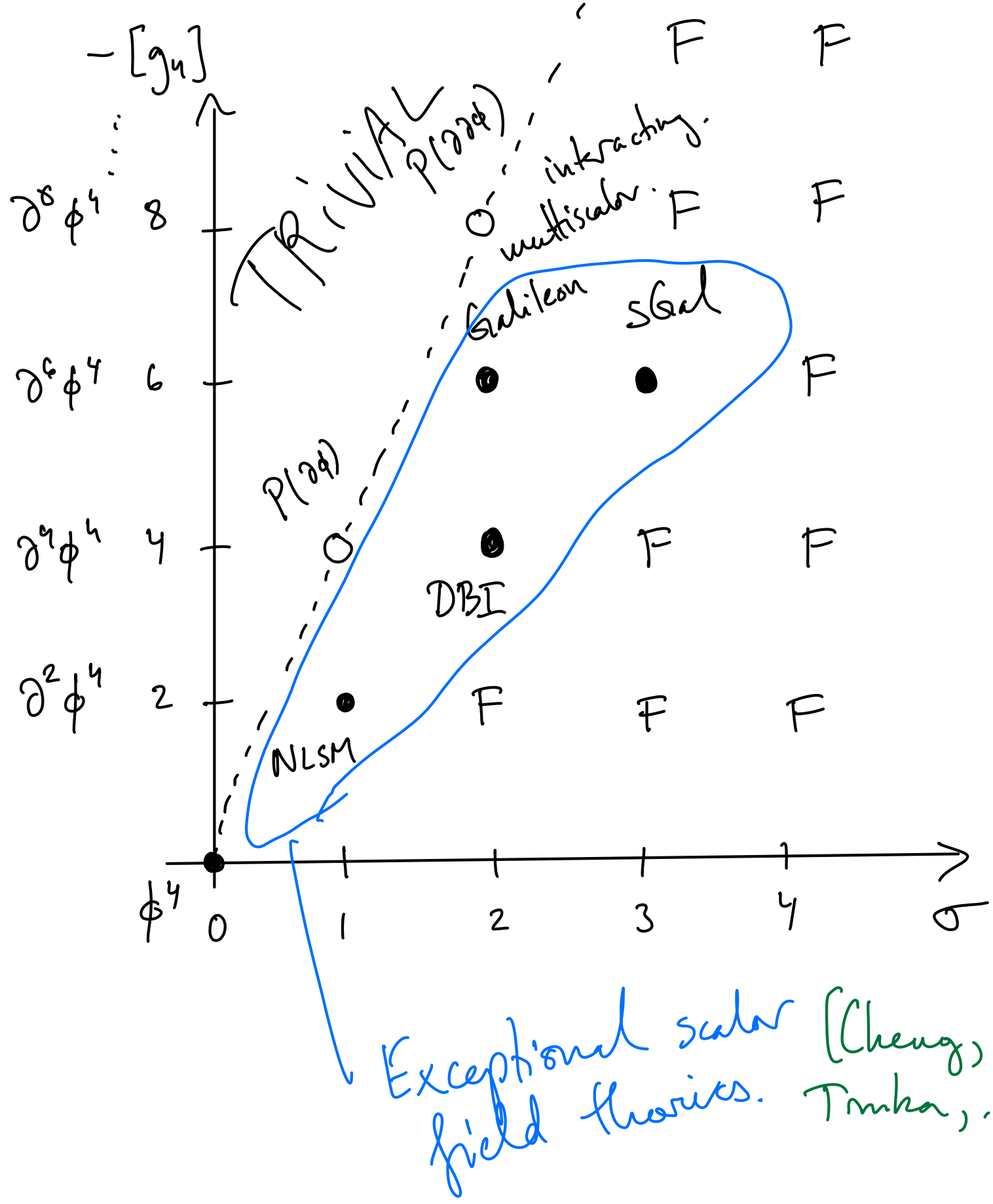
$\sigma = 1$

$\Rightarrow$ NLMS = nonlinear sigma model

$$A_5[\phi\phi\phi\phi\phi] = \lambda_5 \varepsilon(1234) \quad O(p^4)$$

$$\| $$

$$\varepsilon_{\mu\nu\rho\sigma} p_1^\mu p_2^\nu p_3^\rho p_4^\sigma$$

WZW

Soft bootstrap.