# Scale anomaly, non-quantized edge currents, and direct measurement of a beta function in Dirac semimetals

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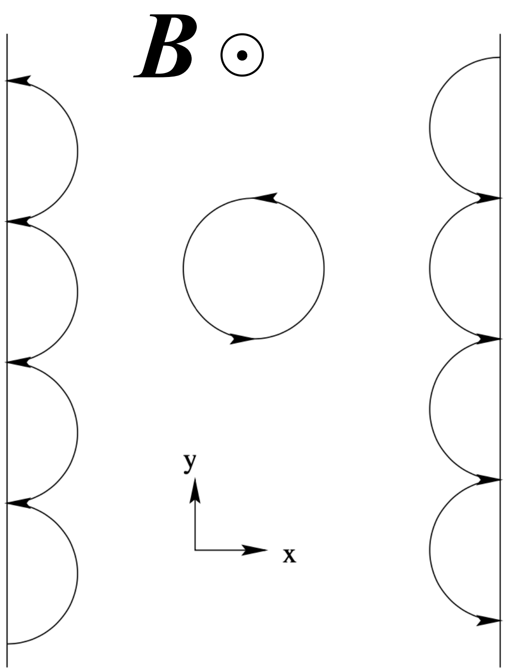
# Integer quantum Hall effect

## Quantization of the Hall conductance

$$\sigma_{xy} = \nu \frac{e^2}{h} \qquad (\nu \in \mathbb{Z})$$

**A universal number independent on microscopic details: type of material, disorder (in reasonable limits), value of magnetic field within a plateau, etc**

$$\mathbf{J} = \sigma \mathbf{E} \qquad \sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix}$$




**Semiclassical skipping orbits**

Taken from S. M. Girvin, Les Houches Lectures (1998)

– Dissipationless edge currents of opposite signs flowing in the ground state in the sample

– No net drift current in the absence of electric field

– Landau level physics, quantized conductance

– Extremely precise and accurate!

$\nu$ is known to $10^{-10}$ precision and $10^{-8}$ accuracy

(≈ statistics) (≈ systematics)

… be provocative …

# Integer quantum Hall effect

Scale anomaly?

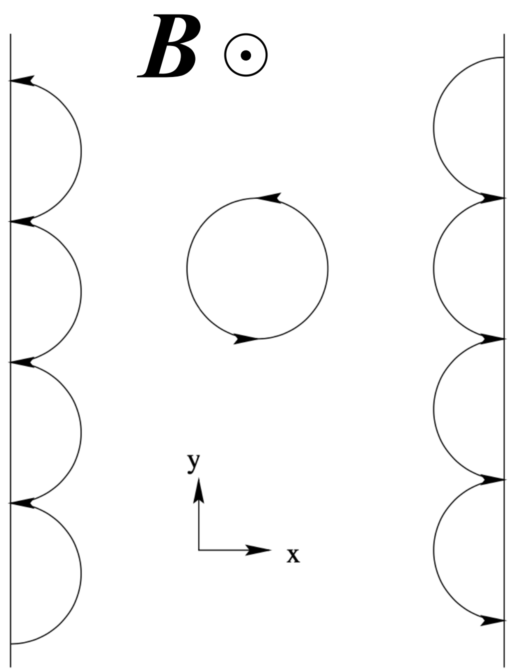
## Quantization of the Hall conductance

$$\sigma_{xy} = \nu \frac{e^2}{h} \qquad (\nu \not\in \mathbb{Z})$$

A universal number independent on microscopic details: type of material, disorder (in reasonable limits), value of magnetic field within a plateau, etc

not integer in the presence of the *relevant* scale anomaly

$$\mathbf{J} = \sigma \mathbf{E} \qquad \sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix}$$

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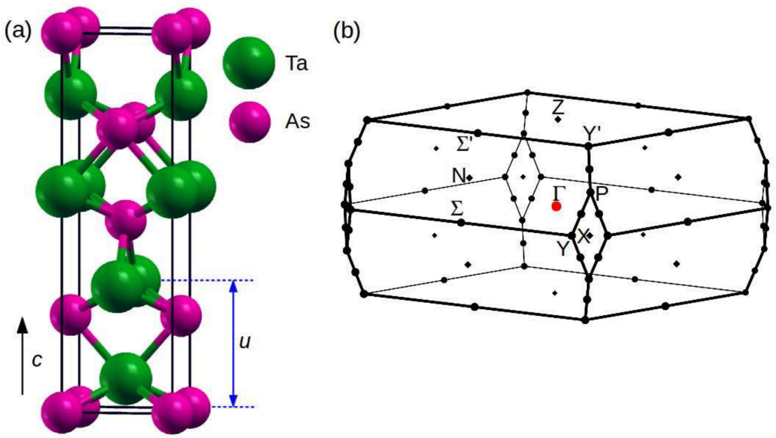
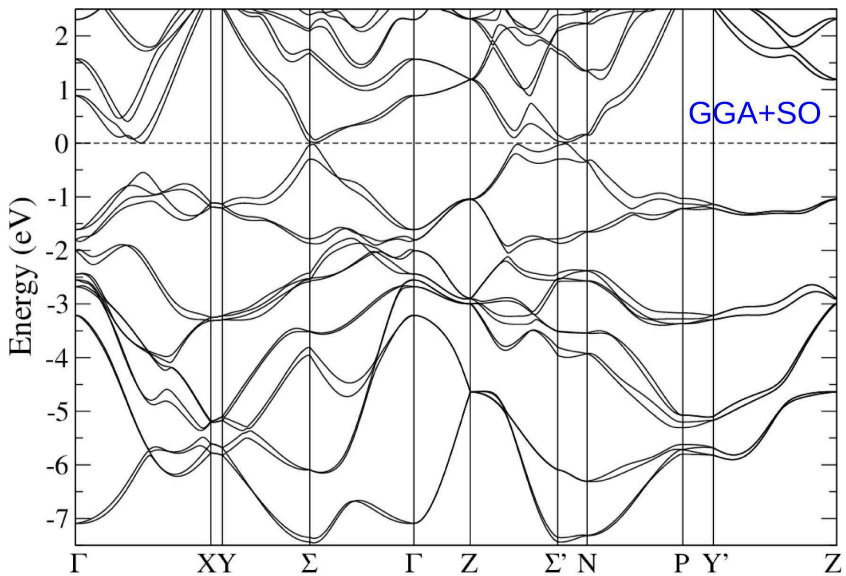
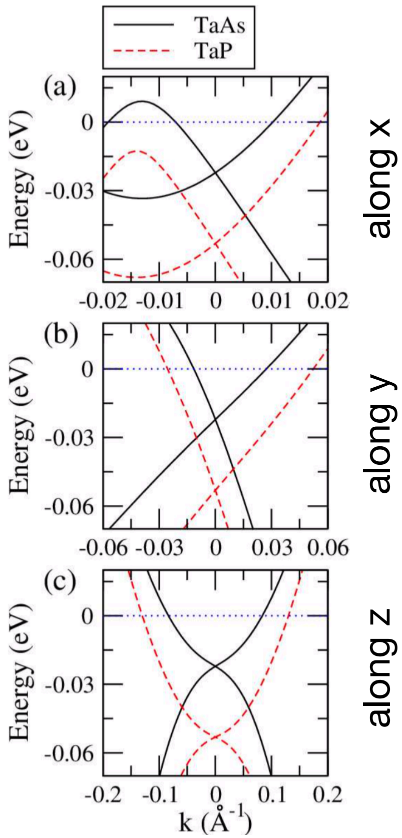
# Chiral fermionic quasiparticles in solid state

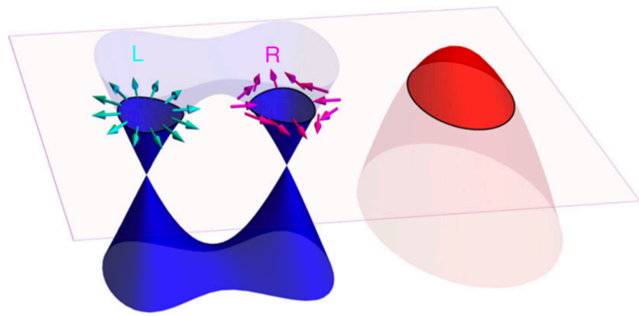
## TaAs as an example (of a Weyl semimetal)

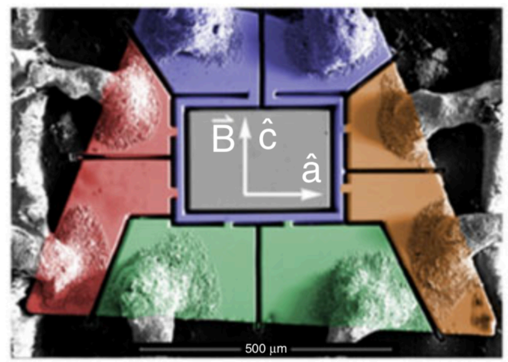



Chi-Cheng Lee et al.
Phys. Rev. B 92, 235104 (2015)




from Nat. Comm. 9, 2217 (2018)
B. J. Ramshaw et al.

# Massless (gapless) Dirac fermions

## A generic system in particle physics, cosmology, solid state …

**Covariant formulation (quantum field theory)**

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} i \not{D} \psi$$

$$\not{D} = \gamma^\mu D_\mu \qquad D_\mu = \partial_\mu + ieA_\mu \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

**Dirac semimetals (solid state):**

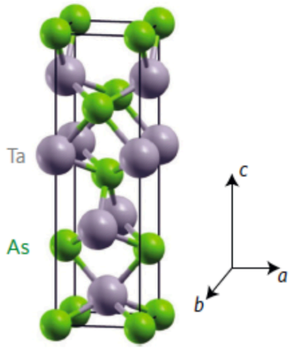
$$\bar{\psi}\left[ i\gamma^0 \hbar \frac{\partial}{\partial t} + v_F \boldsymbol{\gamma}(i\hbar\boldsymbol{\nabla} - e\boldsymbol{A}) \right]\psi$$

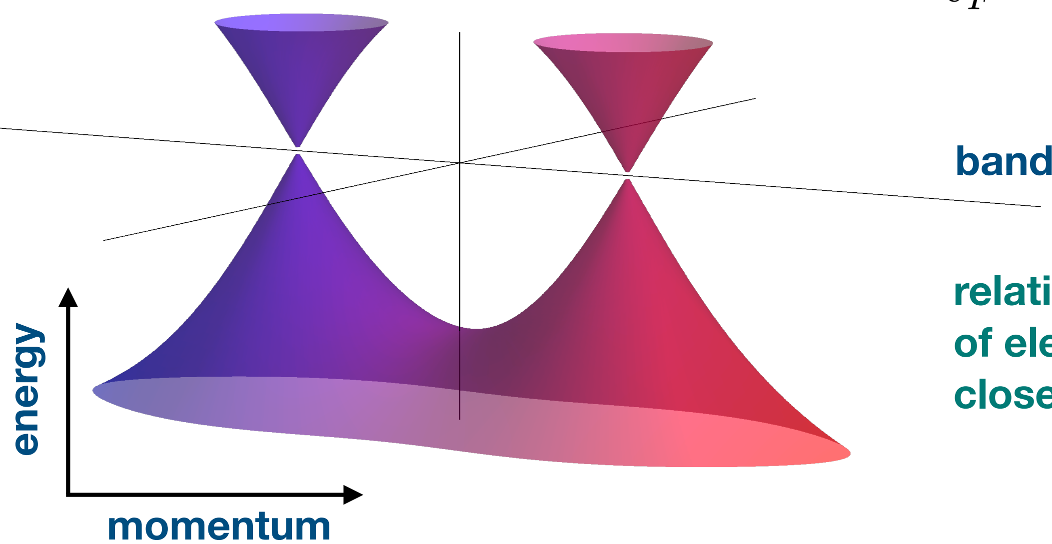
**Fermi velocity:**

$$v_F \simeq 10^5\,\mathrm{m/s} \sim \frac{c}{300}$$

**Dirac/Weyl semimetals**

**example: TaAs**
**(Weyl semimetal)**




**band structure:**

relativistic spectrum
of electronic quasiparticles
close to the Fermi surface

**Effective low energy description around band crossings in 3D crystals.**

# Massless Dirac fermions

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**Covariant formulation (quantum field theory)**

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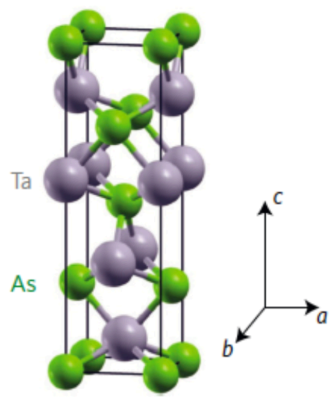
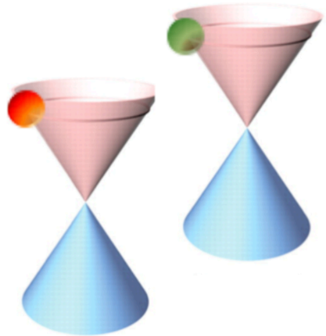



**Dirac semimetals (solid state):**

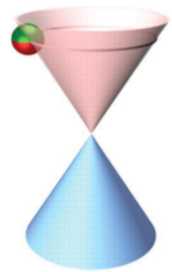
$$\bar{\psi} \left[ i\gamma^0 \hbar \frac{\partial}{\partial t} + v_F \boldsymbol{\gamma} (i\hbar \boldsymbol{\nabla} - e\boldsymbol{A}) \right] \psi$$

**Weyl semimetal**
(non-degenerated bands)



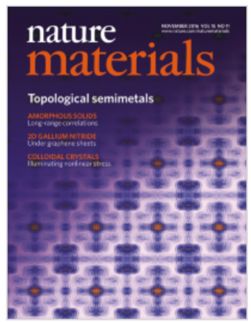
TaAs
NbAs
NbP
TaP

**Dirac semimetal**
(doubly degenerated bands)



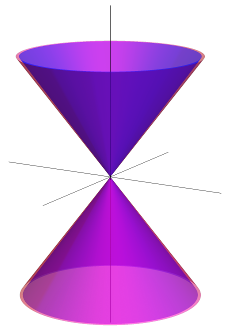
$ZrTe_5$
$Na_3Bi$,
$Cd_3As_2$

nature › nature materials › volumes › volume 15 › issue 11




**Effective low energy description around band crossings in 3D crystals.**

# Classical symmetries



$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi} i \not{D} \psi$$

## Vector

$$\psi \to e^{i\omega_V}\psi$$

local/gauge symmetry

**vector current is classically conserved**

$$j_V^\mu = \bar{\psi}\gamma^\mu\psi \qquad \partial_\mu j_V^\mu = 0$$

## Axial

$$\psi \to e^{i\omega_5 \gamma^5}\psi$$

global symmetry (no axial gauge field)

**axial current is classically conserved**

$$j_A^\mu = \bar{\psi}\gamma^5\gamma^\mu\psi \qquad \partial_\mu j_A^\mu = 0$$

## Scale

global scale transformations

$$x \to \lambda^{-1}x, \qquad A_\mu \to \lambda A_\mu, \qquad \psi \to \lambda^{3/2}\psi$$

**Dilatation current is classically conserved**

$$j_D^\mu = T^{\mu\nu}x_\nu \qquad \partial_\mu j_D^\mu \equiv T^\mu_{\ \mu} \equiv 0$$

$$(T^\mu_{\ \mu})_{\text{cl}} \equiv 0$$

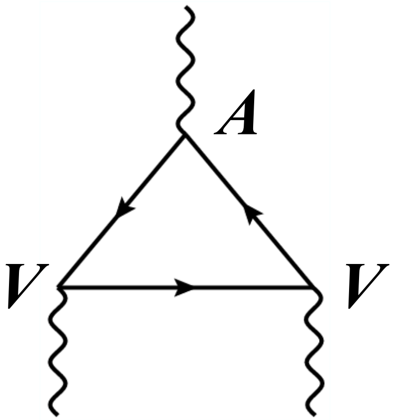
**Energy-Momentum tensor**

$$T^{\mu\nu} = -F^{\mu\alpha}F^{\nu}_{\ \alpha} + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \frac{i}{2}\bar{\psi}(\gamma^\mu D^\nu + \gamma^\nu D^\mu)\psi - \eta^{\mu\nu}\bar{\psi} i \not{D}\psi$$

# Zoo of anomalies

## (three out of six triangular vertices)

### Axial

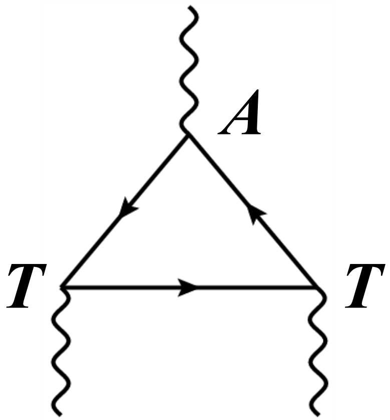



$$\partial_\mu j^\mu_A = \frac{e^2}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}$$

$$\tilde{F}^{\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$$

### Mixed

(axial-gravitational anomaly)

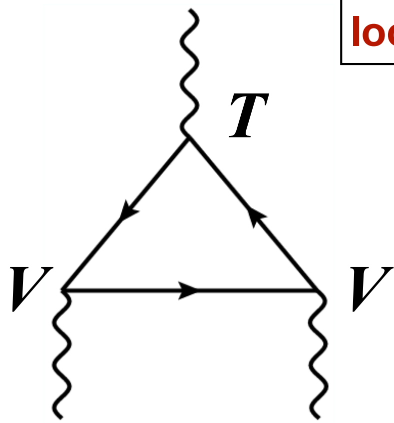



$$\partial_\mu j^\mu_A = -\frac{1}{384\pi^2} R_{\mu\nu\alpha\beta}\tilde{R}^{\mu\nu\alpha\beta}$$

$$\tilde{R}^{\mu\nu\alpha\beta} = \tfrac{1}{2}\varepsilon^{\mu\nu\gamma\lambda}R_{\gamma\lambda}{}^{\alpha\beta}$$

### Conformal

not one-loop exact




$$\partial_\mu j^\mu_D = T^\alpha_\alpha$$

beta function

$$\langle T^\mu{}_\mu \rangle = \frac{\beta(e)}{2e} F_{\mu\nu}F^{\mu\nu}$$

### Currents

$$j^\mu_V = \bar{\psi}\gamma^\mu\psi, \qquad j^\mu_A = \bar{\psi}\gamma^5\gamma^\mu\psi \qquad j^\mu_D = T^{\mu\nu}x_\nu$$

Vector — Axial — Dilatation

Full list: AVV, ATT, TVV, TAA, AAA, TTT (not counting torsion!)

### Energy-Momentum tensor

$$T^{\mu\nu} = -F^{\mu\alpha}F^\nu_\alpha + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$$

$$+\frac{i}{2}\bar{\psi}(\gamma^\mu D^\nu + \gamma^\nu D^\mu)\psi - \eta^{\mu\nu}\bar{\psi}i\not{D}\psi$$

# Scale anomaly and the beta function

**Massless Dirac fermions**

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} i \not{D} \psi$$

**are (classically) invariant under the global (scale) transformations:**

$$x \to \lambda^{-1} x, \qquad A_\mu \to \lambda A_\mu, \qquad \psi \to \lambda^{3/2} \psi$$

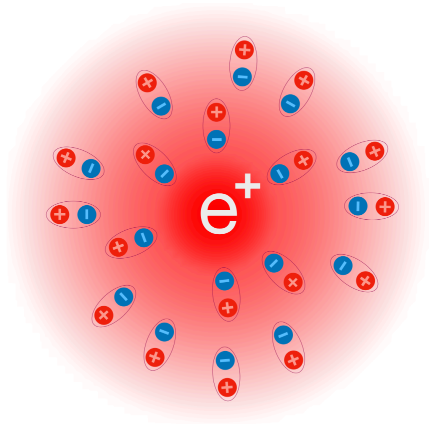
**The quantum theory generates an intrinsic scale due to a renormalization (in this particular case) of the electric charge:**

$$\beta(e) = \frac{de(\mu)}{d \ln \mu}$$

renormalization (energy) scale

**In QED (for one Dirac fermion)**

$$\beta_{\rm QED}^{1-{\rm loop}} = \frac{e^3}{12\pi^2}$$



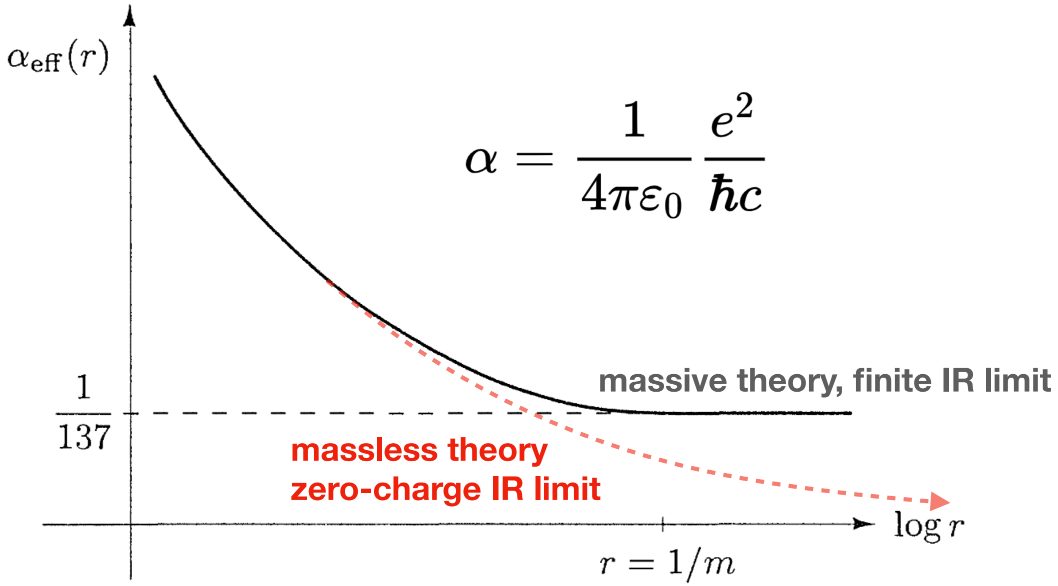

**→ scale symmetry is broken at the quantum level**

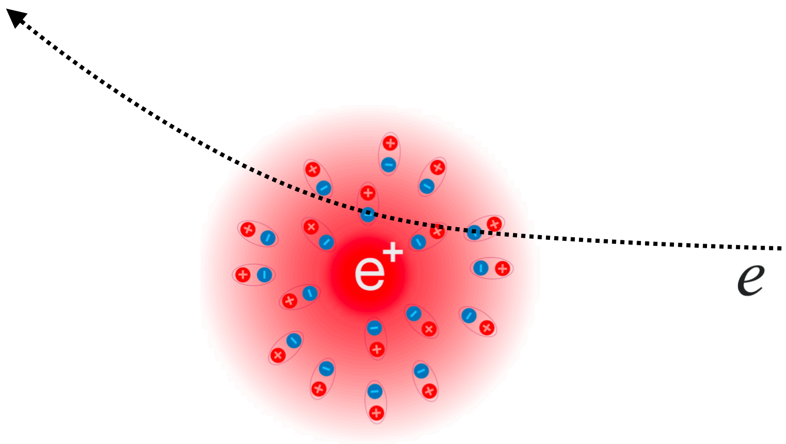
**Known as “scale” ≈ “conformal” ≈ “trace” ≈ “Weyl” anomaly**

# “Running” coupling in QED

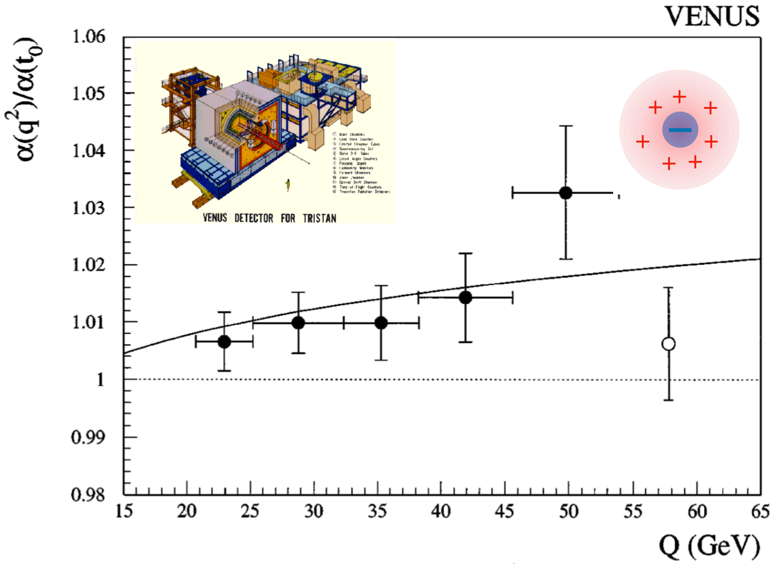
## fundamental QED with a finite mass of fermion (electron)

### fine structure constant

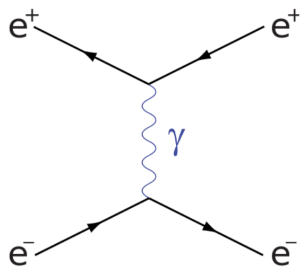



### experiment




S. Odaka et all (VENUS collaboration/KEK), PRL 81, 2428(1998)




Bhabha scattering

Adapted from the book of Peskin and Schroeder “An Introduction To Quantum Field Theory”

# “Running” couplings in topological materials

(solid-state physics)

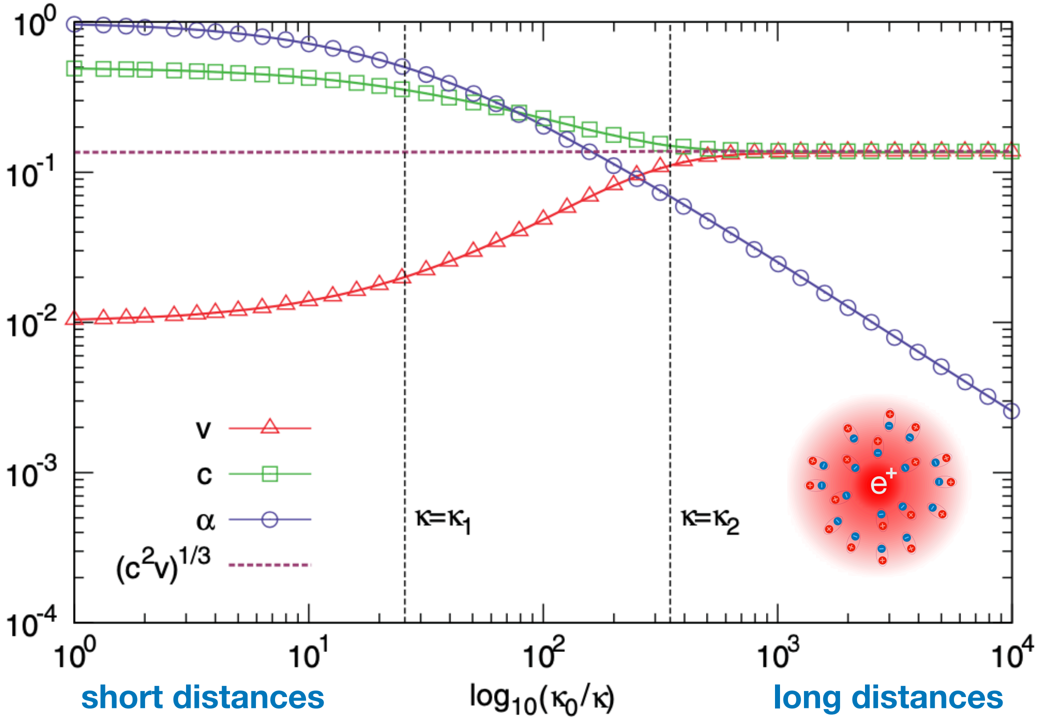
## topological insulator

$$\mathcal{L} = \bar{\psi}(\gamma^0 p_0 - v\vec{\gamma}\cdot\vec{p} - m)\psi + \frac{1}{2}\left(\varepsilon \vec{E}^2 - \frac{1}{\mu}\vec{B}^2\right) - e\bar{\psi}\gamma^0\psi A_0 - e\frac{v}{c}\bar{\psi}\gamma^\alpha\psi A_\alpha$$

Fermi velocity ($v$) — speed of light ($c$)

$$H(\mathbf{k}) \sim f_0(\mathbf{k}_0)\mathbb{I} + \mathbf{v}_0\cdot\delta\mathbf{k}\,\mathbb{I} + \sum_{a=x,y,z} \mathbf{v}_a\cdot\delta\mathbf{k}\,\sigma^a$$

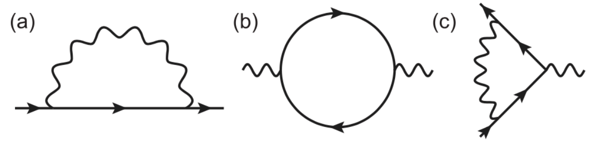
solid-state notations




short distances — long distances

“distance scale”

## renormalization group flow




**Lorentz invariance is restored at large distances (low energies)**

– **Particle physics (Physics of Universe) (Standard Model of particles):**

“One-loop renormalization of Lorentz-violating electrodynamics”
**Kostelecky, Lane, Pickering,** PRD 65, 056006 (2002)

– **Solid state (topological insulators/semimetals):**

– “Theory of a quantum critical phenomenon in a topological insulator: (3 + 1)-dimensional quantum electrodynamics in solids”;
**Isobe, Nagaosa,** PRB 86, 165127 (2012)

– Anisotropic fixed points in Dirac and Weyl semimetals
**Pozo, Ferreiros, Vozmediano**
Phys.Rev.B 98, 11, 115122 (2018)

– **2D systems (graphene) - running Fermi velocity**

“Non-Fermi liquid behavior of electrons in the half-filled honeycomb lattice (A renormalization group approach)”
**Gonzalez, Guinea, Vozmediano**, Nucl. Phys. B 424, 595 (1994).

# Conformal anomaly and transport effects at the edge

## What is about the boundaries? Take massless QED (or scalar QED or similar)

beta function — a normal vector to the boundary — electric current

$$J^\mu = -\frac{2c\beta_e}{e\hbar}\frac{F^{\mu\nu}n_\nu}{x}$$

spatial distance to the reflective boundary

$$V_i^D = -y_+^{3-d}\frac{2\Gamma(\frac{1}{2}d)}{d-1}F^0_{ni} \qquad \text{(C.5)}$$

… somewhere in appendix …

**Current generation at the boundary:**

**A DeWitt expansion of the heat kernel for manifolds with a boundary**

Class. Quantum Grav. 8 (1991) 603–638.

D M McAvity and H Osborn

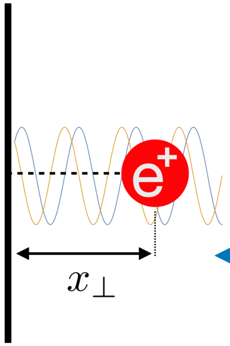
Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge, UK

**Relation to the scale anomaly:**
C.-S. Chu and R.-X. Miao, JHEP 07, 005 (2018), PRL 121, 251602 (2018);

Numerical evidence: V. A. Goy, A. V. Molochkov, M.Ch. PLB 789, 556 (2019);

Relation to Schwinger pair production in semimetals: M. Vozmediano, M.Ch., PRR 1, 032002 (2019).

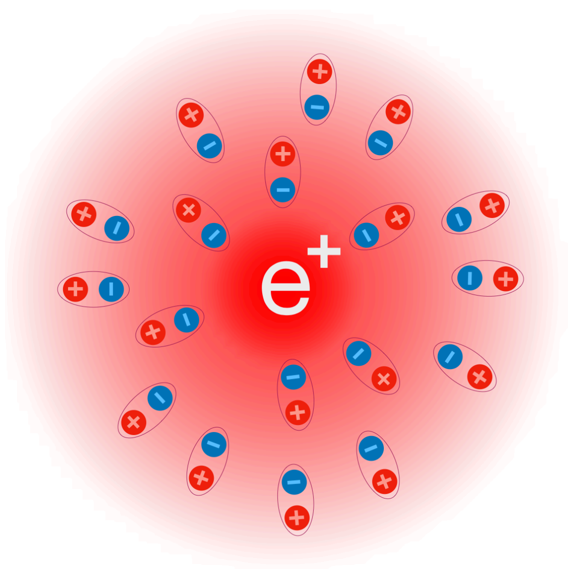
# The scale anomaly and the edge, physical reason: The renormalization is affected by the boundary



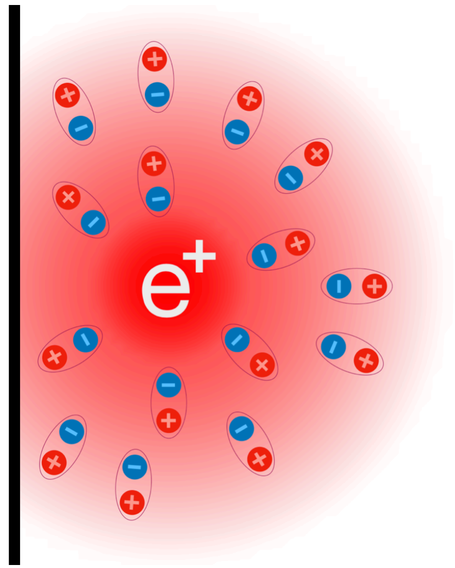

$$e = e(x_\perp)$$

← the distance to the boundary enters as an inverse energy scale

at the bulk (no boundary)




at the boundary




the boundary affects the polarization

→ charge is stronger at the boundary!

→ anomalous (transport, static) effects due to the scale anomaly at the boundary

# Conformal anomaly and transport effects at the edge

**What is about the boundaries? Take massless QED (or any conformal theory …)**

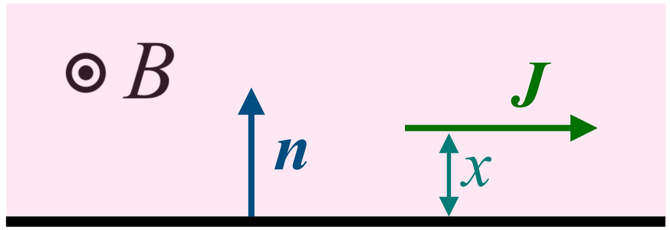
beta function — a normal vector to the boundary

electric current

$$J^\mu = -\frac{2c\beta_e}{e\hbar}\frac{F^{\mu\nu}n_\nu}{x}$$

spatial distance to the reflective boundary

**In the magnetic-field background:**




**Take the spatial components:**

## Scale Magnetic Effect at the Edge (SMEE):

### Electric current along the edge due to tangential magnetic field

$$\boldsymbol{j}(\boldsymbol{x}) = -f_{\boldsymbol{n}}(\boldsymbol{x})\boldsymbol{n}\times\boldsymbol{B} \qquad f_{\boldsymbol{n}}(\boldsymbol{x}) = \frac{2\beta_e}{e\hbar}\frac{1}{|\boldsymbol{n}\cdot\boldsymbol{x}|}$$

diverges at the boundary!

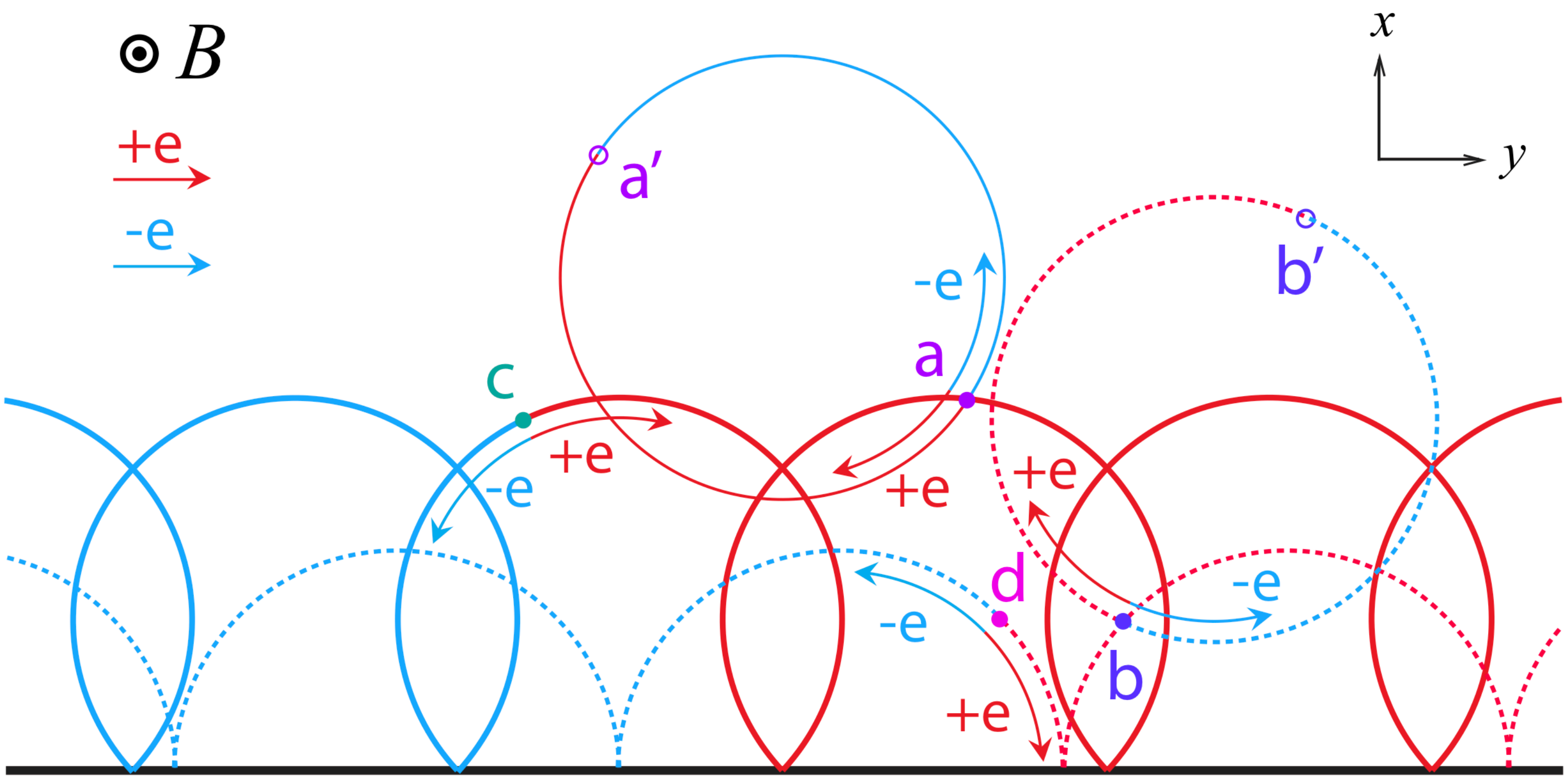
- **Effect due to conformal anomaly**
- **No topology (Berry, Chern, etc)**
- **No matter at all (= quantum vacuum)**
  (= neutrality point, $\mu = 0$ & $T = 0$)

**No topological protection/quantization!**

# Scale Magnetic Effect at the Edge (SMEE)

## A physical picture

**Ingredients: massless particles, vacuum, edge and magnetic field**




C.-S. Chu and R.-X. Miao, PRL 121, 251602 (2018); a slightly more complicated picture.

**Skipping-like orbits (similar but different from the quantum Hall effect, now in the vacuum = "at the neutrality point")**

**No Fermi surface, no temperature needed (works at $T = 0$).**

(However, enhanced by thermal effects: R. Guo and R.-X. Miao, ArXiv:2102.01253)

# SMEE: numerical first-principle check

Generates the current at the boundary?

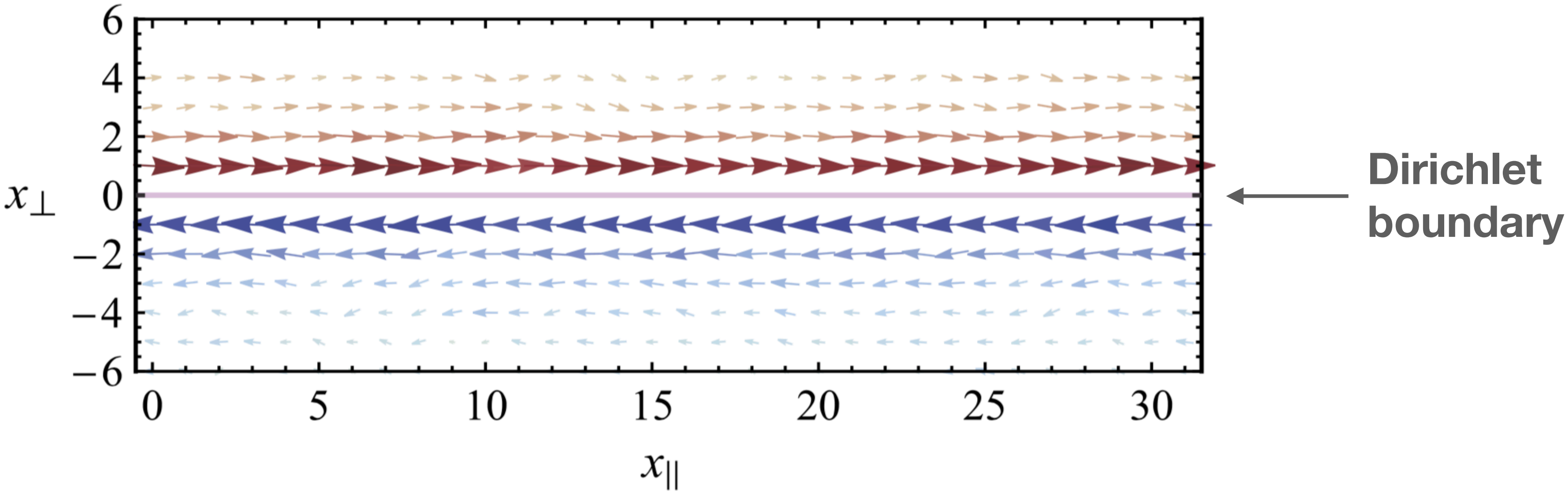
## Scalar electrodynamics at a conformal point in (3+1)D:

$$\mathcal{L}_{\mathrm{sQED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \left[\left(\partial_\mu - ieA_\mu\right)\phi\right]^* \left(\partial^\mu - ieA^\mu\right)\phi$$

Massless one-component electrically-charged scalar field

Numerical Monte-Carlo simulations

We see the generated electric current!

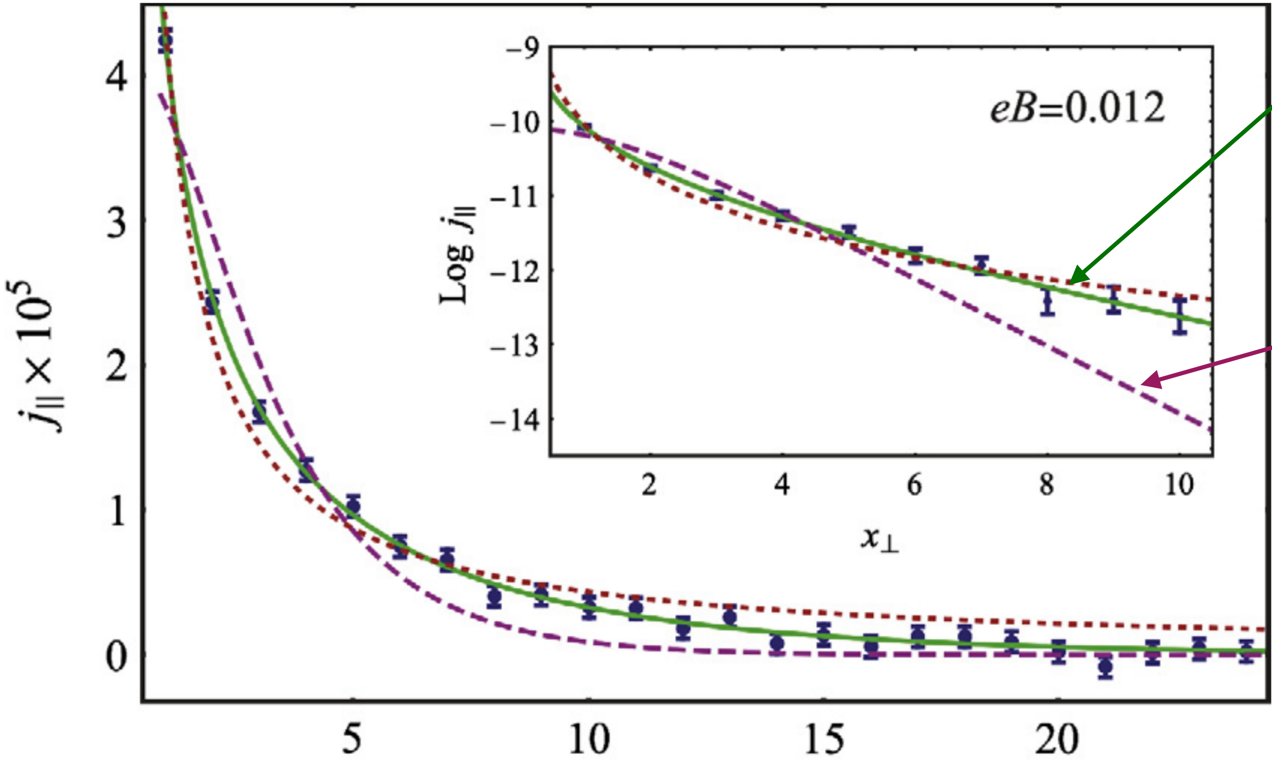



Chemical potential is zero = vacuum

V.A. Goy, A.V. Molochkov, M.Ch., Phys. Lett. B 789, 556 (2019)

# SMEE: numerical first-principle check

**1) We see the *1/x* behavior of the electric current at the boundary**




**conformal behavior (the massless scalar)**

**non-conformal behavior (the scalar is massive)**

**Chemical potential is zero. The current is generated in the vacuum.**

**2) We see the correct coefficient and recover the beta function!**

$$\beta^{\text{1-loop}}_{\text{sQED}} = \frac{e^3}{48\pi^2}$$

(Notice that the beta function of the scalar QED is four times smaller than the beta function in the usual QED)

**V.A. Goy, A.V. Molochkov, M.Ch., Phys. Lett. B 789, 556 (2019)**

**3) No localization at the boundary with the magnetic length**

$$l_B = \sqrt{\frac{\hbar}{eB}}$$

**4) No Landau-level physics → no quantization of conductance**

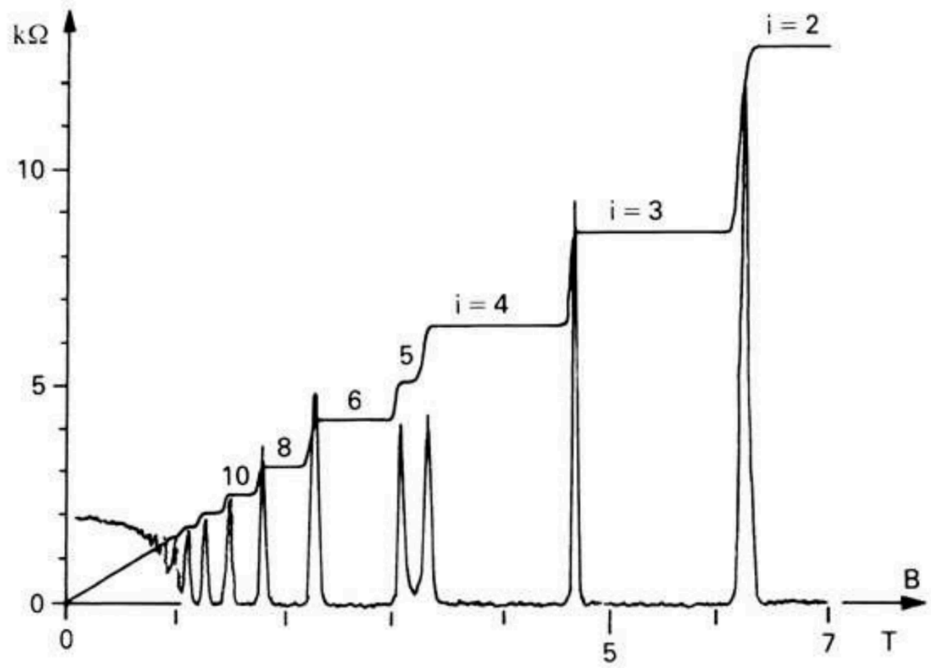
# Integer quantum Hall effect

## Quantization of the Hall conductance

$$\sigma_{xy} = \nu \frac{e^2}{h} \quad (\nu \in \mathbb{Z})$$

It works in systems with charge carriers whose electric charge does not run with the energy scale.

"no scale anomaly"




Examples:

— semiconductors (non-relativistic quantum mechanics, no genuine relativistic renormalization effects)

— graphene (relativistic quasiparticle charge carriers experience the scale anomaly associated with the Fermi velocity, but the electric charge does not run with the scale, $\beta_e = 0$ )

J.González, F.Guinea, M.A.H.Vozmediano, NPB 424, 595 (1994)

→ The quantum Hall is well-quantized in graphene.

# Scale Magnetic Effect at the Edge

## Non-Quantization of the “Hall conductance” for magnetization current

generated by the scale anomaly

Infrared energy scale: size of the system

$$\delta\sigma_{xy} = \frac{2\beta_e}{e\hbar} \ln \frac{R_{\rm IR}}{R_{\rm UV}} \simeq \frac{1}{137} \frac{2}{3\pi\varepsilon} \frac{c}{v_F} \sigma_{xy}^{\rm Hall} \simeq 0.05\, \sigma_{xy}^{\rm Hall}$$

modest = minimalistic estimate assuming ln(...)=1

Ultraviolet energy scale:
nonlinearity in the energy dispersion
interatomic distance

$v_F \simeq c/300$

$\varepsilon \simeq 10$ permittivity

**Should be observed in a 2d or quasi-2d system hosting relativistic (Dirac) quasiparticles with running electric charge**

(N. B.: in Dirac semimetals, the electric charge runs [Isobe, Nagaosa, 2012])

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As contrasted to the quantization of the integer-Hall conductance

$$\sigma_{xy} = \nu \frac{e^2}{h} \qquad (\nu \in \mathbb{Z})$$

$\nu$ is known to $10^{-10}$ precision and $10^{-8}$ accuracy

# Conformal anomaly and transport effects at the edge

## Coming back to the edge:

beta function

a normal vector to the boundary

electric current

$$J^\mu = -\frac{2c\beta_e}{e\hbar}\frac{F^{\mu\nu}n_\nu}{x}$$

spatial distance to the reflective boundary

## Spatial components:

$$\mu = 1, 2, 3$$

**In the magnetic-field background:**


---

## Temporal components?

$$\mu = 0$$

$$\rho = -\frac{2\beta_e}{e\hbar}\frac{\boldsymbol{nE}}{x}$$

**Charge accumulation in the electric field background**

# Scale electric effect at the edge: conformal/scale screening

**Screening of electrostatic field in metals:**

$$E(x) \sim E(0) e^{-x/\lambda}$$

Fermi momentum

$$p_F = (2\pi^2 n)^{1/3} \hbar$$

**Screening lengths:**

Density of carriers $n$

$$\lambda_{\mathrm{D}} = \sqrt{\frac{\varepsilon_0 k_B T}{n e^2}}, \qquad \lambda_{\mathrm{FT}} = \sqrt{\frac{\varepsilon_0 \pi^2 \hbar^3}{m e^2 p_F}}$$

Debye — Fermi-Thomas

**What if the medium is totally conformal and possess no dimensionful parameters?**

**For example, take a Dirac semimetal at particle-hole symmetric point.**

– We have the mobile carriers (massless fermionic quasiparticles)

– Classically, there is no dimensionful scale.

– No classical scale → no screening?

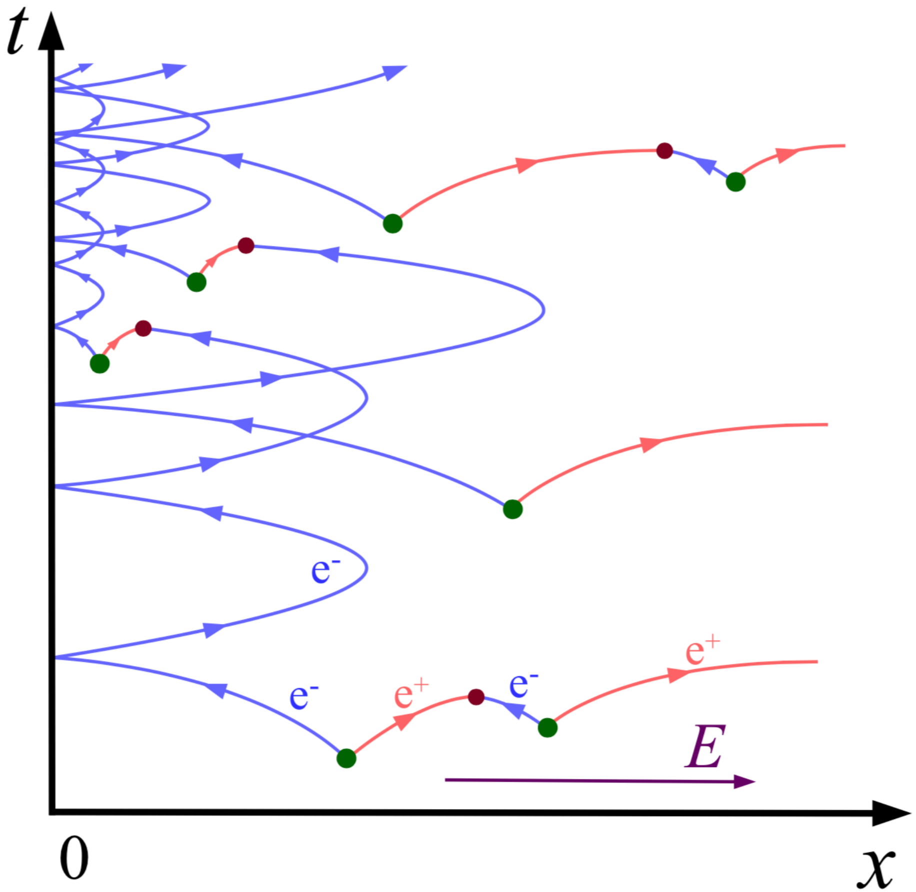
No quantity to construct the screening length from!

(formally, of course)

# Scale electric effect at the edge

$$J^\mu = -\frac{2c\beta_e}{e\hbar}\frac{F^{\mu\nu}n_\nu}{x} \longrightarrow \rho = -\frac{2\beta_e}{e\hbar}\frac{\boldsymbol{nE}}{x}$$

the density of the electric charge accumulated at the boundary




**Physics: the screening is due to the Schwinger effect ("skipping orbits" in time)**

**Works efficiently due to the absence of a mass gap**

**Generated by the conformal anomaly!**
(proportional to the beta function)

**Mechanism in semimetals: creation of electron-hole pairs in the presence of a uniform electric field (the Zener effect)**

# Scale electric effect at the edge of a semimetal

**Consider QED:**

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \sum_{a=1}^{N_f} \bar{\psi}_a i\gamma^\mu D_\mu \psi_a$$

**Charge density due to conformal anomaly:** $\rho = -\frac{2\beta_e}{e\hbar}\frac{\boldsymbol{nE}}{x}$

**Solve the Maxwell equation** $\partial_x E_x(x) = \frac{1}{\varepsilon_0}\rho(x)$

**At the boundary the conformal screening is polynomial:**

| $E_x(x) = \frac{C}{x^\nu}$ | $\rho(x) = -\frac{C\varepsilon_0\nu}{x^{1+\nu}}$ | $\phi(x) = \phi_0 - \frac{Cx^{1-\nu}}{1-\nu}$ |
|---|---|---|
| **Electric field** | **Charge density** | **Electrostatic potential** |

**Similar to the Wilson-Fisher fixed point**

$$\langle\phi(0)\phi(r)\rangle \propto \frac{1}{r^{D-2+\eta}}$$

**Conformal screening exponent:** $\nu = \frac{2\beta_e}{ec\hbar\varepsilon_0}$

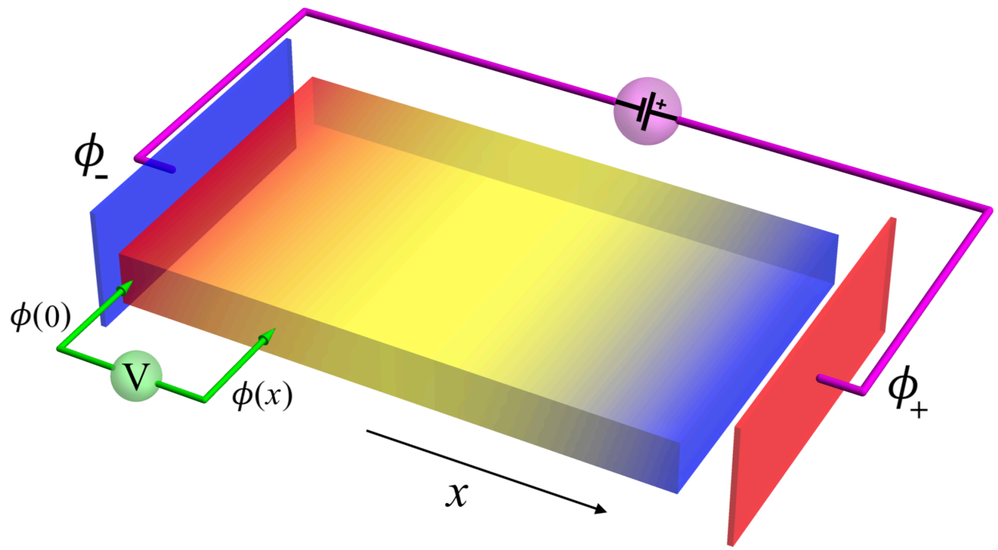
(proportional to a central charge in CFT)

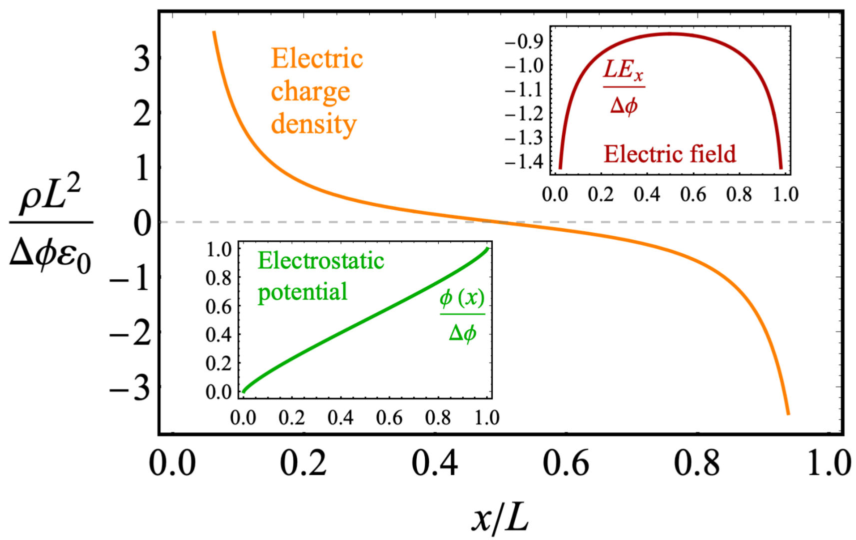
# Scale electric effect at the edge

**Conformal exponent in a Dirac semimetal:** $\nu = \frac{e^2}{6\pi^2 \hbar v_F \varepsilon \varepsilon_0} = \frac{2\beta_e}{e c \hbar \varepsilon_0}$

**Particle density in a finite sample with two boundaries:**

$$\rho(x) = \frac{\Delta\phi}{L^2}\varepsilon_0 \nu h(\nu) \left(1 - \frac{2x}{L}\right) \left[\frac{x}{L}\left(1 - \frac{x}{L}\right)\right]^{-1-\nu}$$




**Direct measurement of the beta function. Indirect evidence of the Schwinger effect.**

M. A. H. Vozmediano, M.Ch., Phys. Rev. Research 1, 032002(R) (2019)

# Accessible experimentally

- direct measurement of the beta function associated with the renormalization of the electric charge

(never done in solid state)

- evidence of the elusive Schwinger effect

(particle-antiparticle production by electric field)

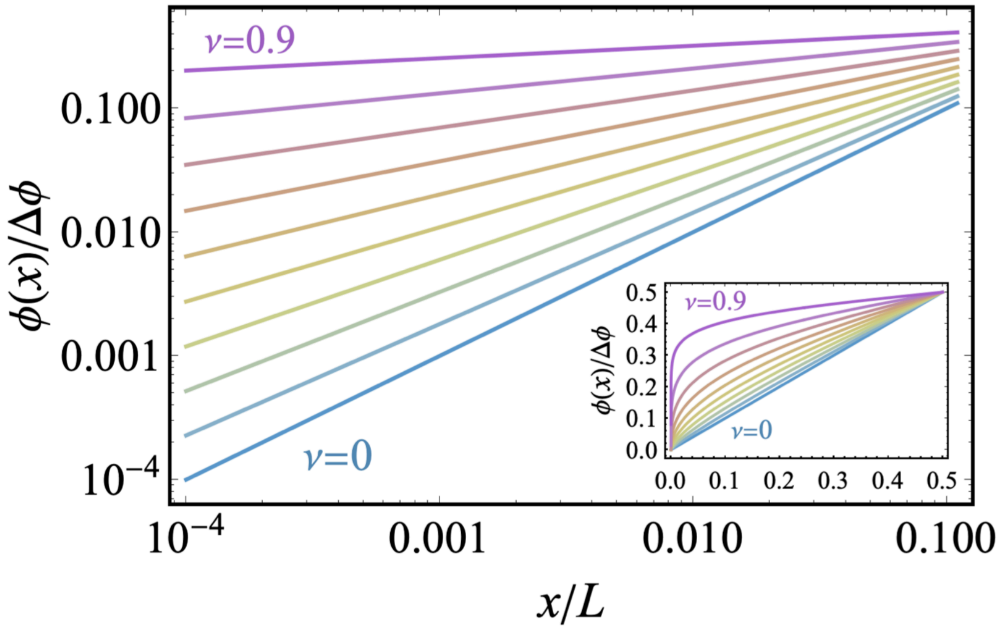
Conformal exponent in a Dirac semimetal: $$\nu = \frac{e^2}{6\pi^2 \hbar v_F \varepsilon \varepsilon_0}$$

In typical Dirac/Weyl materials $v_F \sim 10^{-3} c$ and $\varepsilon \sim 10$

→ large, experimentally accessible conformal exponent: $\nu \sim 10^{-1}$

Electrostatic screening potential vs. distance from the boundary of a Dirac material at T=0 charge neutrality point

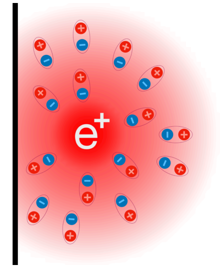
(= at a Lifshitz point at zero temperature)



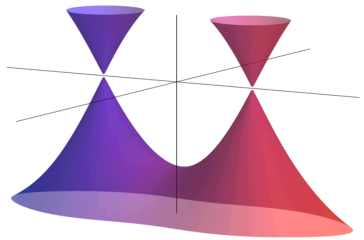
# Summary at the edge

Conformal anomaly leads to a number of new transport effects:

— at reflective boundaries (edges) of bounded systems
— in the bulk (unbounded systems - not covered in the talk)

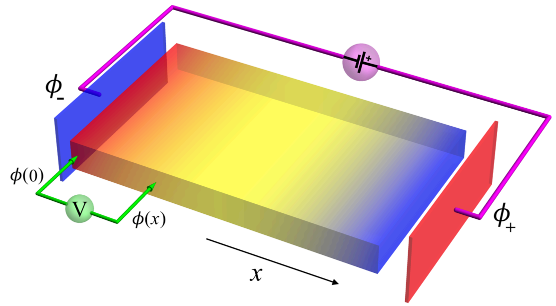



Electric current is proportional to the beta function.
Accessible experimentally in Dirac and Weyl semimetals.



<u>Scale electric effect</u>:

— polynomial screening of electrostatic fields
— particle creation via the Schwinger effect




<u>Scale magnetic effect</u>:

— edge currents in the absence of matter
— non-integer quantum Hall effect.