Duality Relations for Overlaps of Integrable Boundary States in AdS/CFT

Charlotte Kristjansen

Niels Bohr Institute

Based on:

- C.K., D. Müller & K. Zarembo, ArXiv:2005.01392[hep-th], JHEP 08 (2020) 103,
  ArXiv:2011.12192[hep-th], JHEP 03 (2021) 100,
  ArXiv:2106.08116[hep-th], JHEP 09 (2021) 004

Quantum Field Theory at the Boundary

MITP, Mainz, Germany

September 27th, 2021

Motivation

- Exploit ”microscopic” dualities within AdS/CFT (pertaining to underlying psu(2,2|4) integrable super spin chain)
- Overlaps between spin chain eigenstates and Matrix Product States or Valence Bond States encode information about correlation fcts. in AdS/CFT (1-pt fcts, 3-pt fcts and others)
- The same type of overlaps are interesting for the study of quantum quenches in stat. mech.
- Fermionic dualities allow one to move between different Dynkin diagrams of the underlying super Lie algebra
- Bosonic dualities complete the possible set of dualities (QQ-system)
- Duality relations might constrain overlap formulas

Plan of the talk

I. Overlaps and AdS/dCFT

II. The Structure of overlap formulas

III. Fermionic duality relations for overlaps

IV. Bosonic duality relations for overlaps

V. Future directions

AdS/CFT

$$\mathcal{N} = 4 \text{ SYM in 4D} \longleftrightarrow \text{IIB strings on } AdS_5 \times S^5$$

Maldacena '98
Gubser, Klebanov & Polyakov '98
Witten '98

$$\text{Conformal operators} \longleftrightarrow \text{String states}$$

$$\text{Eigenstates of integrable super spin chain: } |\mathbf{u}\rangle$$

Minahan & Zarembo '02
Beisert, C.K. Staudacher '03
Beisert & Staudacher ' 04, '05

AdS/dCFT

Karch & Randall '01

$$\begin{matrix} \mathcal{N} = 4 \text{ SYM in 4D} \\ \text{with co-dimension one defect} \end{matrix} \longleftrightarrow \begin{matrix} \text{IIB strings on } AdS_5 \times S^5 \\ \text{Karch-Randall probe brane} \end{matrix}$$

$|B\rangle$ integrable boundary state describing defect / probe brane

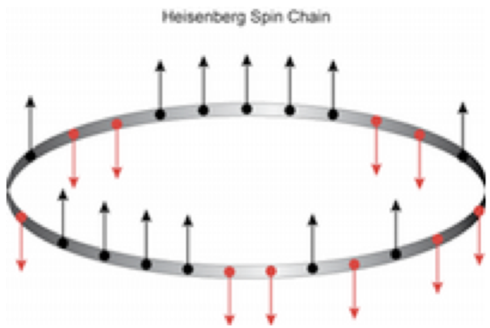
De Leeuw, C.K. Zarembo '15

Similar idea: $|B\rangle \sim$ determinant operator/giant graviton

Jiang, Komatsu Vescovi '19

Example: $SU(2)$ Heisenberg spin-1/2 chain

Encodes conformal single trace operators built from two complex fields X (vacuum) and Y (excitations)



$$H = \sum_{n=1}^{L} (1 - P_{n,n+1})$$

$|\{u_i\}_{i=1}^{K}\rangle \equiv |\mathbf{u}\rangle$: Eigenstates with K excitations where

$$1 = \left(\frac{u_k - \frac{i}{2}}{u_k + \frac{i}{2}}\right)^L \prod_{j \neq k}^{K} \frac{u_k - u_j + \frac{i}{2}}{u_k - u_j - \frac{i}{2}} = e^{i\chi_k}$$

Bethe equations $k = 1, \ldots, K$

Baxter polynomials (Q-functions): $Q(u) = \prod_{j=1}^{K} (u - u_j), \quad Q_\theta(u) = u^L$

$$\langle \mathbf{u} | \mathbf{u} \rangle \propto \det G, \qquad G_{kj} = \frac{\partial \chi_k}{\partial u_j}$$

Gaudin matrix

Integrable boundary states $|B\rangle$ in AdS/dCFT

Bethe eigenstate of integrable spin chain

$\langle B|\mathbf{u}\rangle$ computable in closed form

Matrix product states

$$|B\rangle = |\text{MPS}\rangle = \sum_{\{s_i\}} \text{Tr}(t_{s_1} \ldots t_{s_L})|s_1 \ldots s_L\rangle$$

Valence Bond States

$$|\text{VBS}\rangle = |K\rangle^{\otimes \frac{L}{2}}, \qquad K = \sum_{s_1,s_2} K_{s_1,s_2}|s_1 s_2\rangle$$

Integrability understood in a scattering picture

$$\mathbf{Q}_{2n+1}|B\rangle = 0$$

Conserved parity-odd charges of spin chain

Ghoshal & Zamolodchikov '94

Piroli, Pozsgay Vernier '17

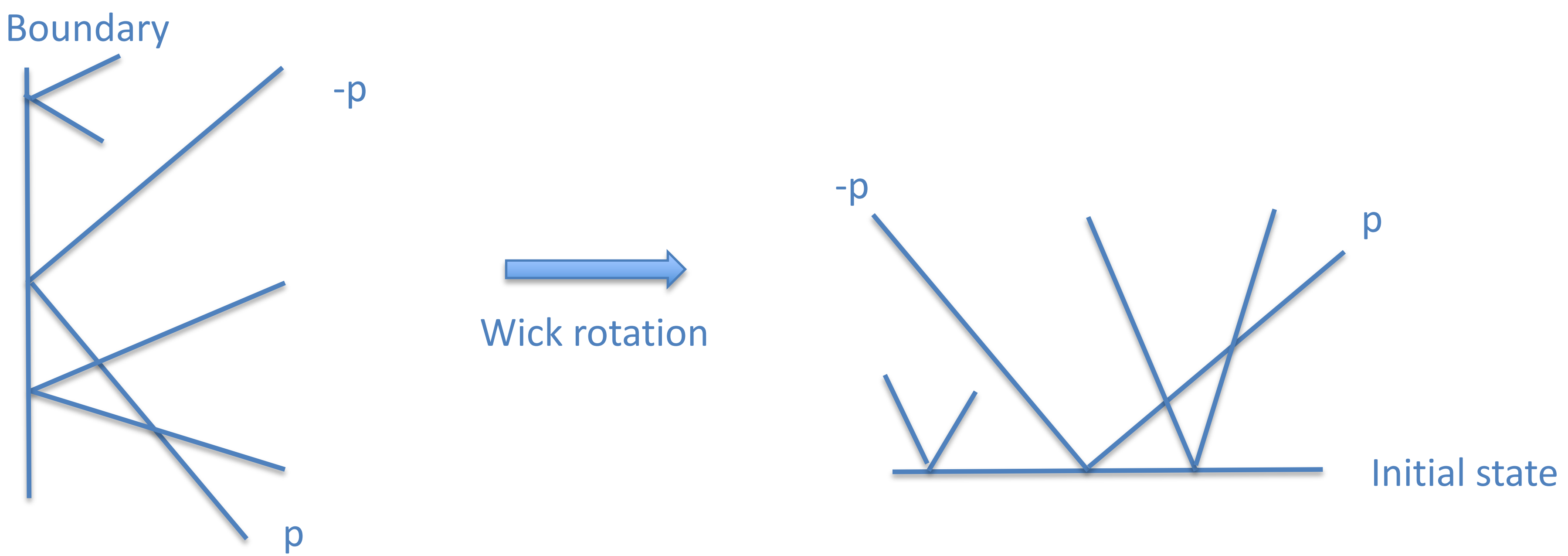
Motivation

(i) Fullfilled for all cases where closed overlap formula is known

(ii) Discrete version of integrable boundary state condition

Piroli, Pozsgay
Vernier '17

Ghoshal,
Zamolodchikov '93



Pure reflection
+BYB for reflection matrix

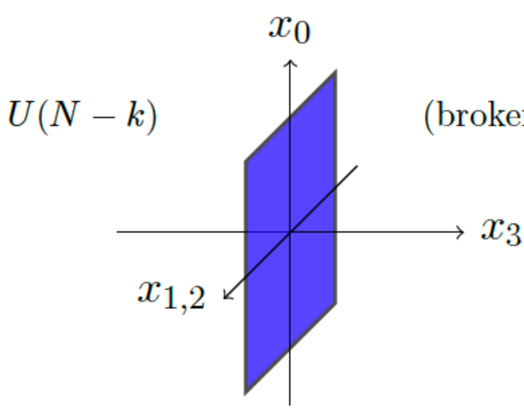
Entangled (p,-p) pairs
+BYB for initial state

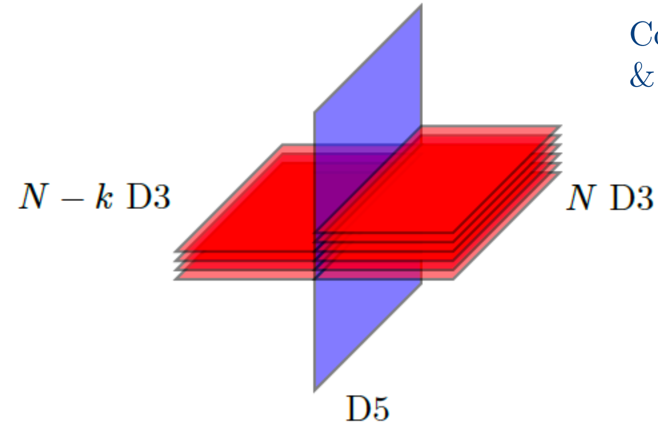
$$Q_{2m+1}|B\rangle = 0$$

AdS/dCFT set-up

Karch & Randall '01

Constable, Myers & Tafjord '99





Gauge Theory

String Theory

$\phi_i, \ \Psi_\alpha, \ A_\mu$

$i = 1, \ldots 6, \ \alpha = 1, \ldots, 4, \ \mu = 1, \ldots 4$

For $x_3 > 0$: $$\phi_i^{\rm cl} = \frac{1}{x_3} \begin{pmatrix} (t_i)_{k\times k} & 0 \\ 0 & 0 \end{pmatrix}, \ i = 1,2,3$$

where t_i, $i = 1,2,3$ constitute a k-dimensional irreducible representation of $\mathfrak{su}(2)$

Set-up supersymmetric 1/2 BPS, dCFT

Gaiotto & Witten, '08

One-point functions and MPS

$$\langle \mathcal{O}^{\text{bulk}}_{\Delta}(x)\rangle = \frac{C}{|x_3|^{\Delta}}$$

Cardy ´84

McAvity & Osborn '95

Due to vevs scalar operators can have non-zero 1-pt fcts at tree-level

$$\langle \mathcal{O}_{\Delta}(x)\rangle = (\text{Tr}(\phi_{i_1}\ldots\phi_{i_\Delta}) + \ldots)\,|_{\phi_i\to\phi_i^{\text{cl}}=\frac{t_i}{x_3}}$$

$\mathcal{O}_{\Delta}(x) \sim$ eigenstate of integrable $SO(6)$ spin chain

Minahan & Zarembo '02

$$\text{Tr}(\phi_{i_1}\phi_{i_2}\ldots\phi_{i_L}) \sim |s_{i_1}s_{i_2}\ldots s_{i_L}\rangle$$

Matrix Product State associated with the defect:

deLeeuw, C.K. & Zarembo '15,

$$|\text{MPS}_{\text{k}}\rangle = \sum_{\vec{i}} \text{tr}[t_{i_i}\ldots t_{i_L}]|\phi_{i_1}\ldots\phi_{i_L}\rangle,$$

Object to calculate:

Bethe eigenstate

$$C_k(\mathbf{u}) = \frac{\langle \text{MPS}_k\,|\mathbf{u}\rangle}{\langle \mathbf{u}\,|\mathbf{u}\rangle^{\frac{1}{2}}}$$

One-point functions and VBS

For $k = 1$: No vevs

Gaiotto & Witten, '08

Quantum fields
$$A_\mu, \Phi_i, \Psi_\alpha = \begin{array}{c} \quad 1 \qquad N-1 \\ \left[\begin{array}{c|ccc} x & y & y & y \\ \hline y & z & z & z \\ y & z & z & z \\ y & z & z & z \end{array}\right] \end{array}$$

Boundary conditions (supersymmetric)

	$\Phi_{4,5,6}$	$\Phi_{1,2,3}$
x, y	Dirichlet	Neumann
z	no BCs	no BCs

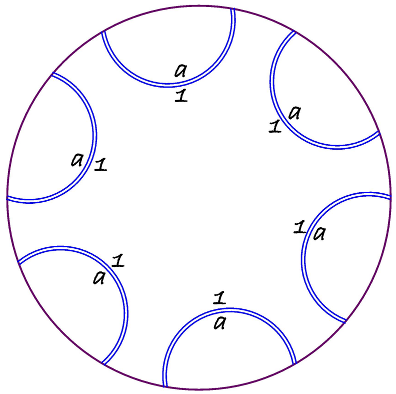
One-point functions require Wick contractions

For complex scalars ($X = \Phi_1 + i\Phi_4$, etc.)

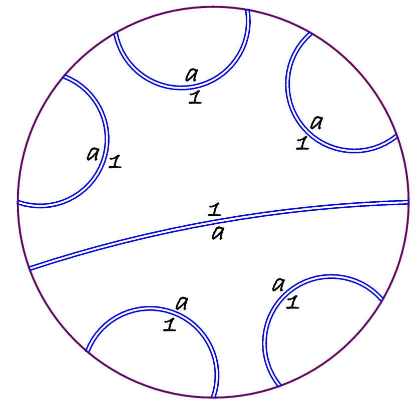
$$\left\langle X^{1a}(x) X^{b1}(y) \right\rangle = \frac{g_{\rm YM}^2 \delta^{ab}}{4\pi^2 |\bar{x} - y|^2} \qquad \bar{x} = (x_0, x_1, x_2, -x_3)$$

For fermions
$$\langle \Psi_\alpha^{1a}(x) \Psi_\beta^{b1}(y) \rangle = \frac{g_{\rm YM}^2}{8\pi^2} \epsilon_{\alpha\beta} \, \delta^{ab} \cdot \frac{\bar{x}_3 - y_3}{|\bar{x} - y|^4}.$$

Feynman diagrams



Leading for large-N



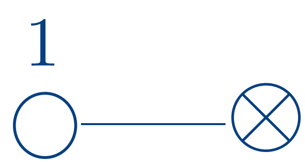
Sub-leading for large-N

C.K., Müller, Zarembo ‘20

Object to calculate $$C_{k=1} = \frac{\langle \mathrm{VBS} | \mathbf{u} \rangle}{\langle \mathbf{u} | \mathbf{u} \rangle^{1/2}}$$

$$\langle \mathrm{VBS} | = \left(\langle XX | + \langle \Psi_1 \Psi_2 | - \langle \Psi_2 \Psi_1 | \right)^{\otimes L/2},$$ $SU(2|1)$ sector

$$C_{k=1} = \frac{Q_1(0) Q_2(0)}{Q_1\left(\frac{i}{2}\right)} \mathrm{SDet} G$$



Integrable overlaps and the Gaudin determinant

$\hat{\mathbf{Q}}_{\mathbf{2n+1}}|B\rangle = 0 \implies$

$\langle B|\mathbf{u}\rangle \neq 0$ iff momentum carrying roots are paired $\{u_i, -u_i\}_{i=1}^{K_u}$
(excluding singular cases)

$\implies$ auxiliary roots paired $\{v_i, -v_i\}_{i=1}^{K_v}$ possibly plus $\{0\}$

Gaudin matrix has block structure
Poszgay 13, Brockmann et al '14

$$\det G = \begin{vmatrix} A & B \\ B & A \end{vmatrix} = \begin{vmatrix} A+B & B \\ B+A & A \end{vmatrix} = \begin{vmatrix} A+B & B \\ 0 & A-B \end{vmatrix} = \det(A+B)\cdot\det(A-B)$$

$$= \det G_+ \cdot \det G_-$$

Quantity entering overlap formulas
C.K., Müller, Zarembo '20

$$\mathrm{SDet}\, G = \frac{\det G_+}{\det G_-} \equiv \mathbb{D} \qquad \left(= e^{\mathrm{Tr}\,\Omega\log G}, \qquad \Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

Integrable Super Spin Chains (of type $SU(M|N)$)

Cartan matrix M_{ab}, Dynkin labels q_a, $a, b = 1, \ldots, M+N-1$

Bethe equations Saleur '99

$$(-1)^{F_a+1} = \left(\frac{u_{a,j} - \frac{iq_a}{2}}{u_{a,j} + \frac{iq_a}{2}}\right)^L \prod_{b,k} \frac{u_{a,j} - u_{b,k} + \frac{iM_{ab}}{2}}{u_{a,j} - u_{b,k} - \frac{iM_{ab}}{2}} \equiv \mathrm{e}^{\,i\chi_{a,j}}$$

$u_{a,j}$: $a = 1, \ldots$ # of nodes in Dynkin diagram
$j = 1, \ldots, K_a$ (# of roots of type a)
momentum carrying if $q_a \neq 0$.

$$G_{aj,bk} = \frac{\partial \chi_{a,j}}{\partial u_{b,k}}$$

$$\frac{\langle \mathrm{VBS}|\mathbf{u}\rangle^2}{\langle \mathbf{u}|\mathbf{u}\rangle} = \prod_a \frac{\prod_{j=1}^{n_a} Q_a(\frac{is_{a,j}}{2})}{\prod_{k=1}^{m_a} Q_a(\frac{ir_{a,k}}{2})}\ \mathrm{SDet}G$$

AdS/CFT: N=M=4

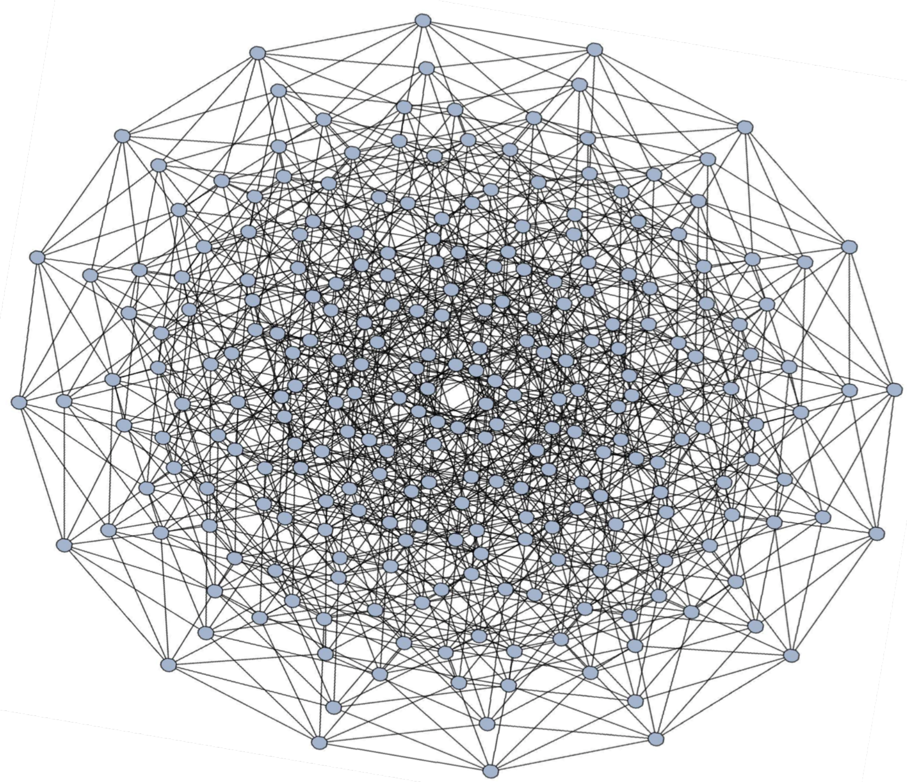
QQ-system

Tsuboi ´98 Gromov, Kazakov, Leurent, Volin ’14

Q-functions and QSC optimal language for the spectral problem

Many equivalent ways of writing the Bethe equations

For $\mathcal{N} = 4$ SYM, # different choices of Q-functions $= 2^8$



Connected via dualities

- Fermionic (Change of Dynkin diagram)
- Bosonic

How does the Gaudin matrix fit into this?

Example: $SU(2|1)$ super spin chain

Encodes conformal single trace operators built from fields
X (bosonic), Ψ_1,Ψ_2 (fermionic)

Cartan matrix   Dynkin label

$$M = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}, \qquad q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$H = \sum_{n=1}^{L} (1 - \Pi_{n,n+1})$$

graded permutation

Baxter polynomials

$$Q_1(u) = \prod_{i=1}^{K_1} (u - u_i), \qquad Q_2(u) = \prod_{j=1}^{K_2} (v - v_j), \qquad Q_\theta = u^L$$

Vacuum: $|\Psi_1 \Psi_1 \ldots \rangle$,   Excitations at level 1 and 2: Ψ_2, X

Fermionic Duality: Ex: $SU(2|1)$

Beisert, Kazakov, ,
Sakai, Zarembo '05

$Q_1, u_i, K_1 \quad Q_2, v_j, K_2$

$$M = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}, \; q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$Q_1, u_i, K_1 \quad \widetilde{Q}_2, \tilde{v}_j, \widetilde{K}_2$

$$\widetilde{M} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; \tilde{q} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$1 = \frac{Q_1^-(v_k)}{Q_1^+(v_k)} \longrightarrow \frac{Q_1^+(\tilde{v}_k)}{Q_1^-(\tilde{v}_k)} = 1$$

$$-1 = \frac{Q_1^{++}(u_k)}{Q_1^{--}(u_k)} \cdot \frac{Q_2^-(u_k)}{Q_2^+(u_k)} \left(\frac{Q_\theta^-(u_k)}{Q_\theta^+(u_k)}\right)^L \longrightarrow \frac{\widetilde{Q}_2^+(u_k)}{\widetilde{Q}_2^-(u_k)} \left(\frac{Q_\theta^-(u_k)}{Q_\theta^+(u_k)}\right)^L = 1$$

Change of variables (from v_j to $\tilde{v}_j$)

$\widetilde{K}_2 = K_1 - K_2 - 1$ roots $\tilde{v}_j$

$\downarrow$

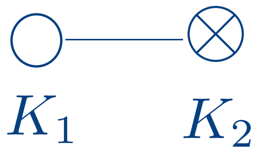
$$Q_1^-(v) - Q_1^+(v) = Q_2(v) \cdot \widetilde{Q}_2(v)$$

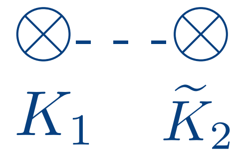
$\uparrow$

K_2 roots v_j

$$Q^\pm(u) \equiv Q(u \pm \frac{i}{2})$$

Transformation formula: Ex: $SU(2|1)$





K_1, K_2 even $\implies$ $\widetilde{K_2} = K_1 - K_2 - 1$ odd, i.e. $\tilde{v}$'s contain a single zero

$Q_1^+(u) - Q_1^-(u) = iK_1\, u\, Q_2(u)\, \widetilde{Q}_2(u)$, with reduced Baxter polynomials

$$\boxed{\widetilde{\mathbb{D}} = K_1 \frac{\widetilde{Q}_2(0) Q_2(0)}{Q_1(\frac{i}{2})} \mathbb{D}}$$

Found numerically C.K., Müller, Zarembo '20

Analytical proof in progress

Notice:

- Holds semi-on-shell (the $\{u_i, -u_i\}$'s can be chosen at random)
- Covariance if the overlap formula involves $Q_2(0)\mathbb{D}$
- Factor K_1 signals that a hws is mapped to a descendent

de Leeuw, Ipsen, C.K., Vardinghus, Wilhelm 17

Fermionic dualities in general

- Allow one to move between any two Dynkin diagrams of a super Lie algebra (of type $SU(N|M)$)

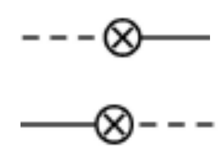
- Involve a fermionic node and its neighbours only 

- Changes the nature of neighbouring nodes $\bigotimes \longleftrightarrow \bigcirc$
  and the connections —— $\longleftrightarrow$ - - -

- Dualized node non-momentum carrying $\implies$ Dynkin labels unchanged

- Dualized node momentum carrying $\implies$ Dynkin labels change

$$\begin{bmatrix} 0 \\ V \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} V \pm 1 \\ -V \\ V \mp 1 \end{bmatrix} \quad \text{for}$$



Dualizing a non-momentum-carrying node

$$M = \begin{bmatrix} \eta_2 & \eta_1 & 0 \\ \eta_1 & 0 & -\eta_1 \\ 0 & -\eta_1 & \eta_3 \end{bmatrix}, \qquad q = \begin{bmatrix} V_l \\ 0 \\ V_r \end{bmatrix}, \qquad \begin{matrix} \eta_1 \in \{-1,+1\} \\ \eta_2 \in \{0,-2\eta_1\} \\ \eta_3 \in \{0, 2\eta_1\} \end{matrix},$$

$$K_l, K_r, K_m \text{ all even} \implies \widetilde{K}_m = K_l + K_r - K_m - 1 \text{ odd}$$

$$Q_l^- Q_r^+ - Q_l^+ Q_r^- = i\eta_1 (K_r - K_l) u\, Q_m \widetilde{Q}_m\,,$$

C.K., Müller, Zarembo '20

$$\widetilde{\mathbb{D}} = \mathrm{J}\,\mathbb{D} = (-\eta_1)^{K_l} \eta_1^{K_r} (\eta_1 K_r - \eta_1 K_l) \frac{Q_m(0)\widetilde{Q}_m(0)}{Q_l\left(\frac{i}{2}\right) Q_r\left(\frac{i}{2}\right)} \mathbb{D}$$

Found numerically
Analytical proof in progress

K_l, K_r even, K_m odd

$$\widetilde{\mathbb{D}} = (-\mathrm{J})^{-1}\,\mathbb{D}\,,$$

Dualizing a momentum-carrying node

K_l K_m K_r

$$M = \begin{bmatrix} \eta_2 & \eta_1 & 0 \\ \eta_1 & 0 & -\eta_1 \\ 0 & -\eta_1 & \eta_3 \end{bmatrix}, \qquad q = \begin{bmatrix} 0 \\ V \\ 0 \end{bmatrix}, \qquad \begin{array}{l} \eta_1 \in \{-1,+1\} \\ \eta_2 \in \{0,-2\eta_1\} \\ \eta_3 \in \{0, 2\eta_1\} \end{array}$$

$$K_l, K_r, K_m, L \text{ all even} \implies \widetilde{K}_m = L + K_l + K_r - K_m - 1 \text{ odd}$$

$$\left(u + V\tfrac{i}{2}\right)^L Q_l^- Q_r^+ - \left(u - V\tfrac{i}{2}\right)^L Q_l^+ Q_r^- = i(VL - \eta_1 K_l + \eta_1 K_r)\, u\, Q_m \widetilde{Q}_m,$$

$$\widetilde{\mathbb{D}} = \left(\frac{2i}{V}\right)^L (VL - \eta_1 K_l + \eta_1 K_r)\, \frac{Q_m(0)\widetilde{Q}_m(0)}{Q_l\left(\frac{i}{2}\right) Q_r\left(\frac{i}{2}\right)}\, \mathbb{D},$$

C.K., Müller, Zarembo '20

Found numerically
Analytical proof
in progress

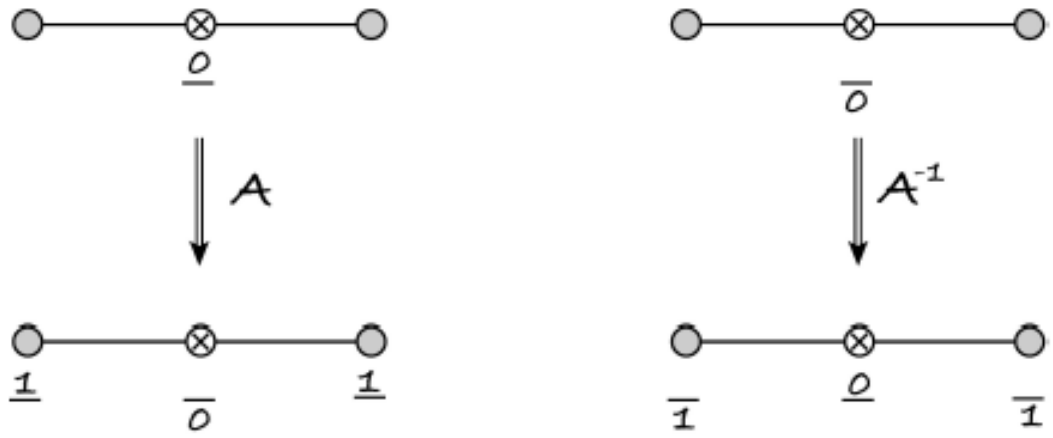
NB: K_r odd or K_l odd requires regularization

Dualizing overlap formulas I

C.K., Müller, Zarembo '20

$$\widetilde{\mathbb{D}} \propto \frac{\widetilde{Q}_a(0) Q_a(0)}{Q_{a-1}\left(\frac{i}{2}\right) Q_{a+1}\left(\frac{i}{2}\right)} \mathbb{D}$$

(Both for momentum carrying and non-momentum carrying nodes)



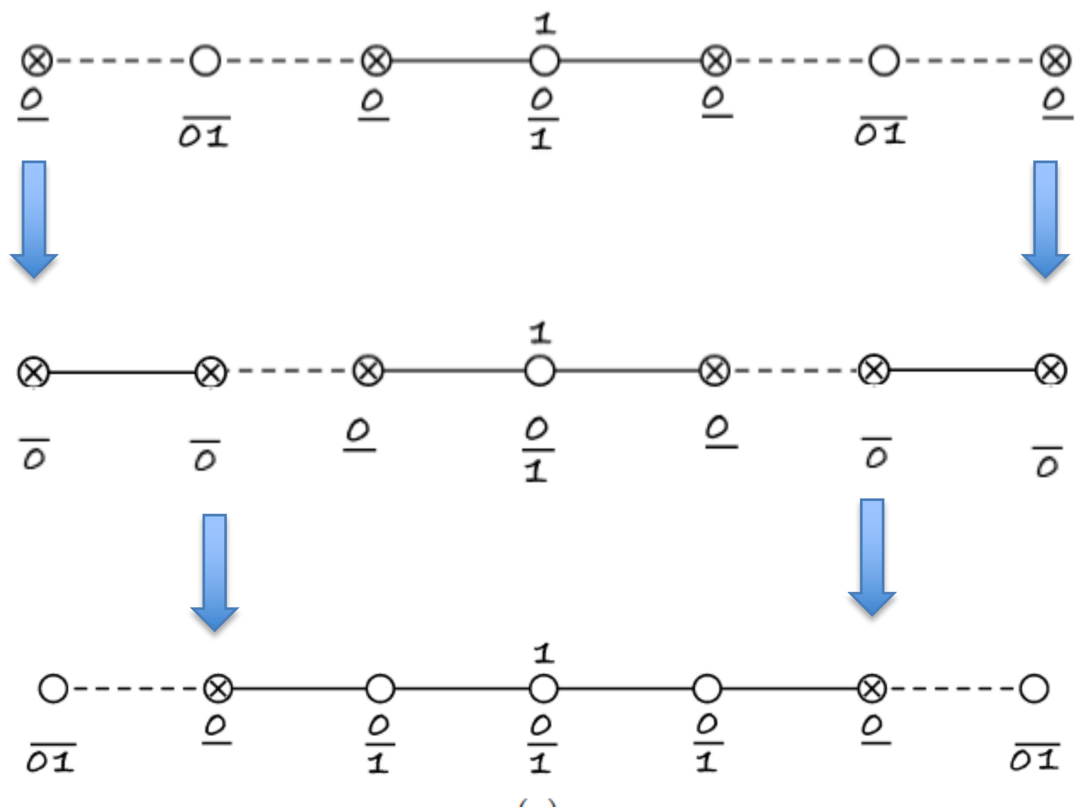
Covariance of overlap formulas very constraining (fully constraining?)

Dualizing overlap formulas II

$PSU(2,2|4)$ overlap formula, alternating grading

Gombor & Bajnok '20

Has exactly the prescribed covariance properties



$PSU(2,2|4)$ overlap formula, beauty grading

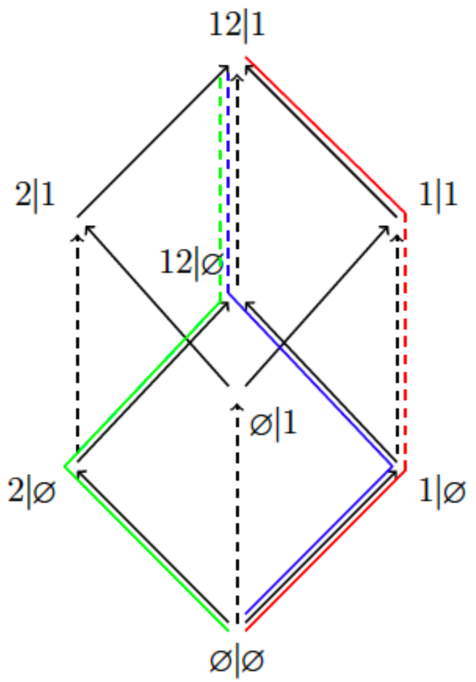
Agrees with field theory result in $SO(6)$ sector

C.K., Müller, Zarembo '20

De leeuw., C.K. Linardopoulos '18

A Web of Dualities: Ex: $SU(1|2)$

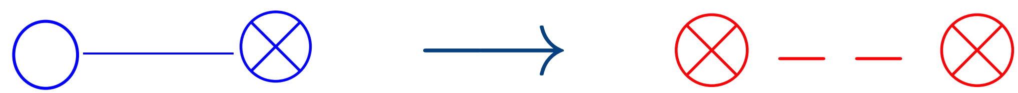
Tsuboi '98



2^3 Q-functions, 2 fixed

$6 = 3 \times 2$ versions of the BE's ($\sim$ paths)

Fermionic Duality considered so far:
(flipping across a vertical face)



$$Q_{12|\emptyset} Q_{1|1} = Q^+_{12|1} Q^-_{1|\emptyset} - Q^-_{12|1} Q^+_{1|\emptyset} = Q^-_{1|\emptyset} - Q^+_{1|\emptyset}$$

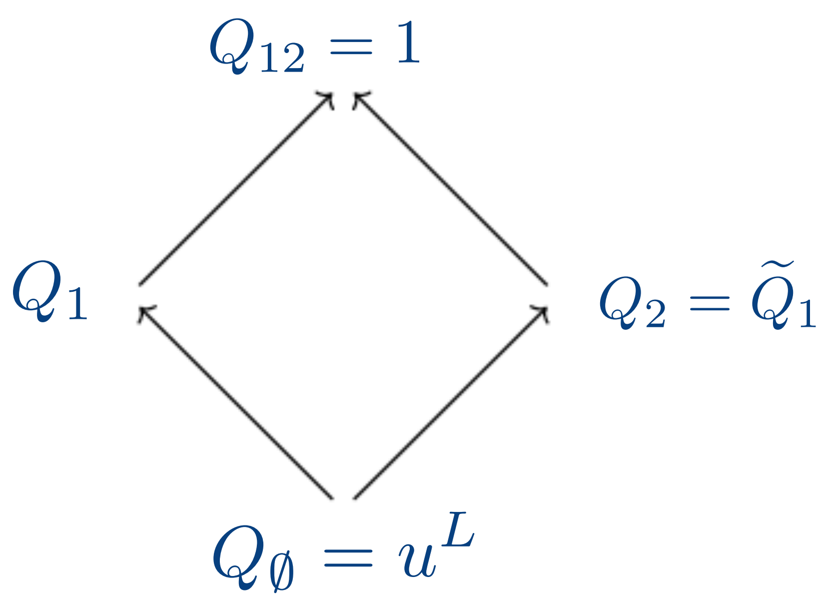
$$Q_{\emptyset|\emptyset} = u^L, \;\; Q_{12|1} = 1$$

Additional bosonic dualities such as

(flipping across a horisontal face)

$$Q^+_{1|\emptyset} Q^-_{2|\emptyset} - Q^-_{1|\emptyset} Q^+_{2|\emptyset} = Q_{\emptyset|\emptyset} Q_{12|\emptyset}$$

Bosonic Dualities: A warm-up example: $SU(2)$



Bosonic duality eqn.

$$Q_1^+ \widetilde{Q}_1^- - Q_1^- \widetilde{Q}_1^+ = u^L$$

$$\widetilde{K} = L - K + 1$$

Dual roots at $0, \pm\frac{i}{2}$ call for regularization of $\det G$

After regularization: Roots at $0, \pm\frac{i}{2}$ left out in $\widetilde{Q}$

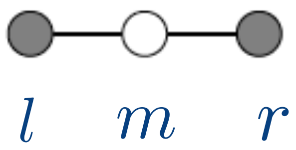
C.K., Müller, Zarembo '21

$$\widetilde{\mathbb{D}} = \mathbb{A}_{L/2-K} \, \frac{Q(0)\widetilde{Q}(i/2)}{Q(i/2)\widetilde{Q}(0)} \, \mathbb{D}, \qquad \mathbb{A}_n = \frac{(2^n n!)^4}{2\,(2n)!\,(2n+1)!}$$

Overlaps with VBS Duality invariant

Bosonic dualities in general

- Involve a bosonic node and its neighbours only



- Do not change the Dynkin diagram or the Dynkin labels
- Transformation formula only involves Q_m and $\widetilde{Q}_m$

C.K., Müller, Zarembo '21

- Momentum carrying bosonic node

$$\widetilde{\mathbb{D}} = \mathbb{A}_{(L+K_r+K_l)/2-K_m} \frac{Q_m(0)\widetilde{Q}_m(i/2)}{Q_m\left(\frac{i}{2}\right)\widetilde{Q}_m(0)} \mathbb{D}$$

- Non-momentum carrying bosonic node

$$\widetilde{\mathbb{D}} = \mathbb{A}_{(K_r+K_l)/2-K_m} \frac{Q_m(0)\widetilde{Q}_m(i/2)}{Q_m\left(\frac{i}{2}\right)\widetilde{Q}_m(0)} \mathbb{D}$$

- Overlaps in the scalar $SO(6)$ sector invariant (up to pre-factor)

Summary

- We have exhausted all fermionic and bosonic spin chain dualities and found their implications for overlap formulas.

Future Directions

- Analytical proof of the duality transformation formulas
  Easy to state --- difficult to prove

- Understand the pre-factors in the transformation formulas

- Express the overlaps entirely in terms of Q-functions
  and treat the overlaps by means of the Quantum Spectral Curve

- Use duality formulas to constrain unknown overlap formulas

- Classify all integrable boundary states in AdS/CFT

Thank you