Evolution equations for B-meson distribution amplitudes

Yao Ji

University of Siegen

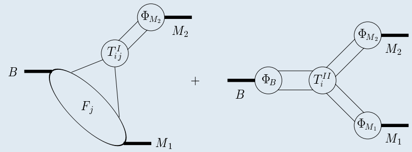
Workshop LCDA 2020, MITP, Mainz

Exclusive B–Decays

- Heavy quark expansion methods ($m_b \gg \Lambda_{\mathrm{QCD}}$)
- Soft-collinear factorization (final state particle energies $\gg \Lambda_{\mathrm{QCD}}$)

Factorization Theorem: [M. Beneke, G. Buchalla, M. Neubert and Sachrajda (1999)]

$$\langle M_1 M_2 | O_i | B \rangle = F^{B\to M_1}(0) \int_0^1 du\, T^{(1)}(u) \Phi_{M_2}(u)$$
$$+ \int_0^\infty d\omega \int_0^1 du\, dv\, T^{(2)}(\omega, u, v) \Phi_+(\omega) \Phi_{M_1}(u) \Phi_{M_2}(v) + \dots$$



u, v — *momentum fractions*
ω — *light quark energy in B-meson*
$\Phi_{M,B}$ — *distribution amplitudes*

$B \to \gamma \ell \nu_\ell$ *provides the cleanest probe for unraveling the B-meson DAs*

Leading-twist distribution amplitude

Definition

[A. Grozin, M. Neubert (1997)]

$$\langle 0|\left[\bar{q}(zn)\, \not{n}[zn,0]\gamma_5 h_v(0)\right]_R|\bar{B}(v)\rangle = iF_B(\mu)\;\Phi_+(z,\mu)$$

- v_μ is the heavy quark velocity
- n_μ is the light-like vector, $n^2 = 0$, such that $n \cdot v = 1$
- The twist-2 LCDA $\Phi_+(z - i0, \mu)$ is an analytic function of z in the lower half-plane

Fourier transform

$$\phi_+(\omega,\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dz\, e^{i\omega z}\Phi_+(z - i0,\mu)\,,$$

$$\Phi_+(z,\mu) = \int_0^{\infty} d\omega\, e^{-i\omega z}\phi_+(\omega,\mu)\,.$$

- $\omega > 0$ is the $(2\times)$ light quark energy in the b–quark rest frame

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Three-particle distribution amplitudes

- **Eight independent Lorentz structures** [B. Geyer and O. Witzel (2005)]

$$
\begin{aligned}
&\langle 0|\Big[\bar q(nz_1) g G_{\mu\nu}(nz_2)\Gamma h_v(0)\Big]_R|\bar B(v)\rangle = \\
&= \tfrac{1}{2} F_B(\mu)\,\mathrm{Tr}\Big\{\gamma_5 \Gamma P_+\Big[(v_\mu\gamma_\nu - v_\nu\gamma_\mu)\big[\Psi_A - \Psi_V\big] - i\sigma_{\mu\nu}\Psi_V - (n_\mu v_\nu - n_\nu v_\mu) X_A \\
&\quad + (n_\mu\gamma_\nu - n_\nu\gamma_\mu)\big[W + Y_A\big] - i\epsilon_{\mu\nu\alpha\beta} n^\alpha v^\beta \gamma_5 \widetilde{X}_A + i\epsilon_{\mu\nu\alpha\beta} n^\alpha \gamma^\beta \gamma_5 \widetilde{Y}_A \\
&\quad - (n_\mu v_\nu - n_\nu v_\mu)\not{n}\, W + (n_\mu\gamma_\nu - n_\nu\gamma_\mu)\not{n}\, Z\Big]\Big\}(z_1, z_2;\mu)
\end{aligned}
$$

blue: [H. Kawamura, J. Kodaira, C. F. Qiao and K. Tanaka (2001)]

red: [V. Braun, YJ and A. Manashov (2017)]

$$
\Psi_A(z_1,z_2) = \int_0^\infty d\omega_1 \int_0^\infty d\omega_2\; e^{-i\omega_1 z_1 - i\omega_2 z_2}\,\psi_A(\omega_1,\omega_2)\,,
$$

$$
\psi_A(\omega_1,\omega_2) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} dz_1\, dz_2\, e^{i\omega_1 z_1 + i\omega_2 z_2}\,\Psi_A(z_1 - i0, z_2 - i0)\,, \qquad \text{etc.}
$$

Convenient for simple Lorentz strucutures.

No definite collinear twist, not suitable for power counting in factorization.

Collinear twist decomposition (1)

- **Twist-three**

$$2F_B(\mu)\Phi_3(z_1, z_2; \mu) = \langle 0|\bar{q}(z_1) g G_{\mu\nu}(z_2) n^\nu \not{n} \gamma_\perp^\mu \gamma_5 h_v(0)|\bar{B}(v)\rangle$$

$$\Phi_3 = \Psi_A - \Psi_V \,,$$

- **Twist-four**

$$2F_B(\mu)\Phi_4(z_1, z_2; \mu) = \langle 0|\bar{q}(z_1) g G_{\mu\nu}(z_2) n^\nu \not{\bar{n}} \gamma_\perp^\mu \gamma_5 h_v(0)|\bar{B}(v)\rangle$$

$$2F_B(\mu)\Psi_4(z_1, z_2; \mu) = \langle 0|\bar{q}(z_1) g G_{\mu\nu}(z_2) \bar{n}^\mu n^\nu \not{n} \gamma_5 h_v(0)|\bar{B}(v)\rangle$$

$$2F_B(\mu)\widetilde{\Psi}_4(z_1, z_2; \mu) = \langle 0|\bar{q}(z_1) i g \widetilde{G}_{\mu\nu}(z_2) \bar{n}^\mu n^\nu \not{n} h_v(0)|\bar{B}(v)\rangle$$

$$\Phi_4 = \Psi_A + \Psi_V \,,$$
$$\Psi_4 = \Psi_A + X_A \,,$$
$$\widetilde{\Psi}_4 = \Psi_V - \widetilde{X}_A \,,$$

Collinear twist decomposition (2)

- **Twist-five**

$$2F_B(\mu)\widetilde{\Phi}_5(z_1,z_2;\mu) = \langle 0|\bar{q}(z_1)gG_{\mu\nu}(z_2)\bar{n}^\nu \not{n}\gamma_\perp^\mu\gamma_5 h_v(0)|\bar{B}(v)\rangle$$
$$2F_B(\mu)\Psi_5(z_1,z_2;\mu) = \langle 0|\bar{q}(z_1)gG_{\mu\nu}(z_2)\bar{n}^\mu n^\nu \not{\bar{n}}\gamma_5 h_v(0)|\bar{B}(v)\rangle$$
$$2F_B(\mu)\widetilde{\Psi}_5(z_1,z_2;\mu) = \langle 0|\bar{q}(z_1)ig\widetilde{G}_{\mu\nu}(z_2)\bar{n}^\mu n^\nu \not{\bar{n}} h_v(0)|\bar{B}(v)\rangle$$

$$\widetilde{\Phi}_5 = \Psi_A + \Psi_V + 2Y_A - 2\widetilde{Y}_A + 2W\,,$$
$$\Psi_5 = -\Psi_A + X_A - 2Y_A\,,$$
$$\widetilde{\Psi}_5 = -\Psi_V - \widetilde{X}_A + 2\widetilde{Y}_A\,,$$

- **Twist-six**

$$2F_B(\mu)\widetilde{\Phi}_6(z_1,z_2;\mu) = \langle 0|\bar{q}(nz_1)gG_{\mu\nu}(nz_2)\bar{n}^\nu \not{\bar{n}}\gamma_\perp^\mu\gamma_5 h_v(0)|\bar{B}(v)\rangle$$

$$\Phi_6 = \Psi_A - \Psi_V + 2Y_A + 2W + 2\widetilde{Y}_A - 4Z$$

Conformal spin and helicity assignment

	Φ_3	Φ_4	$\Psi_4 + \widetilde{\Psi}_4$	$\Psi_4 - \widetilde{\Psi}_4$	Φ_5	$\Psi_5 + \widetilde{\Psi}_5$	$\Psi_5 - \widetilde{\Psi}_5$	Φ_6
twist	3	4	4	4	5	5	5	6
j_q	1	1/2	1	1	1	1/2	1/2	1/2
j_g	3/2	3/2	1	1	1/2	1	1	1/2
chirality	⇅(⇵)	⇈(⇊)	⇈(⇊)	⇅(⇵)	⇈(⇊)	⇈(⇊)	⇅(⇵)	⇅(⇵)

Table : *The twist, conformal spins j_q, j_g of the constituent fields and chirality [same or opposite] of the three-particle B-meson DAs.*

- **Asymptotic behavior at small momenta** [V. Braun, I. Filyanov (1989)]

$$f(\omega_1,\omega_2) \sim \omega_1^{2j_1-1}\omega_2^{2j_2-1}\,. \qquad f \in \{\phi_3, \phi_4, \psi_4, \tilde{\psi}_4\ \ldots\}$$

$$\phi_3(\omega_1,\omega_2) \sim \omega_1\omega_2^2\,, \qquad \phi_4(\omega_1,\omega_2) \sim \omega_2^2\,, \qquad \psi_4(\omega_1,\omega_2) \sim \widetilde{\psi}_4(\omega_1,\omega_2) \sim \omega_1\omega_2$$

— agrees with [A. Khodjamirian, T. Mannel, N. Offen (2006)]

Relations between distribution amplitudes

- **Neglecting four-particle DAs** $qGGh_v$, $q\bar{q}qh_v$

$$2\partial_1 z_1 \Phi_4(\underline{z}) = \Big(z_2\partial_{z_2} + 2\Big)\Big[\Psi_4(\underline{z}) + \widetilde{\Psi}_4(\underline{z})\Big] \qquad \underline{z} = \{z_1, z_2\}$$

as a consequence of Lorentz symmetry. [V. Braun, YJ and A. Manashov, (2017)]

- **Higher moments of** ϕ_+ [A. G. Grozin and M. Neubert, (1997)]

$$\int_0^\infty d\omega\, \omega\, \phi_+(\omega) = \frac{4}{3}\bar{\Lambda}, \quad \int_0^\infty d\omega\, \omega^2 \phi_+(\omega) = 2\bar{\Lambda}^2 + \frac{2}{3}\lambda_E^2 + \frac{1}{3}\lambda_H^2 ,$$

- **Normalization conditions for higher-twist DAs:**

$$\Phi_3(\underline{z}=0) = \frac{1}{3}(\lambda_E^2 - \lambda_H^2) , \qquad \Phi_4(\underline{z}=0) = \frac{1}{3}(\lambda_E^2 + \lambda_H^2) ,$$

$$\Psi_4(\underline{z}=0) = \frac{1}{3}\lambda_E^2 , \qquad \widetilde{\Psi}_4(\underline{z}=0) = \frac{1}{3}\lambda_H^2 ,$$

with $\lambda_E^2/\lambda_H^2 \sim 0.5$ by QCDSR. [A. Grozin and M. Neubert, (1997); T. Nishikawa and K. Tanaka, (2014)]

Two-particle distribution amplitudes of higher-twists

- x^2 **expansion of two-particle matrix element**

$$\langle 0|\,[\bar{q}(x)\Gamma[x,0]h_v(0)]_R\,|\bar{B}(v)\rangle = -\frac{i}{2}F_B\,\mathrm{Tr}\left[\gamma_5\Gamma P_+\right]\int\limits_0^\infty d\omega\, e^{-i\omega(vx)}\left\{\phi_+(\omega,\mu)+x^2 g_+(\omega,\mu)\right\}$$
$$+\frac{i}{4}F_B\,\mathrm{Tr}\left[\gamma_5\Gamma P_+\not{x}\right]\frac{1}{vx}\int\limits_0^\infty d\omega\, e^{-i\omega(vx)}\left\{[\phi_+-\phi_-](\omega,\mu)+x^2[g_+-g_-](\omega,\mu)\right\}+\dots$$

- **Two-particle higher-twist DAs are related to three-particle through EOM:**

$$\left[z\frac{d}{dz}+1\right]\Phi_-(z) = \Phi_+(z)+2z^2\int_0^1 udu\,\Phi_3(z,uz)$$
$$2z^2 \mathrm{G}_+(z) = -\left[z\frac{d}{dz}-\frac{1}{2}+iz\bar{\Lambda}\right]\Phi_+(z)-\frac{1}{2}\Phi_-(z)-z^2\int_0^1 \bar{u}du\,\Psi_4(z,uz)$$
$$2z^2 \mathrm{G}_-(z) = -\left[z\frac{d}{dz}-\frac{1}{2}+iz\bar{\Lambda}\right]\Phi_-(z)-\frac{1}{2}\Phi_+(z)-z^2\int_0^1 \bar{u}du\,\Psi_5(z,uz)$$
$$\Phi_-(z) = \left(z\frac{d}{dz}+1+2iz\bar{\Lambda}\right)\Phi_+(z)+2z^2\int_0^1 du\left[u\Phi_4(z,uz)+\Psi_4(z,uz)\right]$$

[H. Kawamura, J. Kodaira, C. F. Qiao and K. Tanaka, (2001); V. M. Braun, YJ and A. N. Manashov, (2017)]

One-loop evolution of leading twist DA

- **RGE** $$\left(\mu\frac{\partial}{\partial\mu} + \beta(a)\frac{\partial}{\partial a} + \mathcal{H}(a)\right)\Phi_+(z,\mu) = 0\,,$$ [B. Lange, M. Neubert (2003)]

  with $\mathcal{H}$ being the evolution kernel, usually presented as an integral operator.

- **One-loop evolution kernel**

$$\mathcal{H}^{(1)}\Phi_+(z,\mu) = 4C_F\left\{[\ln(i\widetilde{\mu}z) + 1/2]\Phi_+(z,\mu) + \int_0^1 du\frac{\bar{u}}{u}\left[\Phi_+(z,\mu) - \Phi_+(\bar{u}z,\mu)\right]\right\}$$

  where $\widetilde{\mu} = \mathrm{e}^{\gamma_E}\mu_{\overline{\mathrm{MS}}}$ and $\bar{u} = 1-u$. [A. Grozin and M. Neubert, (1997); V. Braun, D. Ivanov and G. Korchemsky, (2004)]

- **Solution to one-loop RGE** [G. Bell, T. Feldmann, Y.-M. Wang and M. W. Y. Yip, (2013); V. Braun and A. Manashov (2014)]

$$\Phi_+(z,\mu) = -\frac{1}{z^2}\int_0^\infty ds\, s\, e^{is/z}\,\eta_+(s,\mu)\,,$$

$$\eta_+(s,\mu) = R(s,\mu,\mu_0)\eta_+(s,\mu_0)\,, \qquad R(s,\mu,\mu_0) \propto s^{\frac{2C_F}{\beta_0}\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}}$$

Come back later!

One-loop evolution of higher-twist DAs (1)

- **One-loop evolution kernels of three-particle DAs are pairwise.**
  Example

$$-2iF(\mu)\Phi_3(z_1, z_2, \mu) = \langle 0|\, \bar{q}(z_1) g G_{\mu\nu}(z_2) n^\nu \sigma^{\mu\rho} n_\rho \gamma_5 h_v(0)\, |\bar{B}(v)\rangle$$

  where the one-loop kernel takes the form $\mathcal{H}^{(1)}_{\Phi_3} = \mathcal{H}^{(1)}_{qg} + \mathcal{H}^{(1)}_{gh} + \mathcal{H}^{(1)}_{qh}$ with

$$[\mathcal{H}^{(1)}_{qh} f](z_1) = \frac{-1}{N_c}\left\{\int_0^1 \frac{d\alpha}{\alpha}\left[f(z_1) - \bar{\alpha} f(\bar{\alpha} z_1)\right] + \left[\ln(i\mu z_1) - \frac{5}{4}\right] f(z_1)\right\},$$

$$[\mathcal{H}^{(1)}_{gh} f](z_2) = N_c\left\{\int_0^1 \frac{d\alpha}{\alpha}\left[f(z_2) - \bar{\alpha}^2 f(\bar{\alpha} z_2)\right] + \left[\ln(i\mu z_2) - \frac{1}{2}\right] f(z_2)\right\},$$

$$[\mathcal{H}^{(1)}_{qg} \varphi](z_1, z_2) = N_c\left\{\int_0^1 \frac{d\alpha}{\alpha}\left[2\varphi(z_1, z_2) - \bar{\alpha}\varphi(z_{12}^\alpha, z_2) - \bar{\alpha}^2 \varphi(z_1, z_{21}^\alpha)\right]\right.$$
$$\left. - \frac{3}{4}\varphi(z_1, z_2)\right\} - \frac{2}{N_c}\int_0^1 d\alpha \int_{\bar{\alpha}}^1 d\beta\, \bar{\beta}\, \varphi(z_{12}^\alpha, z_{21}^\beta),$$

where $z_{12}^\alpha = \bar{\alpha} z_1 + \alpha z_2$. [M. Knödlseder and N. Offen (2011); V. Braun, A. Manashov and J Rohrwild, (2009); YJ and A. Belitsky (2014)]

- **Solution?**

One-loop evolution of higher-twist DAs (2)

- **RGE for $\Phi_3(z_1, z_2, \mu)$ is integrable at large N_c limit.**

  Two conserved charges (hidden symmetries)

$$[\mathbb{Q}_1, \mathbb{Q}_2] = \left[\mathbb{Q}_1, \mathcal{H}^{(1)}_{\Phi_3}\right] = \left[\mathbb{Q}_2, \mathcal{H}^{(1)}_{\Phi_3}\right] = 0$$

  explicitly [V. Braun, A. Manashov and N. Offen (2015)]

$$\mathbb{Q}_1 = i(S_q^+ + S_g^+)\,,$$
$$\mathbb{Q}_2 = \frac{9}{4} i S_g^+ - i S_g^+ \left(S_g^+ S_q^- + S_g^0 S_q^0\right) - i S_g^0 \left(S_q^0 S_g^+ - S_g^0 S_q^+\right)$$

  from **Quantum Inverse Scattering Method (QISM)** [E. Sklyanin (1992)]
  $S_+ = z^2\partial_z + 2jz\,, S_0 = z\partial_z + j\,, S_- = -\partial_z$, j conformal spin.

- **Two DOF in $\Phi^{(1)}_{\Phi_3}$ $\Longrightarrow$ $\mathcal{H}^{(1)}_{\Phi_3}$ and $\{\mathbb{Q}_1, \mathbb{Q}_2\}$ share the same eigenfunction.**

- **Integrability of RGE $\Leftrightarrow$ Integrable spin chains** [V. Braun, YJ and A. Manashov (2018)]

One-loop evolution of higher-twist DAs (3)

- **Solving for eigenfunction of $\{\mathbb{Q}_1, \mathbb{Q}_2\}$ leads to (complete orthonormal basis)**

$$\phi_-(\omega,\mu) = \int_\omega^\infty \frac{d\omega'}{\omega'}\phi_+(\omega',\mu) + \int_0^\infty ds\, J_0(2\sqrt{\omega s})\, \eta_3^{(0)}(s,\mu)$$

$$\phi_3(\underline{\omega},\mu) = \int_0^\infty ds\Big[\eta_3^{(0)}(s,\mu)\, Y_3^{(0)}(s\,|\,\underline{\omega}) + \frac{1}{2}\int_{-\infty}^\infty dx\, \eta_3(s,x,\mu)\, Y_3(s,x\,|\,\underline{\omega})\Big],$$

where $Y_3^{(0)}(s\,|\,\underline{\omega}) = Y_3(s, x = i/2\,|\,\underline{\omega})$ and

$$Y_3(s,x\,|\,\underline{\omega}) = -\int_0^1 du\, \sqrt{us\omega_1}\, J_1(2\sqrt{us\omega_1})\, \omega_2\, J_2(2\sqrt{\bar{u}s\omega_2})\, {}_2F_1\left(\begin{matrix}-\frac{1}{2}-ix, -\frac{1}{2}+ix \\ 2\end{matrix}\,\Big|\, -\frac{u}{\bar{u}}\right)$$

Solving RGE for $\phi_3(\underline{\omega},\mu)$ up to $1/N_c^2$ gives

$$\eta_3(s,x,\mu) = L^{\gamma_3(x)/\beta_0} R(s;\mu,\mu_0)\, \eta_3(s,x,\mu_0)$$

$$\eta_3^{(0)}(s,\mu) = L^{N_c/\beta_0} R(s;\mu,\mu_0)\eta_3^{(0)}(s,\mu_0)$$

where $L = \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}$ and $\gamma_3(x) = N_c[\psi(3/2+ix) + \psi(3/2-ix) + 2\gamma_E]$.

- $1/N_c^2 \sim \mathcal{O}(10^{-1})$ **taken perturbatively**

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One-loop evolution of higher-twist DAs (4)

- **RGEs for twist-4 DAs are also integrable** [V. Braun, YJ and A. Manashov (2017)]

  Light fields mixing: kernels of 2×2 matrices. [V. Braun, A. Manashov and J. Rohrwild (2009); YJ and A. Belitsky (2014)]

  Three conserved charges $\{\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3\}$

$$\Phi_4(\underline{\omega}) = \frac{1}{2}\int_0^\infty ds \int_{-\infty}^\infty dx\, \eta_4^{(+)}(s,x,\mu)\, Y_{4;1}^{(+)}(s,x\,|\underline{\omega}),$$

$$(\Psi_4 + \widetilde{\Psi}_4)(\underline{\omega}) = -\int_0^\infty ds \int_{-\infty}^\infty dx\, \eta_4^{(+)}(s,x,\mu)\, Y_{4;2}^{(+)}(s,x\,|\underline{\omega}),$$

$$(\Psi_4 - \widetilde{\Psi}_4)(\underline{\omega}) = 2\int_0^\infty \frac{ds}{s}\left(-\frac{\partial}{\partial\omega_2}\right)\left\{\eta_3^{(0)}(s,\mu)\, Y_3^{(0)}(s\,|\underline{\omega}) + \frac{1}{2}\int_{-\infty}^{\infty} dx\, \eta_3(s,x,\mu)\, Y_3(s,x\,|\underline{\omega})\right\}$$
$$-\int_0^\infty ds \int_{-\infty}^\infty dx\, \varkappa_4^{(-)}(s,x,\mu)\, Z_{4;2}^{(-)}(s,x\,|\underline{\omega}),$$

$$\eta_4^{(+)}(s,x,\mu) \overset{\mathcal{O}(1/N_c^2)}{=} L^{\gamma_4(x)/\beta_0} R(s;\mu,\mu_0)\, \eta_4^{(+)}(s,x,\mu_0)$$
$$\varkappa_4^{(-)}(s,x,\mu) \overset{\mathcal{O}(1/N_c^2)}{=} L^{\gamma_4(x)/\beta_0} R(s;\mu,\mu_0)\, \varkappa_4^{(-)}(s,x,\mu_0)$$

- **Redundant operators are traded for others using EOMs and Lorentz symmetry.**

Model building for higher-twist DAs

Requirement

— satisfy EOM (at tree level) and other relations
— reasonable low-momentum behavior
— (possible) experimental input for $\bar{\Lambda}$, ratio λ_E^2/λ_H^2 from QCD sum rules

- **General ansatz consistent with EOM** [M. Beneke, V. Braun, YJ and Y. B. Wei (2018)]

$$
\begin{aligned}
\phi_+(\omega) &= c_1\, \omega\, f(\omega) \\
\phi_3(\omega_1,\omega_2) &= -\frac{1}{2}(\lambda_E^2-\lambda_H^2)\,\omega_1\omega_2^2\,\partial_{\omega_2}\, f(\omega_1+\omega_2) \\
\psi_4(\omega_1,\omega_2) &= \lambda_E^2\,\omega_1\omega_2\, f(\omega_1+\omega_2) \\
\widetilde{\psi}_4(\omega_1,\omega_2) &= \lambda_H^2\,\omega_1\omega_2\, f(\omega_1+\omega_2) \\
2\omega_1\,\phi_4(\omega_1,\omega_2) &= \omega_2\left[\psi_4(\omega_1,\omega_2)+\widetilde{\psi}_4(\omega_1,\omega_2)\right]
\end{aligned}
$$

where $c_1 = \bar{\Lambda}^2 + \frac{1}{6}(2\lambda_E^2+\lambda_H^2)$ fixed by Grozin-Neubert relations

Hard-collinear contribution of $B\to\gamma\ell\nu_\ell$ at tree-level independent of $f(\omega)$.

Model building for twist-2 DA

Models are constructed at $\mu_0 = 1\ \mathrm{GeV}$ and evolved to different scales

- **Exponential Model:** [A. Grozin and M. Neubert, (1997); A. Khodjamirian, T. Mannel and N. Offen (2007)]

$$\phi_+(\omega, \mu_0) = \frac{\omega}{\omega_0^2}\, e^{-\omega/\omega_0}$$

  simple, easy to implement, expected $\omega \sim 0$ behavior, analytical one-loop evolution
  only one free parameter ω_0

- **Duality Model:** [A. Khodjamirian, T. Mannel and N. Offen (2007); V. Braun, YJ and A. Manashov (2017)]

$$\phi_+(\omega, \mu_0) \propto \omega(2\omega_0 - \omega)^p \theta(2\omega_0 - \omega)$$

  larger parameter space (ω_0, p)
  evolution can only be done numerically

- **Generalized Exponential Model:** [M. Beneke, V. M. Braun, YJ and Y-B. Wei (2018)] *Discussion Session?*

$$\phi_+(\omega, \alpha, \beta, \mu_0) = \frac{\Gamma(\beta)}{\Gamma(\alpha)} \frac{\omega}{\omega_0^2}\, e^{-\omega/\omega_0} U(\beta - \alpha, 3 - \alpha, \omega/\omega_0)\,, \quad \alpha, \beta > 1$$

  easy to implement, analytical one-loop evolution, larger parameter space
  !! wider range of σ_1 for fixed $\lambda_B \propto \omega_0$

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Scale dependence of twist-4 DAs

- **Evolution of twist-4 DAs in exponential model with $\omega_1 = u\omega$ and $\omega_2 = \bar{u}\omega$.**

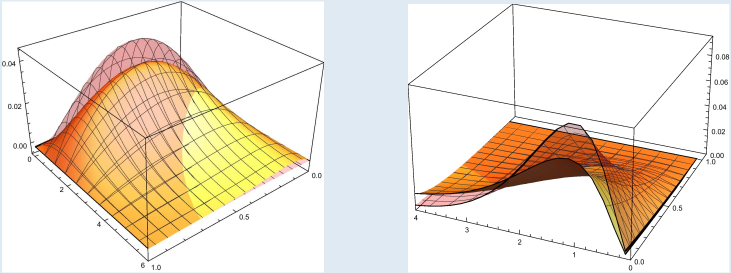

Figure : Evolution of $\psi_4 + \widetilde{\psi}_4$ (left) and ϕ_4 (right) from 1 GeV to 2.5 GeV.

- **The development of a large momentum tail from evolution is rather general.**

Two-loop evolution of twist-2 DA

- **Motivations**
  - Last missing piece for a complete NNLL resummation for charmless B-decays
  - Theoretically interesting in its own right
    - Conformal symmetry in light-heavy systems (*new method*!)
    - Test for light-heavy relation at two-loop [V. Braun, YJ. and A. Manashov (2018)]

Conformal symmetry of light kernels

- **One-loop light kernels are $SL(2)$ invariant** [Bukhvostov, Frolov, Kuraev, Lipatov (1985)]

generators
$$S_+^{(i)} = z_i^2\partial_{z_i} + 2j_i z_i, \quad S_0^{(i)} = z_i\partial_{z_i} + j_i, \quad S_-^{(i)} = -\partial_{z_i}$$
j : conformal spin

algebra
$$[S_+^{(i)}, S_-^{(i)}] = 2S_0^{(i)}, \quad [S_0^{(i)}, S_+^{(i)}] = S_+^{(i)}, \quad [S_0^{(i)}, S_-^{(i)}] = -S_-^{(i)}$$

Example $\mathcal{O}(z_1, z_2) = \bar{q}(nz_1)\gamma^+ q(nz_2)$ $(j_{1,2} = 1)$ $SL(2)$ generators $S_{0,\pm}^{(12)} = S_{0,\pm}^{(1)} + S_{0,\pm}^{(2)}$

$SL(2)$ invariance $\Longrightarrow [S_{0,\pm}^{(12)}, \mathcal{H}_{\bar{q}q}^{(1)}] = 0 \Longrightarrow \mathcal{H}_{\bar{q}q}^{(1)} = h(S_{12}^2)$

$$S_{12}^2 = S_+^{(12)} S_-^{(12)} + S_0^{(12)}(S_0^{(12)} - 1)$$
quadratic Casimir operator

explicitly,

$$\mathcal{H}_{\bar{q}q}^{(1)} = 2C_F\left[\psi(\widehat{J}+1) + \psi(\widehat{J}-1) - 2\gamma_E - \frac{3}{2}\right], \qquad S_{12}^2 = \widehat{J}(\widehat{J}-1)$$

- **Light kernels up to three-loops available** [V. Braun, A. Manashov, S. Moch and M. Strohmaier (2016)-(2019)]

Conformal symmetry of heavy kernels

What about heavy kernels??

- $\mathcal{H}_h^{(1)}$ commute with special conformal generator of light field $S_+ \sim v^\mu \mathbf{K}_\mu$ but *not* S_0

$$[S_+, \mathcal{H}_h^{(1)}] = 0\,, \quad [S_0, \mathcal{H}_h^{(1)}] = 4C_F = \Gamma^{(1)}_{\text{cusp}}$$

[M. Knödlseder and N. Offen (2011)]

Solution: $$\mathcal{H}_h^{(1)} = \Gamma^{(1)}_{\text{cusp}} \ln(i\mu S_+) + \text{const}$$ (for $\mathcal{O}_{\bar q h} = \bar q(nz)\gamma^+ h(0)$, $j=1$)

- **Light $\mapsto$ heavy reduction**

$$S_+^{(h)} \mapsto \lambda^{-1} S_+^{(h)}\,, \qquad S_-^{(h)} \mapsto \lambda S_-^{(h)} \mapsto \mu\,, \qquad S_0^{(h)} \mapsto S_0^{(h)}\,, \qquad \lambda \sim m_b \to \infty$$

$$\Longrightarrow \quad \mathcal{H}^{(1)}_{\bar q q} \mapsto \mathcal{H}_{\bar q h} = \Gamma^{(1)}_{\text{cusp}} \ln(i\mu S_+) + \text{const}$$

- **Eigenfunction of $\mathcal{H}^{(1)}_{\bar q h}$ coincides with that of S_+:**

$$Q_s(z) = -\frac{e^{is/z}}{z^2}$$

[V. Braun and A. Manashov (2014)]

Conformal symmetry of heavy kernels

- **Proposition:** $\mathcal{H}_h^{t=2} = \Gamma_{\text{cusp}}(a) \ln(i\bar{\mu} S_+) + \Gamma_+(a)$ **to all orders** **why?**

Evolution kernels in the MS-like schemes are ϵ-independent

Exact conformal symmetry in $d = 4 - 2\epsilon$ at the critical point $\beta(a_*) = 0$

$$(1) \quad \left[S_+^{\text{full}}, \mathcal{H}_h^{t=2}(a_*)\right] = 0$$

Conformal generators receive quantum corrections:

$$S_+^{(0)} = z^2\partial_z + 2z \mapsto S_+^{\text{full}}(a_*) = S_+^{(0)} + z\left[-\epsilon + \Delta(a_*)\right],$$
$$S_0^{(0)} = z\partial_z + 1 \mapsto S_0^{\text{full}}(a_*) = S_0^{(0)} - \epsilon + \mathcal{H}_h^{t=2}(a_*)$$

$\Delta(a_*) = a_*\Delta^{(1)} + a_*^2\Delta^{(2)} + \ldots$ is called conformal anomaly satisfying

$[z\partial_z, \Delta(a_*)] = 0$ from (1) and $SL(2)$ algebra $\Longrightarrow$ $(2) \quad [z\partial_z, S_+^{\text{full}}(a_*)] = S_+^{\text{full}}(a_*)$

$\ln \mu z$ enters $\mathcal{H}_h^{t=2}$ only linearly with coefficient Γ_{cusp} [G. Korchemsky, A. Radyushkin (1992)]

$$(3) \quad \left[z\partial_z, \mathcal{H}_h^{t=2}(a_*)\right] = \Gamma_{\text{cusp}}(a_*)$$

$$(1) \Longrightarrow \mathcal{H}_h^{t=2}(a_*) = f(S_+^{\text{full}}(a_*)) \overset{(2),(3)}{\Longrightarrow} z f'(z) = \Gamma_{\text{cusp}}(a_*) \Longrightarrow$$ **Proposition**

Conformal generators at one-loop

- **Two-loop evolution of twist-2 DA** [V. Braun, YJ and A. Manashov (2019)]

$$\mathcal{H}_h^{(2)}(a_*) = \Gamma_{\text{cusp}}^{(2)}(a_*)\ln(i\bar{\mu}S_+^{(1)}(a_*)) + \Gamma_+^{(2)}(a_*)\,,$$
$$S_+^{(1)}(a_*) = S_+^{(0)} + z\big(-\epsilon(a_*) + a_*\Delta^{(1)}\big)$$

$$\bar{\mu} = \widetilde{\mu}e^{\gamma_E} = \mu_{\overline{\text{MS}}}\,e^{2\gamma_E}$$

$$\epsilon(a_*) = -\beta_0 a_* + O(a_*^2)$$

One-loop conformal anomaly *four one-loop diagrams*

$$\Delta^{(1)}\mathcal{O}(z) = C_F\left\{3\mathcal{O}(z) + 2\int_0^1 d\alpha\left(\frac{2\bar{\alpha}}{\alpha} + \ln\alpha\right)[\mathcal{O}(z) - \mathcal{O}(\bar{\alpha}z)]\right\}$$

- **The scheme-dependent constant $\Gamma_+^{(2)}(a)$ is found from Feynman diagrams**

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Two-loop kernel in integral representation

- **Integral representation for $\mathcal{H}_h^{t=2}$ is usually preferred**

Ansatz

$$\mathcal{H}(a)\mathcal{O}(z) = \Gamma_{\text{cusp}}(a)\left[\ln(i\widetilde{\mu}z)\mathcal{O}(z) + \int_0^1 d\alpha\,\frac{\bar{\alpha}}{\alpha}\big(1+h(a,\alpha)\big)\big(\mathcal{O}(z)-\mathcal{O}(\bar{\alpha}z)\big)\right] + \gamma_+(a)\mathcal{O}(z)$$

- $\Delta^{(1)}$ **and** $\epsilon(a_*) = -\beta_0 a_* + O(a_*^2)$ **dictate** $h(a,\alpha)$ *going to Mellin space*

$$h(a,\alpha) = a\ln\bar{\alpha}\left\{\beta_0 - 2C_F\left(\frac{3}{2} + \ln\frac{\alpha}{\bar{\alpha}} + \frac{\ln\alpha}{\bar{\alpha}}\right)\right\} + O(a^2)$$

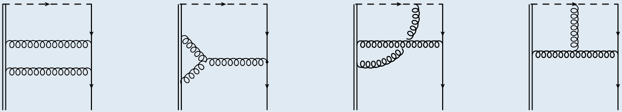
- γ_+ **requires additional calculation** *scheme-dependent*, $\gamma_{\phi_+} = \gamma_+ - \gamma_F$

$$\gamma_+^{\overline{\text{MS}}}(a) = -aC_F + a^2C_F\left\{4C_F\left[\frac{21}{8} + \frac{\pi^2}{3} - 6\zeta_3\right] + C_A\left[\frac{83}{9} - \frac{2\pi^2}{3} - 6\zeta_3\right] + \beta_0\left[\frac{35}{18} - \frac{\pi^2}{6}\right]\right\} + \dots$$

Two-loop kernel from Feynman diagrams

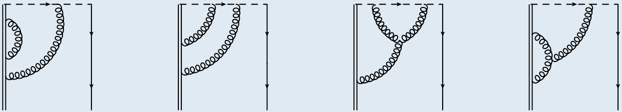
There are ~ 30 diagrams in three categories:

- **Exchange diagrams**



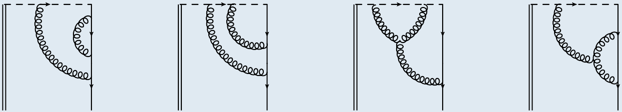
. . .

- **Cusp diagrams**



. . .

- **Light vertices**



. . .

Two-loop kernels from Feynman diagrams

- **Exchange diagrams contribute to both $h(a, \alpha)$ and γ_+** *(many are UV-finite)*
- **Cusp diagrams generate $\sim \ln z$ and contribute to γ_+**
- **Light vertices contribute to $h(a, \alpha)$ only, known**

[V. Braun, A. Manashov, S. Moch, and M. Strohmaier (2016)]

- $h(a, \alpha)$ **confirmed by explicit Feynman diagram calculation!**

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Light-heavy reduction

- **Evolution kernel of $\mathcal{O} = \bar{q}(nz_1)\gamma^+ q(nz_2)$ in integral form**

$$[\mathcal{H}_l\varphi](z_1,z_2) \propto \int_0^1 du\, h(u)\Big[2\varphi(z_1,z_2) - \varphi(z_{12}^u, z_2) - \boxed{\varphi(z_1, z_{21}^u)}\Big] + \boxed{\int_0^1 du \int_0^{\bar{u}} dv\, \chi(u,v)[\varphi(z_{12}^u, z_{21}^v) + \varphi(z_{12}^v, z_{21}^u)]} + c\varphi(z_1,z_2)$$

- **drop terms in boxes and $z_2 \to 0$ to obtain $\mathcal{H}_h^{\text{ex}} + \mathcal{H}_h^{\text{lv}}$.**
  ↖ **Location of the heavy quark is fixed!**

  Explicit expressions for $\mathcal{H}_l^{(2)}$ available [V. Braun, A. Manashov, S. Moch and M. Strohmaier (2016)]
- **Adding contribution of cusp diagrams again gives us $\mathcal{H}_h^{(2)}$.**

Evolution of the coefficient function at two-loop

Reminder ($Q_s(z)$ form a complete orthonormal basis)

$$\Phi_+(z,\mu) = \int_0^\infty ds\, s\, Q_s(z)\, \eta_+(s,\mu) = -\frac{1}{z^2}\int_0^\infty ds\, s\, \mathrm{e}^{is/z}\, \eta_+(s,\mu)$$

- **RGE of $\phi_+(z,\mu) \mapsto$ integro-differential eq. over $\eta_+(s,\mu)$** [V. Braun, YJ and A. Manashov (2019)]

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(a)\frac{\partial}{\partial a} + \Gamma_{\text{cusp}}(a)\ln(\widetilde{\mu}\mathrm{e}^{\gamma_E}s) + \gamma_\eta(a)\right)\eta_+(s,\mu) = 4C_F a\int_0^1 du\,\frac{\bar{u}}{u}h(a,u)\eta_+(\bar{u}s,\mu)$$

$$\gamma_\eta = \gamma_+^{\overline{\text{MS}}} - \gamma_F - \Gamma_{\text{cusp}}^{(2)}\left[1 - a\left(C_F\left(\frac{\pi^2}{6}-3\right) + \beta_0\left(1-\frac{\pi^2}{6}\right)\right)\right]$$

- **NNLL resummation requires Γ_{cusp} to $O(a^3)$ since numerically $\ln(s)\sim 1/a$**

Analytic solution of the two-loop RGE

- **Operator $\mathcal{O}(z)$ in Mellin space**

$$\mathcal{O}(z) = \int_{-i\infty}^{+i\infty} dj\, (i\mu_{\overline{\text{MS}}}\, e^{\gamma_E}\, z)^j\, \mathcal{O}(j)$$

gives rise to the Mellin-space RGE:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(a)\frac{\partial}{\partial a} - \Gamma_{\text{cusp}}(a)\frac{\partial}{\partial j} + V(j,a)\right)\mathcal{O}(j,a,\mu) = 0$$

explicit expression for $V(j,a)$ up to $\mathcal{O}(a^3)$ available in [V. Braun, YJ and A. Manashov, 1912.03210]

- **Mellin moment j as the second coupling, with Γ_{cusp} as the β-function**

Numerical study

- **Numerically solving the integro-differential equation of $\eta_+(s,\mu)$**

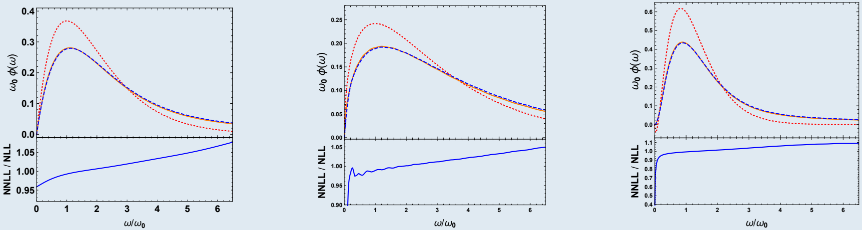

Figure : Models at $\mu_0^{\overline{\text{MS}}} = 1$ GeV (dots) evolved to $\mu_1^{\overline{\text{MS}}} = 2$ GeV at NLL (solid) and NNLL (dashed) for exponential model (left), Model II with σ_1^{max} (middle), and Model III with σ_1^{min} [Details of each model found in Ref. M. Beneke, V. Braun, YJ and Y-B. Wei (2018)]

- **Two-loop evolution has a smaller effect than its one-loop counterpart**
- **Nonlinear behaviors of Model II, III at $\omega \sim 0$ generate larger NNLL corrections**

Implication of the two-loop evolution

- **Smallness of the two-loop effect explains why scale dependence of $B \to \gamma \ell \nu_\ell$ decay is weak** [M. Beneke, V. Braun, YJ and Y-B. Wei (2018)]
- **More prominent effect by ratio at the small ω region**
- **Two-loop correction further $\downarrow$ ($\uparrow$) the amplitude at $\omega \sim 0$ ($\omega \gg \Lambda_{\mathrm{QCD}}$)**
- **Large effects on parameters λ_B, $\widehat{\sigma}_1, \widehat{\sigma}_2$ to $\omega \sim 0$ are possible** [V. Braun, YJ and A. Manashov (2019)]

ω_0, MeV	$\lambda_B^{\mathrm{NLL}}/\omega_0$	$\lambda_B^{\mathrm{NNLL}}/\omega_0$	$\widehat{\sigma}_1^{\mathrm{NLL}}$	$\widehat{\sigma}_1^{\mathrm{NNLL}}$
200	1.29	1.31	+0.011	−0.042
300	1.22	1.24	−0.043	−0.116
400	1.18	1.18	−0.082	−0.172

Table : *NNLL effects on λ_B, $\widehat{\sigma}_1$ in simple exponential model evolved from* 1 GeV *to* 2 GeV.

$$\widehat{\sigma}_n = \int_0^\infty d\omega \, \frac{\lambda_B}{\omega} \ln^n \frac{\lambda_B e^{-\gamma_E}}{\omega} \, \phi_+(\omega)$$

Conclusion and Outlook

Conclusion

- Three-particle DAs classified by collinear and conformal twists
- Integrability of RGEs for higher twist DAs at one-loop
- Two-loop kernel of twist-2 DA from conformal symmetry
- The DAs are subject to several constraints, general models proposed

Outlook for future work

- Treat EOMs and large momentum behavior of $\phi_+(\omega)$ in the same manner
- An updated estimate of local higher twist matrix elements
- Update charmless B-decays to the full NNLL accuracy

Parameters in generalized exponential model

Details of the generalized exponential model

$$\phi_+(\omega,\alpha,\beta,\mu_0) = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\frac{\omega}{\omega_0^2}\, e^{-\omega/\omega_0} U(\beta-\alpha, 3-\alpha, \omega/\omega_0)\,,\;\; \alpha,\beta > 1$$

- **Inverse Moments**

$$\lambda_B = \frac{\alpha-1}{\beta-1}\omega_0\,, \qquad \widehat{\sigma}_1 = \psi(\beta-1) - \psi(\alpha-1) + \ln\frac{\alpha-1}{\beta-1}$$

covering $-0.693... < \widehat{\sigma}_1 < 0.693...$ with $\widehat{\sigma}_1 = 0$ corresponding to the SE model

- $\phi_+(\omega\sim 0) \sim \omega$ **is broken!** In particular, $\phi_+(\omega \sim 0) \sim \omega^{0.5\sim 1.5}$
  motived by pion LCDA from BaBar and BELLE measurements of $\gamma^* \to \pi\gamma$ transition FF
  [S. Agaev, V. Braun, N. Often and F. Porkert (2012), (2013); I. Cloët, L. Chang, C. Roberts, S. Schmidt and P. C. Tandy (2013)]
  [N. Stefanis and A. V. Pimikov (2016)]

One-loop evolution

- **Analytical NLL resummation**

$$\begin{aligned}\eta_+(s,\mu_0) &= {}_1F_1(\alpha,\beta,-s\omega_0)\,,\\ \eta_+(s,\mu) &= U_+(s,\mu,\mu_0)\eta_+(s,\mu_0)\,,\\ U_+(s;\mu,\mu_0) &= \exp\left\{-\frac{\Gamma_0}{4\beta_0^2}\left(\frac{4\pi}{\alpha_s(\mu_0)}\left[\ln r - 1 + \frac{1}{r}\right]\right.\right.\\ &\left.\left. -\frac{\beta_1}{2\beta_0}\ln^2 r + \left(\frac{\Gamma_1}{\Gamma_0}-\frac{\beta_1}{\beta_0}\right)[r-1-\ln r]\right)\right\}\left(se^{2\gamma_E}\mu_0\right)^{\frac{\Gamma_0}{2\beta_0}\ln r}\, r^{\frac{\gamma_0}{2\beta_0}}\end{aligned}$$

$$r = \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}$$

- **s-space $\mapsto$ ω-space**

$$\begin{aligned}&\omega_0\int_0^\infty ds\,(\omega_0 s)^p\sqrt{\omega s}J_1(2\sqrt{\omega s})\,{}_1F_1(\alpha,\beta,-\omega_0 s) =\\ &= \frac{\omega}{\omega_0}\frac{\Gamma(\beta)\Gamma(2+p)\Gamma(\alpha-p-2)}{\Gamma(\alpha)\Gamma(\beta-p-2)}\,{}_2F_2(p+2,p+3-\beta;2,p+3-\alpha,-\omega/\omega_0)\\ &+\left(\frac{\omega}{\omega_0}\right)^{\alpha-p-1}\frac{\Gamma(\beta)\Gamma(p+2-\alpha)}{\Gamma(\beta-\alpha)\Gamma(\alpha-p)}\,{}_2F_2(\alpha,\alpha-\beta+1;\alpha-p-1,\alpha-p,-\omega/\omega_0)\end{aligned}$$