Inferring hadron LCDAs using lattice Q(C+E)D

Davide Giusti

Universität Regensburg

INFN Istituto Nazionale di Fisica Nucleare

JOHANNES GUTENBERG UNIVERSITÄT MAINZ

LCDA of Hadrons in QCD and their Applications

Mainz

21st January 2020

OUTLINE

- Motivations
- Leptonic decays of pseudoscalar mesons

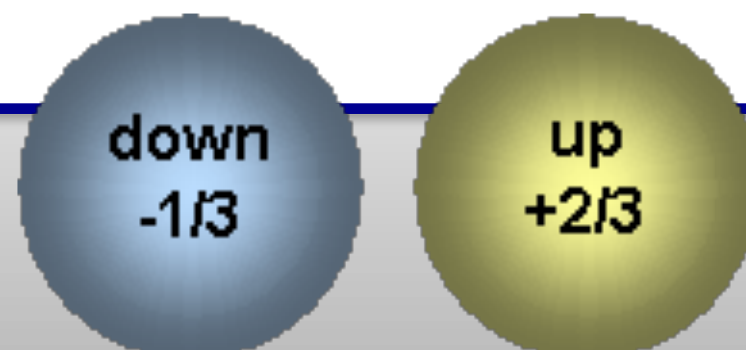
  $P^- \to \ell^- \bar{\nu}_\ell \gamma$
- Outlook

First lattice calculations

- C. Kane, C. Lehner, S. Meinel, A. Soni, arXiv:1907.00279
- RM123 & Soton Collaboration, arXiv:1908.10160

Lattice 2019 Wuhan·China June 16-22

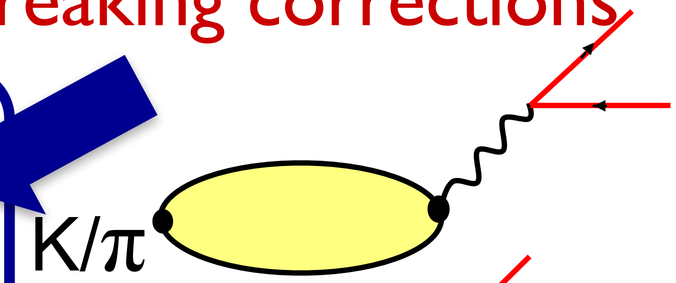
Phenomenological motivations



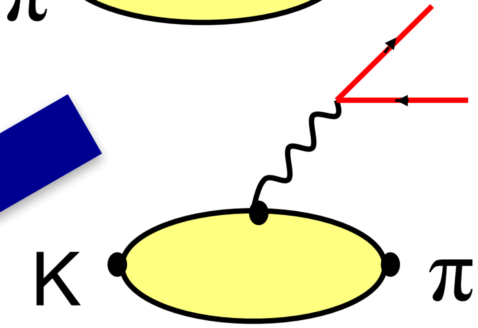
Electromagnetic and isospin-breaking effects

Given the present exper. and theor. (LQCD) accuracy, an important source of uncertainty are long distance electromagnetic and SU(2)-breaking corrections

$$\frac{\Gamma\left(K^+ \to \ell^+ \nu_\ell(\gamma)\right)}{\Gamma\left(\pi^+ \to \ell^+ \nu_\ell(\gamma)\right)} = \left(\frac{|V_{us}|}{|V_{ud}|}\frac{f_K}{f_\pi}\right)^2 \frac{M_{K^+}\left(1 - m_\ell^2 / M_{K^+}^2\right)^2}{M_{\pi^+}\left(1 - m_\ell^2 / M_{\pi^+}^2\right)^2} \boxed{\left(1 + \delta_{EM} + \delta_{SU(2)}\right)}$$



$$\Gamma\left(K^{+,0} \to \pi^{0,-} \ell^+ \nu_\ell(\gamma)\right) = \frac{G_F^2 M_{K^{+,0}}^5}{192\pi^3} C_{K^{+,0}}^2 \left|V_{us} f_+^{K^0\pi^-}(0)\right|^2 I_{K\ell}^{(0)} S_{EW} \boxed{\left(1 + \delta_{EM}^{K^{+,0}\ell} + \delta_{SU(2)}^{K^{+,0}\pi}\right)}$$



For $\Gamma_{Kl2}/\Gamma_{\pi l2}$ At leading order in ChPT both δ_{EM} and $\delta_{SU(2)}$ can be expressed in terms of physical quantities (e.m. pion mass splitting, f_K/f_π, …)

- $\boxed{\delta_{EM} = -0.0069\,(17)}$ 25% of error due to higher orders ➡ 0.2% on $\Gamma_{Kl2}/\Gamma_{\pi l2}$

  M.Knecht *et al.*, 2000; V.Cirigliano and H.Neufeld, 2011

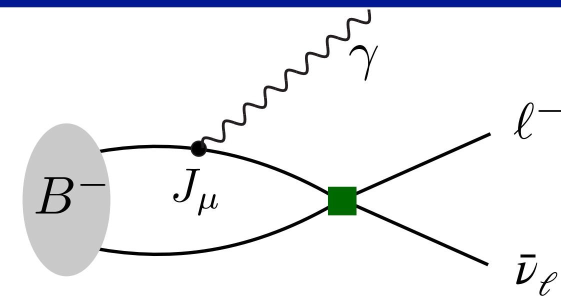
- $\boxed{\delta_{SU(2)} = \left(\frac{f_{K^+}/f_{\pi^+}}{f_K/f_\pi}\right)^2 - 1 = -0.0044\,(12)}$ 25% of error due to higher orders ➡ 0.1% on $\Gamma_{Kl2}/\Gamma_{\pi l2}$

  J.Gasser and H.Leutwyler, 1985; V.Cirigliano and H.Neufeld, 2011

ChPT is not applicable to D and B decays

Radiative corrections to leptonic B-meson decays

$B^- \to \ell^- \bar{\nu}_\ell \gamma$



- The emission of a real hard photon removes the $(m_\ell/M_B)^2$ helicity suppression
- This is the simplest process that probes (for large E_γ) the first inverse moment of the B-meson LCDA

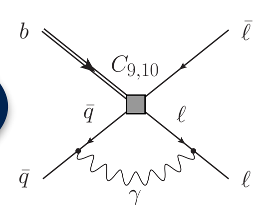
$$\frac{1}{\lambda_B(\mu)} = \int_0^\infty \frac{d\omega}{\omega} \Phi_{B+}(\omega, \mu)$$

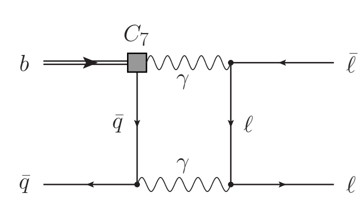
λ_B is an important input in QCD-factorization predictions for non-leptonic B decays but is poorly known

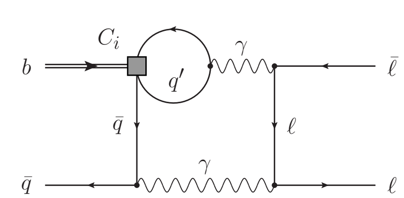
M. Beneke, V. M. Braun, Y. Ji, Y.-B. Wei, 2018

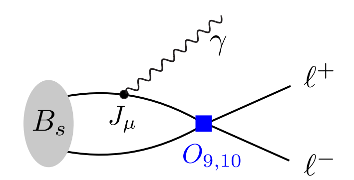
- Belle 2018: $\mathscr{B}(B^- \to \ell^- \bar{\nu}_\ell \gamma, E_\gamma > 1 \text{ GeV}) < 3.0 \cdot 10^{-6}$ $\longrightarrow$ $\lambda_B > 0.24$ GeV
- QCD sum rules in HQET: $\lambda_B(1 \text{ GeV}) = 0.46\,(11)$ GeV

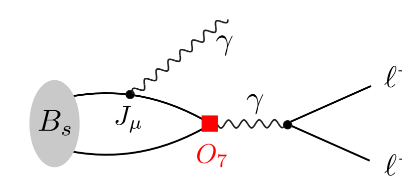
$B_q \to \ell^+ \ell^- (\gamma)$











- Enhancement of the virtual corrections by a factor M_B/Λ_{QCD} and by large logarithms

M. Beneke, C. Bobeth, R. Szafron, 2019

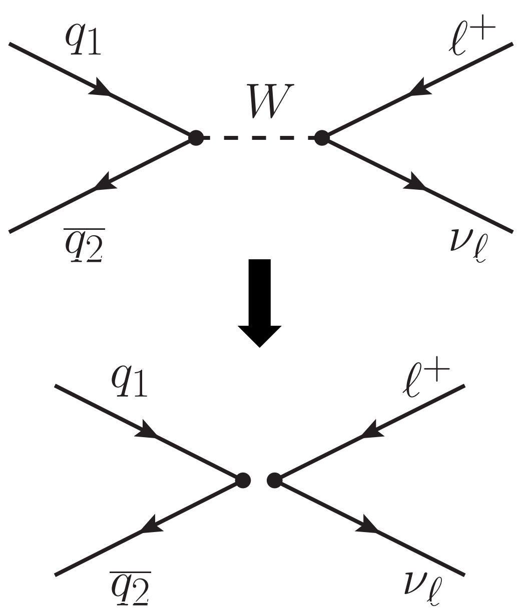
- The real photon emission process is a clean probe of NP: sensitiveness to $F_{V,A,TV,TA}(E_\gamma)$

Lattice calculations of

$$P^- \to \ell^- \bar{\nu}_\ell(\gamma)$$

Leptonic decays at tree level

Since the masses of the pion and kaon are much smaller than M_W we use the effective Hamiltonian



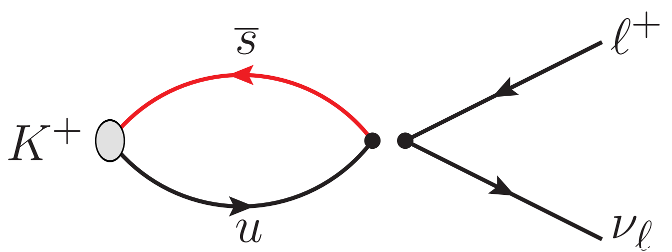
$$H_{eff} = \frac{G_F}{\sqrt{2}} V^*_{q_1 q_2} \left(\overline{q_2}\gamma^\mu (1-\gamma_5) q_1\right)\left(\overline{\nu_\ell}\gamma_\mu (1-\gamma_5)\ell\right)$$

This replacement is necessary in a lattice calculation, since $1/a \ll M_W$

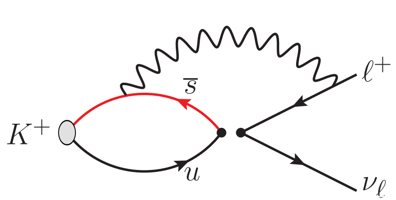
The rate is:

$$\Gamma^{(tree)}_{P^\pm}\left(P^\pm \to \ell^\pm \nu_\ell\right) = \frac{G_F^2}{8\pi}\left|V_{q_1 q_2}\right|^2 \left[f_P^{(0)}\right]^2 M_{P^\pm} m_\ell^2 \left(1 - \frac{m_\ell^2}{M_{P^\pm}^2}\right)^2$$

In the absence of electromagnetism, the non-perturbative QCD effects are contained in a single number, the pseudoscalar decay constant



$$A_P^{(0)} \equiv \langle 0 | \overline{q}_2 \gamma_4 \gamma_5 q_1 | P^{(0)} \rangle = f_P^{(0)} M_P^{(0)}$$

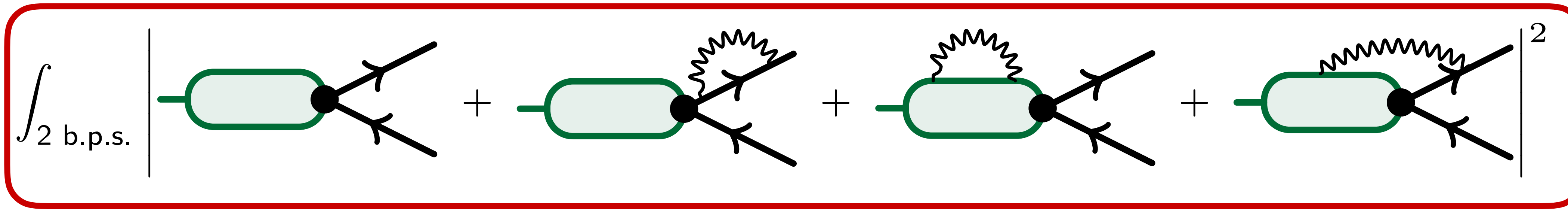


In the presence of electromagnetism it is not even possible to give a physical definition of f_P

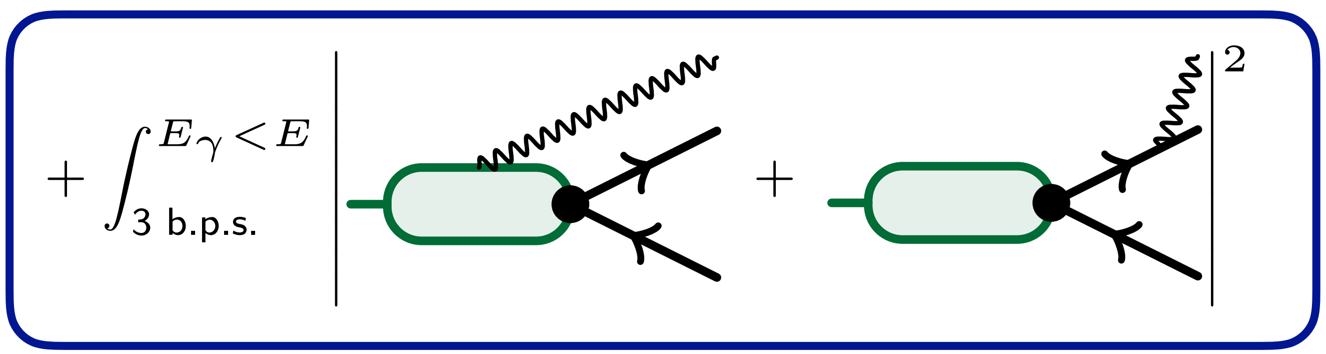
J. Gasser and G.R.S. Zarnauskas, PLB 693 (2010) 122

Leptonic decays at O(α)

$\Gamma(E) =$



Γ_0



$\Gamma_1(E)$

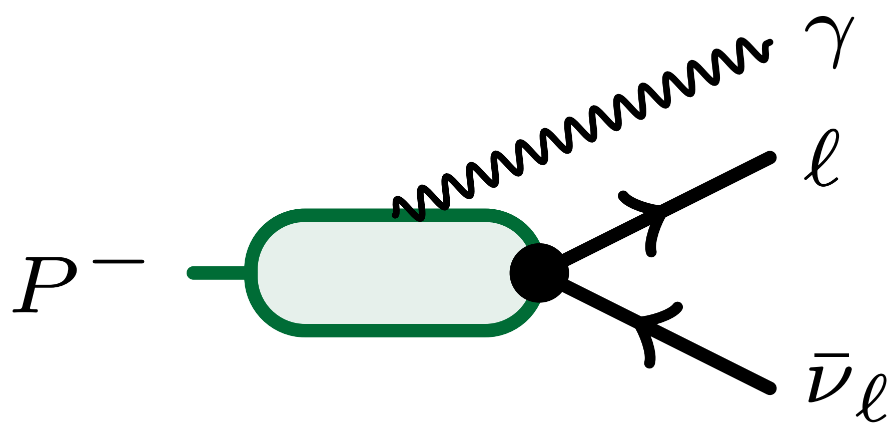
$\Gamma(E) = \Gamma_0 + \Gamma_1(E)$ with $0 \leq E_\gamma \leq E$ is infrared finite

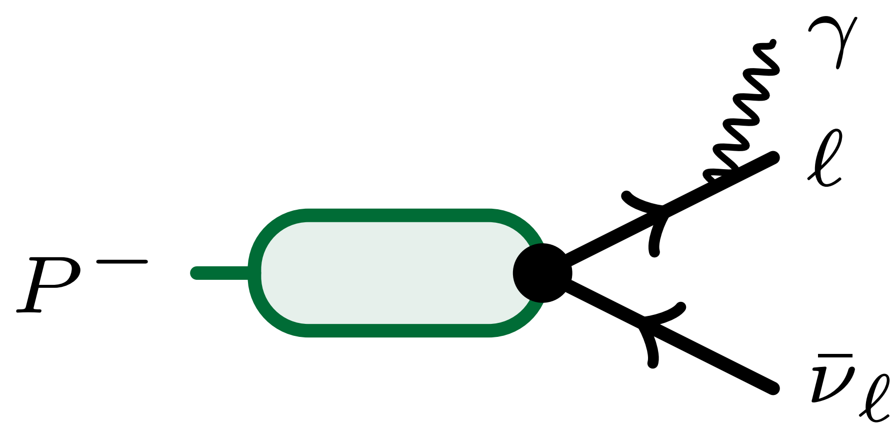
F. Bloch and A. Nordsieck, 1937

- Both Γ_0 and $\Gamma_1(E)$ can be evaluated in a fully non-perturbative way in lattice simulations
- The first lattice calculations of $\Gamma[\pi, K \to \ell\nu(\gamma)]$ have been finalized

N. Carrasco *et al.* arXiv:1502.00257   V. Lubicz *et al.* arXiv:1611.08497   DG *et al.* arXiv:1711.06537   M. Di Carlo *et al.* arXiv:1904.08731

Real photon emission amplitude





$$\boxed{H_W^{\alpha r}(k,p) = \epsilon_\mu^r(k)\, H_W^{\alpha\mu}(k,p) = \epsilon_\mu^r(k) \int d^4y\, e^{ik\cdot y}\, \mathtt{T}\langle 0|j_W^\alpha(0) j_{em}^\mu(y)|P(\boldsymbol{p})\rangle}$$

$$H_W^{\alpha r}(k,p) = \epsilon_\mu^r(k) \Big\{ H_1 \left[k^2 g^{\mu\alpha} - k^\mu k^\alpha\right] + H_2 \left[(p\cdot k - k^2)k^\mu - k^2(p-k)^\mu\right](p-k)^\alpha$$

$$- i\frac{F_V}{m_P}\varepsilon^{\mu\alpha\gamma\beta}k_\gamma p_\beta + \frac{F_A}{m_P}\left[(p\cdot k - k^2)g^{\mu\alpha} - (p-k)^\mu k^\alpha\right] \quad H_{SD}^{\alpha\mu}(k,p)$$

$$+ \boxed{f_P \left[g^{\mu\alpha} + \frac{(2p-k)^\mu (p-k)^\alpha}{2p\cdot k - k^2}\right]} \Big\} \quad H_{pt}^{\alpha\mu}(k,p)$$

$$k_\mu\, H_W^{\alpha\mu}(k,p) = k_\mu\, H_{pt}^{\alpha\mu}(k,p) = i\langle 0|j_W^\alpha(0)|P(\boldsymbol{p})\rangle = f_P\, p^\alpha$$

Real photon emission amplitude

By setting $k^2 = 0$, at fixed meson mass, the form factors depend on $p \cdot k$ only. Moreover, by choosing a *physical* basis for the polarization vectors, *i.e.* $\epsilon_r(\mathbf{k}) \cdot k = 0$, one has

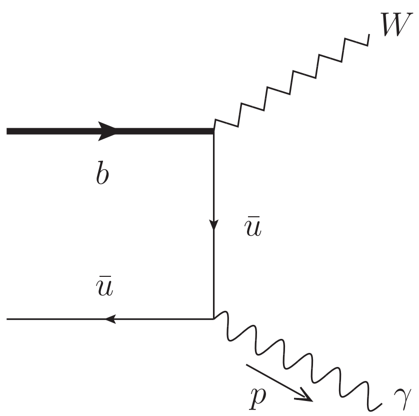
$$H_W^{\alpha r}(k,p) = \epsilon_\mu^r(\boldsymbol{k})\left\{ -i\frac{F_V}{m_P}\varepsilon^{\mu\alpha\gamma\beta}k_\gamma p_\beta + \left[\frac{F_A}{m_P} + \frac{f_P}{p\cdot k}\right](p\cdot k\, g^{\mu\alpha} - p^\mu k^\alpha) + \frac{f_P}{p\cdot k}p^\mu p^\alpha \right\}$$

In the case of off-shell photons ($k^2 \neq 0$) $\longrightarrow$ $\Gamma[K \to \ell\nu_\ell \ell^+\ell^-]$

For large photon energies and in the B-meson rest frame the form factors can be written as

$$F_V(E_\gamma) = \frac{e_u M_B f_B}{2E_\gamma \lambda_B(\mu)} R(E_\gamma, \mu) + \xi(E_\gamma) + \Delta\xi(E_\gamma)$$

$$F_A(E_\gamma) = \frac{e_u M_B f_B}{2E_\gamma \lambda_B(\mu)} R(E_\gamma, \mu) + \xi(E_\gamma) - \Delta\xi(E_\gamma)$$



M. Beneke and J. Rohrwild, 2011

Euclidean correlators

$$C_W^{\alpha r}(t,\boldsymbol{p},\boldsymbol{k}) = -i\,\varepsilon_\mu^r(\boldsymbol{k}) \int d^4y \int d^3\boldsymbol{x}\, \langle 0|\mathrm{T}\{j_W^\alpha(t,\boldsymbol{0})\, j_{em}^\mu(y)\}P(0,\boldsymbol{x})|0\rangle\, e^{E_\gamma t_y - i\boldsymbol{k}\cdot\boldsymbol{y} + i\boldsymbol{p}\cdot\boldsymbol{x}}$$

The convergence of the integral over t_y is ensured by the safe analytic continuation from Minkowsky to Euclidean spacetime, because of the absence of intermediate states lighter than the pseudoscalar meson

$$H_W^{\alpha r}(k,p) = \int d^4y\, e^{ik\cdot y}\, \epsilon_\mu^r(k)\mathrm{T}\langle 0|j_W^\alpha(0) j_{em}^\mu(y)|P(p)\rangle$$

$$\Downarrow$$

$$H_W^{\alpha r}(k,p) = H_{W,1}^{\alpha r}(k,p) + H_{W,2}^{\alpha r}(k,p)$$

$$j^r(\boldsymbol{k}) = \int d^3y\, e^{-i\boldsymbol{k}\cdot\boldsymbol{y}}\, \epsilon_\mu^r(k)\, j_{em}^\mu(0,\boldsymbol{y})$$

$$H_{W,1}^{\alpha r}(k,p) = -\,i\int_{-\infty}^{0} dt_y\, \langle 0\,|\,j_W^\alpha(0) e^{(H+E_\gamma-E_P)t_y}\, j^r(\mathbf{k})\,|\,P(p)\rangle$$

$$H_{W,2}^{\alpha r}(k,p) = -\,i\int_{0}^{\infty} dt_y\, \langle 0\,|\,j^r(\mathbf{k}) e^{-(H-E_\gamma)t_y}\, j_W^\alpha(0)\,|\,P(p)\rangle$$

$$\sqrt{m_P^2 + (\boldsymbol{p}-\boldsymbol{k})^2} + E_\gamma > \sqrt{m_P^2 + \boldsymbol{p}^2}\,, \qquad |\boldsymbol{k}| \neq 0$$

Form factors from Euclidean correlators

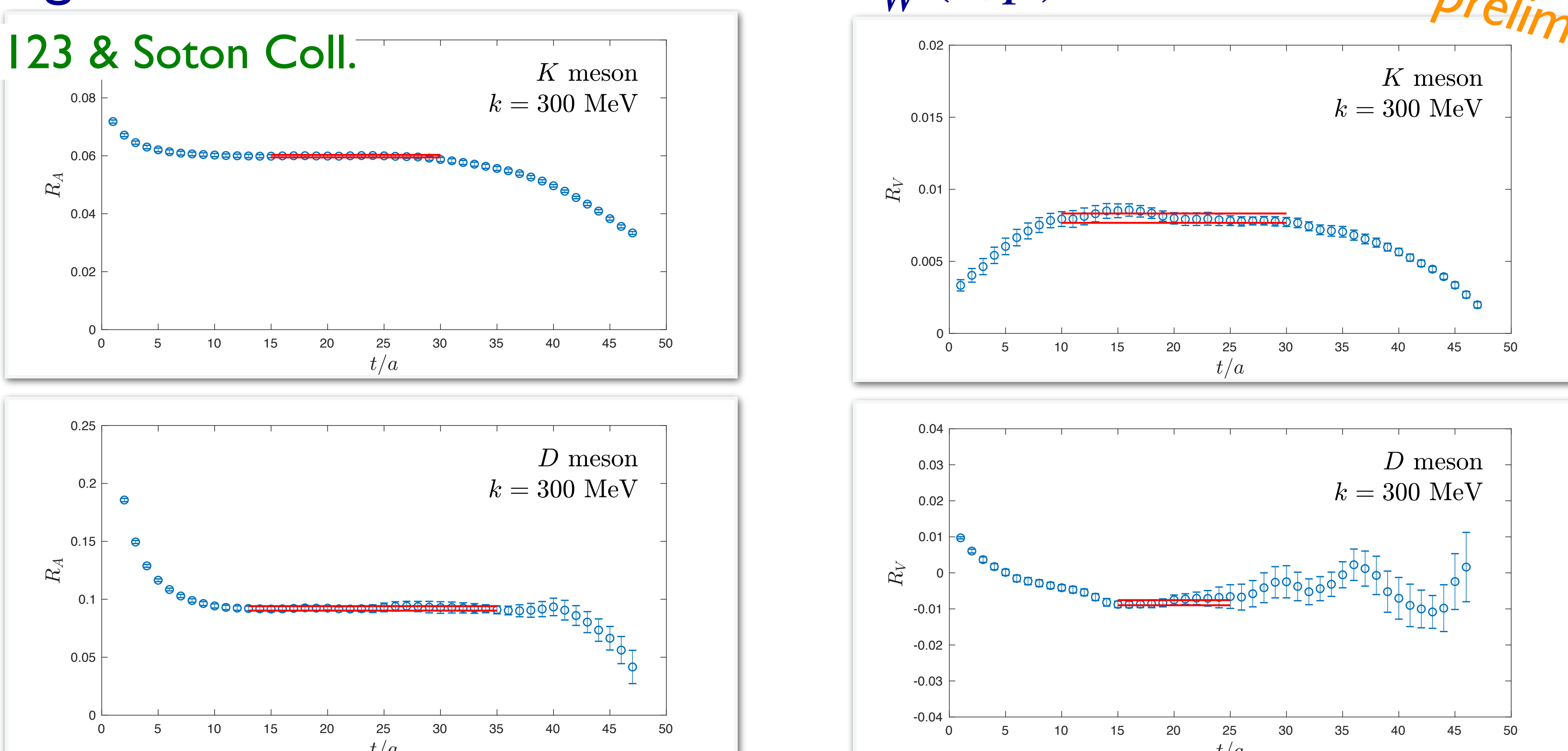
The physical form factors can be extracted directly from the Euclidean correlation functions (in the infinite-*T* limit)

$$R_W^{\alpha r}(t;\boldsymbol{p},\boldsymbol{k}) = \frac{2E_P}{e^{-t(E_P-E_\gamma)}\langle P(\boldsymbol{p})|P|0\rangle} C_W^{\alpha r}(t;\boldsymbol{p},\boldsymbol{k})$$

The numerical ratios $R_W^{\alpha r}(t;\mathbf{p},\mathbf{k})$ are expected to exhibit plateaux for $0 \ll t \ll T/2$, where exponentially-suppressed contributions can be neglected. In that region the above ratios give access to matrix elements $H_W^{\alpha r}(k,p)$

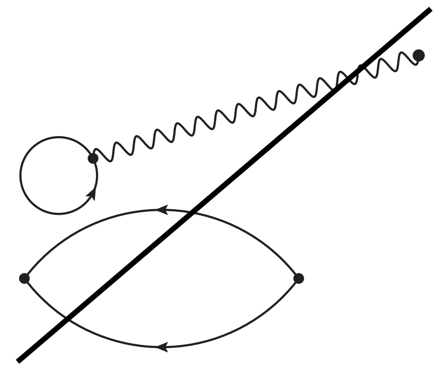
RM123 & Soton Coll.

preliminary



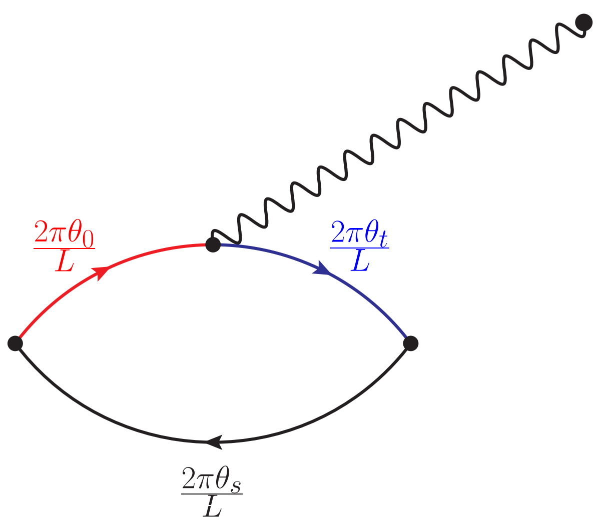
Form factors from Euclidean correlators

RM123 & Soton Coll., arXiv:1908.10160

Within the electro-quenched approximation  it is possible to

choose arbitrary values of the spatial momenta by using different spatial b.c. for the quark fields

$$\psi(x+\hat{\boldsymbol{k}}L) = \exp(2\pi i\hat{\boldsymbol{k}}\cdot\boldsymbol{\theta}_s/L)\psi(x)$$



$$\boldsymbol{p} = \frac{2\pi}{L}\left(\boldsymbol{\theta}_0 - \boldsymbol{\theta}_s\right)$$

$$\boldsymbol{k} = \frac{2\pi}{L}\left(\boldsymbol{\theta}_0 - \boldsymbol{\theta}_t\right)$$

Finite-volume effects are exponentially suppressed

We set: $\boldsymbol{p} = (0,0,|\boldsymbol{p}|)\ ,\qquad \boldsymbol{k} = (0,0,E_\gamma)\qquad \epsilon_1^\mu = \left(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)\ ,\qquad \epsilon_2^\mu = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$

Thus

$$H_A^{ir}(k,p) = \frac{\epsilon_r^i\, m_P}{2}\, x_\gamma \left[F_A + \frac{2f_P}{m_P x_\gamma}\right]\ ,\qquad H_V^{ir}(k,p) = \frac{i\left(E_\gamma\, \boldsymbol{\epsilon}_r \wedge \boldsymbol{p} - E\, \boldsymbol{\epsilon}_r \wedge \boldsymbol{k}\right)^i}{m_P}\, F_V$$

$$x_\gamma = \frac{2p\cdot k}{m_P^2} \qquad \text{with} \qquad 0 \le x_\gamma \le 1 - \frac{m_\ell^2}{m_P^2}$$

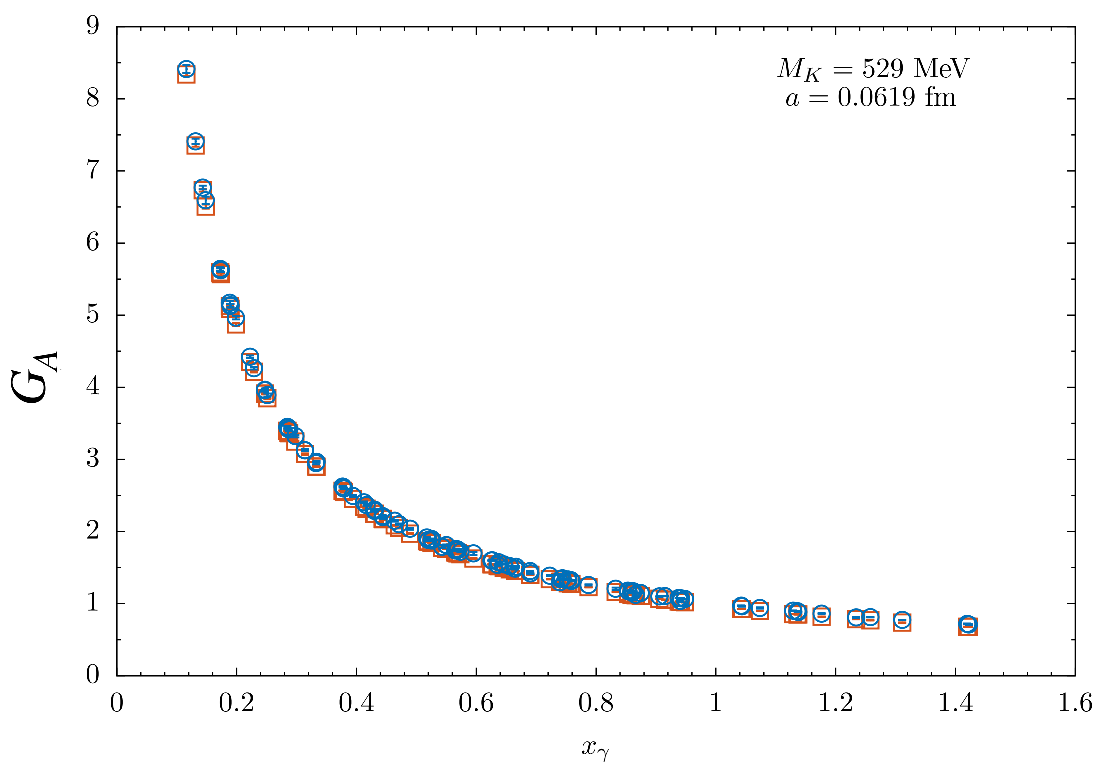
Form factors from Euclidean correlators

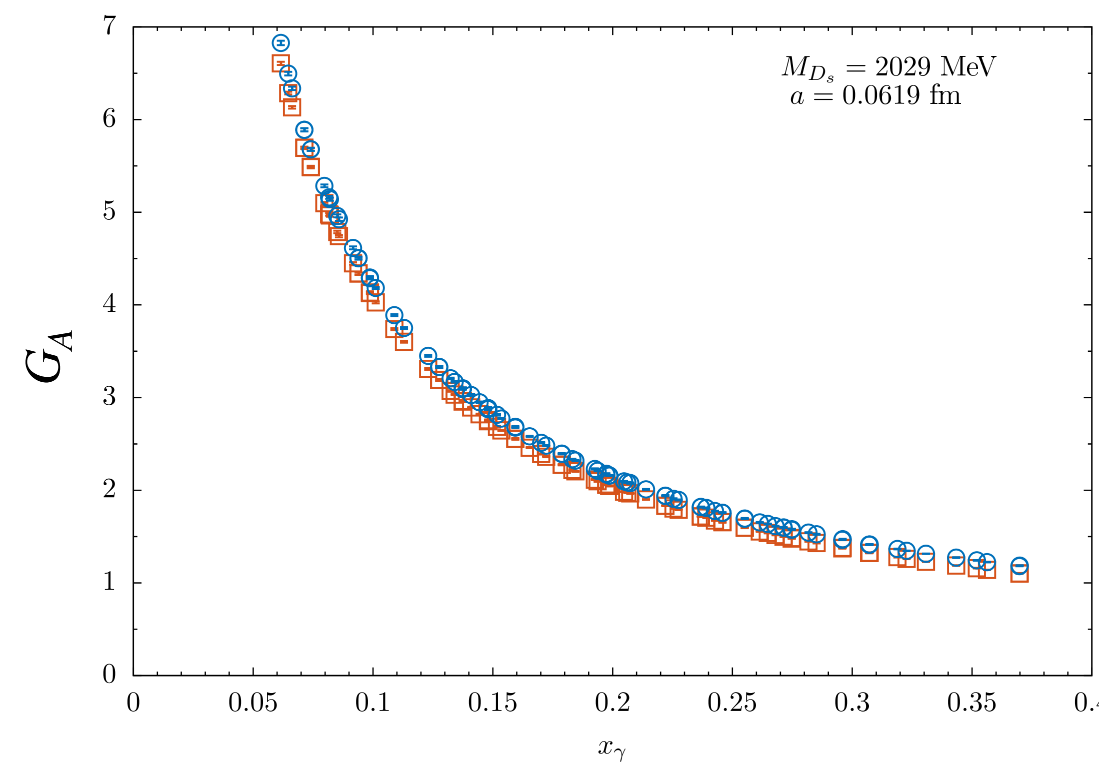
We build the following estimators

$$R_A(t) = \frac{1}{2m_P} \sum_{r=1,2} \sum_{j=1,2} \frac{R_A^{jr}(t;\boldsymbol{p},\boldsymbol{k})}{\epsilon_r^j} \quad \rightarrow G_A(x_\gamma) = \left[x_\gamma F_A(x_\gamma) + \frac{2f_P}{m_P}\right]$$

$$R_V(t) = \frac{m_P}{4} \sum_{r=1,2} \sum_{j=1,2} \frac{R_V^{jr}(t;\boldsymbol{p},\boldsymbol{k})}{i\left(E_\gamma\, \boldsymbol{\epsilon}_r \wedge \boldsymbol{p} - E_P \boldsymbol{\epsilon}_r \wedge \boldsymbol{k}\right)^j} \quad \rightarrow F_V(x_\gamma)$$

In our numerical calculation the above ratios are built in terms of finite-*T* correlators and time-reversal symmetries are exploited

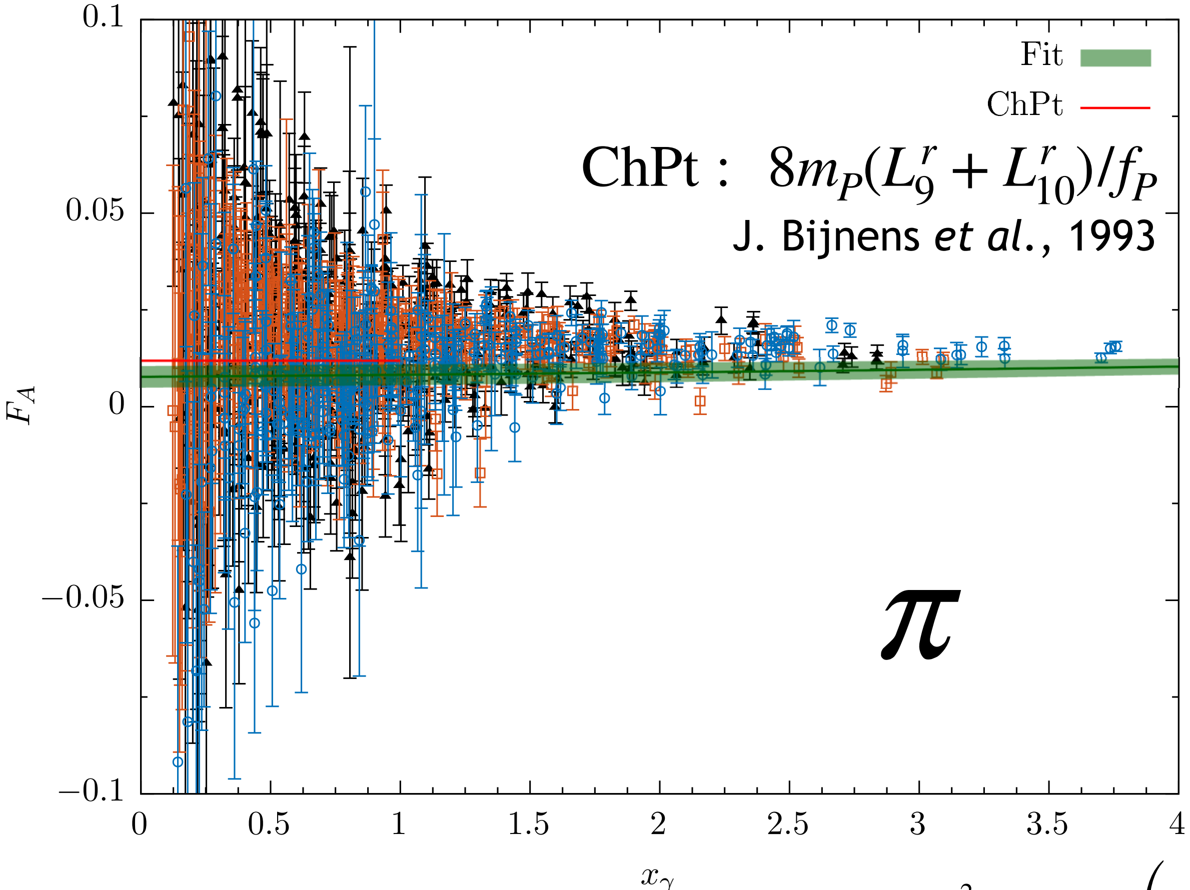


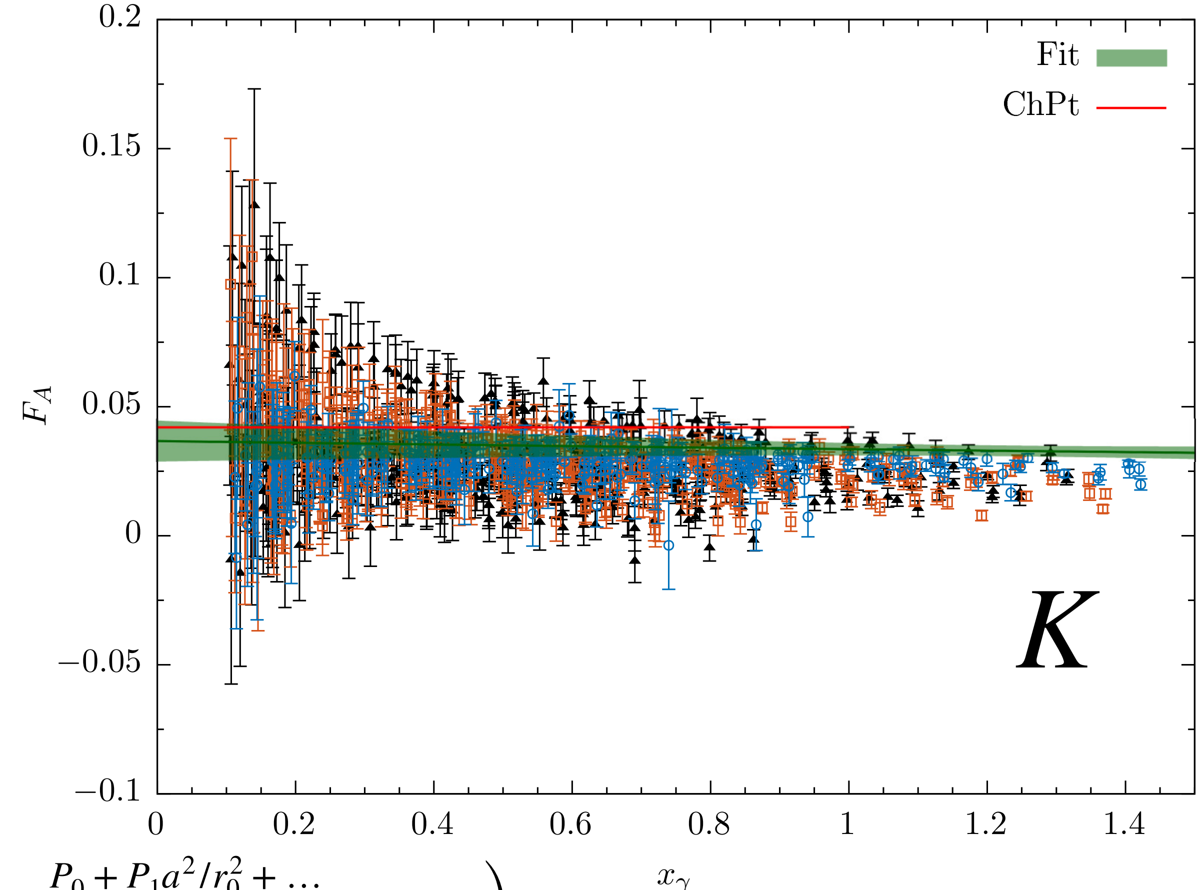


Preliminary results

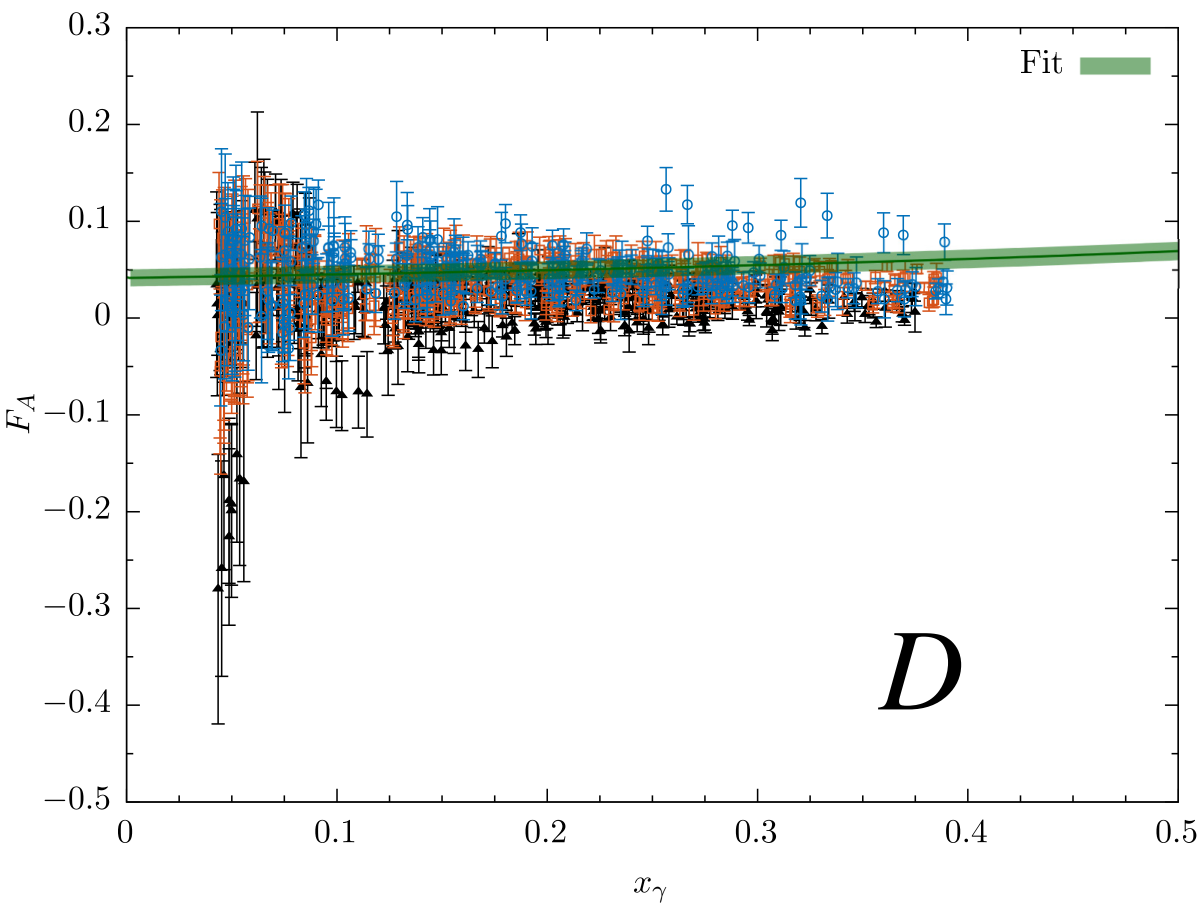
RM123 & Soton Coll., arXiv:1908.10160

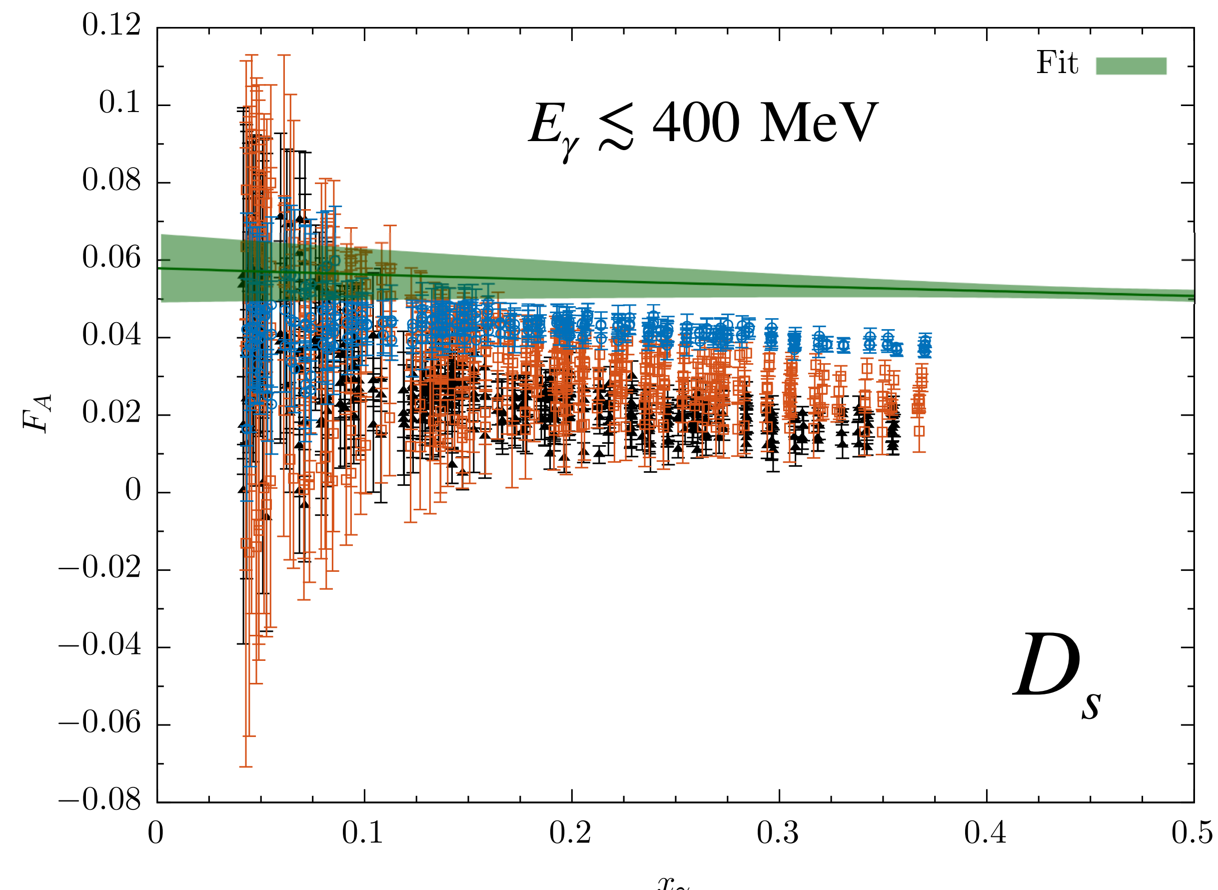
- $N_f = 2 + 1 + 1$ twisted mass fermions
- $a[\mathrm{fm}] = 0.0885\,(36),\ 0.0815\,(30),\ 0.0619\,(18)$
- 17 ensembles; 3-4 different pion masses for each lattice spacing
- Pion masses down to $\simeq 230$ MeV
- 125 different momentum configurations
- $O(100)$ gauge configurations for each ensemble
- Conserved electromagnetic current

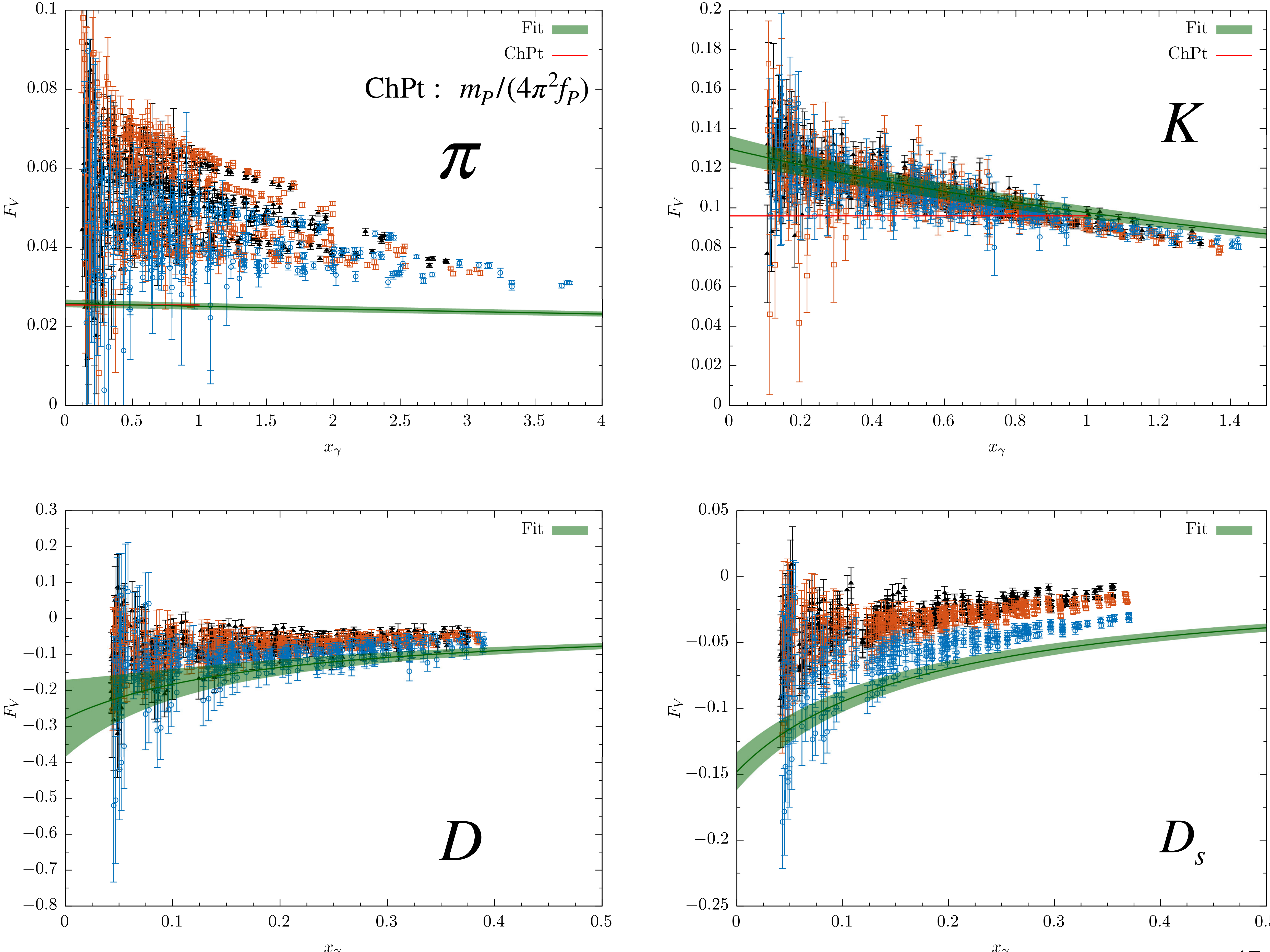




$$F_{(A,V)}(x_\gamma, m_{ud}, a^2, L) = m_P r_0 \left(\frac{P_0 + P_1 a^2/r_0^2 + \ldots}{1 - (P_2 + P_3 a^2/r_0^2 + \ldots)(m_P r_0)^2 (1 - x_\gamma)} \right)$$

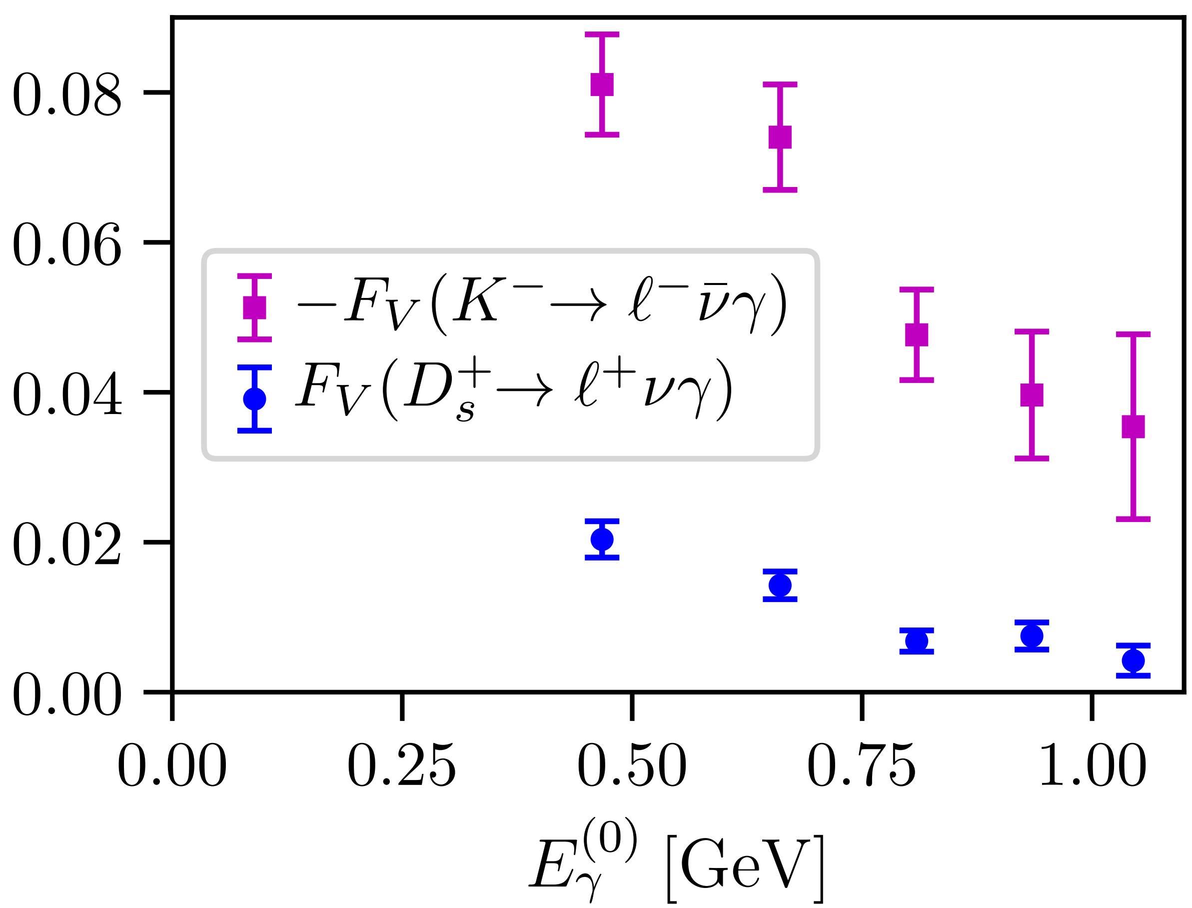


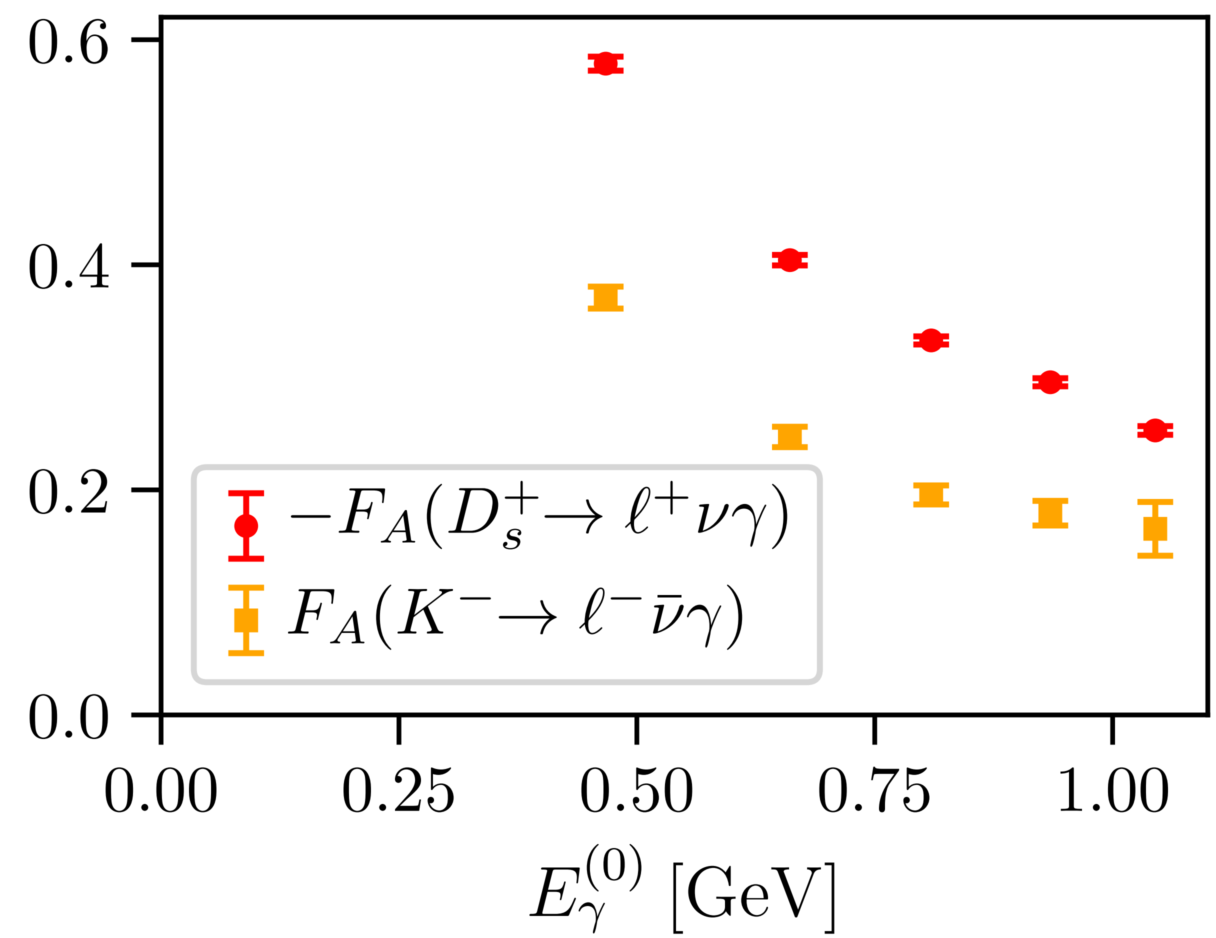



Fit
ChPt
ChPt : $m_P/(4\pi^2 f_P)$
π
F_V
x_γ
Fit
ChPt
K
F_V
x_γ
Fit
D
F_V
x_γ
Fit
D_s
F_V
x_γ

C. Kane, C. Lehner, S. Meinel, A. Soni, arXiv:1907.00279

- $N_f = 2 + 1$ domain wall fermions
- $M_\pi = 340\,(1)$ MeV, $a =$ fm
- $\mathbf{p}_{K/D_s} = 0$ and $\mathbf{p}_\gamma^2 \in \{1,2,3,4,5\}\left(\frac{2\pi}{L}\right)^2$
- 25 gauge configurations
- Local electromagnetic current + non-perturbative RCs





$$\frac{4\pi}{\alpha\,\Gamma_0^{\rm tree}}\frac{d\Gamma_1^{\rm SD}}{dx_\gamma} = \frac{m_P^2}{6 f_P^2\, r_\ell^2\,(1-r_\ell^2)^2}\left[F_V(x_\gamma)^2 + F_A(x_\gamma)^2\right] f^{\rm SD}(x_\gamma)$$

$$\frac{4\pi}{\alpha\,\Gamma_0^{\rm tree}}\frac{d\Gamma_1^{\rm INT}}{dx_\gamma} = -\frac{2\,m_P}{f_P\,(1-r_\ell^2)^2}\left[F_V(x_\gamma)\, f_V^{\rm INT}(x_\gamma) + F_A(x_\gamma)\, f_A^{\rm INT}(x_\gamma)\right]$$

$$K^{\rm SD}(x_\gamma) = \frac{m_P^2}{6 f_P^2\, r_\ell^2\,(1-r_\ell^2)^2}\, f^{\rm SD}(x_\gamma)$$

$$K_{V,A}^{\rm INT}(x_\gamma) = -\frac{2\,m_P}{f_P\,(1-r_\ell^2)^2}\, f_{V,A}^{\rm INT}(x_\gamma)$$

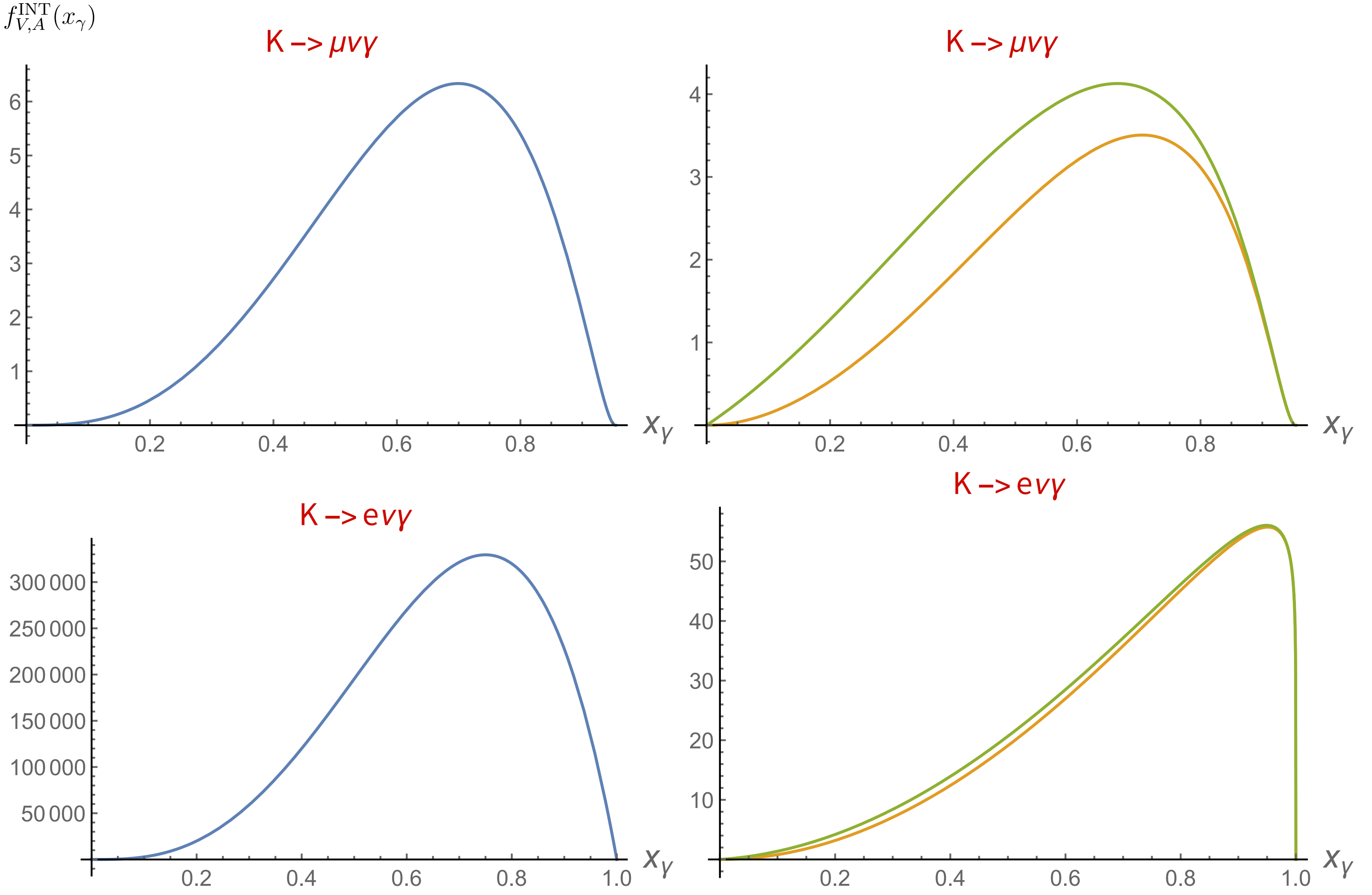


Figure 2: The kernels $K^{\rm SD}(x_\gamma)$ (blue, left plots), $K_V^{\rm INT}(x_\gamma)$ (orange, right plots) and $-K_A^{\rm INT}(x_\gamma)$ (green, right plots) for the $K \to \mu\nu\gamma$ and $K \to e\nu\gamma$ decays.

Conclusions and future perspectives

- Full lattice calculations of radiative corrections to leptonic decay rates are possible

- The form factors for real emissions are accessible from Euclidean correlators

- Preliminary results are very encouraging. Still more work is needed and refined analyses are ongoing

- Lattice calculations of radiative leptonic B-meson decays at high photon energy could provide useful information to constrain the first inverse moment of the B-meson LCDA

Supplementary slides

A strategy for Lattice QCD:

The isospin-breaking part of the Lagrangian is treated as a perturbation

Expand in: $m_d - m_u$ + α_{em}

JHEP

PUBLI... FOR SISSA BY SPRINGER

RECEIVED: *November 7, 2011*
REVISED: *March 16, 2*
ACCEPTED: *April 2, 2*
PUBLISHED: *April 26, 2*

arXiv:1110.6294

Isospin breaking effects due to the up-down mass difference in lattice QCD

RM123 collaboration

G.M. de Divitiis,[a,b] P. Dimopoulos,[c,d] R. Frezzotti,[a,b] V. Lubicz,[e,f] G. Martinelli,[g,d] R. Petronzio,[a,b] G.C. Rossi,[a,b] F. Sanfilippo,[c,d] S. Simula,[f] N. Tantalo[a,b] and C. Tarantino[e,f]

PHYSICAL REVIEW D **87,** 114505 (2013)

Leading isospin breaking effects on the lattice

G. M. de Divitiis,[1,2] R. Frezzotti,[1,2] V. Lubicz,[3,4] G. Martinelli,[5,6] R. Petronzio,[1,2] G. C. Rossi,[1,2] F. Sanfilippo,[7] S. Simula,[4] and N. Tantalo[1,2]

(RM123 Collaboratio

[1]*Dipartimento di Fisica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica 1, I-00133 Rome, Italy*
[2]*INFN, Sezione di Roma "Tor Vergata", Via della Ricerca Scientifica 1, I-00133 Rome, Italy*
[3]*Dipartimento di Matematica e Fisica, Università Roma Tre, Via della Vasca Navale 84, I-00146 Rome, Italy*
[4]*INFN, Sezione di Roma Tre, Via della Vasca Navale 84, I-00146 Rome, Italy*
[5]*SISSA, Via Bonomea 265, 34136 Trieste, Italy*
[6]*INFN, Sezione di Roma, Piazzale Aldo Moro 5, I-00185 Rome, Italy*
[7]*Laboratoire de Physique Théorique (Bâtiment 210), Université Paris Sud, F-91405 Orasay-Cedex, France*

arXiv:1303.4896

(Received 3 April 2013; published 7 June 2013)

RM123 Collaboration

① The (md-mu) expansion

- Identify the isospin-breaking term in the QCD action

$$S_m = \sum_x \left[m_u \bar{u}u + m_d \bar{d}d \right] = \sum_x \left[\frac{1}{2}\left(m_u + m_d\right)\left(\bar{u}u + \bar{d}d\right) - \frac{1}{2}\left(m_d - m_u\right)\left(\bar{u}u - \bar{d}d\right) \right] =$$

$$= \sum_x \left[m_{ud}\left(\bar{u}u + \bar{d}d\right) - \Delta m\left(\bar{u}u - \bar{d}d\right)\right] = S_0 - \Delta m \hat{S} \longleftarrow \boxed{\hat{S} = \Sigma_x(\bar{u}u - \bar{d}d)}$$

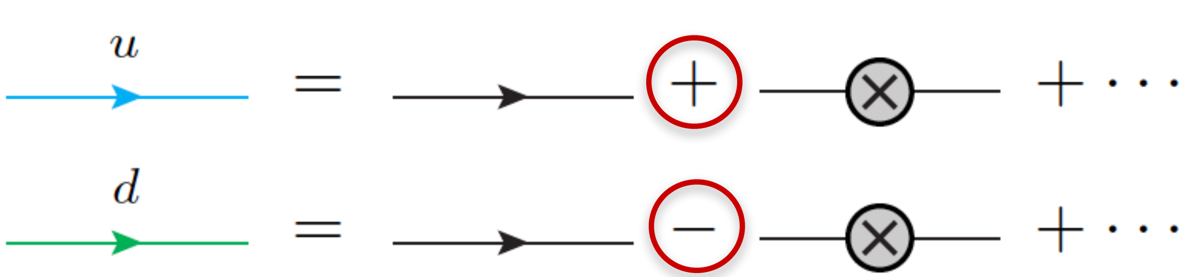
- Expand the functional integral in powers of Δm

Advantage: factorized out

$$\langle O \rangle = \frac{\int D\phi\, O e^{-S_0 + \Delta m \hat{S}}}{\int D\phi\, e^{-S_0 + \Delta m \hat{S}}} \overset{1st}{\simeq} \frac{\int D\phi\, O e^{-S_0}\left(1 + \Delta m \hat{S}\right)}{\int D\phi\, e^{-S_0}\left(1 + \Delta m \hat{S}\right)} \simeq \frac{\langle O \rangle_0 + \Delta m \langle O \hat{S} \rangle_0}{1 + \Delta m \cancel{\langle \hat{S} \rangle_0}} = \langle O \rangle_0 + \Delta m \langle O \hat{S} \rangle_0$$

for isospin symmetry

- At leading order in Δm the corrections only appear in the valence-quark propagators:



(disconnected contractions of ūu and d̄d vanish due to isospin symmetry)

② The QED expansion

- Non-compact QED: the dynamical variable is the gauge potential $A_\mu(x)$ in a fixed covariant gauge ($\nabla^-_\mu A_\mu(x) = 0$)

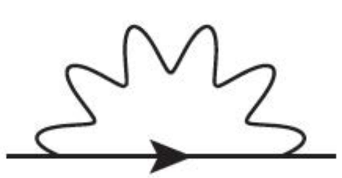
$$\boxed{S_{QED} = \frac{1}{2}\sum_{x;\mu\nu} A_\nu(x)\left(-\nabla^-_\mu \nabla^+_\mu\right) A_\nu(x) \overset{(p.b.c.)}{=} \frac{1}{2}\sum_{k;\mu\nu} \tilde{A}^*_\nu(k)\left(2\sin(k_\mu/2)\right)^2 \tilde{A}_\nu(k)}$$

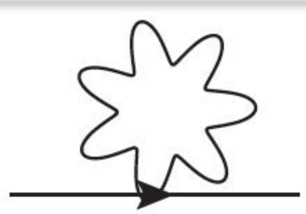
- The photon propagator is IR divergent $\rightarrow$ subtract the zero momentum mode

- Full covariant derivatives are defined introducing QED and QCD links:

$$\boxed{A_\mu(x) \rightarrow E_\mu(x) = e^{-iaeA_\mu(x)}} \qquad \boxed{D^+_\mu q_f(x) = \underbrace{\left[E_\mu(x)\right]^{e_f}}_{\text{QED}} \underbrace{U_\mu(x)}_{\text{QCD}}\, q_f(x+\hat{\mu}) - q_f(x)}$$

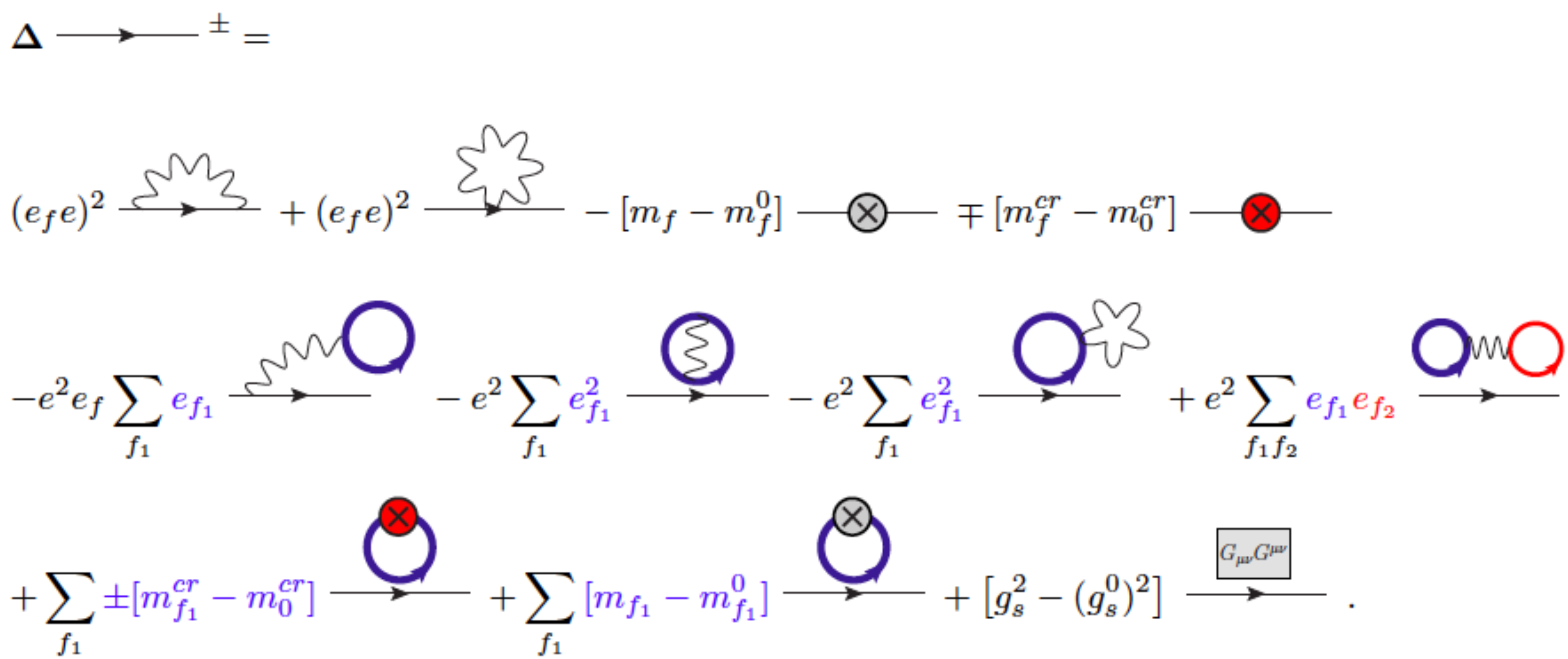
- Since $E_\mu(x) = e^{-ieA_\mu(x)} = 1 - ieA_\mu(x) - 1/2\, e^2 A^2_\mu(x) + \ldots$ the expansion leads to:

$(e_f e)^2$ 

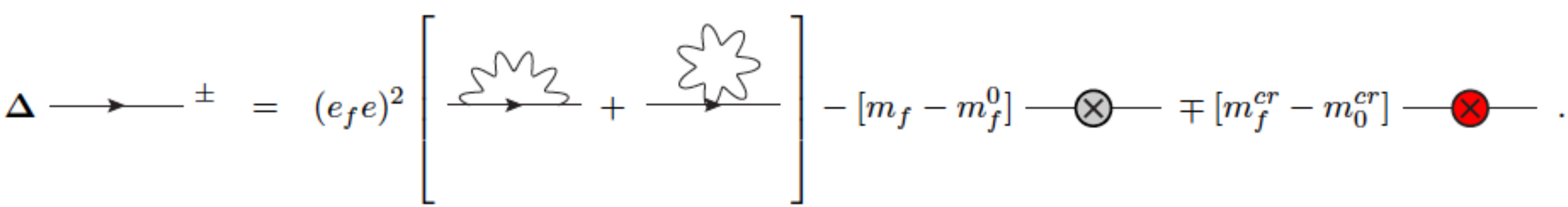
$(e_f e)^2$ 

+ counterterms

The QED expansion for the quark propagator



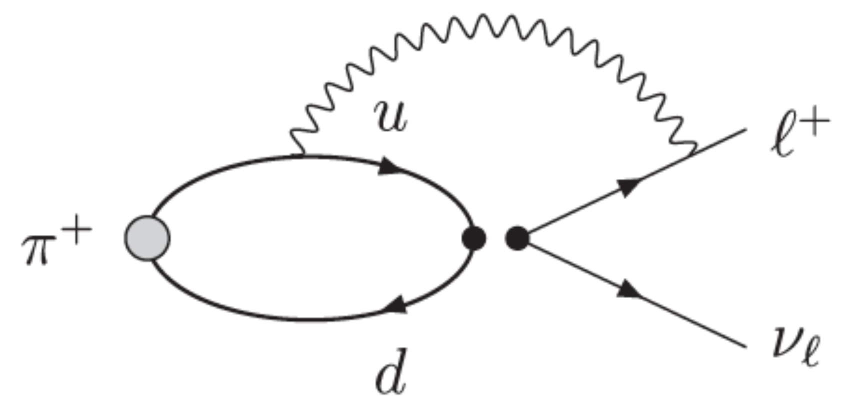
In the electro-quenched approximation:



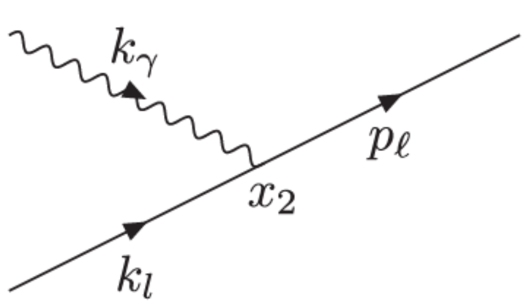
Lattice calculation of Γ_0 at $O(\alpha)$

- <u>A technical but important point</u>:

$$\delta C^{(q\ell)}(t)_{\alpha\beta} = -\int d^3\vec{x}\, d^4x_1\, d^4x_2\, \langle 0|T\left\{J_W^{\nu}(0)\, j_\mu(x_1)\, \phi^\dagger(\vec{x},-t)\right\}|0\rangle \times \Delta(x_1,x_2)\left(\gamma_\nu(1-\gamma^5)S(0,x_2)\gamma_\mu\right)_{\alpha\beta} e^{E_\ell t_2 - i\,\vec{p}_\ell\cdot\vec{x}_2}$$



We need to ensure that the t_2 integration converges as $t_2 \to \infty$. The large t_2 behavior is given by the factor $\exp\left[\left(E_\ell - \omega_\ell - \omega_\gamma\right)t_2\right]$



$$E_\ell = \sqrt{\vec{p}_\ell^{\,2} + m_\ell^2} \qquad \omega_\ell = \sqrt{\vec{k}_\ell^{\,2} + m_\ell^2} \qquad \omega_\gamma = \sqrt{\vec{k}_\gamma^{\,2} + m_\gamma^2} \qquad \vec{k}_\ell + \vec{k}_\gamma = \vec{p}_\ell$$

$$\Rightarrow \quad \boxed{\left(\omega_\ell + \omega_\gamma\right)_{\min} = \sqrt{\left(m_\ell^2 + m_\gamma^2\right) + \vec{p}_\ell^{\,2}} > E_\ell}$$

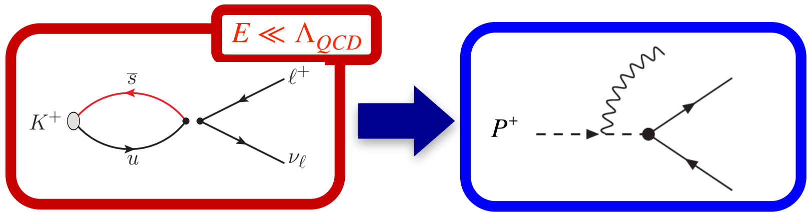
The integral is convergent and the continuation from Minkowski to Euclidean space can be performed (same if we set $m_\gamma=0$ but remove the photon zero mode in FV).

<u>CONDITIONS</u>: - mass gap between the decaying particle and the intermediate states
- absence of lighter intermediate states

The strategy

$$\Gamma[P_{\ell 2}] = (\Gamma_0 - \Gamma_0^{pt}) + (\Gamma_0^{pt} + \Gamma_1^{pt}(E))$$

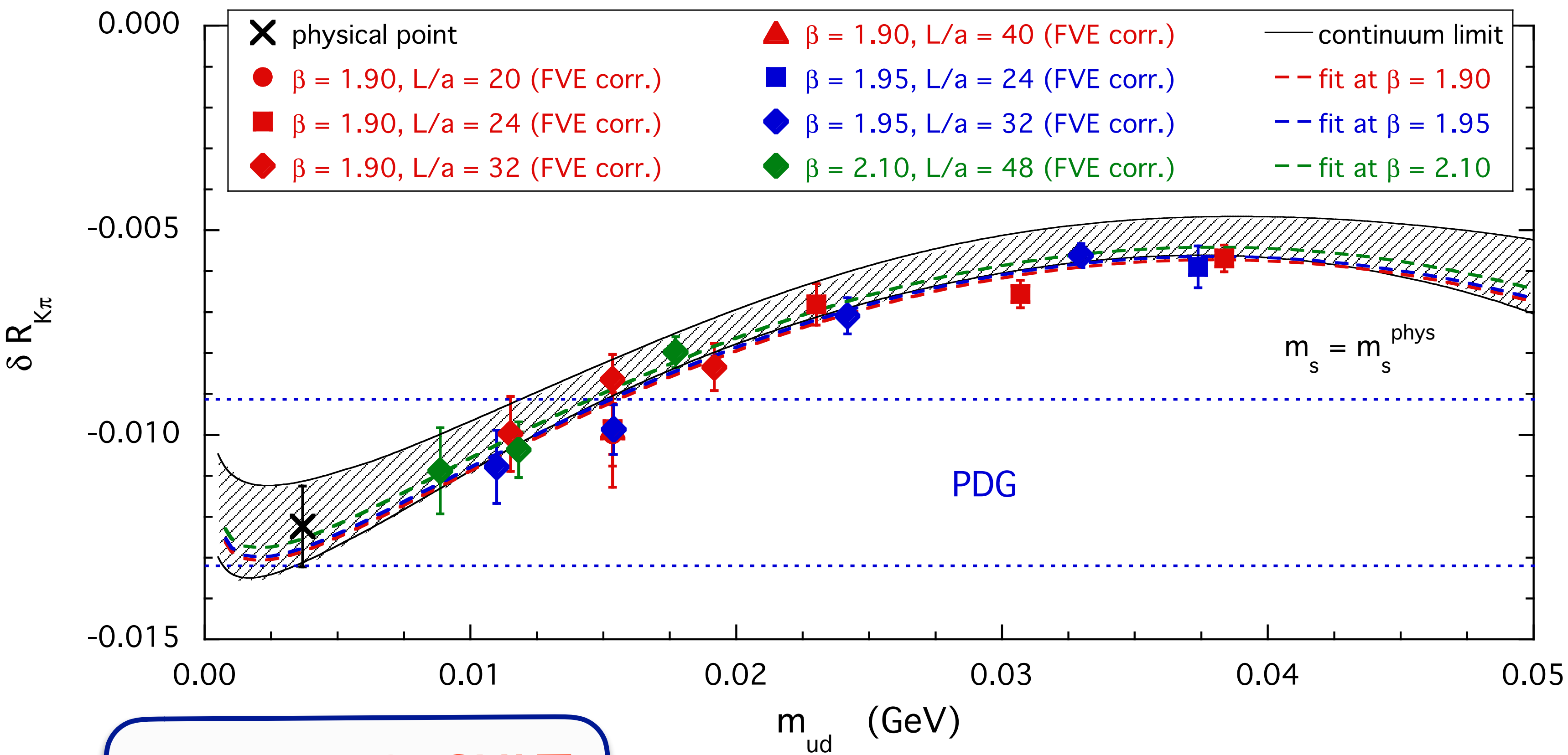
- The contributions from soft virtual photon to Γ_0 and Γ_0^{pt} in the first term are exactly the same and the IR divergence cancels in the difference $\Gamma_0 - \Gamma_0^{\mathrm{pt}}$.
- The sum $\Gamma_0^{pt} + \Gamma_1^{pt}(E)$ in the second term is also IR finite since it is a physically well defined quantity. This term can be thus calculated in perturbation theory with a different IR cutoff.
- The two terms are also separately gauge invariant.



$$\Delta\Gamma_0(L) = \Gamma_0(L) - \Gamma_0^{pt}(L)$$

$$\Gamma^{pt}(E) = \lim_{m_\gamma \to 0} \left[\Gamma_0^{pt}(m_\gamma) + \Gamma_1^{pt}(E, m_\gamma)\right]$$

Leptonic decays at O(α): RESULTS



RM123 & Soton Coll., 2017

$$\delta R_{K\pi} = C_0 + C_\chi \log(m_{ud}) + C_1 m_{ud} + C_2 m_{ud}^2 + Da^2$$
$$+\frac{K_2}{L^2}\left[\frac{1}{M_K^2}-\frac{1}{M_\pi^2}\right]+\frac{K_2^\mu}{L^2}\left[\frac{1}{(E_\mu^K)^2}-\frac{1}{(E_\mu^\pi)^2}\right]$$
$$+\delta\Gamma^{pt}\left(\Delta E_\gamma^{\max,K}\right)-\delta\Gamma^{pt}\left(\Delta E_\gamma^{\max,\pi}\right)$$

LATTICE RESULT

$$\delta R_{K\pi} = -0.0126(10)_{stat}(2)_{input}(5)_{chir}(5)_{FVE}(4)_{disc}(6)_{qQED}$$
$$= -0.0126(14)$$

ChPT

$$\delta R_K - \delta R_\pi = -0.0112(21)$$

V.Cirigliano and H.Neufeld, PLB 700 (2011) 7

$$\left|\frac{V_{us}}{V_{ud}}\right|\frac{f_K^{(0)}}{f_\pi^{(0)}} = 0.27683(29)_{exp}(20)_{th}$$

$$\left|\frac{V_{us}}{V_{ud}}\right| = 0.23135(24)_{exp}(39)_{th}$$

FLAG(2019) Nf=2+1+1 $\frac{f_K^{(0)}}{f_\pi^{(0)}} = 1.1966(18)$

$|V_{ud}|$ from

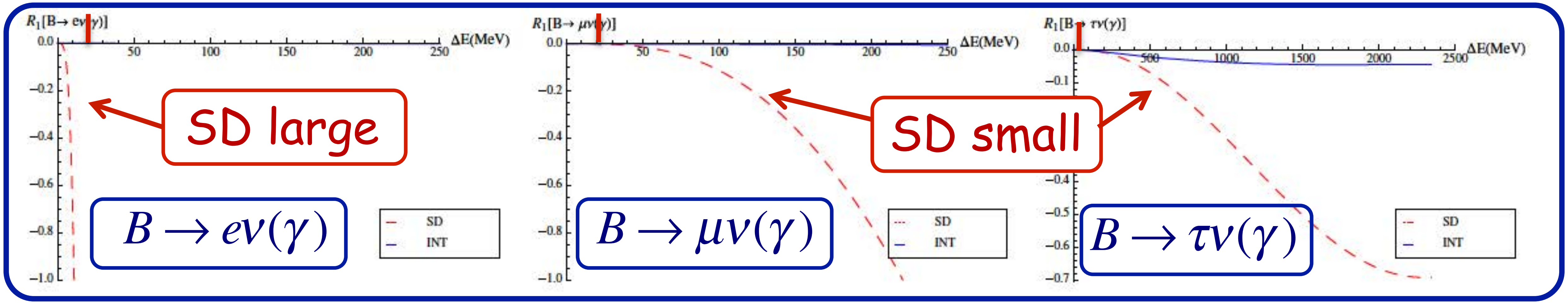
$|V_{us}| = 0.22538(46)$ Hardy and Towner, 2016

$|V_{us}| = 0.22526(46)$ Seng *et al.*, 2018

Structure dependent contributions to decays of D and B mesons

- For the studies of D and B mesons decays we cannot apply ChPT
- For B mesons in particular we have another small scale, $m_{B^*} - m_B \simeq 45 \text{ MeV}$
  ➡ the radiation of a soft photon may still induce sizeable SD effects
- A phenomenological analysis based on a simple pole model for F_V and F_A confirms this picture

D. Becirevic *et al.*, PLB 681 (2009) 257



$$F_V \simeq \frac{\tilde{C}_V}{1-(p_B-k)^2/m_{B^*}^2}$$

$$F_A \simeq \frac{\tilde{C}_A}{1-(p_B-k)^2/m_{B_1}^2}$$

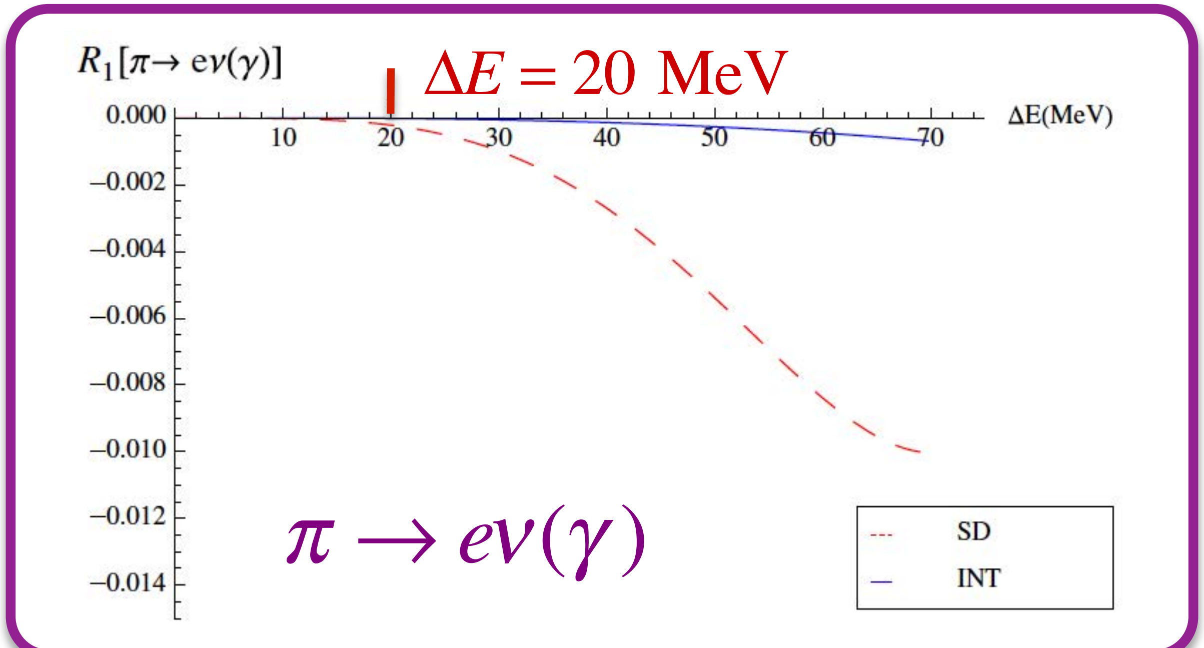
Under this assumption the SD contributions to $B \to e\nu(\gamma)$ for $E_\gamma \simeq 20$ MeV can be very large, but are small for $B \to \mu\nu(\gamma)$ and $B \to \tau\nu(\gamma)$

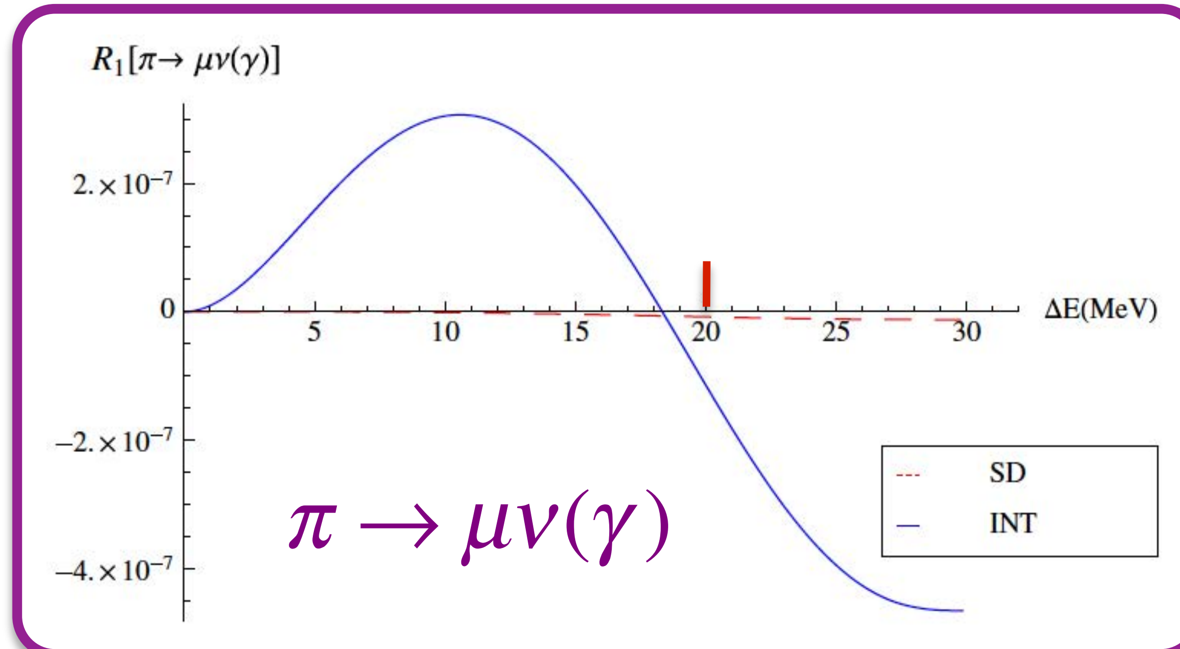
A lattice calculation of F_V and F_A would be very useful

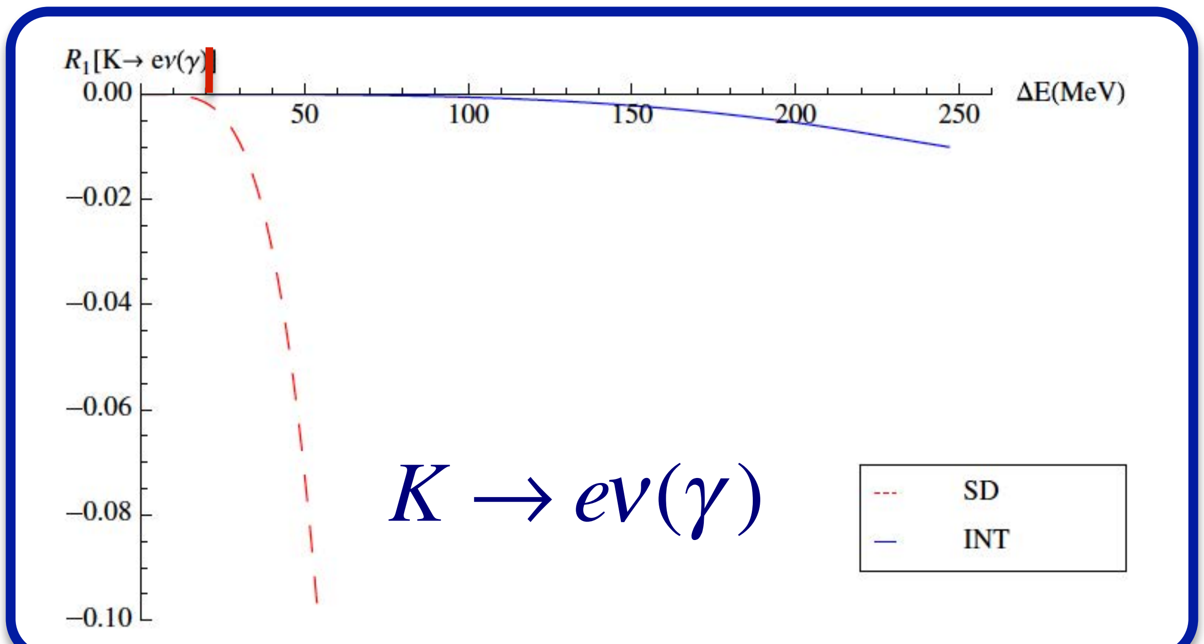
$$R_1^A(\Delta E) = \frac{\Gamma_1^A(\Delta E)}{\Gamma_0^{\alpha,\mathrm{pt}} + \Gamma_1^{\mathrm{pt}}(\Delta E)} \quad , \quad A = \{\mathrm{SD}, \mathrm{INT}\}$$

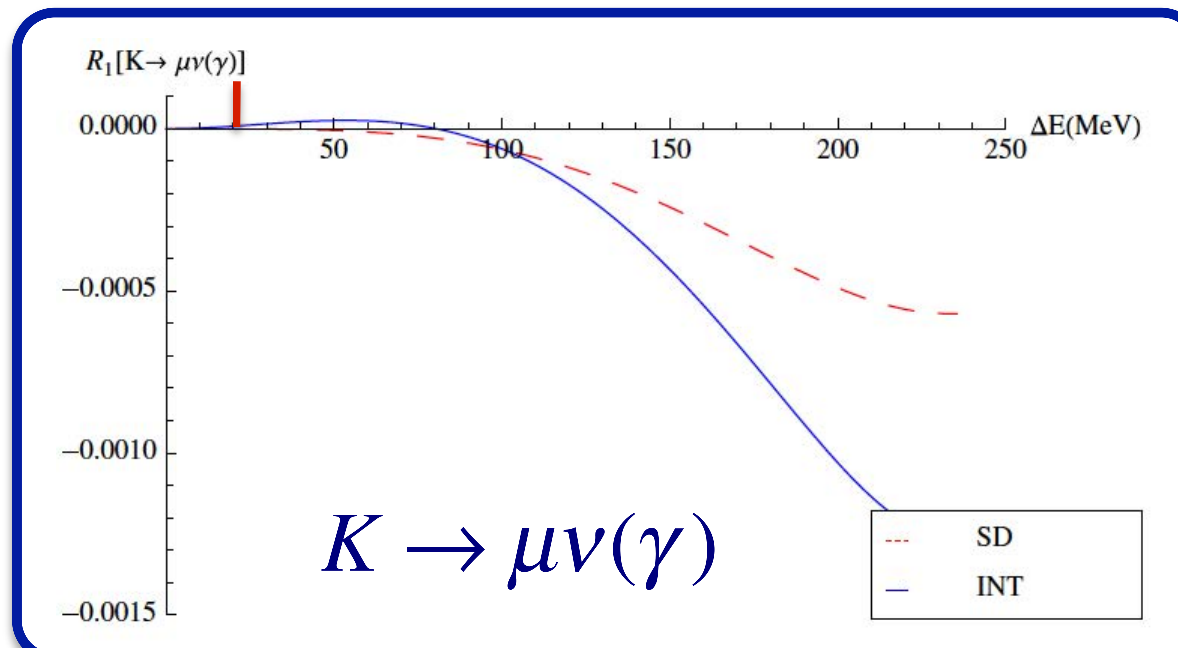
SD = structure dependent

INT = interference









- Interference contributions are negligible in all the decays
- Structure-dependent contributions can be sizable for $K \to e\nu(\gamma)$ but they are negligible for $\Delta E < 20$ MeV (which is experimentally accessible)