Applications of dispersive theory

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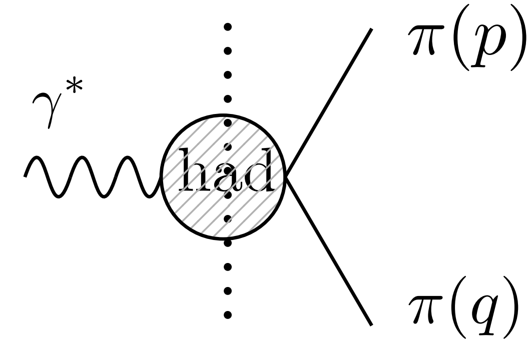
THE LOW-ENERGY FRONTIER
OF THE STANDARD MODEL

JOHANNES GUTENBERG UNIVERSITÄT MAINZ

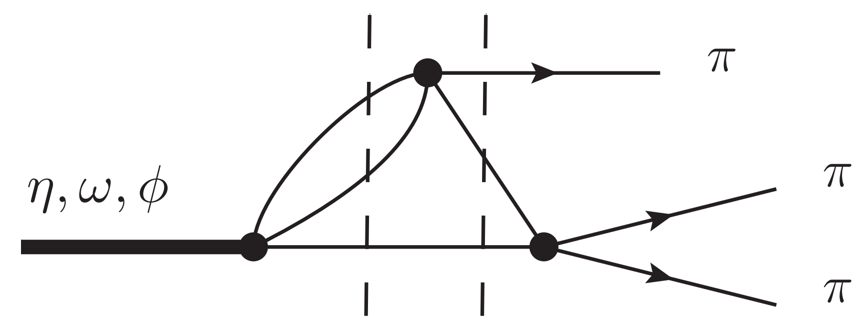
- Pion vector Form Factor ($\pi\pi\gamma^*$)

  important building block to (g-2)
  data from $e^+e^-\to 2\pi$ (c.f also $\tau^-\to\pi^-\pi^0\nu_\tau$)

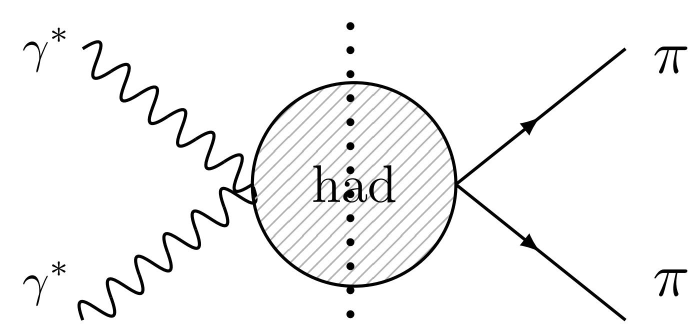


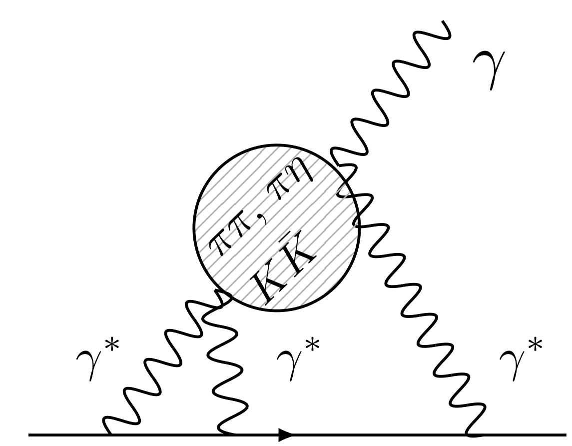
- Light mesons decays:

  $\eta\to 3\pi$ and light quark masses
  $\omega, \varphi\to 3\pi$ and TFF of $\omega\pi\gamma^*$

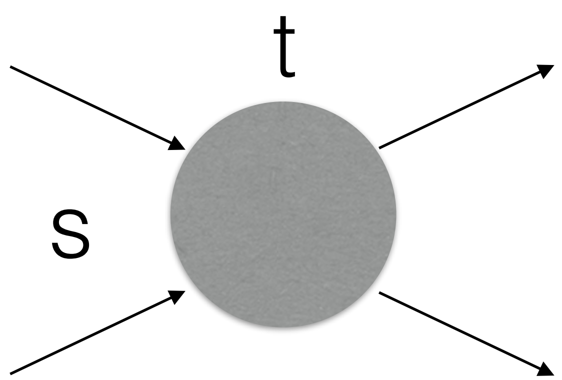


- Photon fusion reactions and (g-2)





First principle constraints



$$A(s,t) = \sum_{J=0}^{\infty} (2J+1)\, P_J(z)\, a_J(s)$$

Unitarity: for low energy unitarily is "simple"

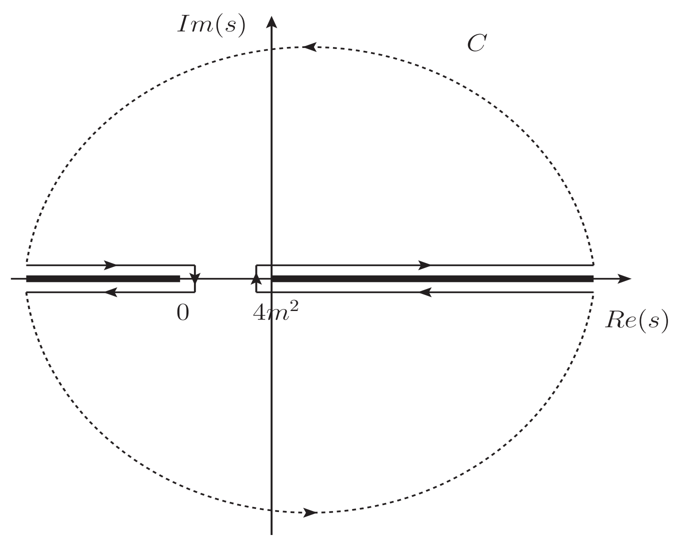
$$\begin{aligned}\text{Disc}\, a_J(s) &\equiv \frac{a_J(s+i\epsilon) - a_J(s-i\epsilon)}{2\,i} \\ &= \text{Im}\, a_J(s) = \rho(s)|a_J(s)|^2\end{aligned}$$

Crossing symmetry

- the same function A(s,t) should describe different processes (rotate the diagram by 90° or flip the leg)

Analyticity

- relates scattering amplitude at different energies



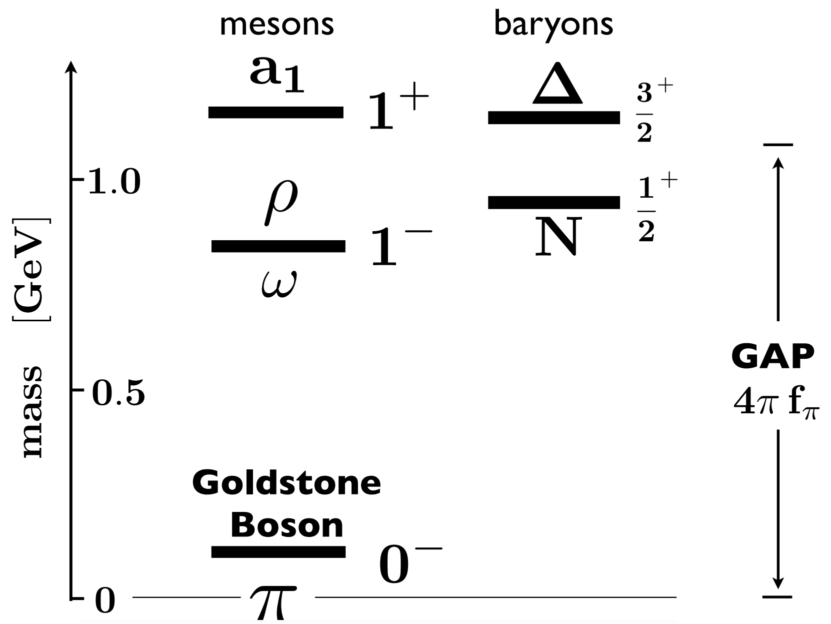
$$a_J(s) = \frac{1}{2\pi i} \int_C ds' \frac{a_J(s')}{s'-s} = \int_{-\infty}^{0} \frac{ds'}{\pi} \frac{\text{Disc}\, a_J(s')}{s'-s} + \int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{\text{Disc}\, a_J(s')}{s'-s}$$

First principle constraints

$$\mathcal{L}_{QCD} = \sum_{f=\substack{u,d,s,\\c,b,t}} \bar{q}_f \left(i\gamma^\mu D_\mu - m_f\right) q_f - \frac{1}{4} G^{(a)}_{\mu\nu} G^{(a)\,\mu\nu}$$

- at high energies: asymptotic freedom → perturbative QCD
- at low energies: **chiral symmetry**

$$SU(3)_L \times SU(3)_R \to SU(3)_V$$



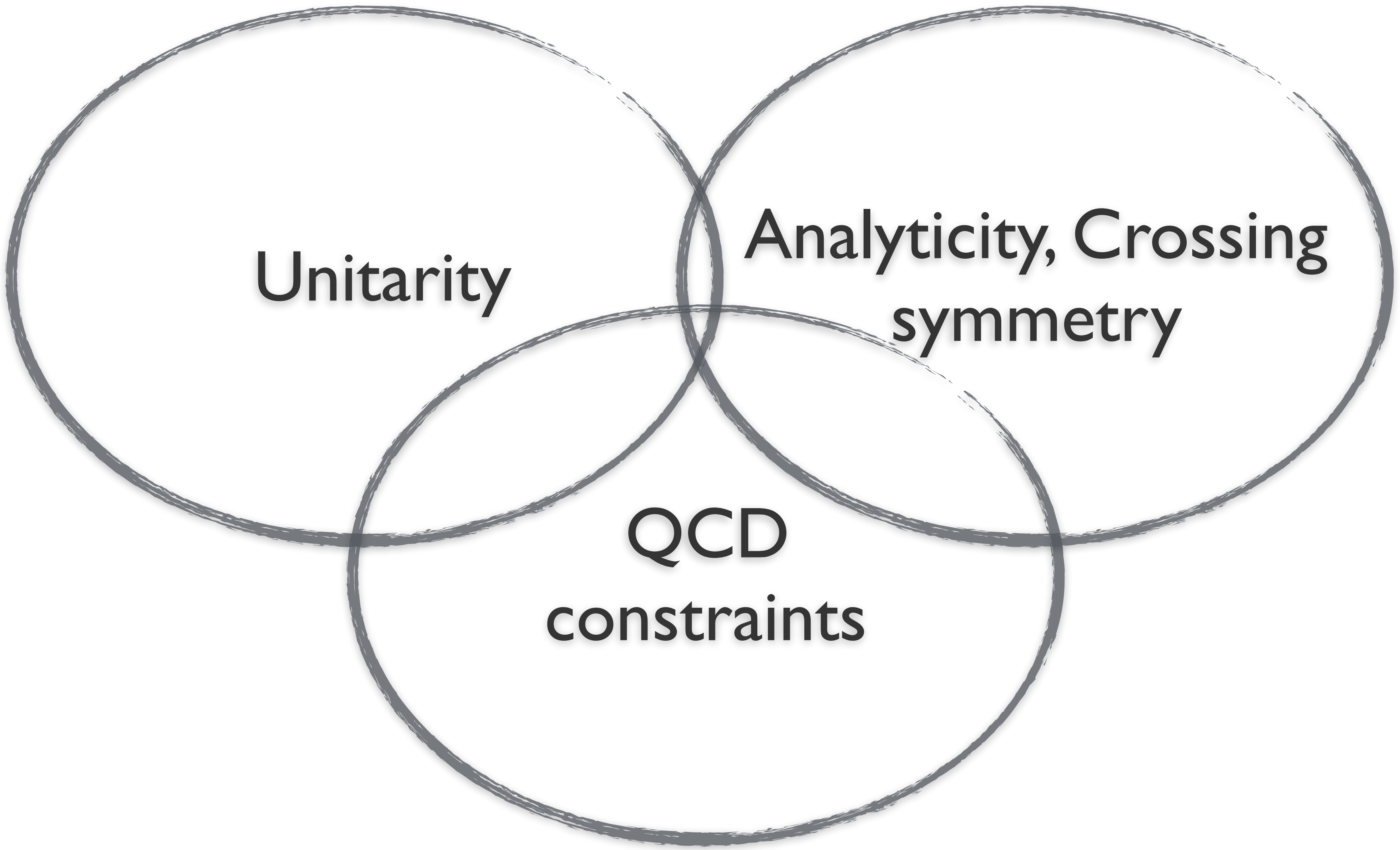
Wolfram Weise

Chiral perturbation theory (ChPT)

- d.o.f. - hadrons
- expansion in mass and momenta
- Unknown coupling constants (L_i) fitted to the data

Weinberg
Gasser & Leutwyler

First principle constraints



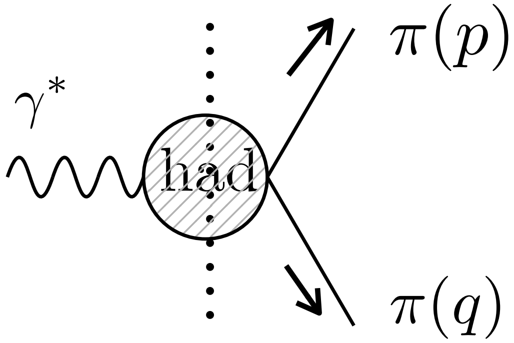
In practice rigorous implementation of these principles is **very hard**. However, for a given reaction it is possible to kinematically isolate regions where specific processes dominate.

Roy (Steiner): $\pi\pi$, πK, πN

Pion vector form factor

- Pion vector Form Factor

$$\langle \pi^+(p)\pi^-(q)|J_\mu(0)|0\rangle = (p-q)_\mu \, F_\pi^V(s)$$



- Unitarity relation

$$\mathrm{Im}\, F_\pi^V(s) = \rho(s)\, t_{\pi\pi}^*(s)\, F_\pi^V(s)$$

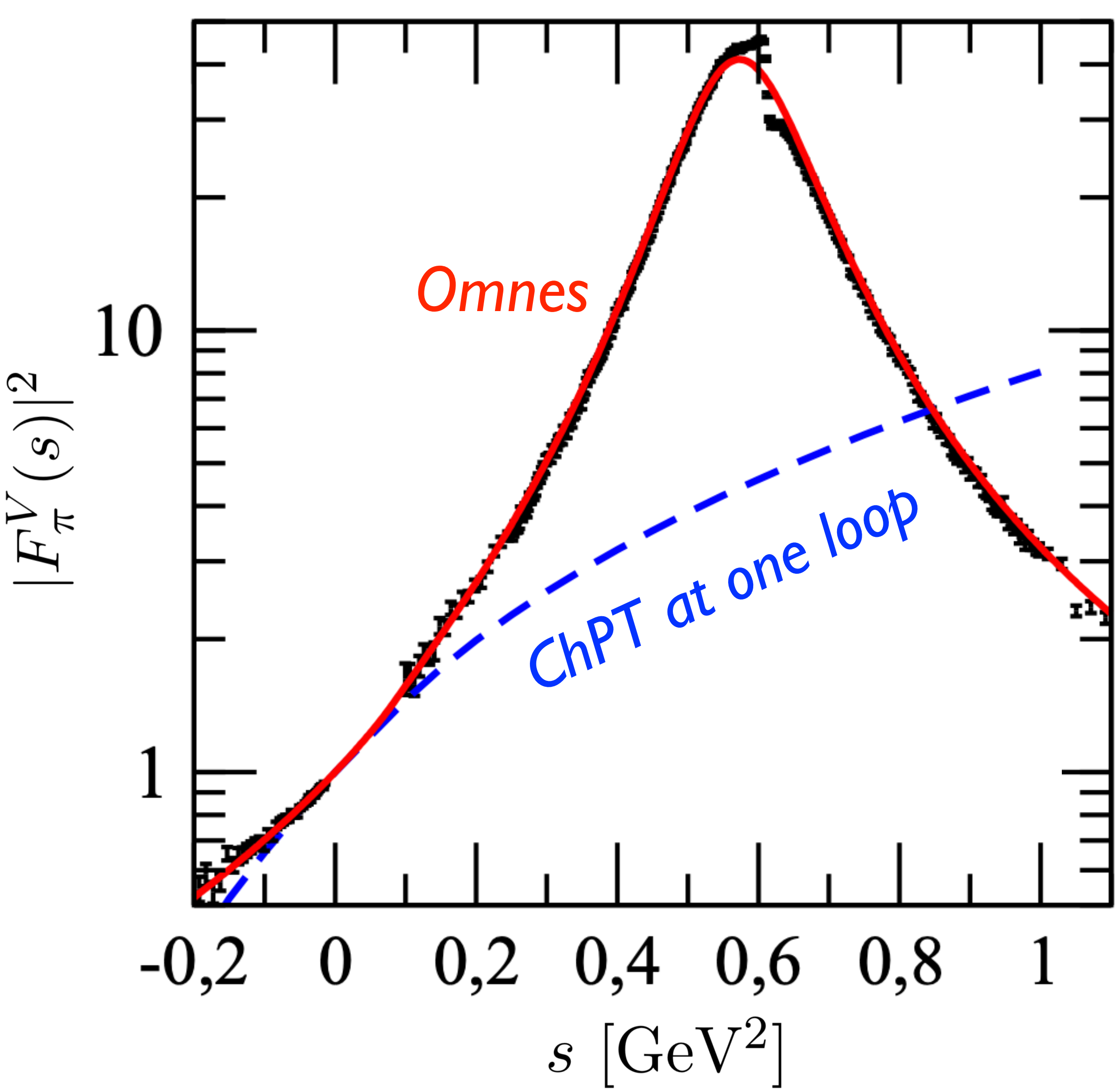
- Watson theorem (below inelastic threshold)

$$\mathrm{Arg}\, F_\pi^V(s) = \delta_{\pi\pi}(s)$$

- Omnès solution

$$F_\pi^V(s) = P(s)\,\Omega(s)$$

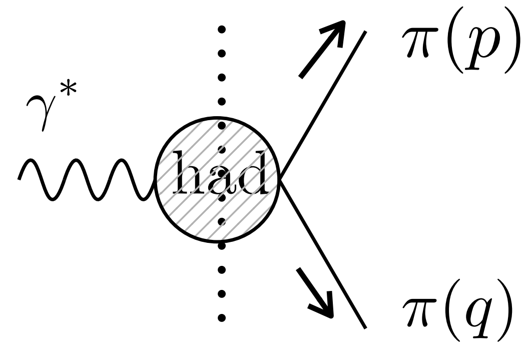
$$\Omega(s) = \exp\left(\frac{s}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{s'}\frac{\delta_{\pi\pi}(s')}{s'-s}\right)$$



Stollenwerk et al. (2012)

Pion vector form factor

- Defined as a matrix element of EM current



$$\langle \pi^+(p)\pi^-(q)|J_\mu(0)|0\rangle = (p-q)_\mu \, F_\pi^V(s)$$

1). Gauge invariance

$$p^\mu_{\gamma^*}\, \langle \pi^+(p)\pi^-(q)|J_\mu(0)|0\rangle = (\overbrace{p^2}^{m_\pi^2} - \overbrace{q^2}^{m_\pi^2})\, F_\pi^V(s) = 0$$

2). For zero momentum transfer nothing happens: $F_\pi^V(0) = 1$

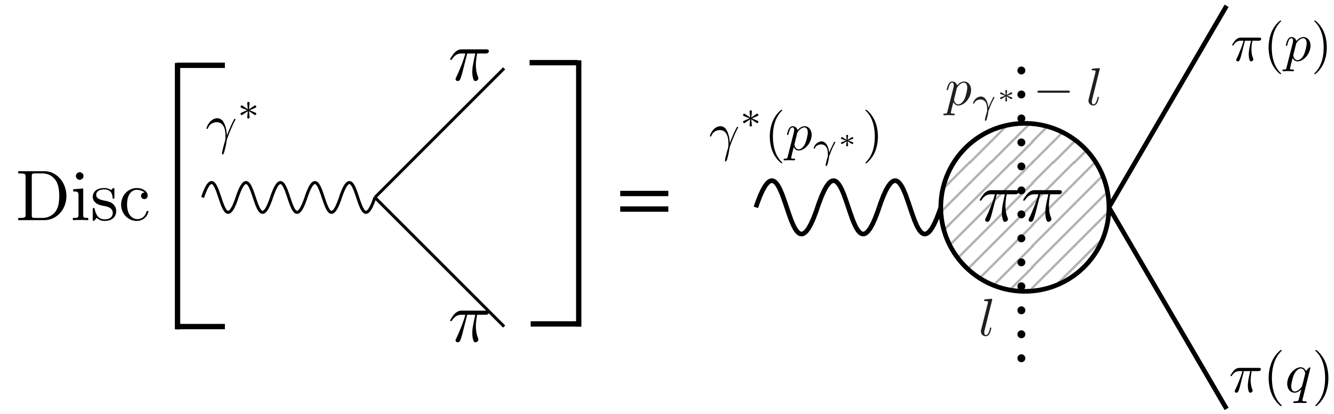
3). At very low energy it can be calculated in ChPT

$$F_\pi^V(s) = 1 + \frac{1}{6}\frac{1}{(4\pi f_\pi)^2}(L_6 - 1)\, s + \frac{1}{6\, f_\pi^2}(s - 4m_\pi^2)\bar{J}(s) + O(s^2)$$

fails very soon
(parametrize only tail of rho meson)

Pion vector form factor

- Unitarity relation (the function is analytic with only right-hand cut)



sym. factor

$z = \cos\theta$

$$(p-q)_\mu \mathrm{Im}\, F_\pi^V(s) = \frac{1}{2}\int d\Phi_2\, (\overbrace{p_{\gamma^*}}^{p+q} - 2\,l)_\mu\, F_\pi^V(s)\, T_{\pi\pi}^*(s,z) \times \frac{1}{2}$$

$$\int d\Omega\, (p+q-2\,l)_\mu\, T_{\pi\pi}^*(s,z) = \underbrace{L_1}_{=0}\, (p+q)_\mu + \underbrace{L_2}_{\int d\Omega\, T_{\pi\pi}^*(s,z)\, z}\, (p-q)_\mu \;\; = 2\pi \int_{-1}^{1} dz\, T_{\pi\pi}^*(s,z)\, z\, (p-q)_\mu$$

- Partial wave expansion

only J=1 survive
(ok, photon has spin-1)

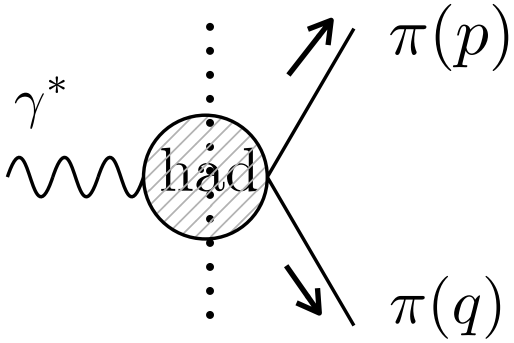
$$T_{\pi\pi}(s,z) = \sum_{J=0}^{\infty}(2J+1)\, t_{\pi\pi}^{(J)}(s)\, P_J(z), \quad \int_{-1}^{+1} dz\, P_J(z)\, z = \frac{2\delta_{J,1}}{2J+1}$$

$$\boxed{\mathrm{Im}\, F_\pi^V(s) = \rho(s)\, t_{\pi\pi}^{(1)*}(s)\, F_\pi^V(s)\, \theta(s > 4m_\pi^2)}$$

Pion vector form factor

- Pion vector Form Factor

$$\langle \pi^+(p)\pi^-(q)|J_\mu(0)|0\rangle = (p-q)_\mu\, F_\pi^V(s)$$



- Unitarity relation

$$\mathrm{Im}\, F_\pi^V(s) = \rho(s)\, t^*_{\pi\pi}(s)\, F_\pi^V(s)$$

- Watson theorem (below inelastic threshold)

$$t_{\pi\pi}(s) = \frac{\sin\delta_{\pi\pi}(s)\, e^{i\delta_{\pi\pi}(s)}}{\rho(s)}$$

$$F_\pi^V(s) = |F_\pi^V(s)|e^{i\phi(s)}$$

$$|F_\pi^V(s)|\, \sin\phi(s) = \sin\delta_{\pi\pi}(s)\, e^{-i\delta_{\pi\pi}(s)}\, |F_\pi^V(s)|e^{i\phi(s)}$$

$$\boxed{\begin{gathered}\phi(s) = \delta_{\pi\pi}(s)\,(+n\pi) \\ \mathrm{Arg}\, F_\pi^V(s) = \delta_{\pi\pi}(s)\end{gathered}}$$

Pion vector form factor

- Watson theorem (below inelastic threshold)

$$\mathrm{Im}\, F_\pi^V(s) = \rho(s)\, t_{\pi\pi}^*(s)\, F_\pi^V(s) = \sin\delta_{\pi\pi}(s)\, e^{-i\delta_{\pi\pi}(s)}\, F_\pi^V(s),$$

$$F_\pi^V(s) = |F_\pi^V(s)| e^{i\phi(s)}$$

$$\mathrm{Arg}\, F_\pi^V(s) = \delta_{\pi\pi}(s)$$

- Muskhelishvili-Omnès solution

$$F_\pi^V(s) = P(s)\, \Omega(s)$$

polynomial

$$\frac{1}{2i}\left(\Omega(s+i\epsilon) - \Omega(s-i\epsilon)\right) = \sin\delta_{\pi\pi}(s)\, e^{-i\delta_{\pi\pi}(s)}\, \Omega(s+i\epsilon),$$

$$\Omega(s+i\epsilon)\left(\frac{1}{2i} - \sin\delta_{\pi\pi}\, e^{-i\delta_{\pi\pi}(s)}\right) = \Omega(s-i\epsilon)\frac{1}{2i}$$

$$\Omega(s+i\epsilon)\, e^{-2i\delta_{\pi\pi}(s)} = \Omega(s-i\epsilon)$$

$$\mathrm{Disc}\left(\ln\, \Omega(s)\right) = 2i\, \delta_{\pi\pi}(s)$$

$$\ln\, \Omega(s) = \frac{1}{2i}\int_{4m_\pi^2}^{\infty} \frac{ds'}{\pi} \frac{\mathrm{Disc}(\ln\, \Omega(s'))}{s'-s} = \int_{4m_\pi^2}^{\infty} \frac{ds'}{\pi} \frac{\delta(s')}{s'-s}$$

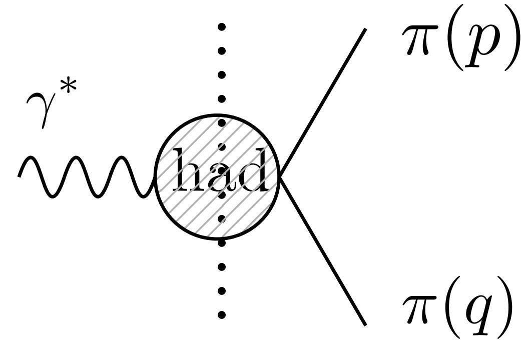
$$\boxed{\begin{array}{l} \Omega(s) = \exp\left(\dfrac{s}{\pi}\displaystyle\int_{4m^2}^{\infty} \frac{ds'}{s'} \frac{\delta_{\pi\pi}(s')}{s'-s}\right) \\ \Omega(s=0) = 1 \end{array}}$$

- High energy behaviour: $\delta(s\to\infty) \to \alpha\,\pi\,, \quad \Omega(s\to\infty) \to \dfrac{1}{s^\alpha}$

Pion vector form factor

- Muskhelishvili-Omnès solution

$$F_\pi^V(s) = P(s)\,\Omega(s)\,, \quad \Omega(s) = \exp\left(\frac{s}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{s'}\frac{\delta_{\pi\pi}(s')}{s'-s}\right)$$



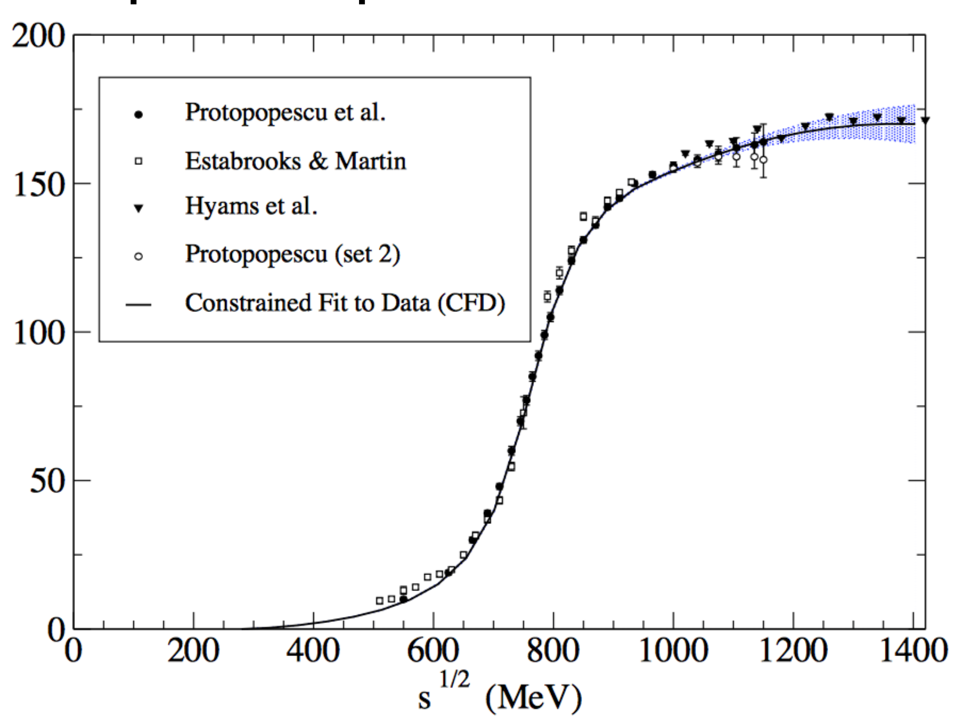
- perturbative QCD

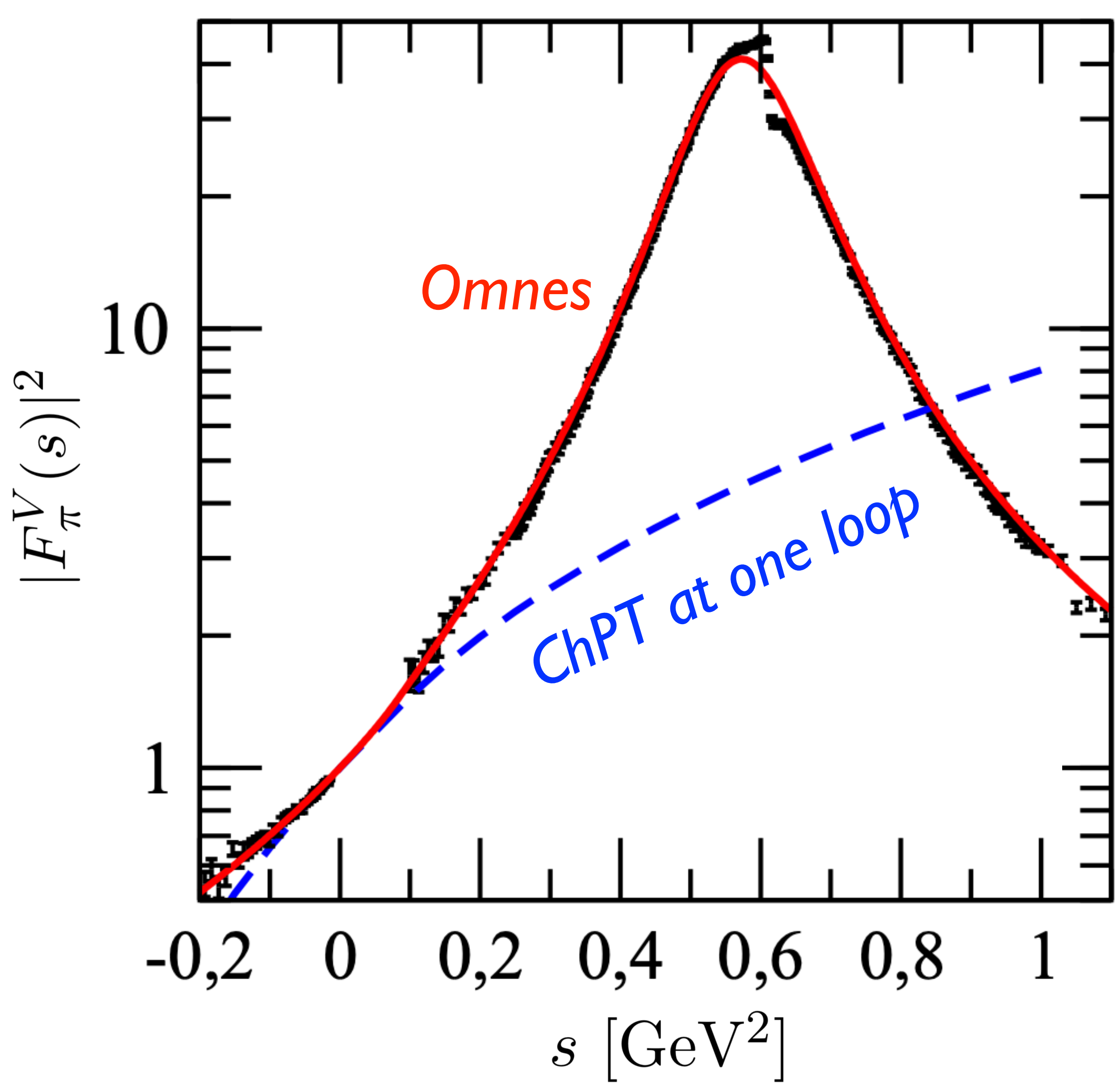
$$F_\pi^V(s\to\infty) = \frac{1}{s}$$

assume

$$\delta_{\pi\pi}(s\to\infty)\to\pi$$
$$P(s) = 1$$

- Main input: ππ phase shift





Stollenwerk et al. (2012)

Pion vector form factor

- State-of-art dispersive parametrisation

$$F_\pi^V(s) = \Omega(s)\, G_\omega(s)\, G_{in}^N(s)$$

- Omnès function

$$\Omega(s) = \exp\left(\frac{s}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{s'}\frac{\delta_{\pi\pi}(s')}{s'-s}\right)$$

- ρ-ω mixing (isospin breaking effects)

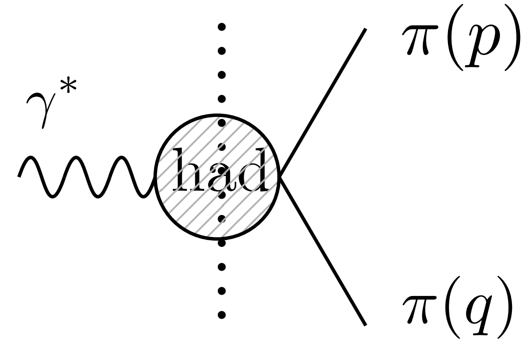
$$G_\omega(s) = 1 + \epsilon_\omega \frac{s}{(M_\omega - \frac{i}{2}\Gamma_\omega)^2 - s}$$

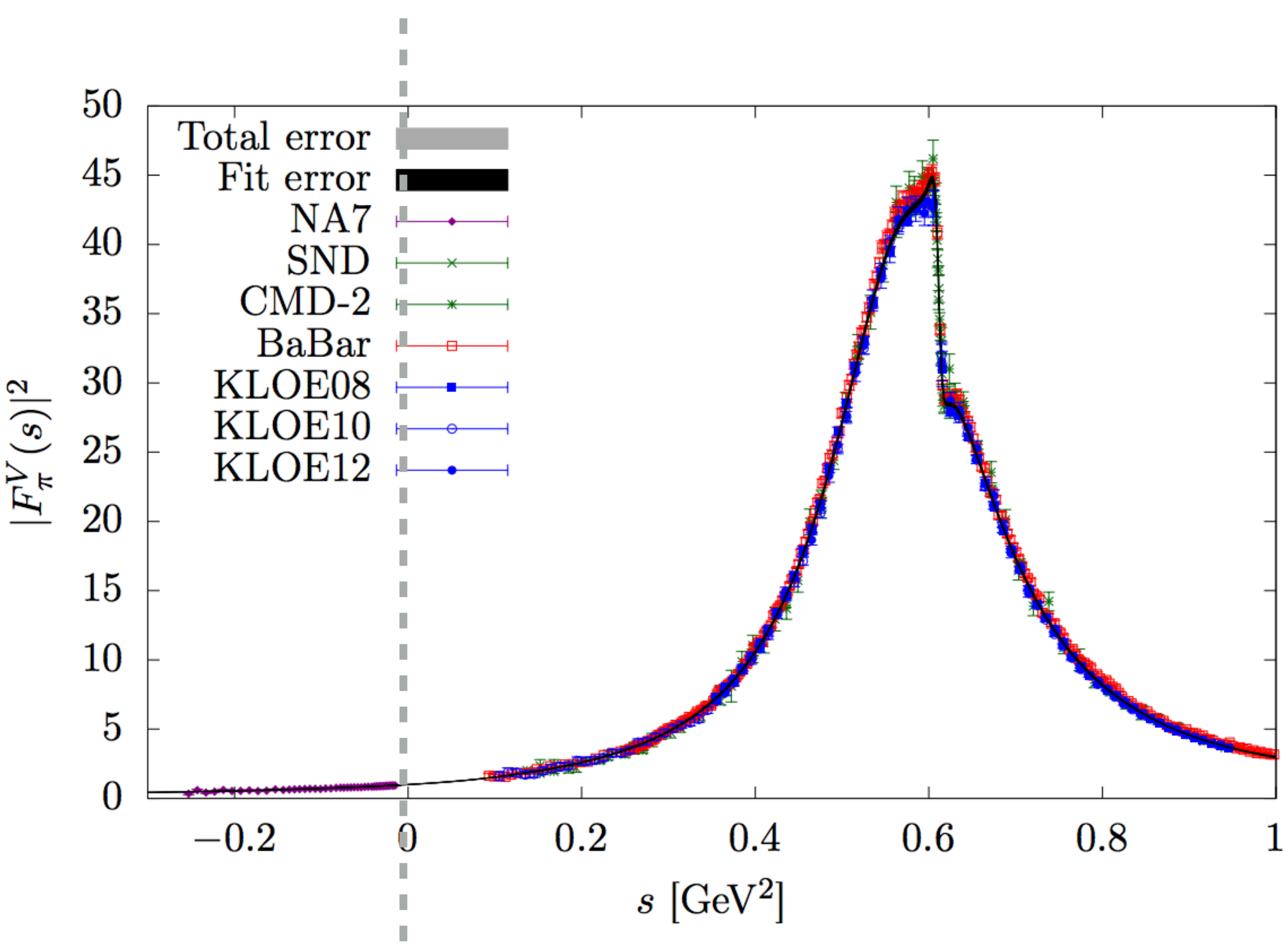
- Further inelastic contributions

$$G_{in}^N = 1 + \sum_{k=1}^{N} c_k(\xi(s)^k - \xi(0)^k)$$

$$\xi(s) = \frac{\sqrt{s_{in}-s_c} - \sqrt{s_{in}-s_c}}{\sqrt{s_{in}-s_c} + \sqrt{s_{in}-s_c}}$$

$$s_{in} = (m_\pi + m_\omega)^2$$



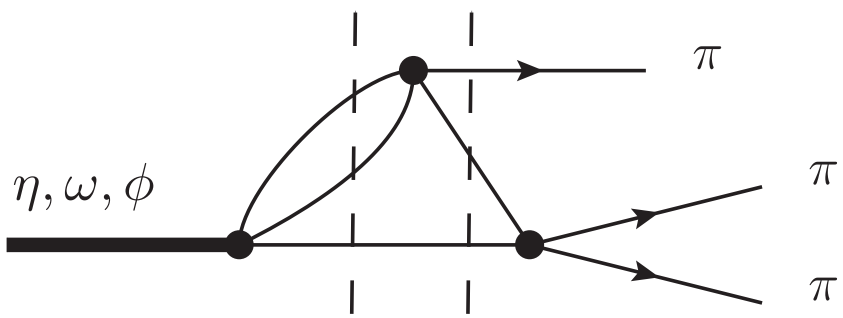


Colangelo et al. (2019)

Three body decays

- Light mesons decays:

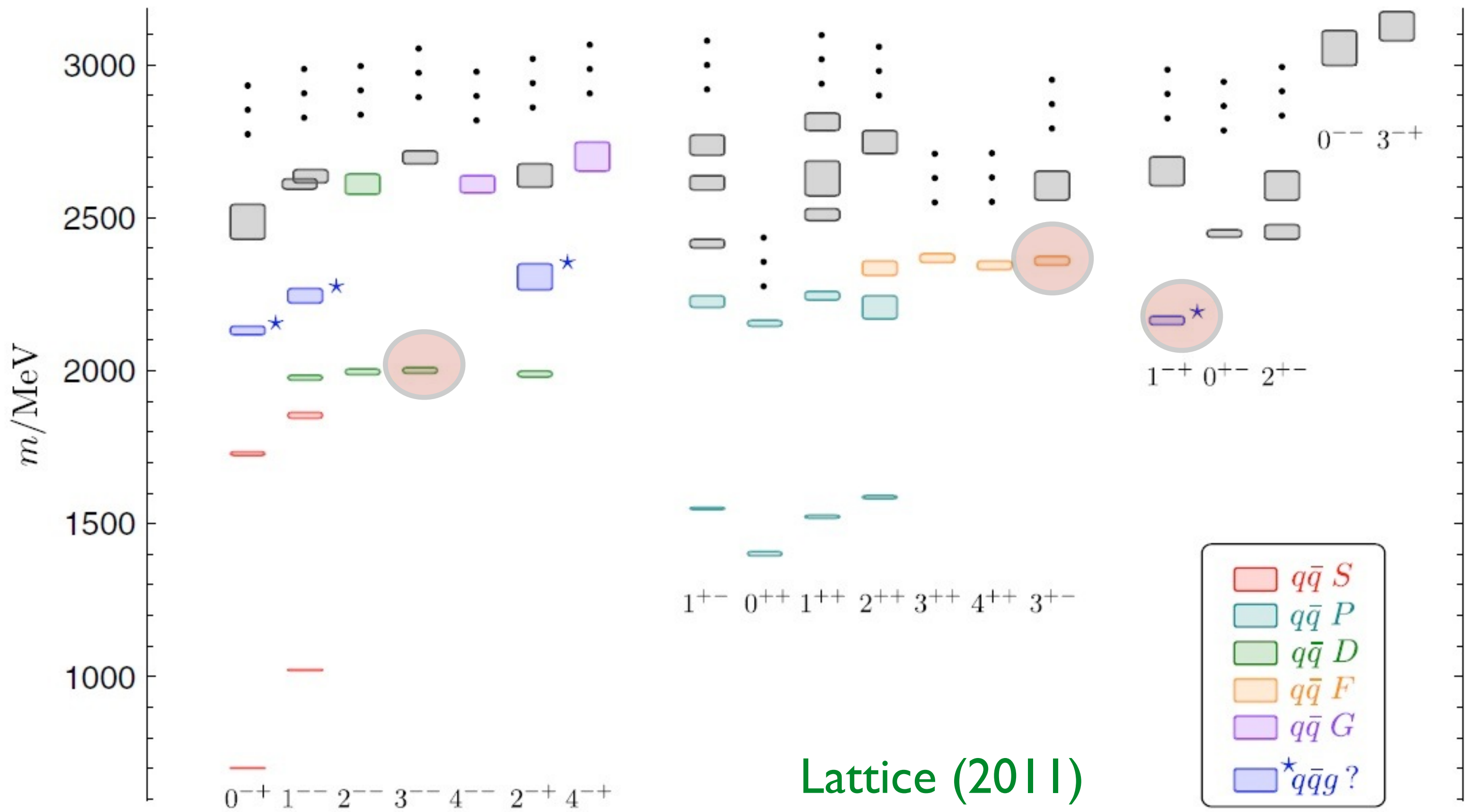
  η→3π and light quark masses
  ω, φ→3π and TFF of ωπγ*



Three body decays (motivation)

Spectrum of QCD: complete understanding, discover new resonances, exotics ...

Decay properties of the known states



Lattice (2011)
Dudek et al.

- multi-body (final state) interactions are expected to play a crucial role for the hadron spectroscopy
- analysis of the precision exp. data
- test/develop/cross-check tools on conventional states → move to exotic

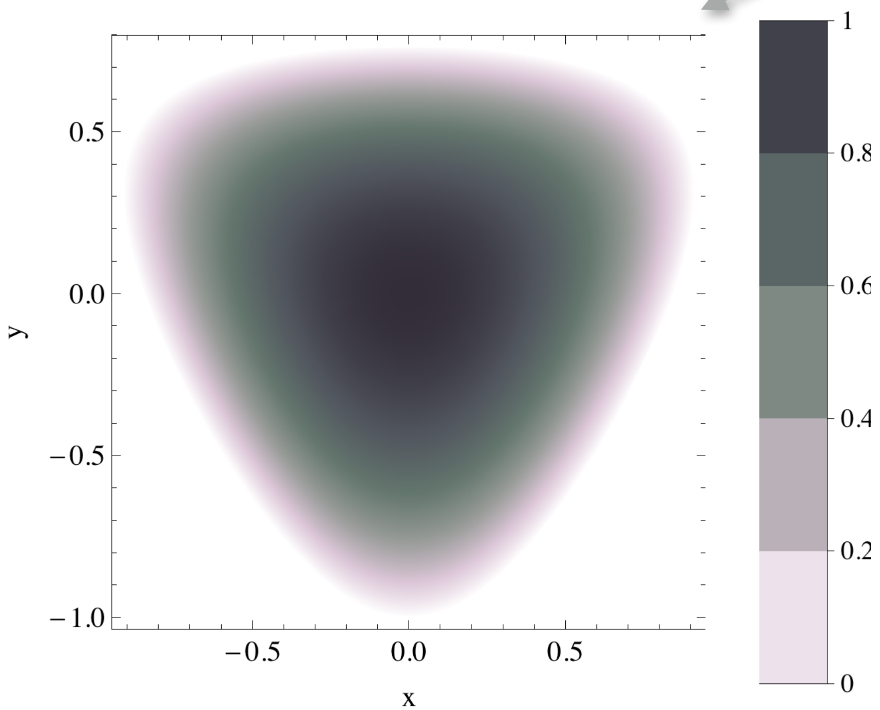
ω-meson discovered in ~1960th

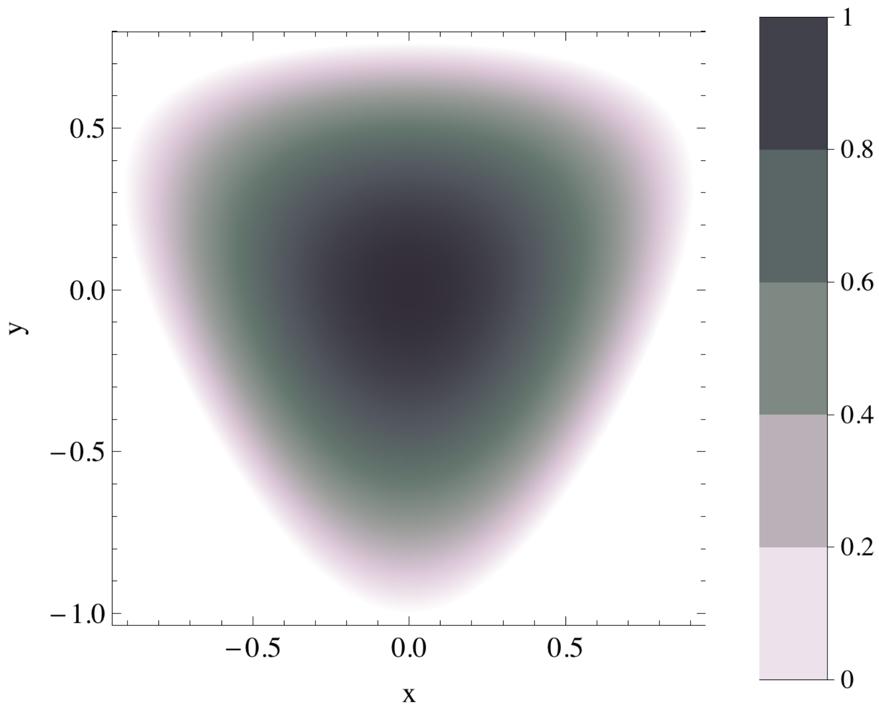
π p → ω p, Kp → ωΛ, e^+e^- → 3π, pp → ωππ, … **number of events:** 10^3 - 10^4

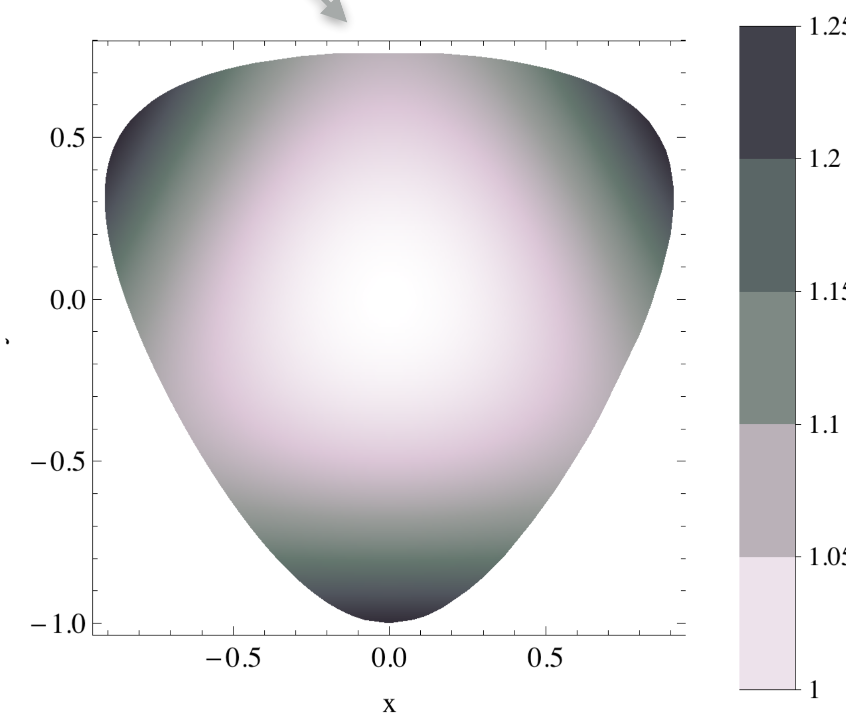
Dalitz plot: ω→3π

$$\frac{d^2\Gamma}{ds\,dt} \propto |\vec{p}_+ \times \vec{p}_-|^2 |F(s,t)|^2$$

$$x \quad \propto \quad (t-u)$$
$$y \quad \propto \quad (s_c - s)$$







WASA-at-COSY (2016)
BESIII (2018)
number of events: 10^6 - 10^7

Three body decays (motivation)

$$\Gamma_{\eta\to\pi^+\pi^-\pi^0} = 66_{[\mathrm{LO}]} + 94_{[\mathrm{NLO}]} + \ldots = 296 \pm 16\,\mathrm{eV}_{[\mathrm{Exp}]}$$

- Isospin violating decay $\eta\to3\pi$: sensitive to quark mass difference

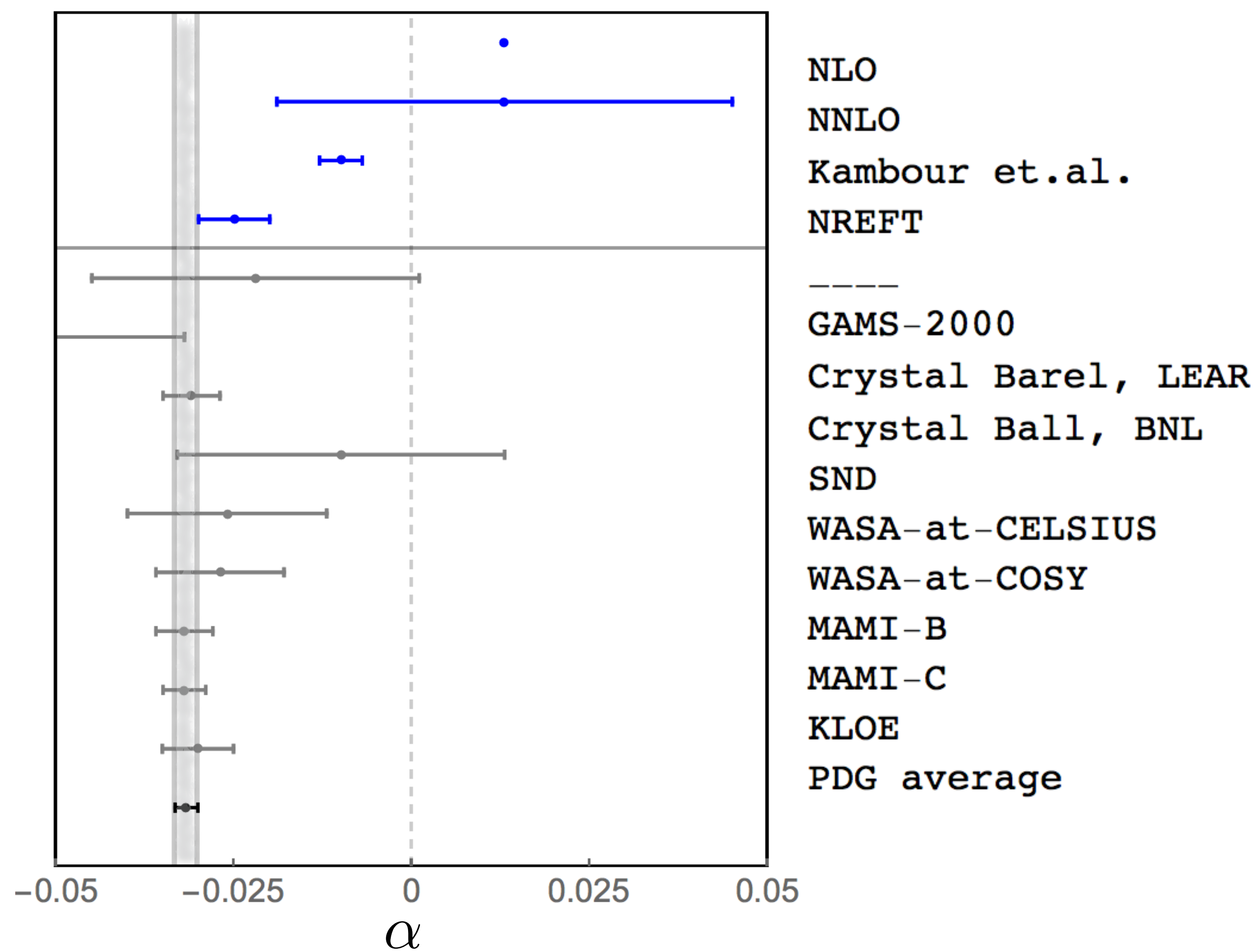
$$\frac{1}{Q^2} = \frac{m_d^2 - m_u^2}{m_s^2 - \hat{m}^2}$$

- Slow convergence of ChPT (importance of FSI)
- Slope parameter puzzle for $\eta\to3\pi^0$

$$|A_{\eta\to3\pi^0}|^2 \propto 1 + 2\alpha z + \ldots$$

- New data on $\eta\to\pi^+\pi^-\pi^0$

WASA-at-COSY (2014)
KLOE-2 (2016)



Method

- P.w. expansion

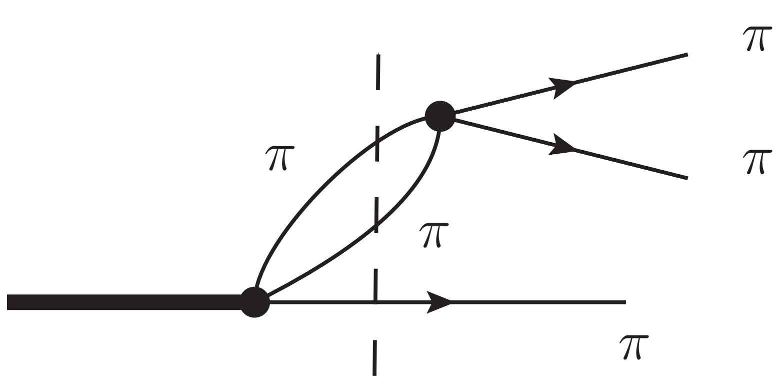
$$A(s,t,u) = \sum_{J=0}^{\cancel{\infty}\;\color{red}{J_{max}}} (2J+1)\, P_J(\cos\theta)\, f_J(s)$$

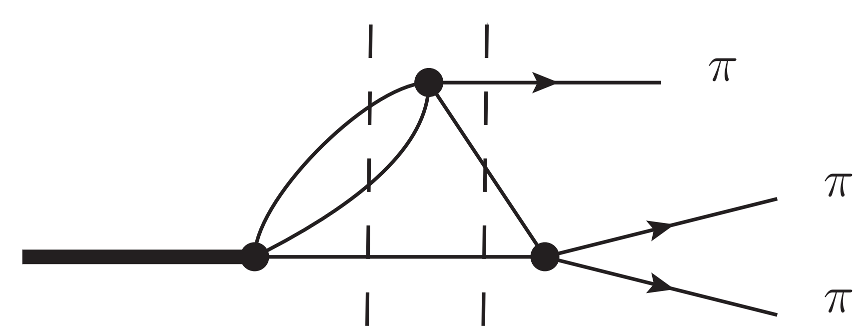
- Reconstruction theorem:
  crossing symmetry, analyticity up to NNLO

$$A(s,t,u) = \sum_{J}^{J_{max}} ...a_J(s) + \sum_{J}^{J_{max}} ...a_J(t) + \sum_{J}^{J_{max}} ...a_J(u)$$

ππ scattering
Fuchs, Sazdjian, Stern (1993)

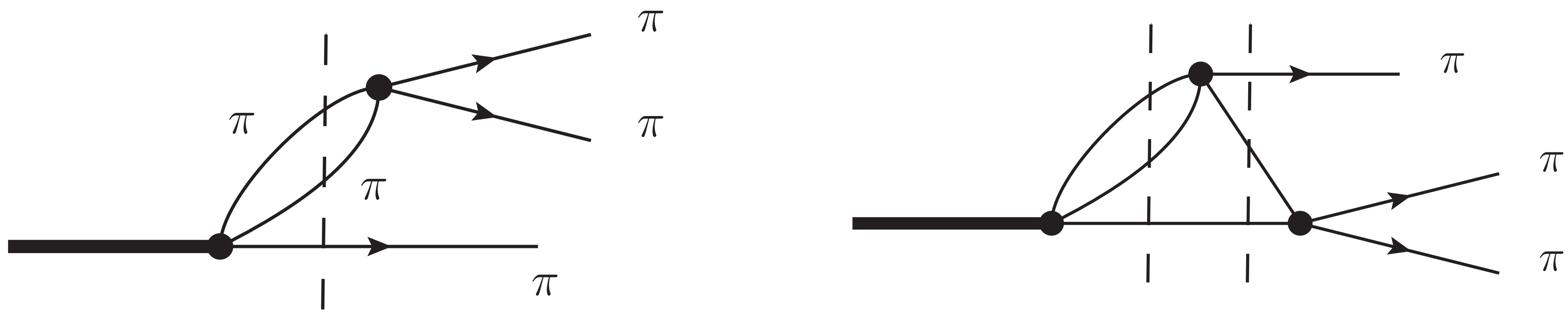
- Unitarity





Khuri, Treiman (1960)
Aitchison (1977)

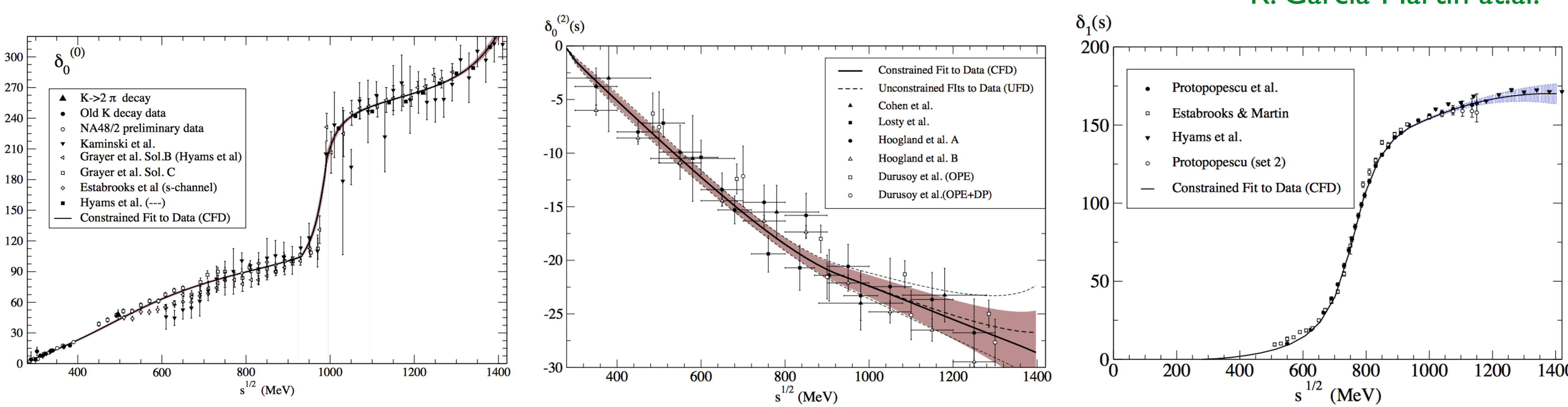
Unitarity



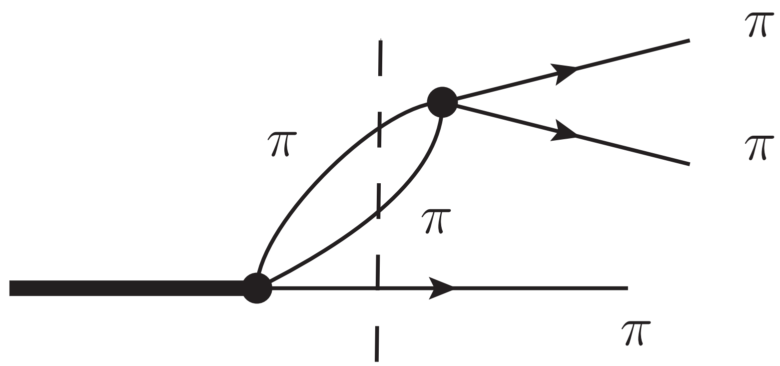
$$\mathrm{Disc}\, a_J(s) = t_J^*(s)\, \rho(s) \left(a_J(s) + \int_{-1}^{+1} \frac{d\cos\theta}{2} \ldots a_J(t) \right)$$

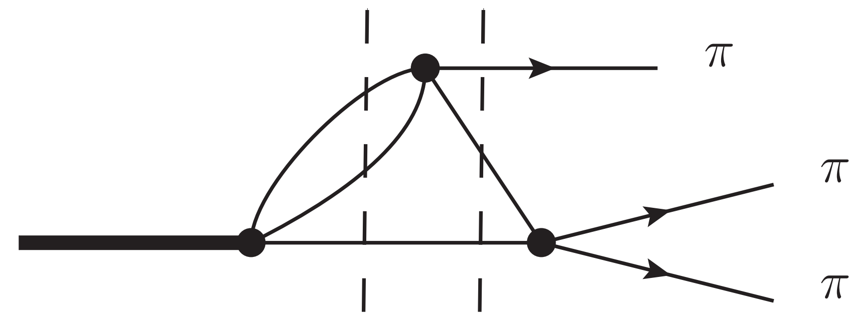
input

Roy analysis (2011)
R. Garcia-Martin at.al.



Unitarity



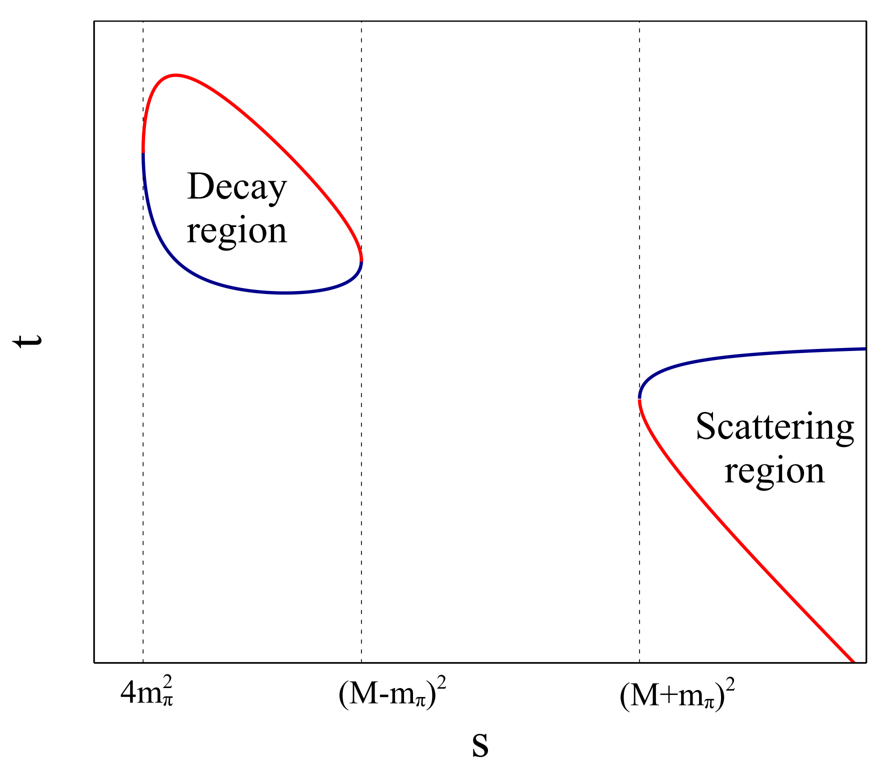


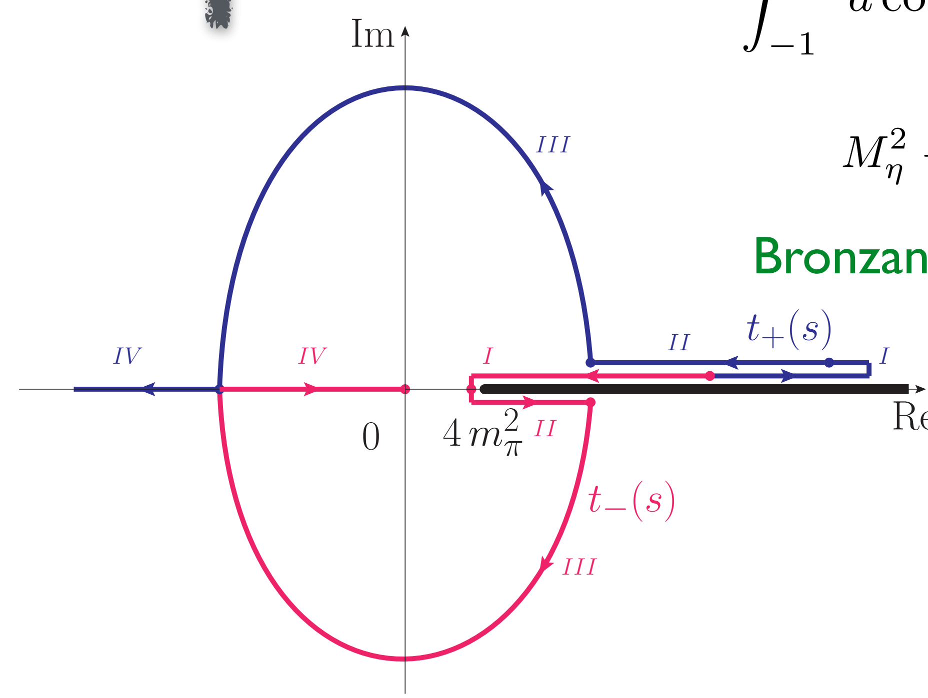
$$\mathrm{Disc}\, a_J(s) = t_J^*(s)\,\rho(s)\left(a_J(s) + \int_{-1}^{+1}\frac{d\cos\theta}{2}\ldots a_J(t)\right)$$

$$\int_{-1}^{+1} d\cos\theta \rightarrow \int_{t_-(s)}^{t_+(s)} dt$$

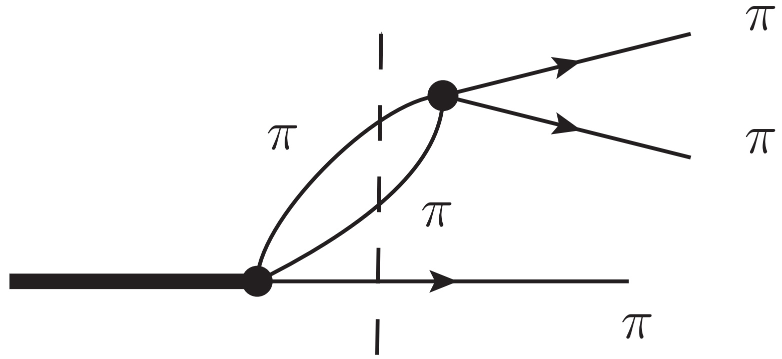
$$M_\eta^2 \rightarrow M_\eta^2 + i\,\epsilon$$

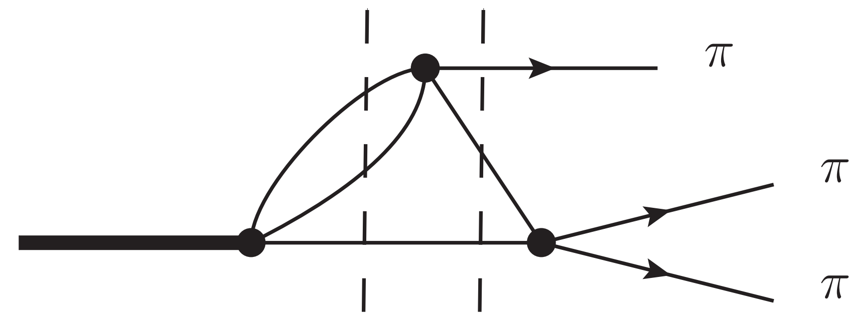
Bronzan, Kacser (1963)





Pasquier inversion





$$\mathrm{Disc}\, a_J(s) = t_J^*(s)\,\rho(s)\left(a_J(s) + \int_{-1}^{+1} \frac{d\cos\theta}{2} ... a_J(t)\right)$$

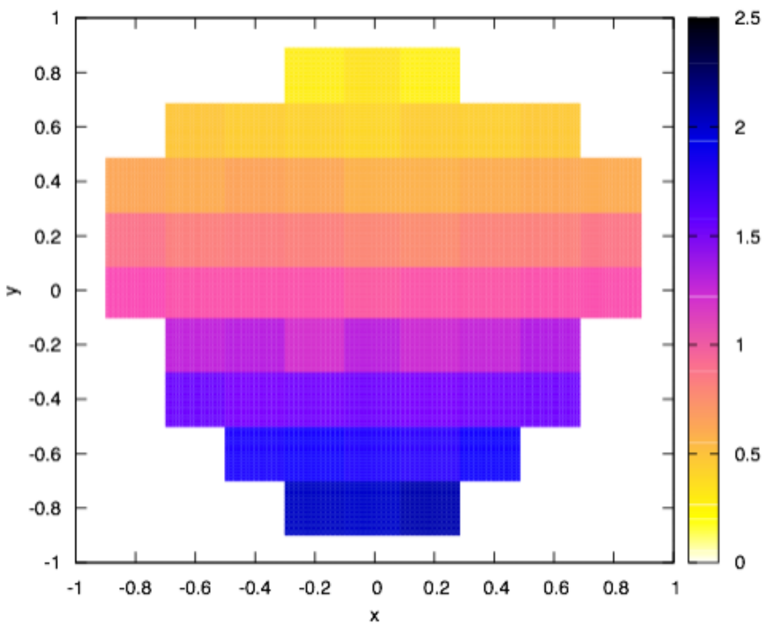
$$a_J(s) = \int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{\mathrm{Disc}\, a_J(s')}{s' - s}$$

$$\int_{-1}^{+1} d\cos\theta \rightarrow \int_{t_-(s)}^{t_+(s)} dt$$

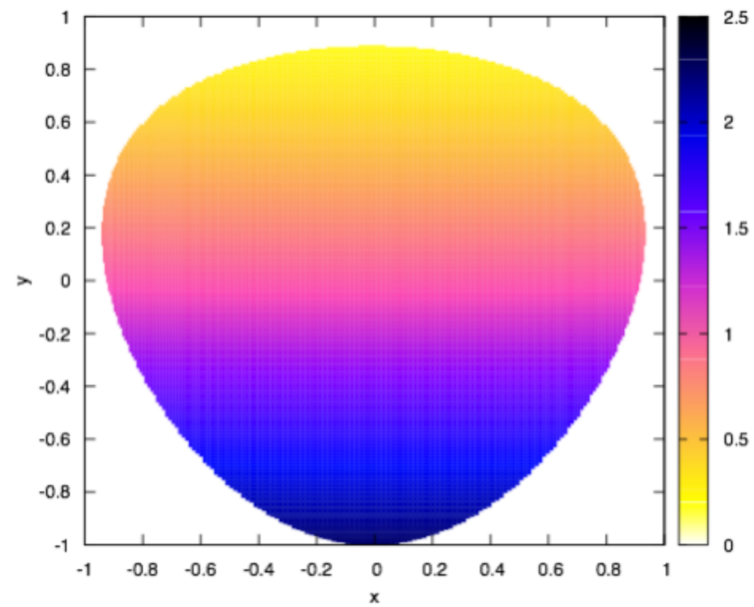
$$\int_{4\,m_\pi^2}^{\infty} \frac{ds'}{\pi} \int_{t_-(s)}^{t_+(s)} dt \; ... \quad \longrightarrow \quad = \int_{\Gamma'} dt \int_{s_-(t)}^{s_+(t)} ds' \, ...$$

Pasquier et al. (1968)
Guo, I.D., Szczepaniak (2015)

$\eta \to \pi^+\pi^-\pi^0$ (WASA-at-COSY fit)



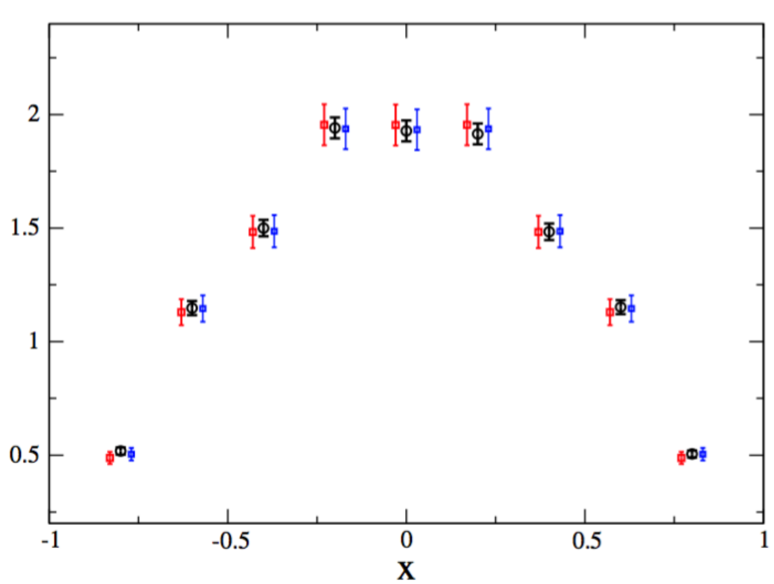
WASA-at-COSY (2014)

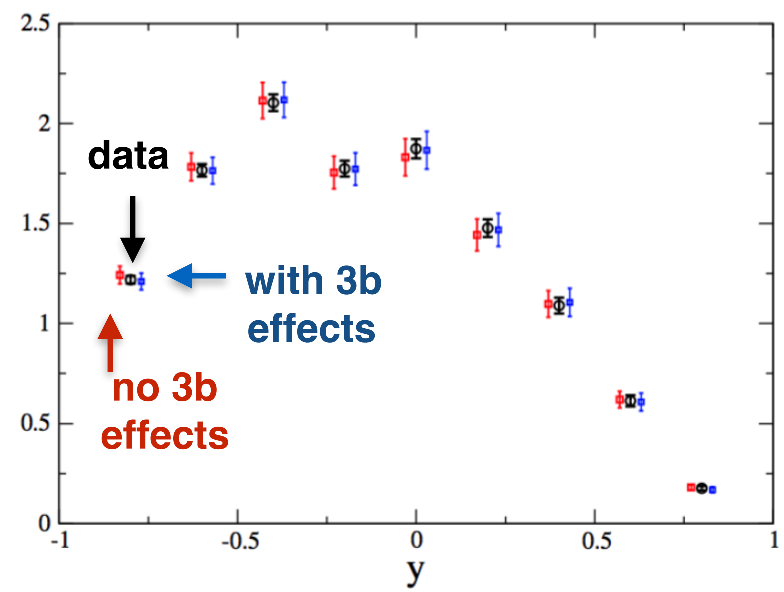


Guo et al. [JPAC] (2015)

Fit to WASA

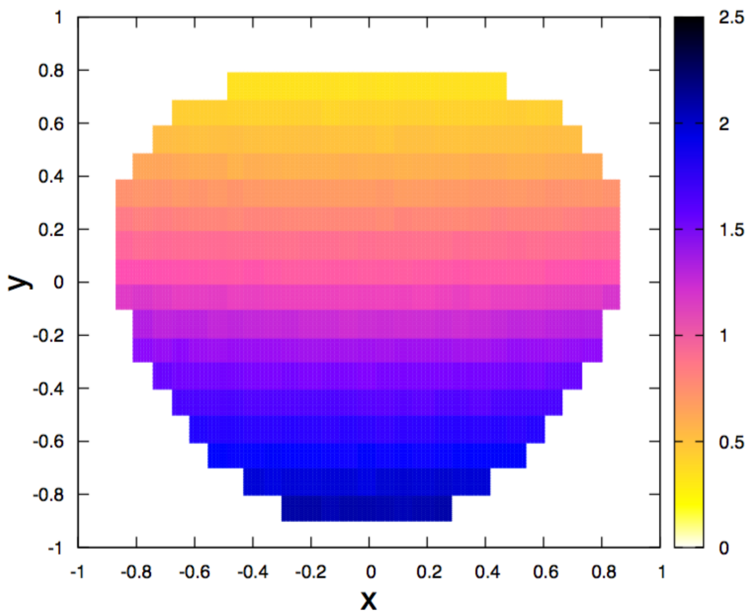
$\chi^2/d.o.f.$	*no 3b*	*with 3b*
(L,I)=(0,0), (1,1) 1 real par.	*1,45*	*0,95*
(L,I)=(0,0),(1,1), (0,2) 2 real par.	*0,94*	*0,90*

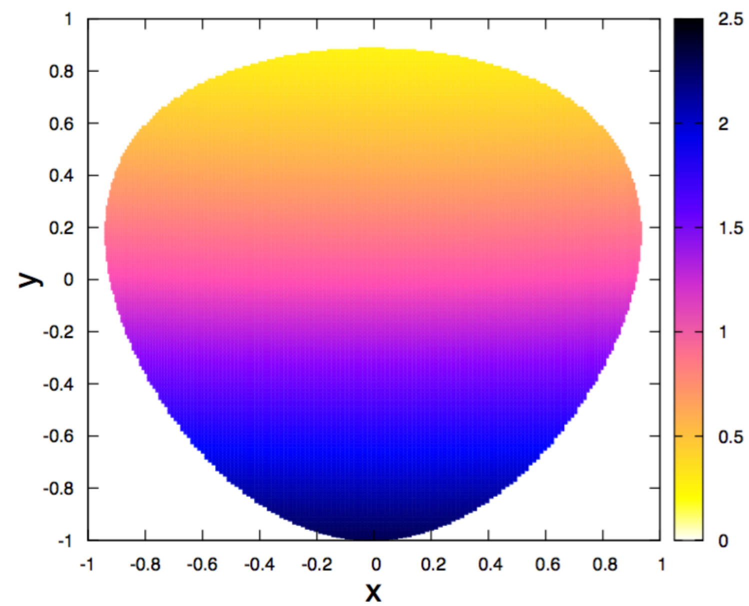




$$x \quad \propto \quad (t-u)$$
$$y \quad \propto \quad (s_c - s)$$

$\eta \to \pi^+\pi^-\pi^0$ (KLOE-2 fit)





KLOE-2 (2016)

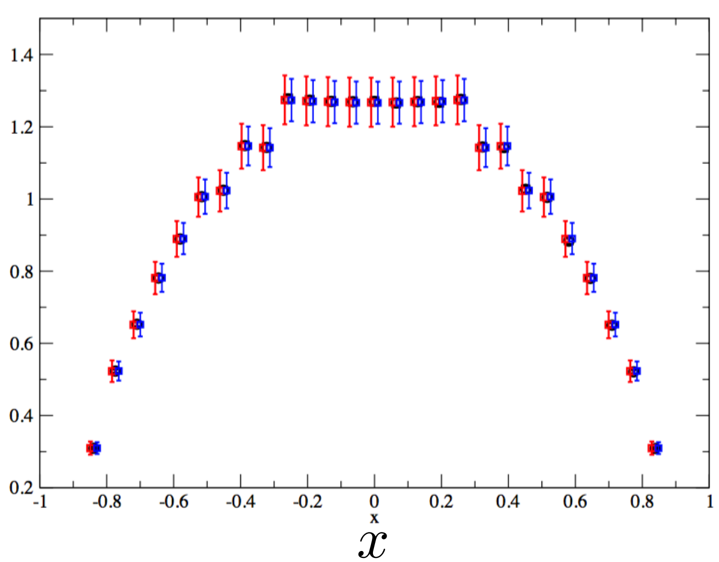
Guo et al. [JPAC] (2017)

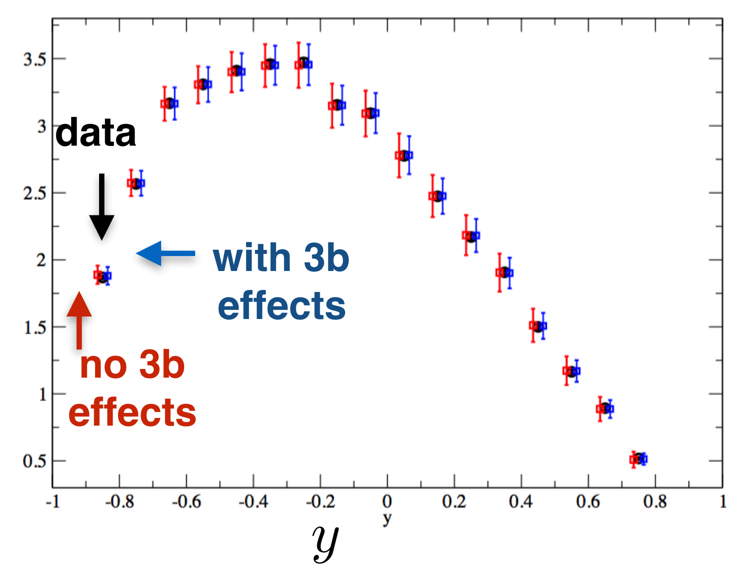
Fit to KLOE-2

$\chi^2/d.o.f.$	*no 3b*	*with 3b*
(L,I)=(0,0), (1,1) 1 real par.	*10,4*	*2,61*
(L,I)=(0,0),(1,1), (0,2) 2 real par.	*1,21*	*1,29*

Combined fit: WASA & KLOE-2

$\chi^2/d.o.f.$	*no 3b*	*with 3b*
(L,I)=(0,0), (1,1) 1 real par.	*9,5*	*1,64*
(L,I)=(0,0),(1,1), (0,2) 2 real par.	*1,54*	*1,61*





$$x \propto (t-u)$$
$$y \propto (s_c - s)$$

$\eta \to 3\pi^0$

- Dalitz plot expansion:

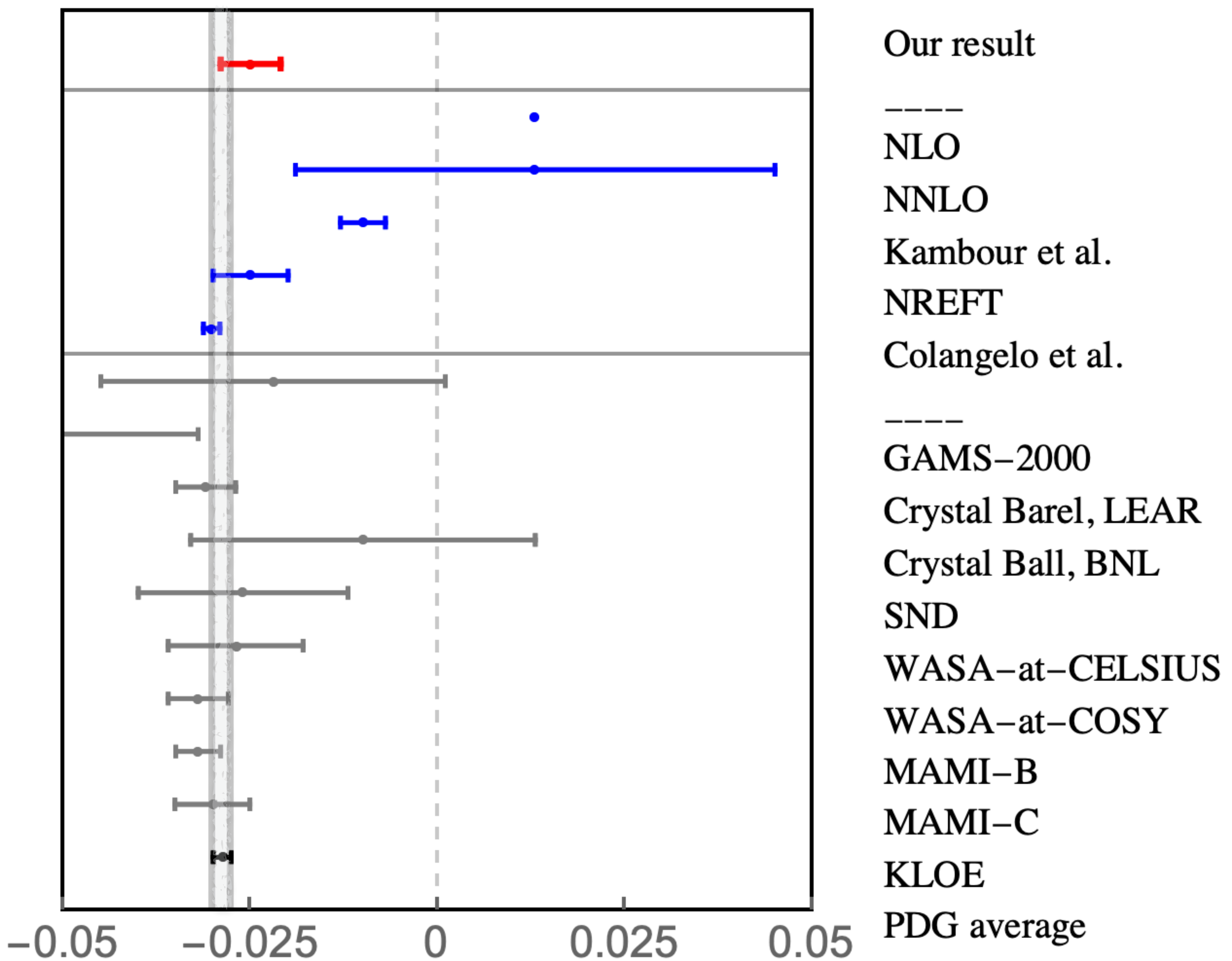
$$|A_{\eta\to\pi^+\pi^-\pi^0}|^2 \propto 1 + a\,y + b\,y^2 + d\,x^2 + f\,y^3 + \ldots$$

$$|A_{\eta\to 3\pi^0}|^2 \propto 1 + 2\alpha z + \ldots$$

$$x = \sqrt{z}\cos\phi$$
$$y = \sqrt{z}\sin\phi$$

- Prediction:

$$\begin{aligned} \alpha &= -0.025 \pm 0.004 \\ \alpha^{\mathrm{PDG}} &= -0.0288 \pm 0.0012 \end{aligned}$$



Matching to ChPT

We described Dalitz distribution normalised to unity at $x=y=0$

$$|A_{\eta\to\pi^+\pi^-\pi^0}|^2 \propto 1 + a\,y + b\,y^2 + d\,x^2 + f\,y^3 + \ldots$$

$$\Gamma^{exp}_{\eta\to\pi^+\pi^-\pi^0} \propto \int |A_{\eta\to\pi^+\pi^-\pi^0}|^2 \propto \boxed{\frac{N^2}{Q^4}} \qquad \frac{1}{Q^2} = \frac{m_d^2 - m_u^2}{m_s^2 - \hat{m}^2}$$

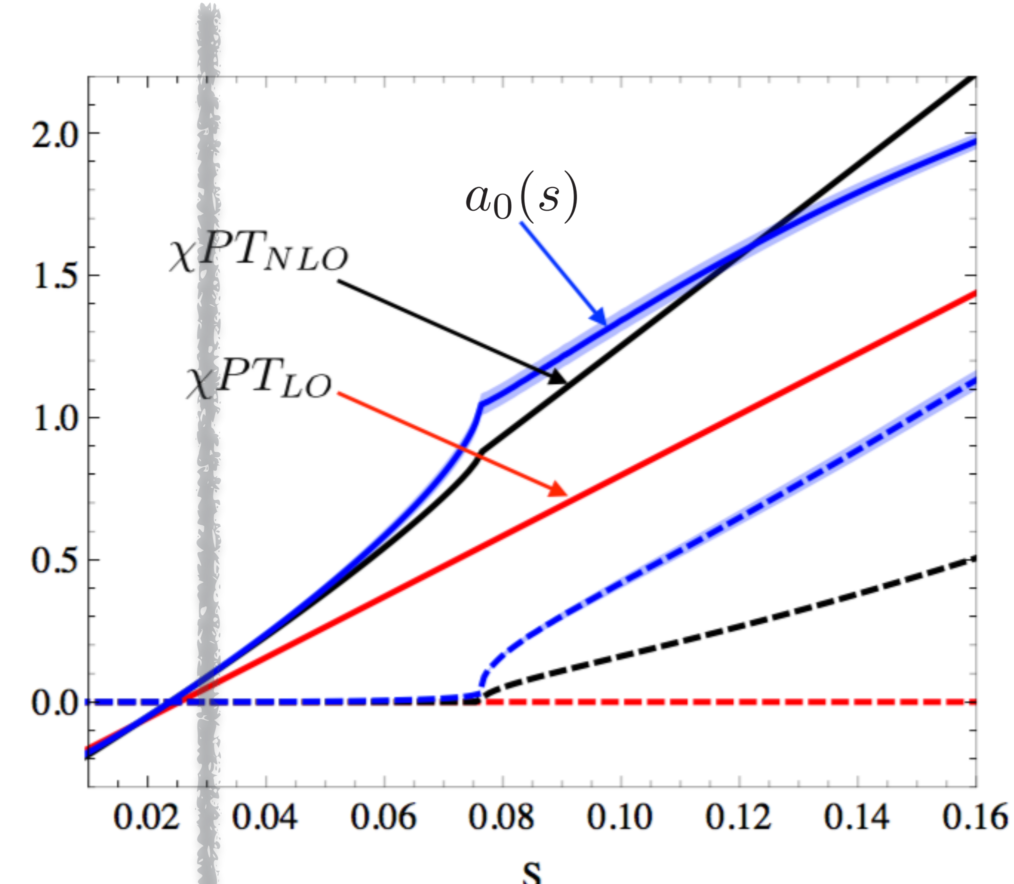
fix overall normalisation

Matching to ChPT

$$A(s,t,u) = \sum_J^{J_{max}} \boxed{\ldots a_J(s)} + \sum_J^{J_{max}} \ldots a_J(t) + \sum_J^{J_{max}} \ldots a_J(u)$$

$$A^{\chi PT}(s,t,u) = -\frac{1}{Q^2}\frac{m_K^2(m_K^2 - m_\pi^2)}{3\sqrt{3}\,m_\pi^2 f_\pi^2}\left(\sum_J^{J_{max}} \boxed{\ldots a_J^{\chi PT}(s)} + \ldots\right)$$

Match individual (I, J) components of the full amplitude near Adler zero $s=4/3\ m_\pi^2$



Matching point

Q-value predictions

Quark mass double ratio:

$$\frac{1}{Q^2} = \frac{m_d^2 - m_u^2}{m_s^2 - \hat{m}^2}$$

	Q
Our result (fit to WASA@COSY)	21.4 ± 1.1
Our result (fit to KLOE-2)	21.7 ± 1.1
Our result (combined fit)	21.6 ± 1.1
Lattice, FLAG, 2016 ($N_f = 2+1$)	22.5 ± 0.8
Lattice, FLAG, 2016 ($N_f = 2+1+1$)	22.2 ± 1.6
NLO	20.1
NNLO	22.9
Dispersive (Kambor *et al.*)	22.4 ± 0.9
Dispersive (Kampf *et al.*)	23.1 ± 0.7
Dispersive (Colangelo *et al.*)	22.0 ± 0.7

Quark masses

$$\hat{m} = 3.42 \pm 0.09 \text{ MeV} \qquad m_s = 93.8 \pm 0.24 \text{ MeV}$$

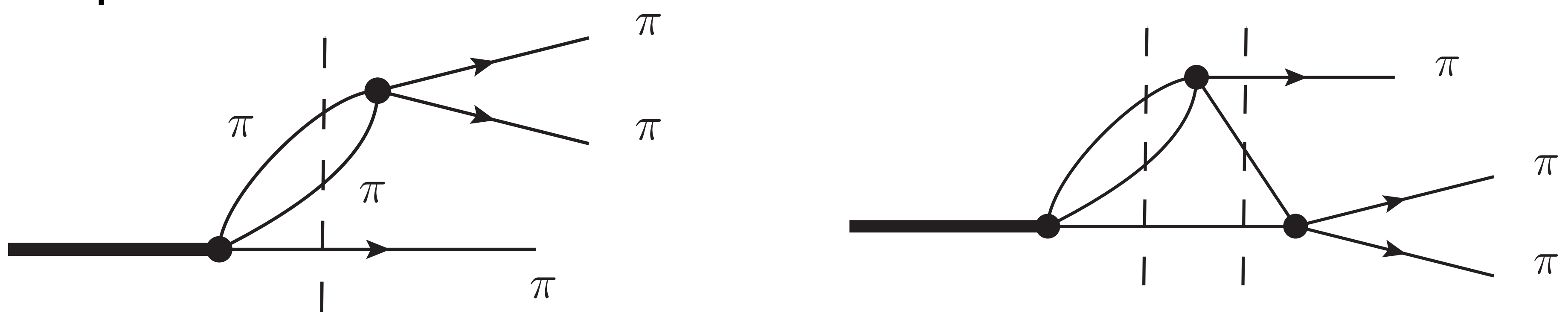
$$\longrightarrow$$

$$m_u = 2.04 \pm 0.14 \text{ MeV} \qquad m_d = 4.80 \pm 0.08 \text{ MeV}$$

Lattice, FLAG, (Nf=2+1), 2014

ω, φ→3π

ω/φ is spin 1 particle:



$$\mathrm{Disc}\, a_J(s) = t_J^*(s)\,\rho(s)\left(a_J(s) + \int_{-1}^{+1}\frac{d\cos\theta}{2}\ldots a_J(t)\right)$$

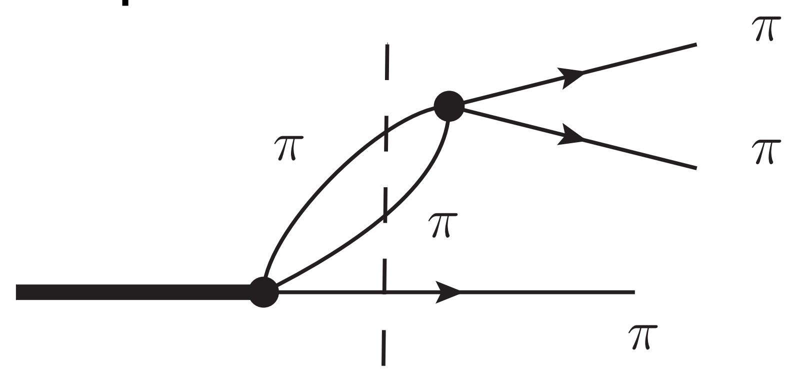
$$a_J(s) = \int_{4m^2}^{\infty}\frac{ds'}{\pi}\frac{\mathrm{Disc}\, a_J(s')}{s'-s}$$

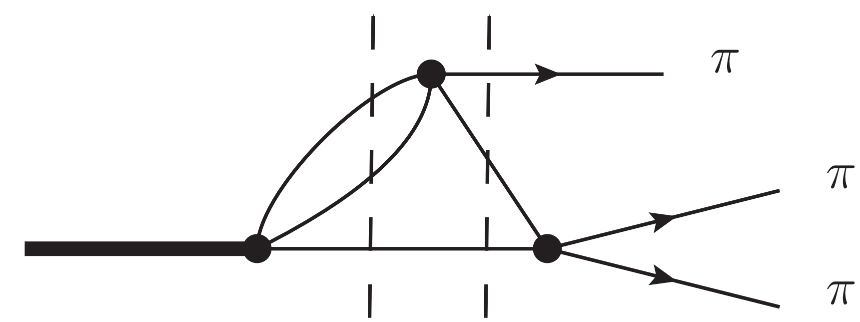
Over subtraction technique (suppress high energy input)

$$a_J(s) = \alpha + \beta\, s + \frac{s^2}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{s'^2}\frac{\mathrm{Disc}\, a_J(s')}{s'-s}\,, \qquad \beta = \int_{4m^2}^{\infty}\frac{ds'}{\pi}\frac{\mathrm{Disc}\, a_J(s')}{s'^2}$$

Niecknig at al. (2012)

ω/φ is spin 1 particle:



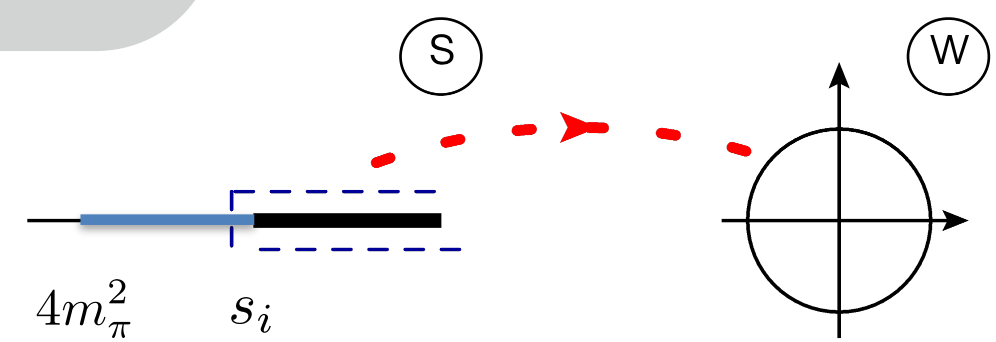


$$\mathrm{Disc}\, a_J(s) = t_J^*(s)\,\rho(s)\left(a_J(s) + \int_{-1}^{+1} \frac{d\cos\theta}{2} ... a_J(t)\right)$$

$$a_J(s) = \int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{\mathrm{Disc}\, a_J(s')}{s' - s}$$

inelastic contributions parametrise with a conformal mapping expansion $\sum_{i=0}^{N} C_i\, \omega(s)^i$

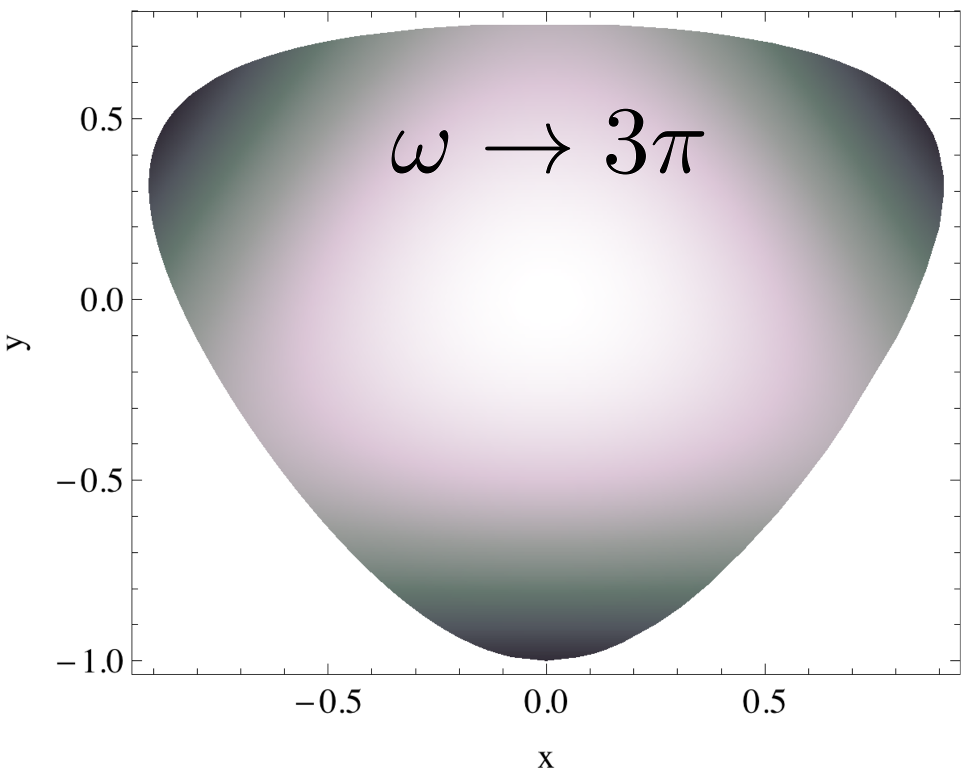
$$= \int_{4m_\pi^2}^{s_i} ... + \int_{s_i}^{\infty} ...$$



Coefficients C_i play the role of subtraction constants in conventional approach

Danilkin at al. (JPAC) (2015)

ω→3π



$$\frac{d^2\Gamma}{ds\,dt} \propto |\vec{p}_+ \times \vec{p}_-|^2 |F(s,t)|^2$$

$$x \propto (t-u)$$
$$y \propto (s_c - s)$$

Dalitz plot parameters

$$|F(s,t)|^2 \simeq |N|^2 \big(1 + 2\,\alpha\, z + 2\,\beta\, z^{3/2} \sin(3\phi) + 2\,\gamma\, z^2 + 2\,\delta\, z^{5/2} \sin(3\phi) + \mathcal{O}(z^3)\big)$$

ω→3π: fit event by event g12 CLAS data in progress

Carlos Salgado, Volker Crede, etc.

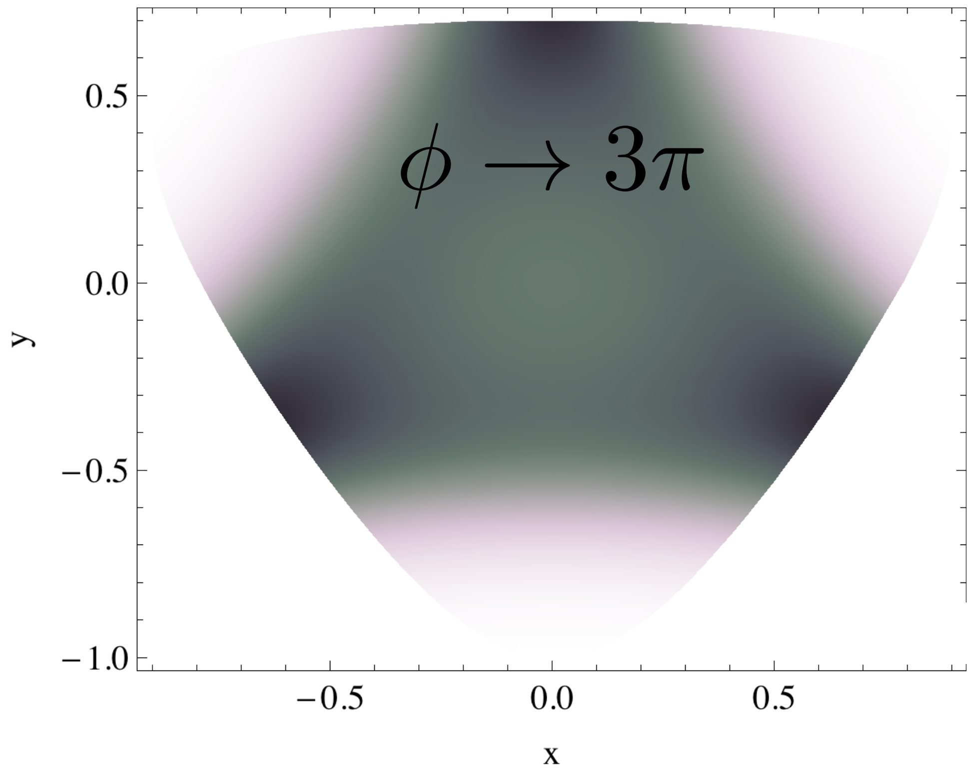
WASA-at-COSY (2016)

$\alpha \times 10^3$	$\beta \times 10^3$	χ^2/d.o.f.
–	–	90.6 / 60
147(36)	–	71.5 / 59
133(41)	37(54)	71.0 / 58

BESIII (2018)

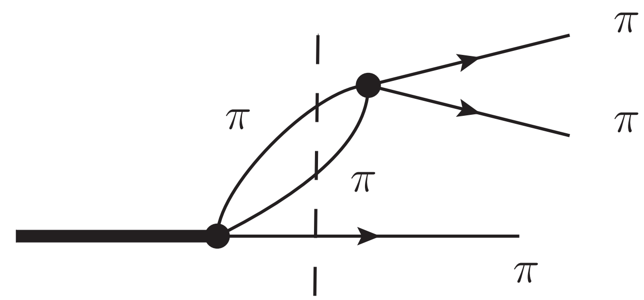
	Para. $\times 10^3$	Theoretical Predictions					Experiment
		Ref. [4]		Ref. [5]		Ref. [19]	BESIII
		w/o	**w**	**w/o**	**w**		
Fit I	α	136	94	(137, 148)	(84, 96)	202	$132.1 \pm 6.7 \pm 4.6$
Fit II	α	125	84	(125, 135)	(74, 84)	190	$120.2 \pm 7.1 \pm 3.8$
	β	30	28	(29, 33)	(24, 28)	54	$29.5 \pm 8.0 \pm 5.3$

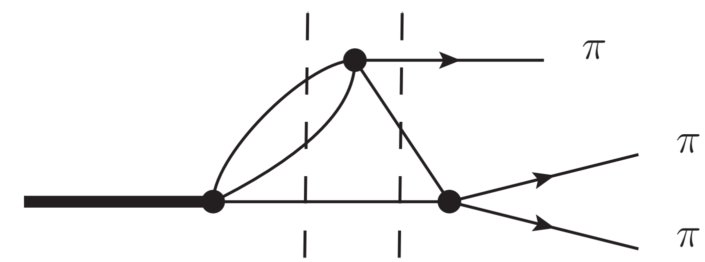
$\varphi \to 3\pi$

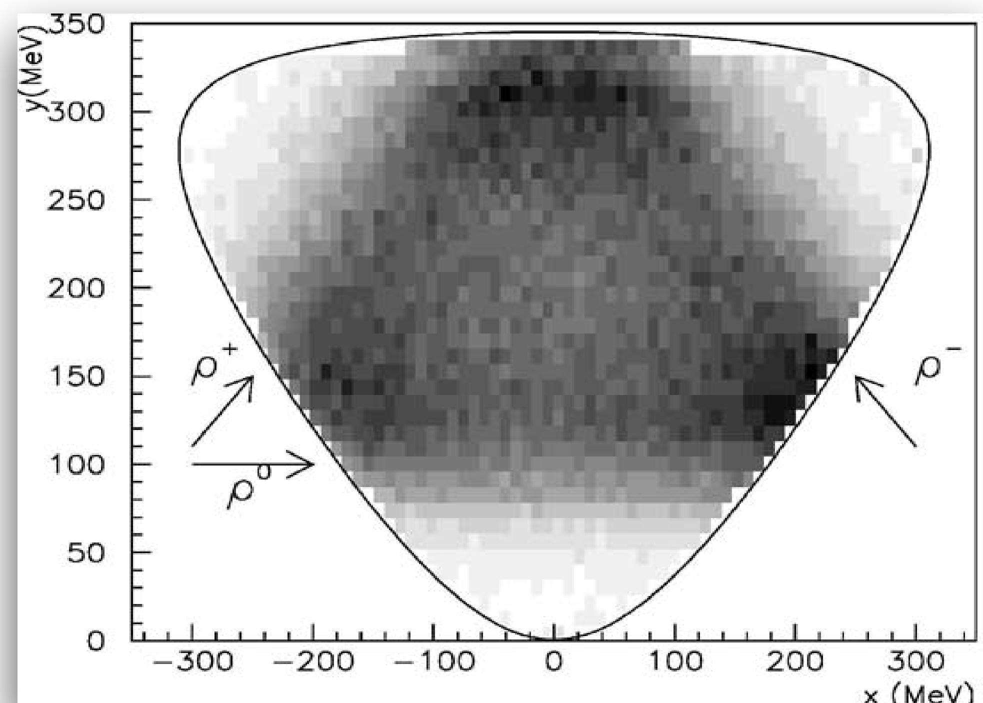


$$\frac{d^2\Gamma}{ds\,dt} \propto |\vec{p}_+ \times \vec{p}_-|^2 |F(s,t)|^2$$

$$x \propto (t-u)$$
$$y \propto (s_c - s)$$







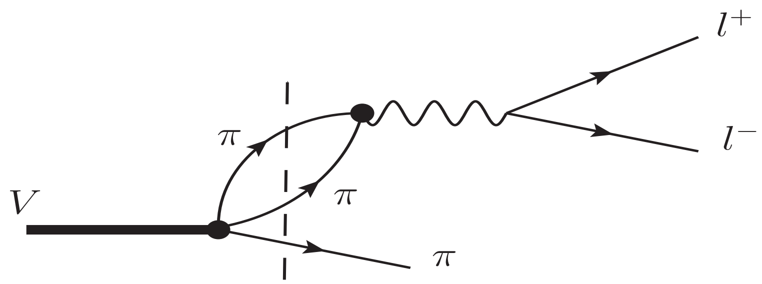
χ^2/ndof	$1.7\ldots2.1$	$1.2\ldots1.5$	1.0

Niecknig at al. (2012)

I.D. & JPAC (2019)

KLOE (2003)

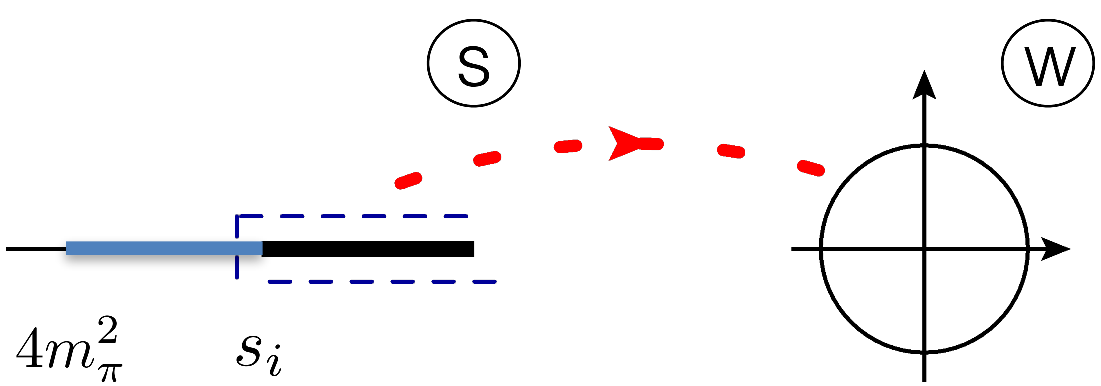
Discontinuity relation: $\omega/\varphi \to \pi^0\gamma^*$

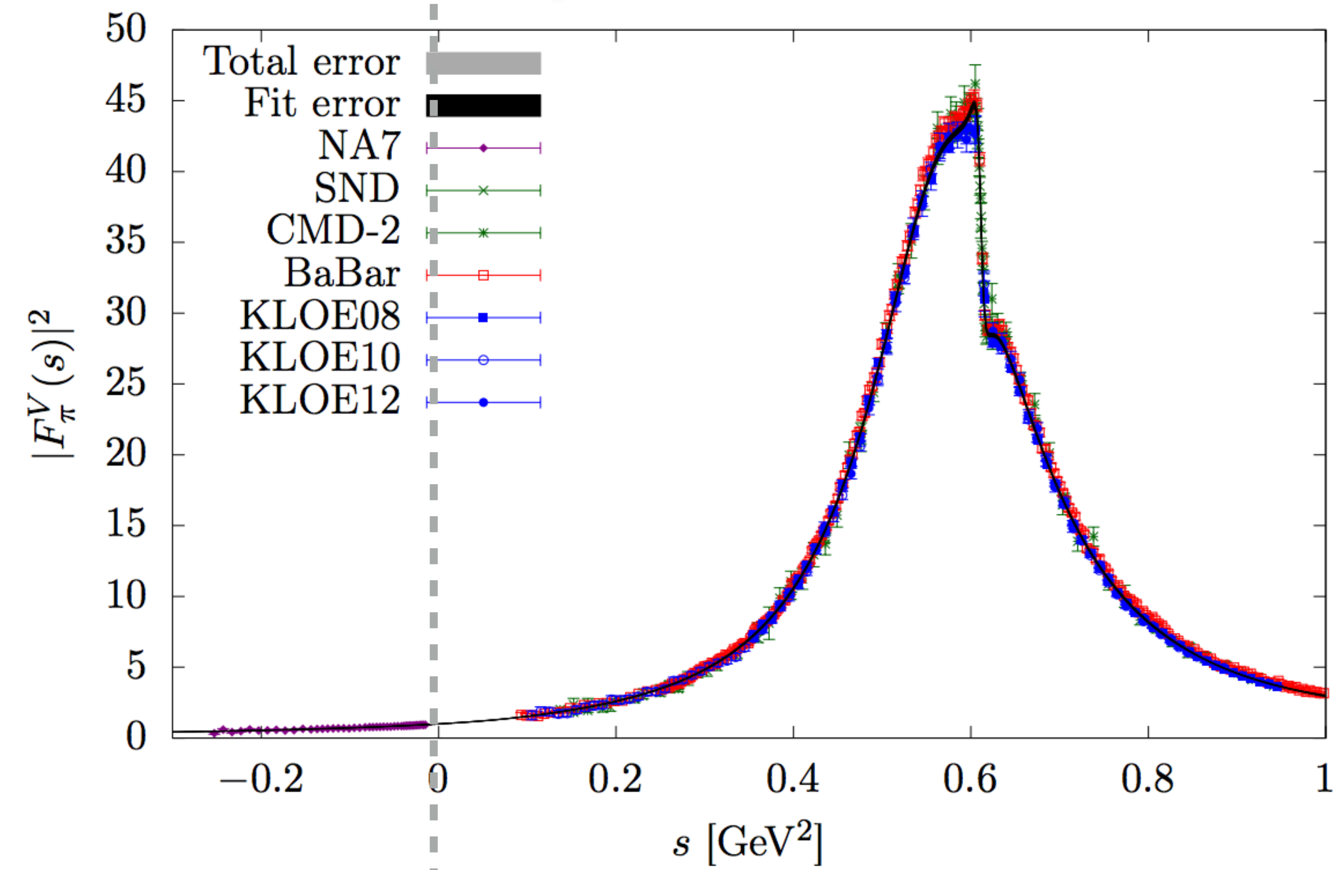


pion vector form factor $\omega/\varphi \to 3\pi$

$$\mathrm{Disc}\, f_{V\pi}(s) = \frac{\rho^3(s)\, s}{128\,\pi}\, \overbrace{F_\pi^V(s)^*}^{\text{pion vector form factor}} \int_{-1}^{1} dz'(1-z'^2)\, \overbrace{F(s,t',u')}^{\omega/\varphi\to 3\pi}$$

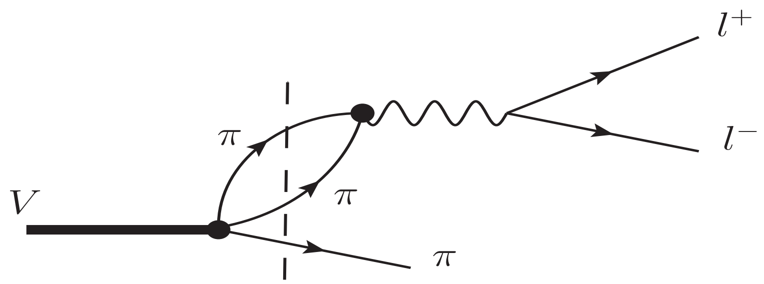
$$f_{V\pi}(s) = \int_{4m_\pi^2}^{s_i} \frac{ds'}{\pi} \frac{\mathrm{Disc}\, f_{V\pi}(s')}{s'-s} + \sum_{k=0}^{N} b_k\,(\omega(s))^k$$





Colangelo et al. (2019)

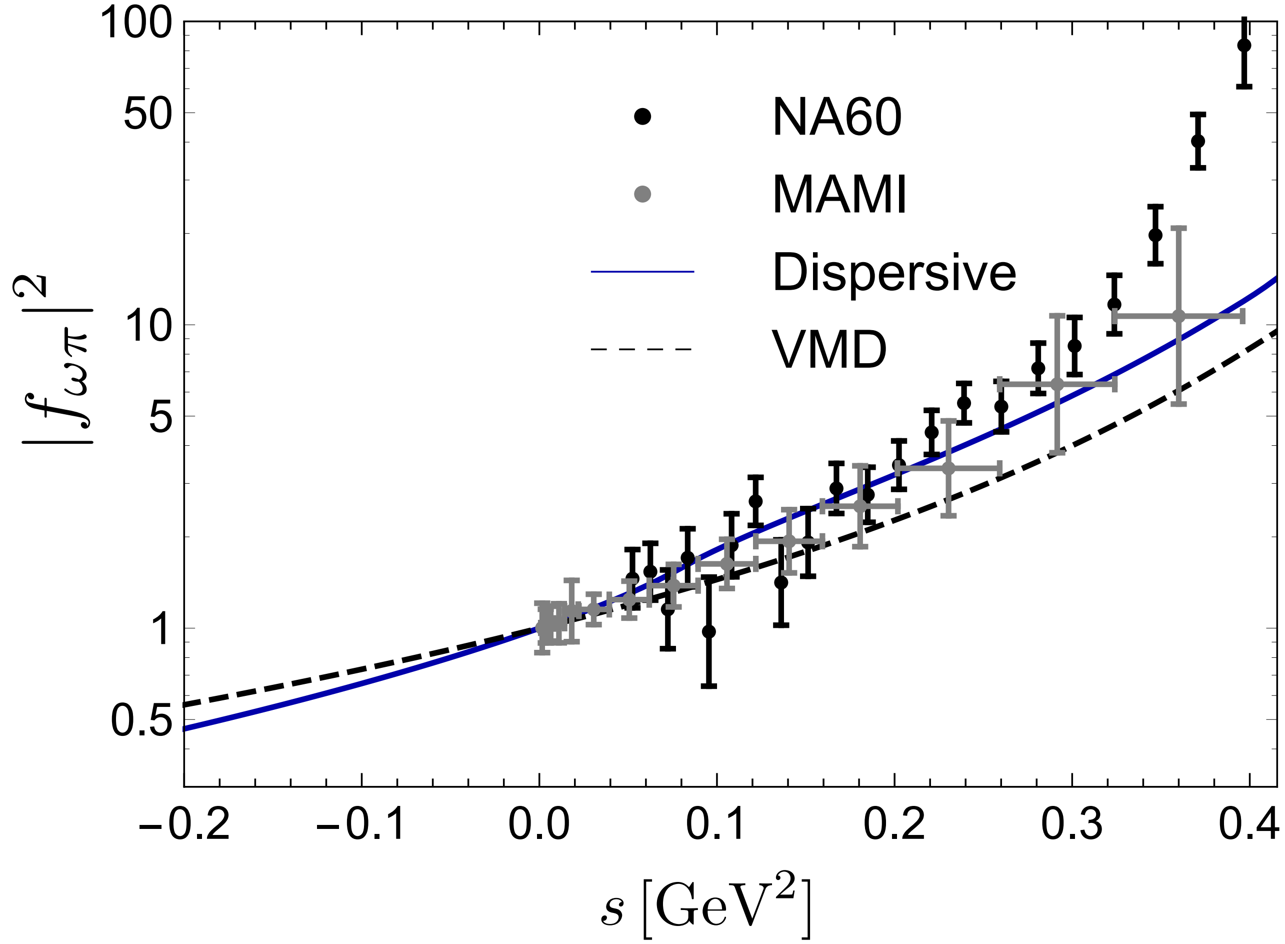
ω→π⁰γ*



pion vector form factor $\qquad$ $\omega/\varphi\to 3\pi$

$$\mathrm{Disc}\, f_{V\pi}(s) = \frac{\rho^3(s)\, s}{128\,\pi}\, F_\pi^V(s)^* \int_{-1}^{1} dz'(1-z'^2)\, F(s,t',u')$$

$$f_{V\pi}(s) = \int_{4m_\pi^2}^{s_i} \frac{ds'}{\pi} \frac{\mathrm{Disc}\, f_{V\pi}(s')}{s'-s} + \sum_{k=0}^{N} b_k\,(\omega(s))^k$$



b_0 fixed from $\Gamma_{exp}(\omega\to\pi\gamma)$

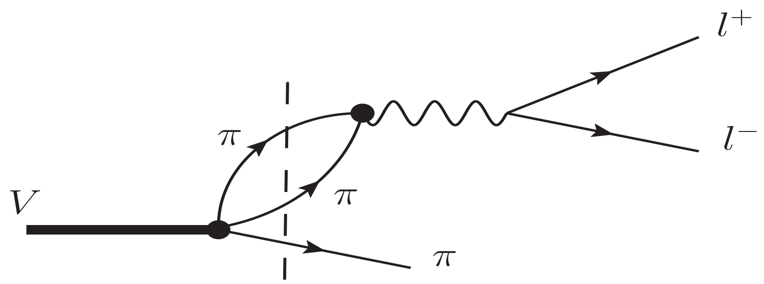
NA60: Nature of the steep rise?

Exp. analysis of $\varphi\to\pi\gamma$ is very important

Schneider et al. (2012)

Danilkin at al. (JPAC) (2015)

$\varphi \to \pi^0 \gamma^*$



pion vector form factor $\omega/\varphi \to 3\pi$

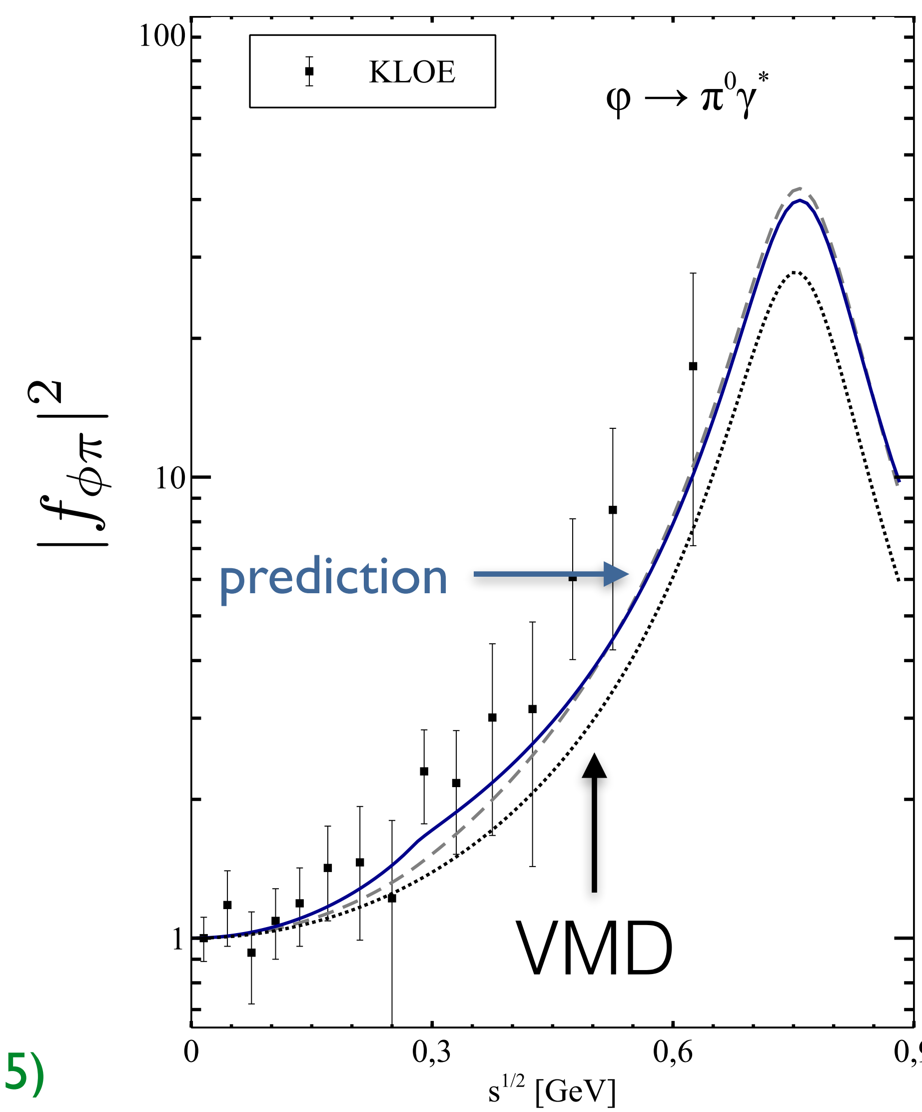
$$\text{Disc}\, f_{V\pi}(s) = \frac{\rho^3(s)\, s}{128\,\pi} F_\pi^V(s)^* \int_{-1}^{1} dz'(1-z'^2)\, F(s,t',u')$$

$$f_{V\pi}(s) = \int_{4m_\pi^2}^{s_i} \frac{ds'}{\pi} \frac{\text{Disc}\, f_{V\pi}(s')}{s'-s} + \sum_{k=0}^{N} b_k\, (\omega(s))^k$$

b_0 fixed from $\Gamma_{exp}(\varphi \to \pi\gamma)$

Grey: no 3b effects

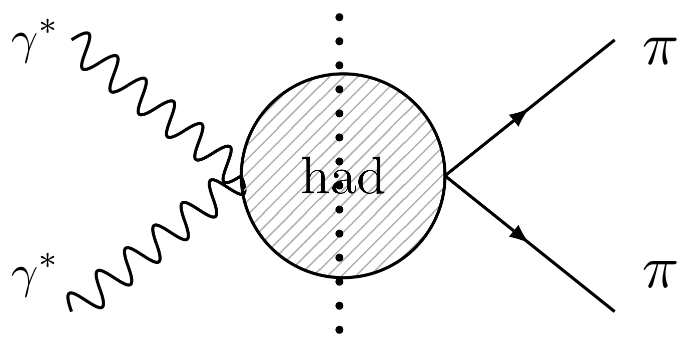
Our prediction [2014] is consistent with new **KLOE** data [2016]

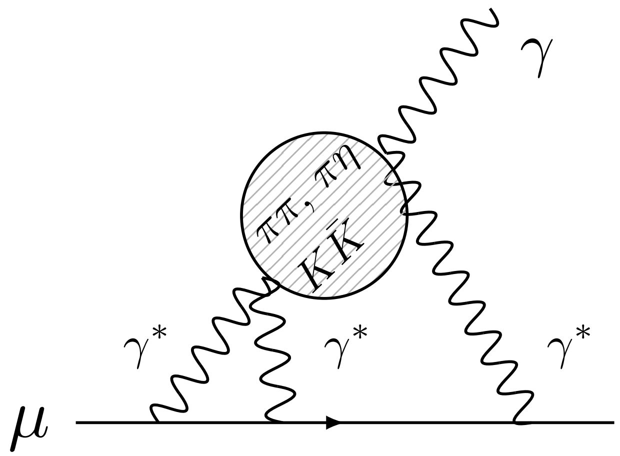


Schneider et al. (2012)
Danilkin at al. (JPAC) (2015)

Applications

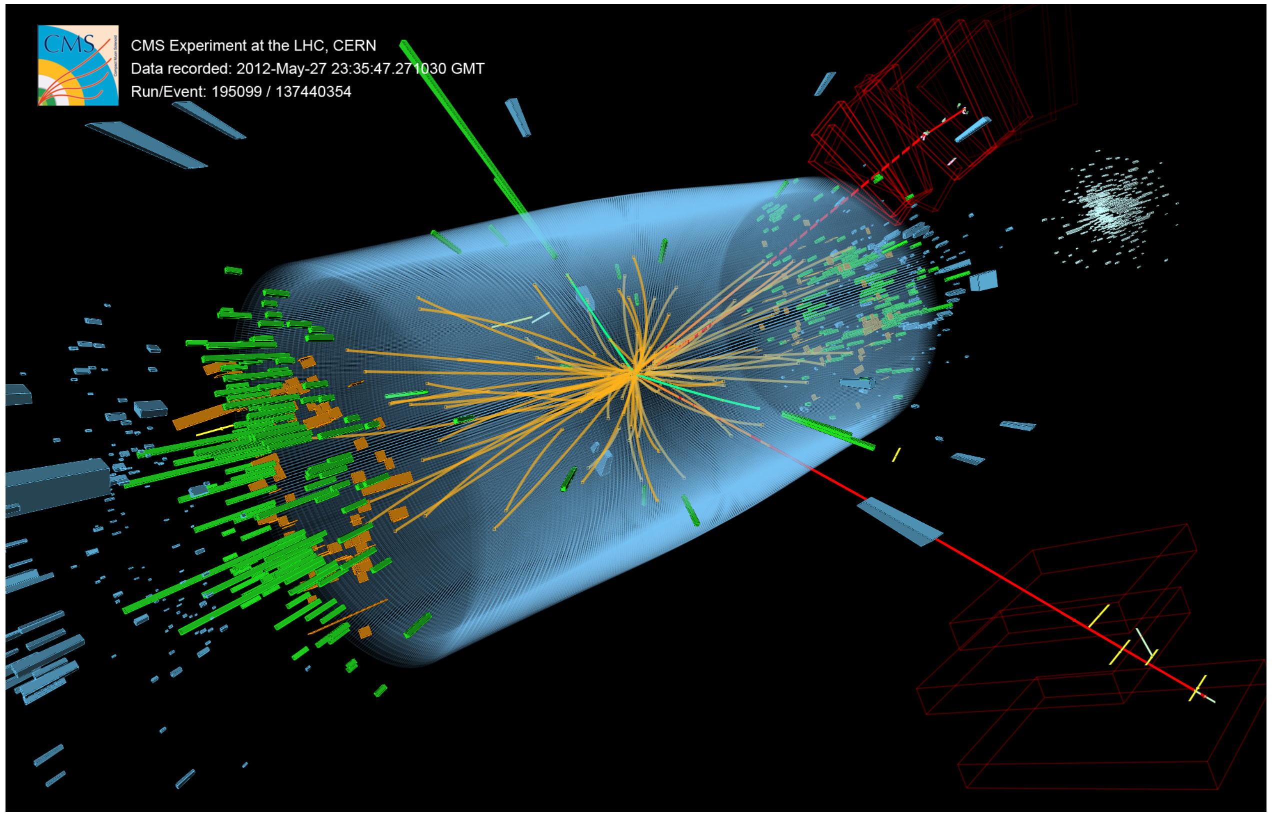
- Photon fusion reactions and (g-2)





Exploring the frontier of knowledge

- Collide high energy beams

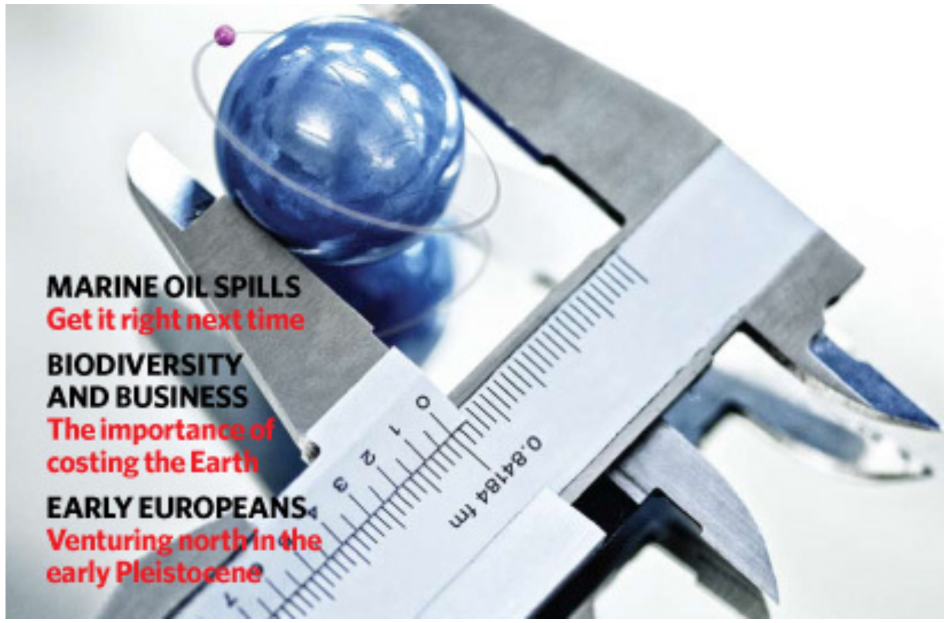


- Astroparticle physics/ cosmology

- Ultra-precise predictions
  vs measurements:

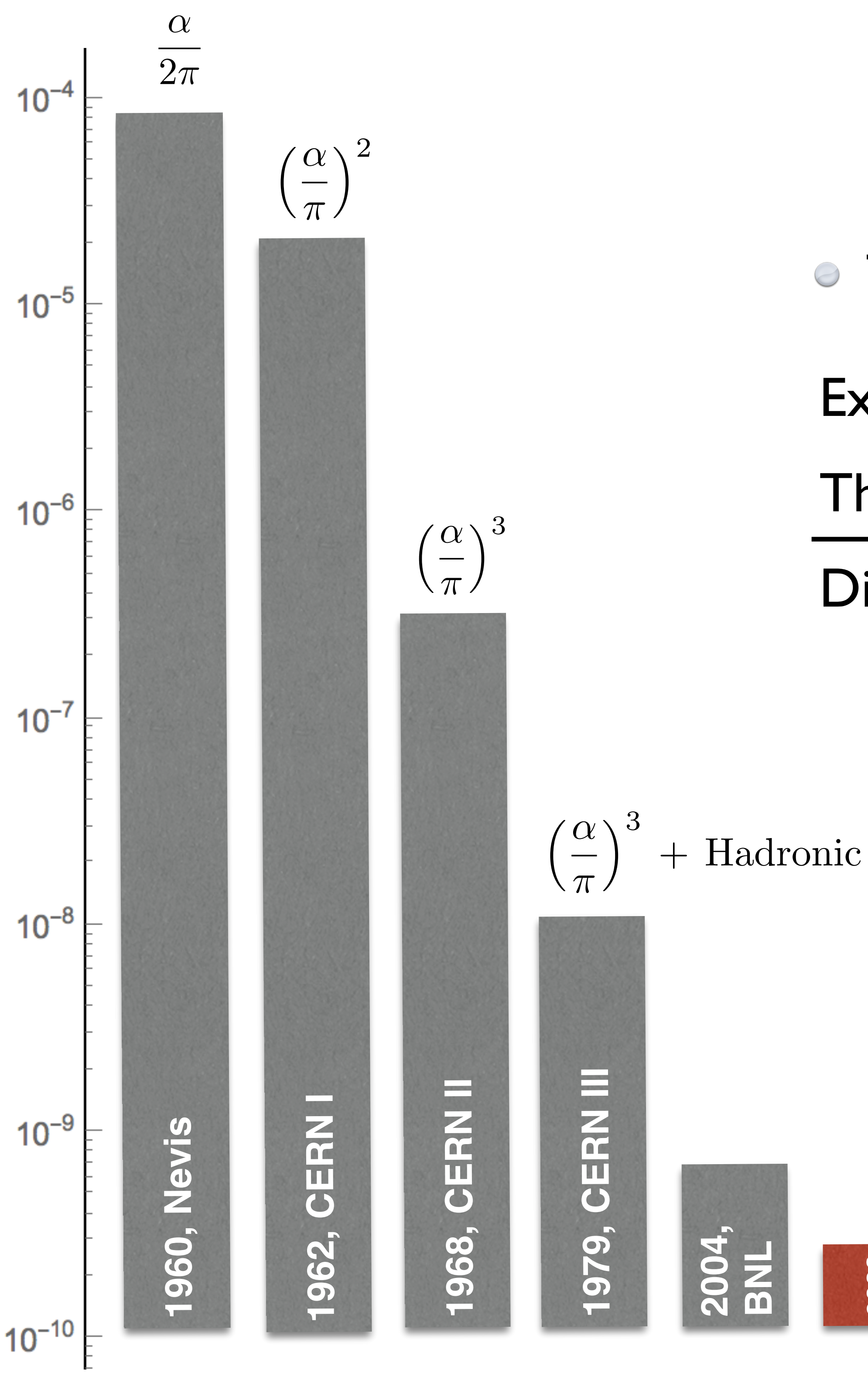
  the more precise the comparison,
  the more subtle the theory

  Disagreement: we might be talking
  about a **discovery**



Motivation

- Accuracy



- Magnetic moment of the muon

$$\vec{\mu} = \frac{Q}{2m}\, g\, \vec{S}$$

- Anomalous part

$$a_\mu = \frac{(g-2)_\mu}{2}$$

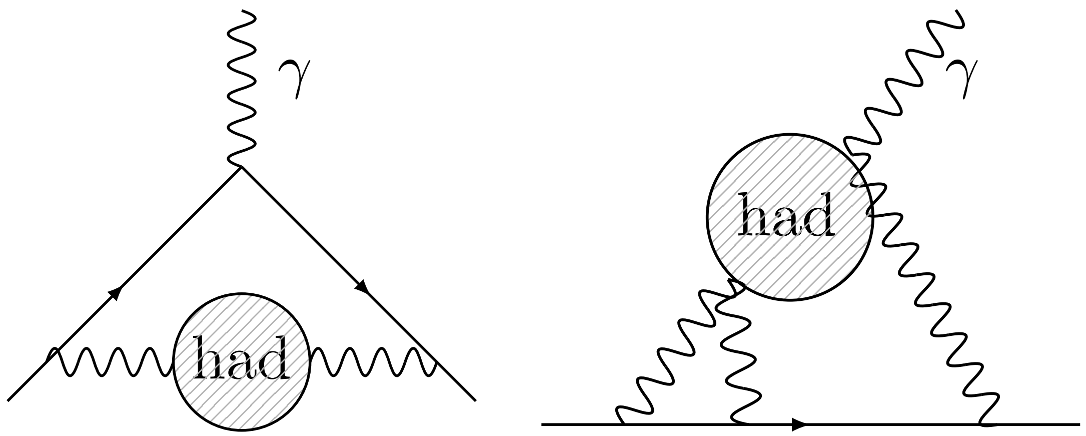
- Today

Experiment (BNL 2004)	$a_\mu^{\mathrm{exp}} = 0.0011659209(6)$
Theory (Standard model)	$a_\mu^{\mathrm{th}} = 0.0011659182(4)$
Difference	$\sim (3-4)\,\sigma$

Keshavarzi et al. (2018)

$$\Delta a_\mu^{\mathrm{exp}} = 6.3 \times 10^{-10}$$
$$\to 1.6 \times 10^{-10}$$

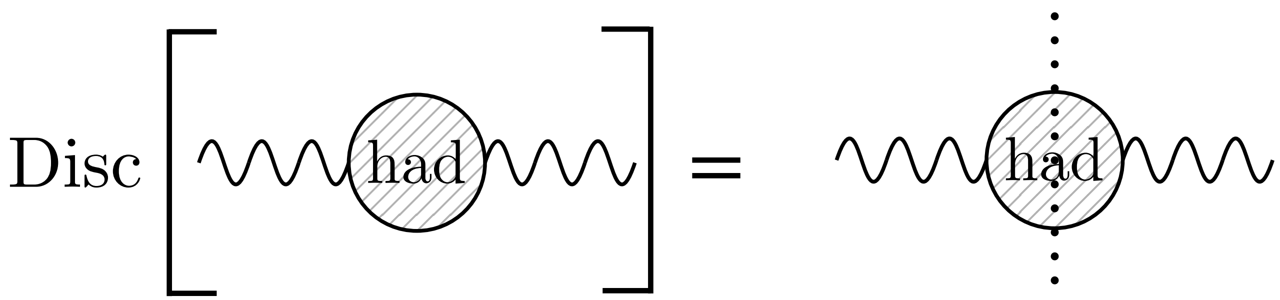
QCD contribution to (g-2)



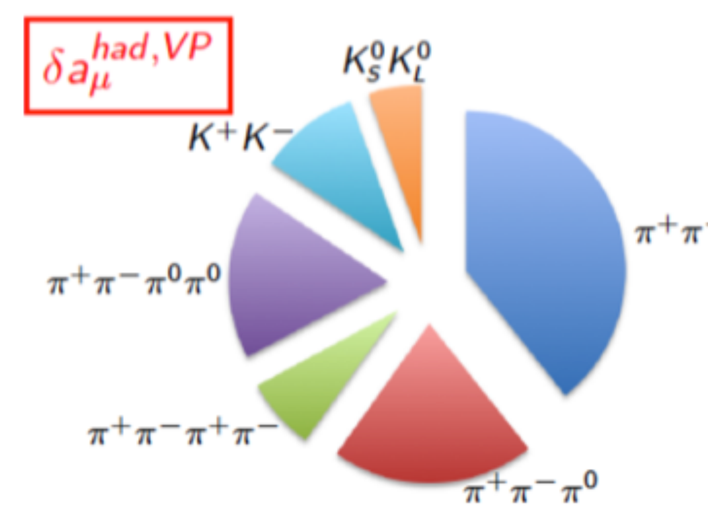
$$a_\mu^{\text{had, VP}} = 685.1(4.3) \times 10^{-10}$$

$$a_\mu^{\text{had, LbL}} = 10.5(2.6) \times 10^{-10}$$
$$= 10.2(3.9) \times 10^{-10}$$

- **Dispersion theory**: method that relies on unitarity and analyticity (model independent)

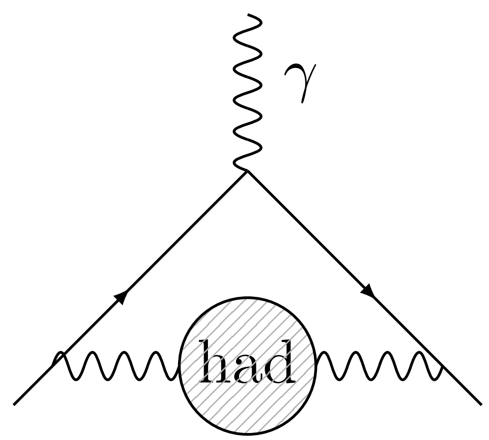


$$a_\mu^{\text{had, VP, LO}} = \frac{1}{4\pi^3} \int_{s_{\text{thr}}}^{\infty} ds\, K(s)\, \sigma_{e^+e^-\to\text{had}}$$



- Energy range up to 3 GeV is essential: ongoing ISR analyses BESIII
- Aim: reduction of current error by factor of 2

QCD contribution to (g-2)



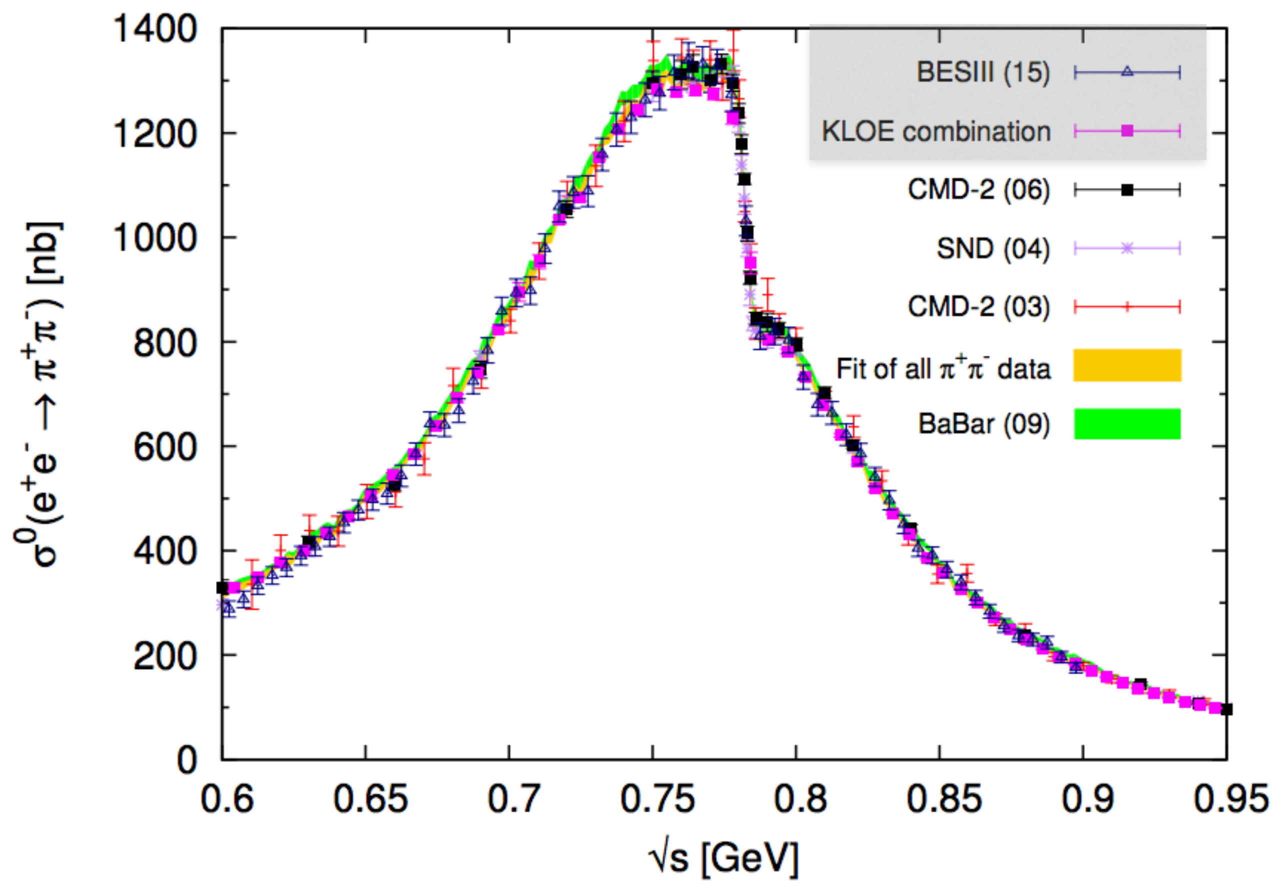
$$a_\mu^{\text{had, VP}} = 685.1\boxed{(4.3)} \times 10^{-10}$$

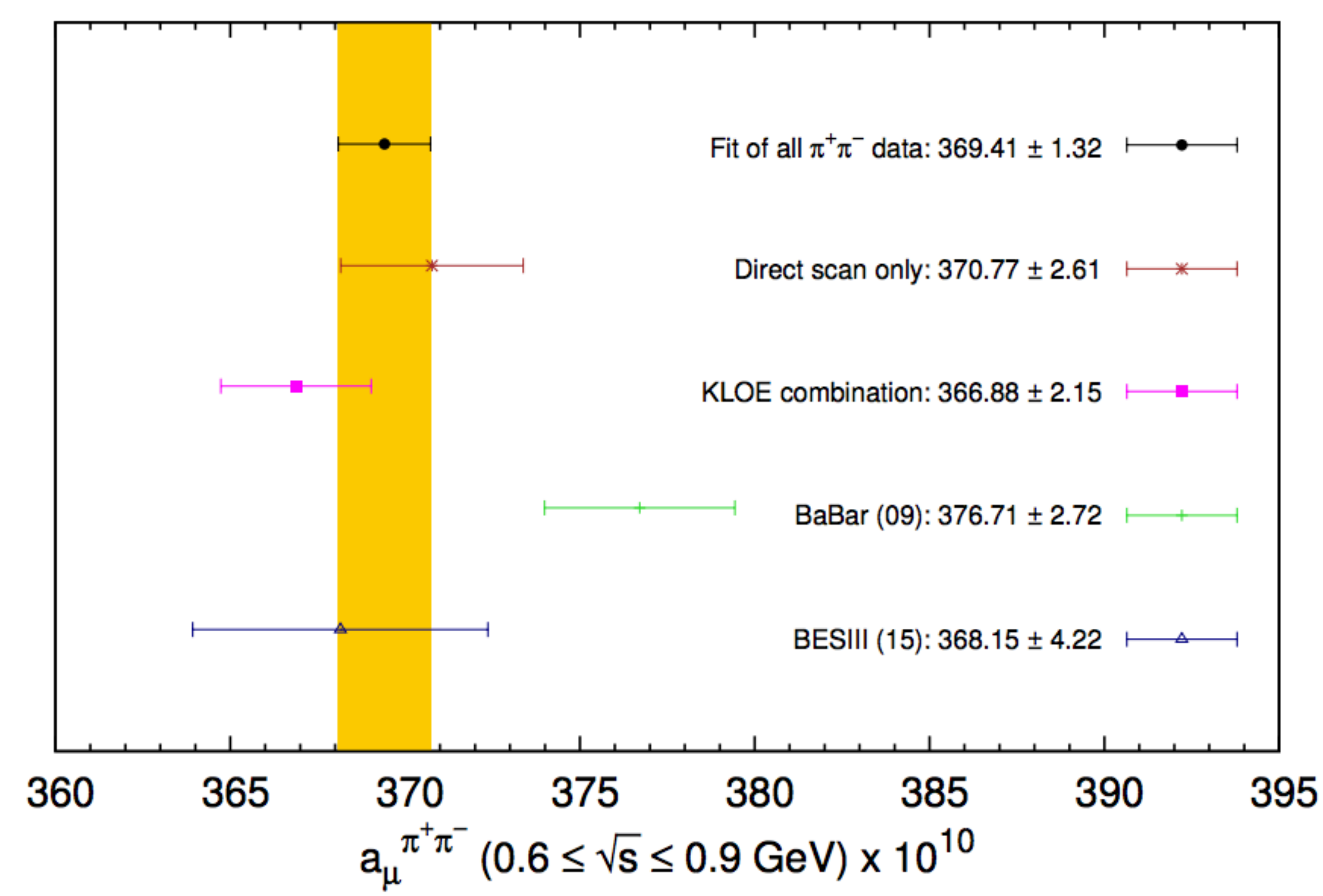
Hagiwara et al. (2011)

$$a_\mu^{\text{had, VP}} = 683.4\boxed{(2.5)} \times 10^{-10}$$

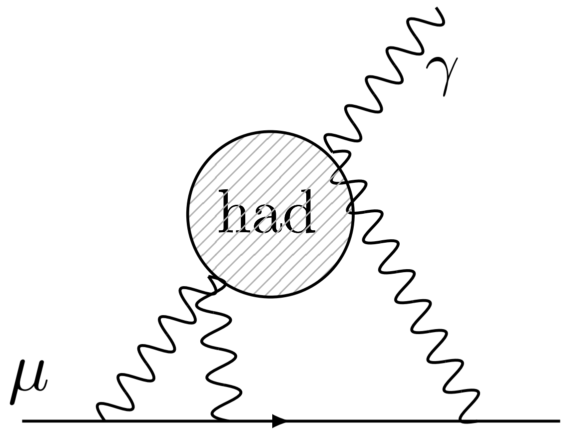
Keshavarzi et al. (2018)

- **Dispersion theory**: method that relies on unitarity and analyticity (model independent)

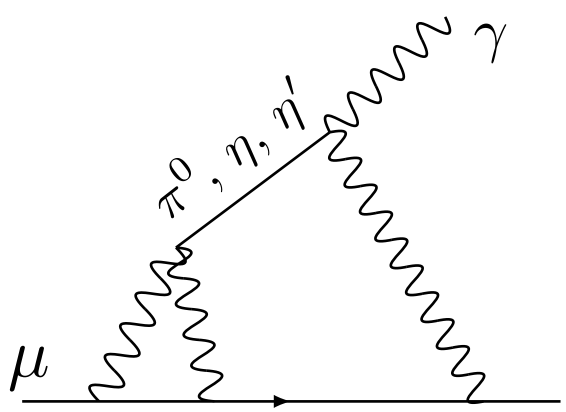




QCD contribution to (g-2)

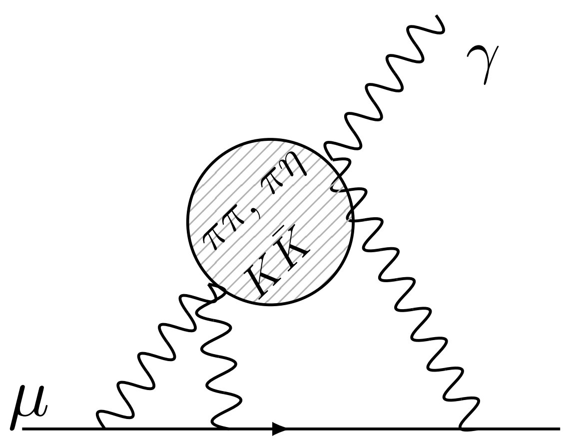


$=$



$+$

Relies on measurements of **TFF** $\pi^0\gamma^*\gamma^{(*)}, \eta\gamma^*\gamma^{(*)}, \ldots$ to reduce the model dependence



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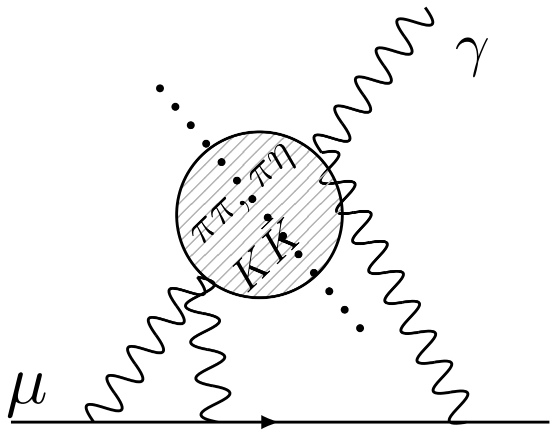
Dispersive analysis for ππ, πη, … loops is needed

…

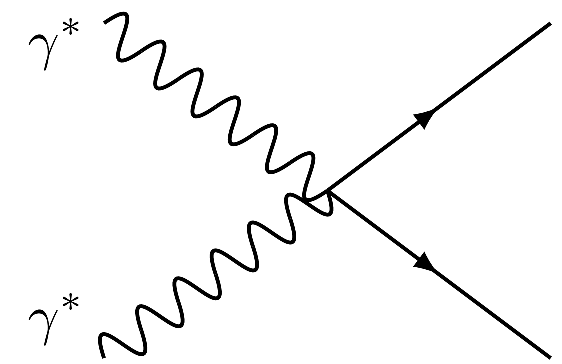
Multi-meson production

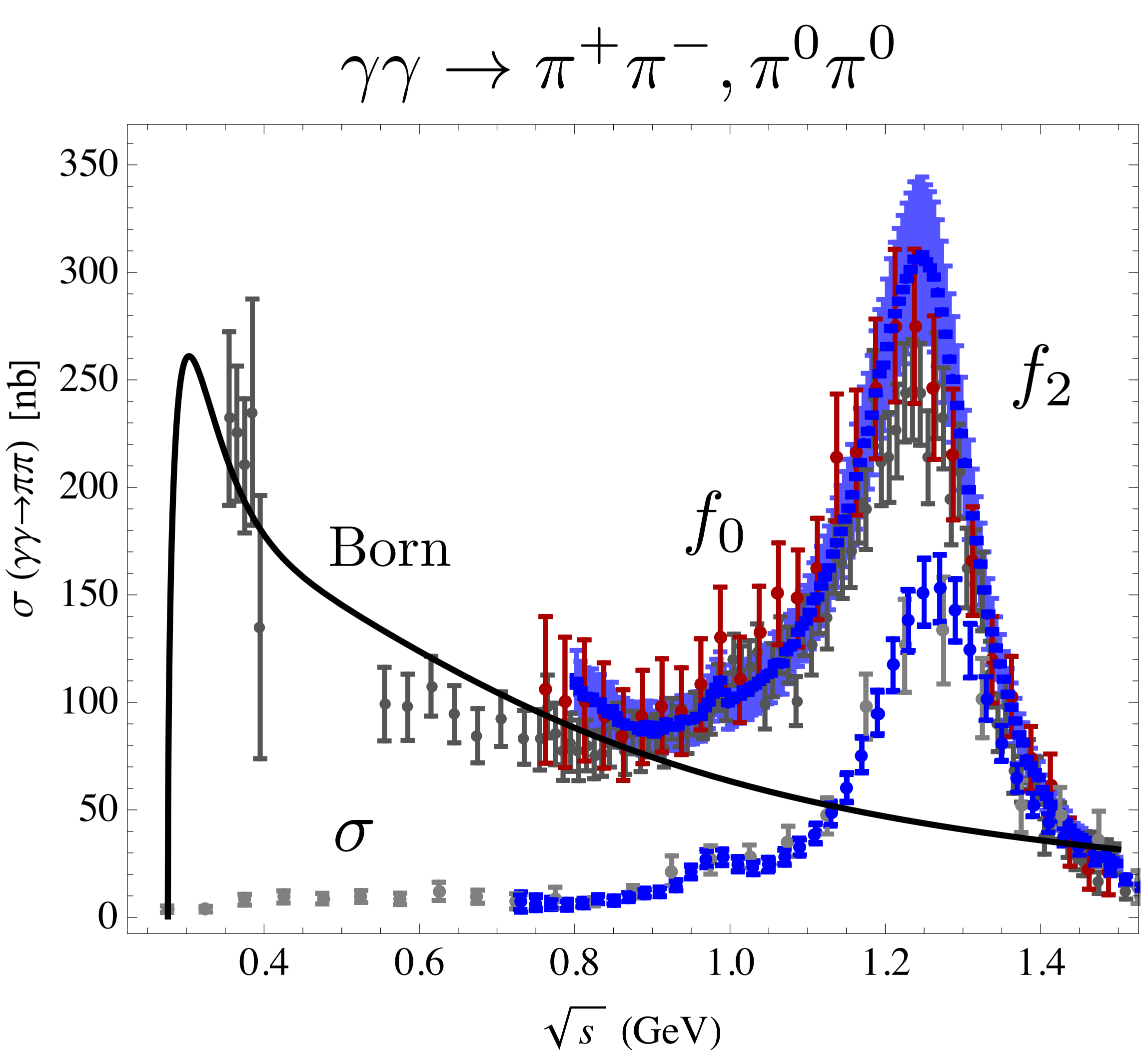


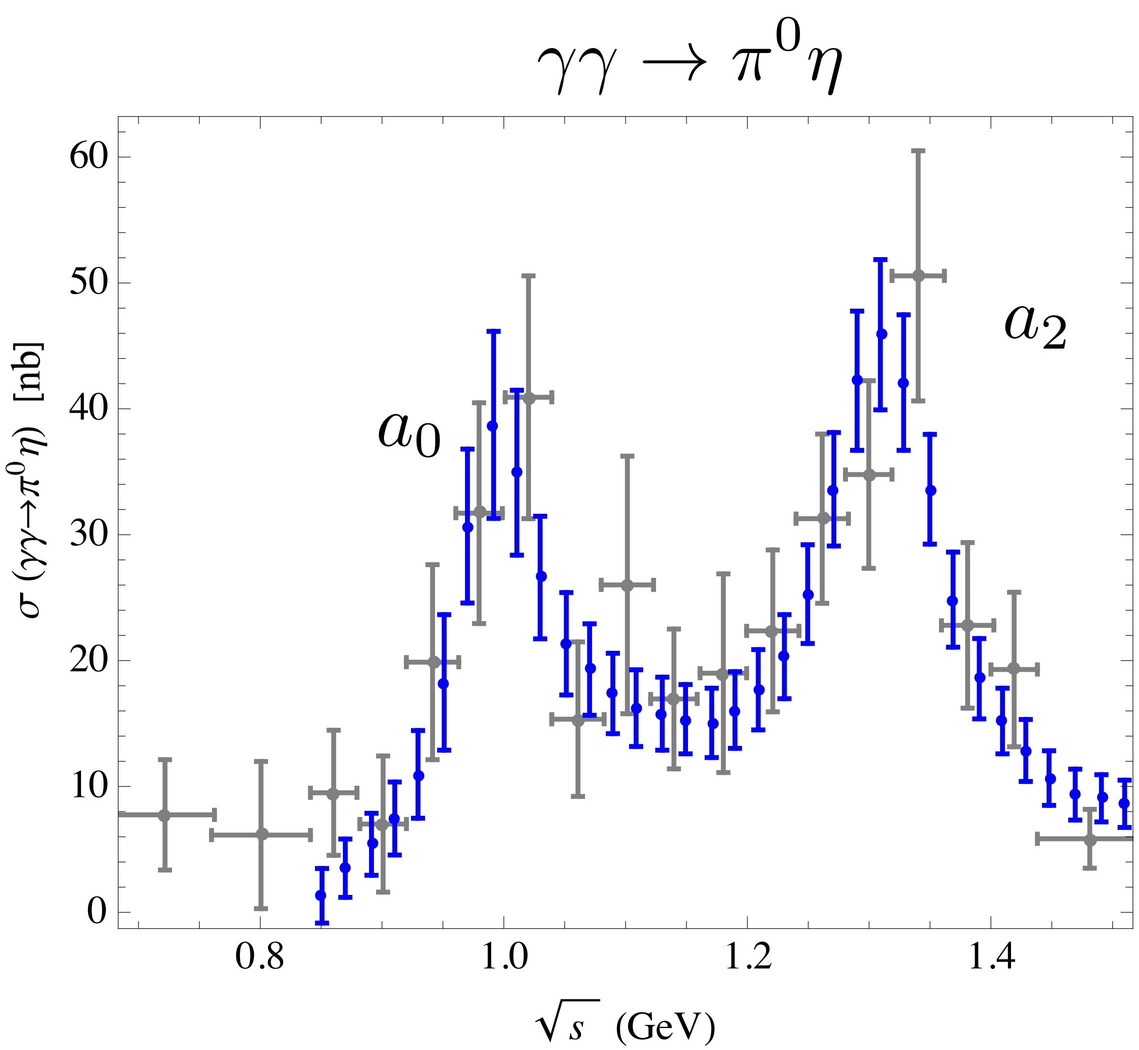
Important ingredient:

$\gamma^*\gamma^* \to \pi\pi, \pi\eta, \ldots$

$q^2 = -Q^2 < 0$ space-like γ^*

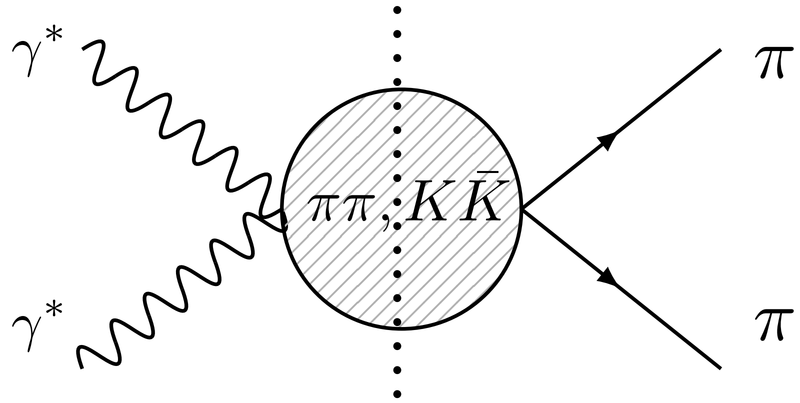


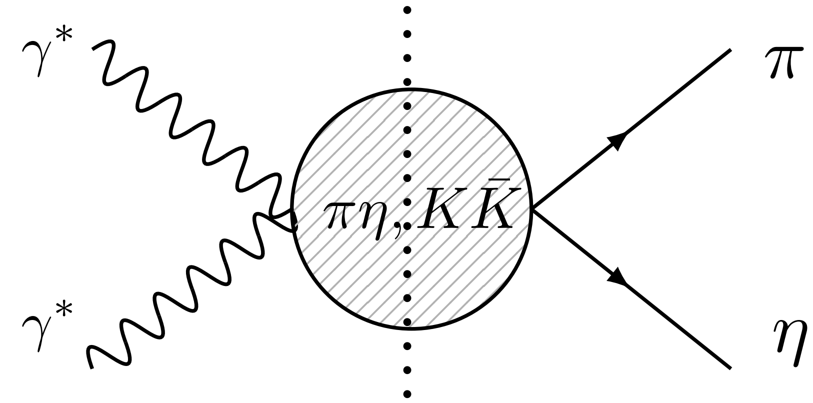




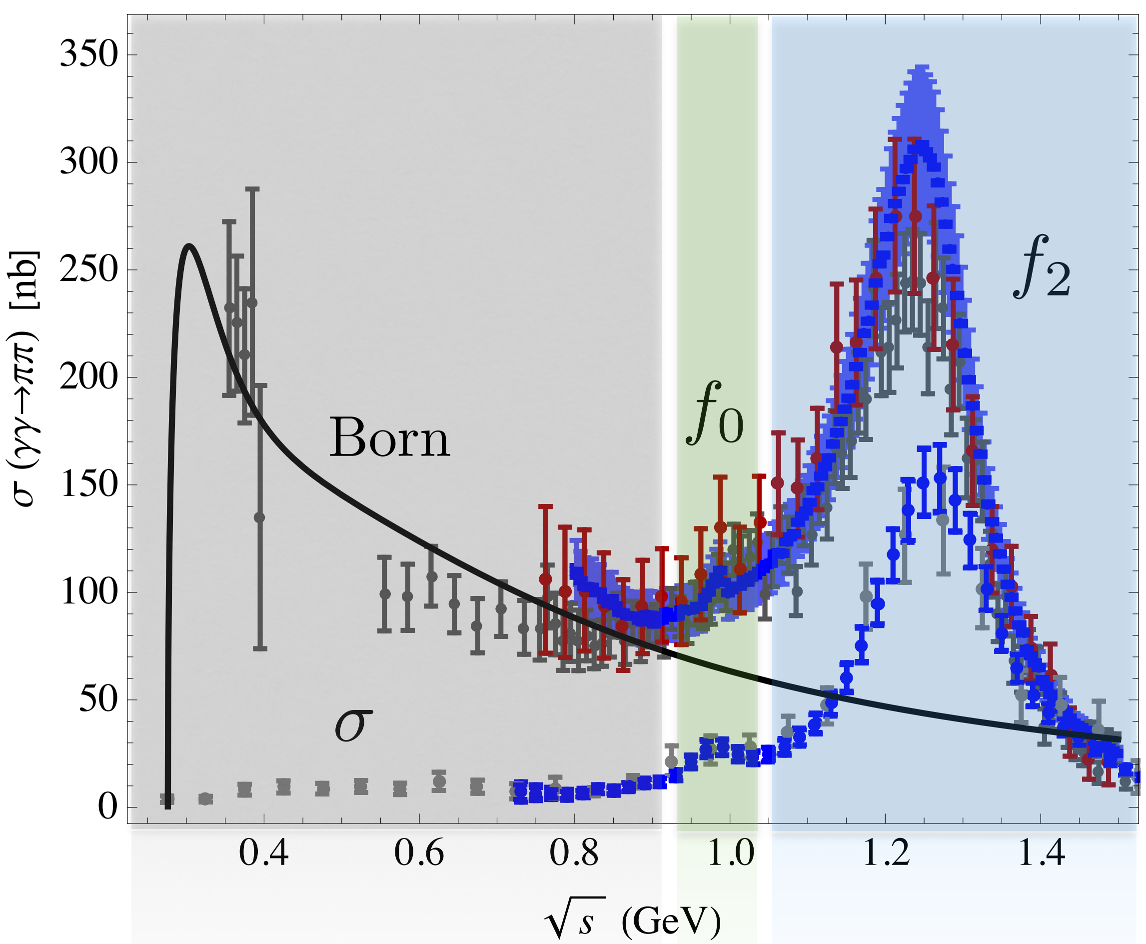
MarkII('90), CELLO ('92), Crystal Ball ('90), Belle ('07 '09)
Crystal Ball ('90), Belle ('09)

Inelastic contributions

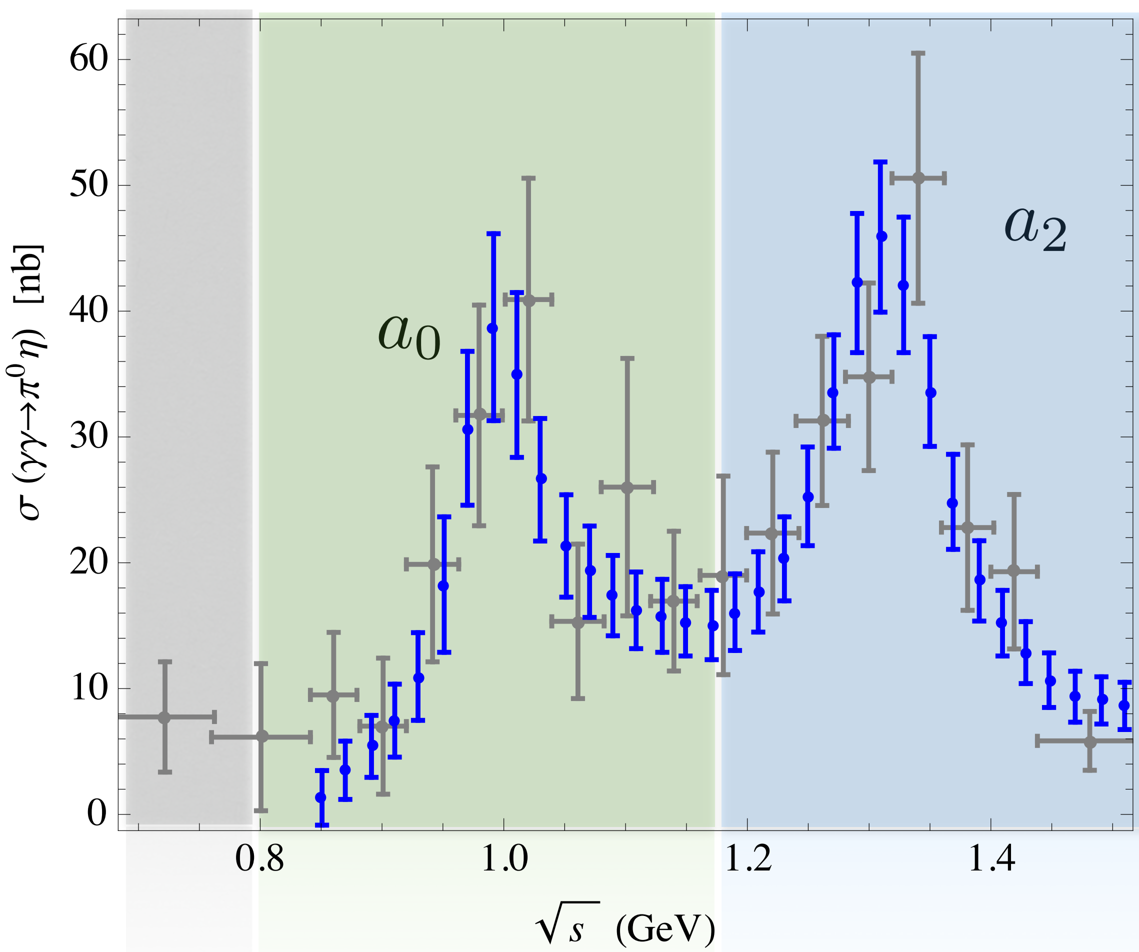




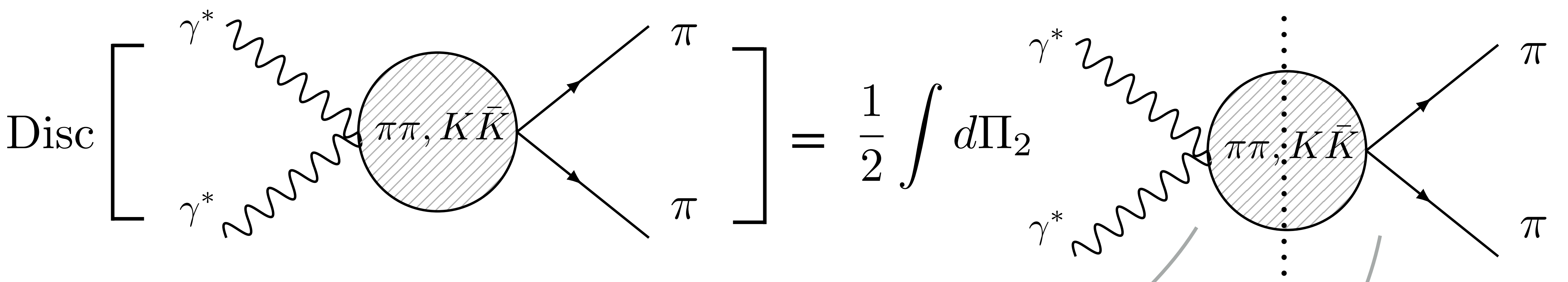
$\gamma\gamma \to \pi^+\pi^-, \pi^0\pi^0$



$\gamma\gamma \to \pi^0\eta$



MarkII('90), CELLO ('92), Crystal Ball ('90), Belle ('07 '09)
Crystal Ball ('90), Belle ('09)



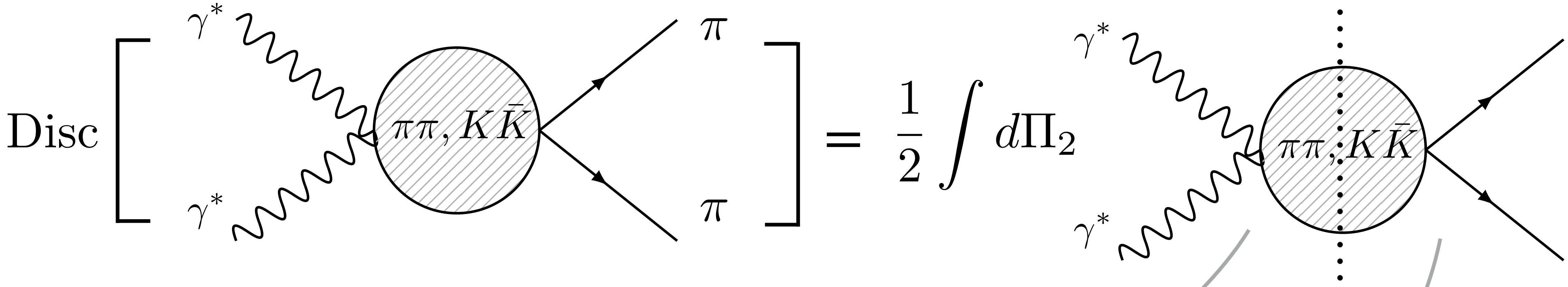
Partial wave expansion

$$H_{\lambda_1\lambda_2}(s,t) = \sum_{J=0}^{\overset{J_{max}=2}{\cancel{\infty}}} (2J+1)\, h^{(J)}_{\lambda_1\lambda_2}(s)\, d^J_{\lambda_1-\lambda_2,0}(\theta)$$

$$T(s,t) = \sum_{J=0}^{\overset{J_{max}=2}{\cancel{\infty}}} (2J+1)\, t^{(J)}(s)\, P_J(\theta)$$

These “diagonalise unitarity” and contain resonance information

$$\mathrm{Disc}\, h^{(J)}_{\lambda_1\lambda_2}(s) = h^{(J)}_{\lambda_1\lambda_2}(s)\, \rho_{\pi\pi}(s)\, t^{(J)*}_{\pi\pi\to\pi\pi}(s)$$



Partial wave expansion

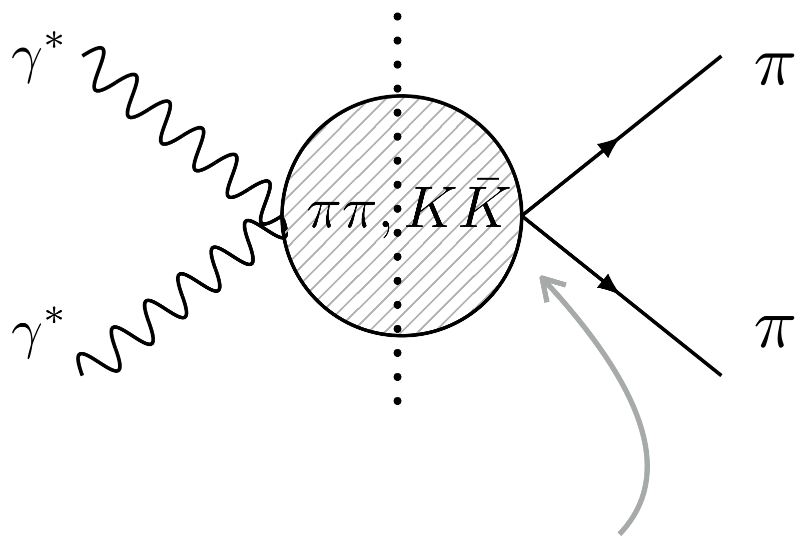
$$H_{\lambda_1\lambda_2}(s,t) = \sum_{J=0}^{\overset{J_{max}=2}{\cancel{\infty}}} (2J+1)\, h^{(J)}_{\lambda_1\lambda_2}(s)\, d^J_{\lambda_1-\lambda_2,0}(\theta)$$

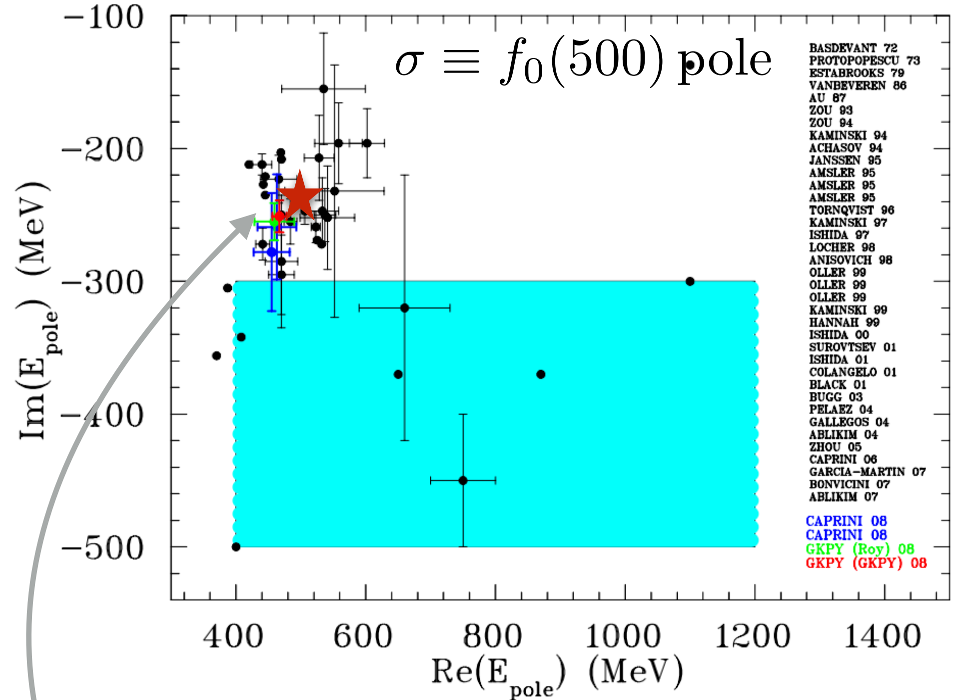
$$T(s,t) = \sum_{J=0}^{\overset{J_{max}=2}{\cancel{\infty}}} (2J+1)\, t^{(J)}(s)\, P_J(\theta)$$

These "diagonalise unitarity" and contain resonance information (coupled-channel unitarity)

$$\mathrm{Disc}\, h^{(J)}_{\lambda_1\lambda_2}(s) = h^{(J)}_{\lambda_1\lambda_2}(s)\, \rho_{\pi\pi}(s)\, t^{(J)*}_{\pi\pi\to\pi\pi}(s) + k^{(J)}_{\lambda_1\lambda_2}(s)\, \rho_{K\bar{K}}(s)\, t^{(J)*}_{K\bar{K}\to\pi\pi}(s)$$

Right-hand cuts (hadronic input)





Pelaez (2016)

$$\sqrt{s^{\rm II}_{\rm Roy}} = (449^{+22}_{-16}) \pm i\,(275 \pm 12)\ \text{MeV}$$
$$\sqrt{s^{\rm II}_{N/D}} = (496 \pm 48) \pm i\,(226 \pm 30)\ \text{MeV}$$

- Coupled-channel Omnès formalism

$$t(s) = U(s) + \int_R \frac{ds'}{\pi}\,\frac{\rho(s')\,|t(s')|^2}{s' - s}$$

$$t(s) = \frac{N(s)}{D(s)} = \Omega(s)\,N(s)$$ Chew, Mandelstam (1960)

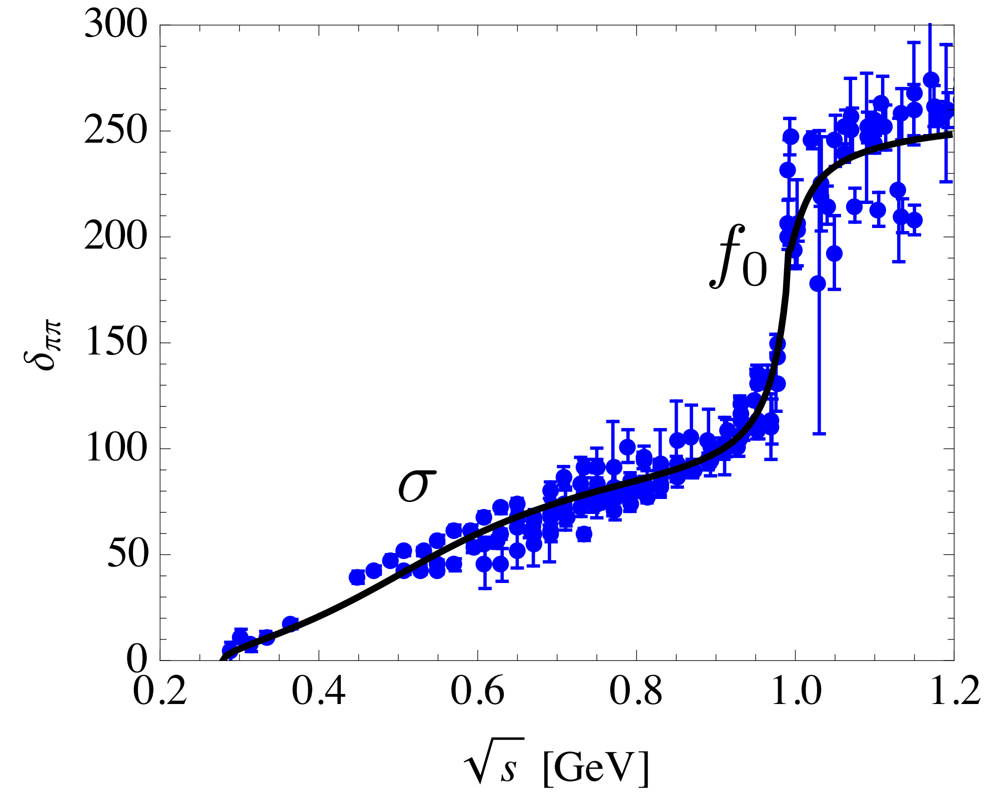
- Model independent form of the left-hand cuts: conformal mapping expansion

$$U(s) = \sum_k C_k\,\xi(s)^k$$

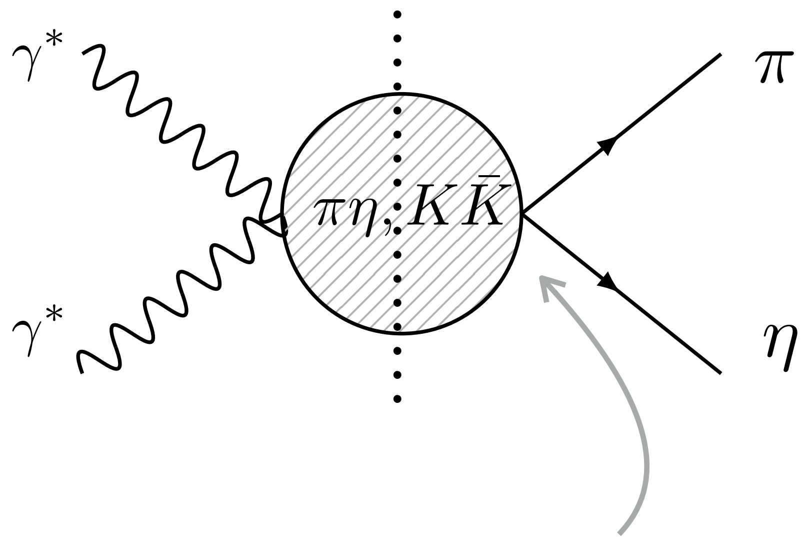
I.D., Lutz, Gasparyan (2011)

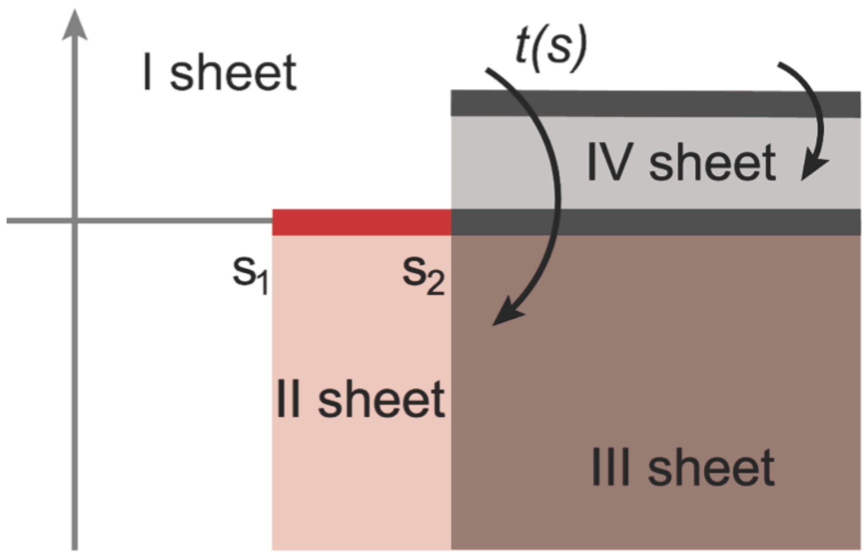
- Coefficients C_k determined from Exp. data and Roy Eq. solutions (Madrid)

I.D., Vanderhaeghen (2018)



Right-hand cuts (hadronic input)





$$\sqrt{s_{a_0}^{\mathrm{IV}}} = (1.12^{-0.07}_{+0.02}) \pm \frac{i}{2}\,(0.28^{+0.08}_{-0.13})\ \mathrm{GeV}$$

- Coupled-channel Omnès formalism

$$t(s) = U(s) + \int_R \frac{ds'}{\pi}\,\frac{\rho(s')\,|t(s')|^2}{s'-s}$$

$$t(s) = \frac{N(s)}{D(s)} = \Omega(s)\,N(s)$$ Chew, Mandelstam (1960)

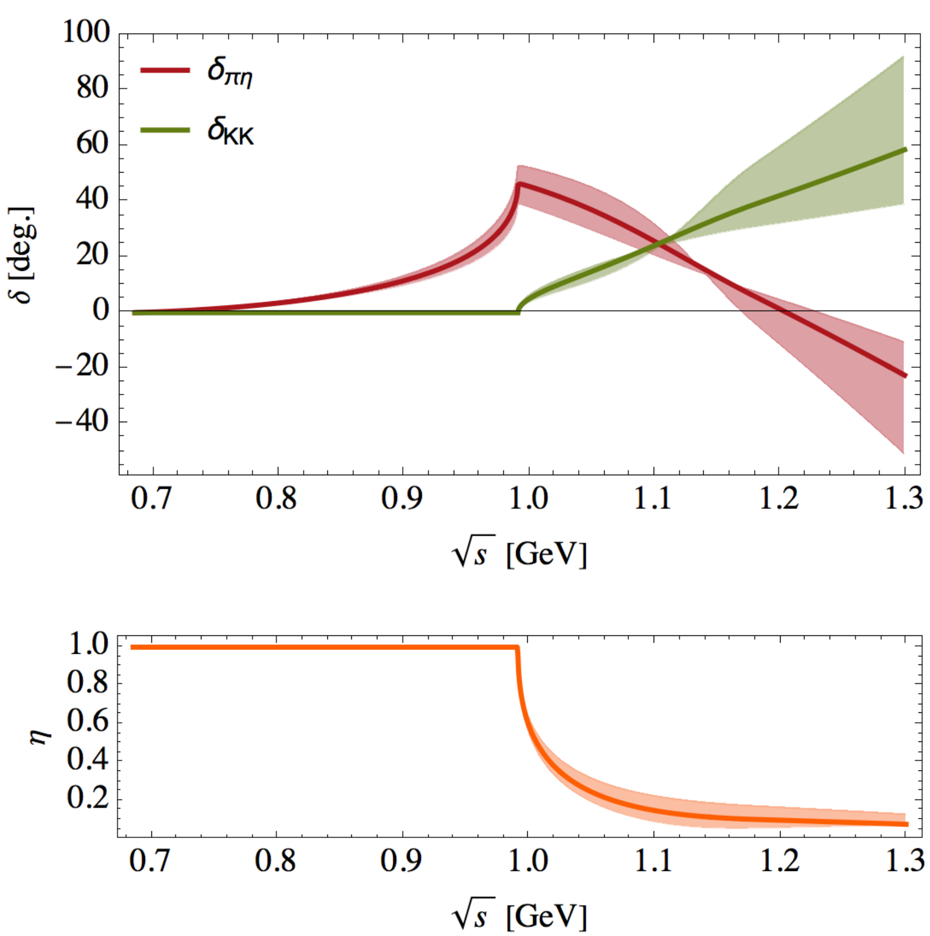
- Model independent form of the left-hand cuts: conformal mapping expansion

$$U(s) = \sum_k C_k\,\xi(s)^k$$

I.D., Lutz, Gasparyan (2011)

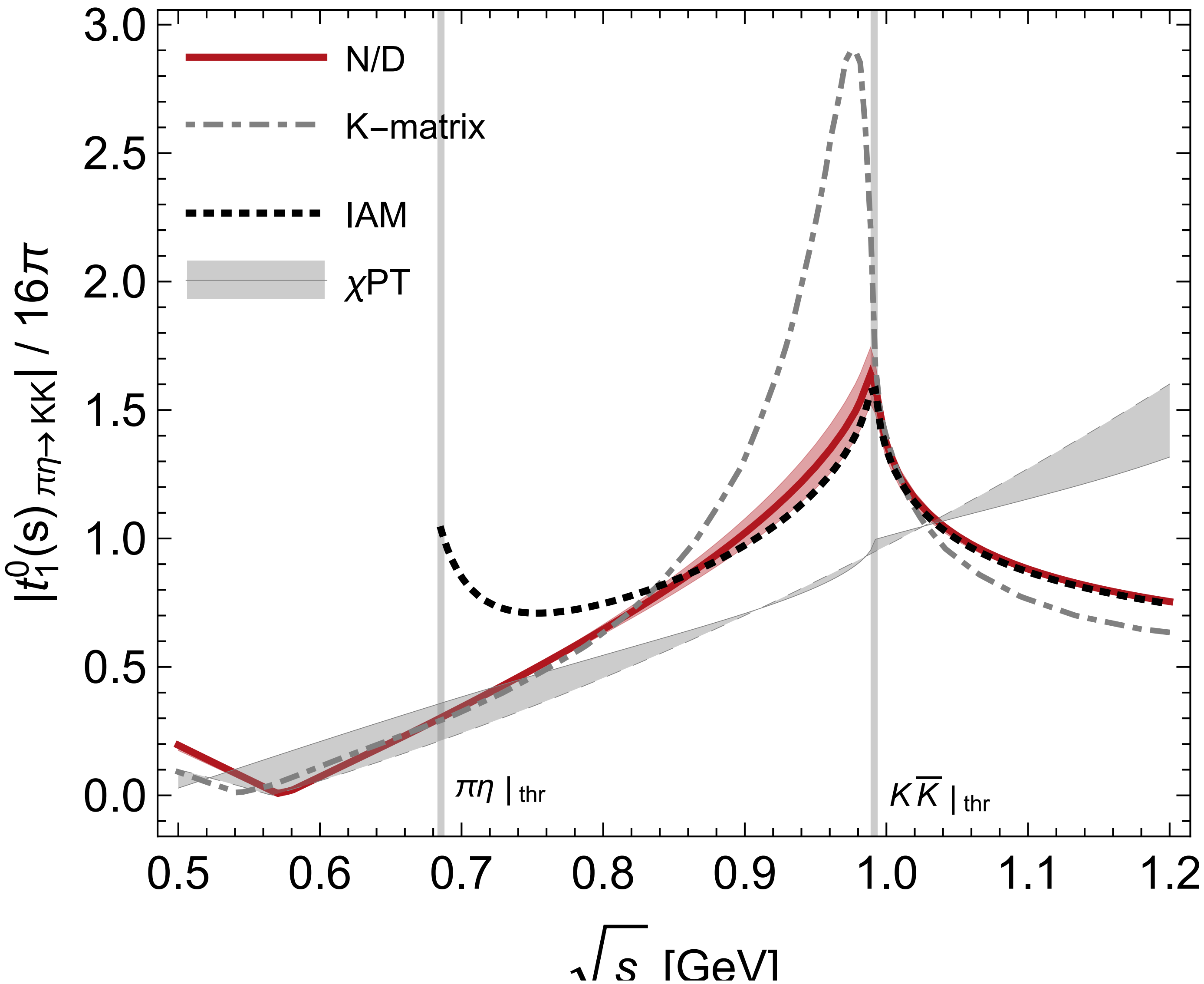
- Coefficients C_k matched to SU(3) Chiral Perturbation Theory at threshold

I.D., Gil, Lutz (2011, 2013)



cf. also HadSpec Coll. (2016)

Scattering amplitude πη→KK



N/D
I.D., Gil, Lutz (2013), I.D., Deineka, Vanderhaeghen (2017)

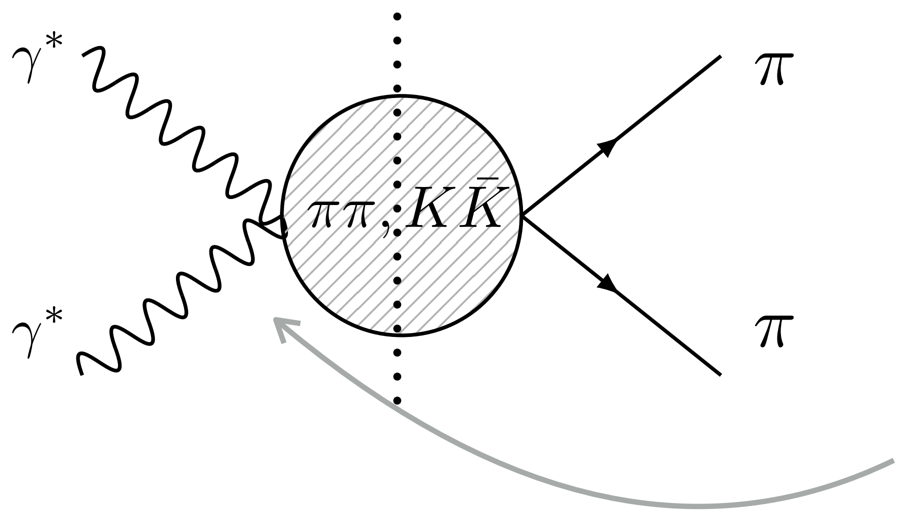
K-matrix
Albaladejo et. al. (2017)

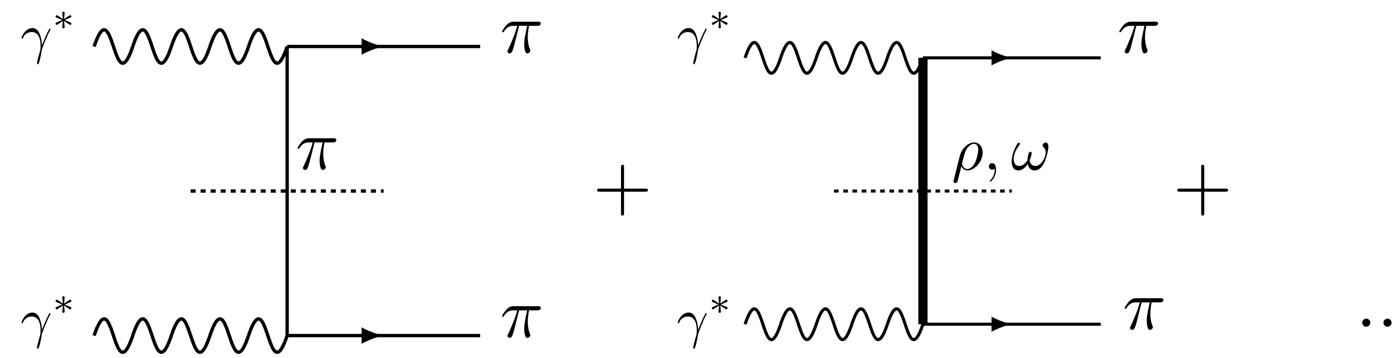
Inverse Amplitude Method (IAM)
Gomez Nicola et.al. (2002)

Chiral Perturbation Theory
Gasser et. al. (1985)

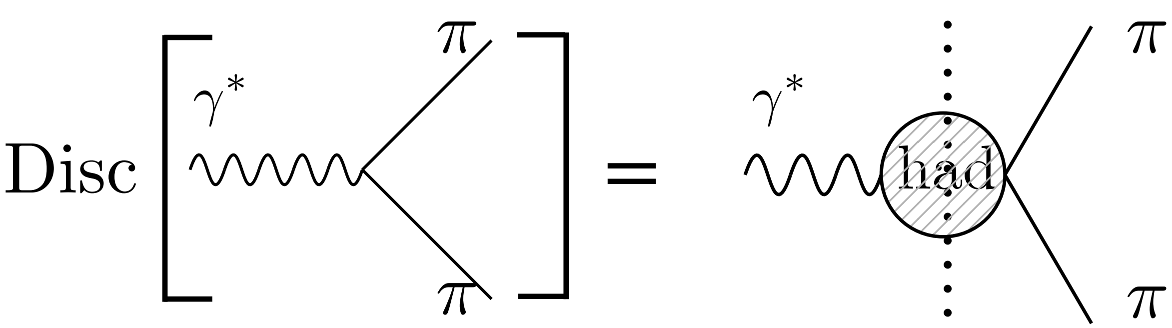
- First lattice analysis for m_π=391 MeV HadSpec Coll. (2016)
- Chiral extrapolation of the lattice results Zhi-Hui Guo et. al. (2017)

Left-hand cuts (pion pole)

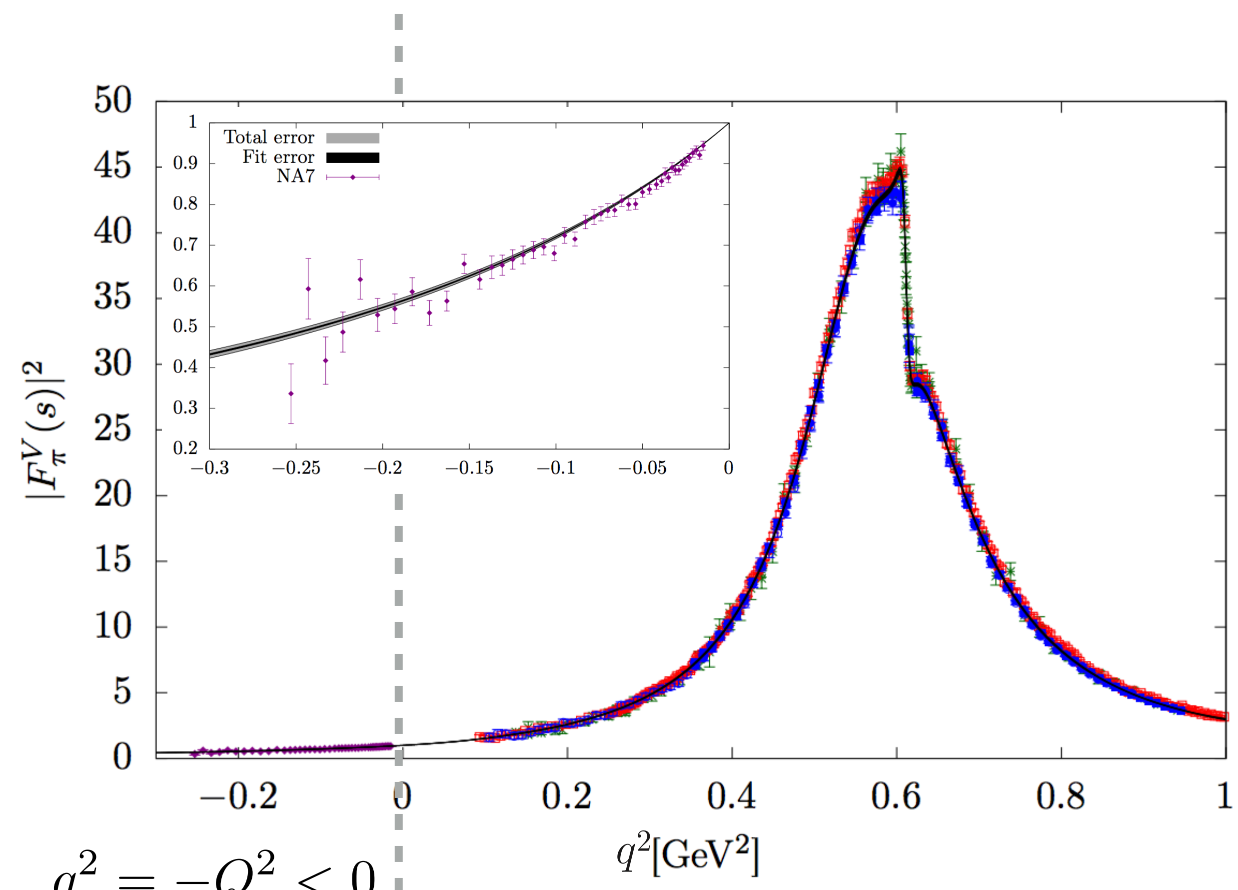




- Left-hand cuts requires knowledge from $\gamma^*\pi\pi,\ \gamma^*\pi\omega,\ \gamma^*\pi\rho$ transition form factors

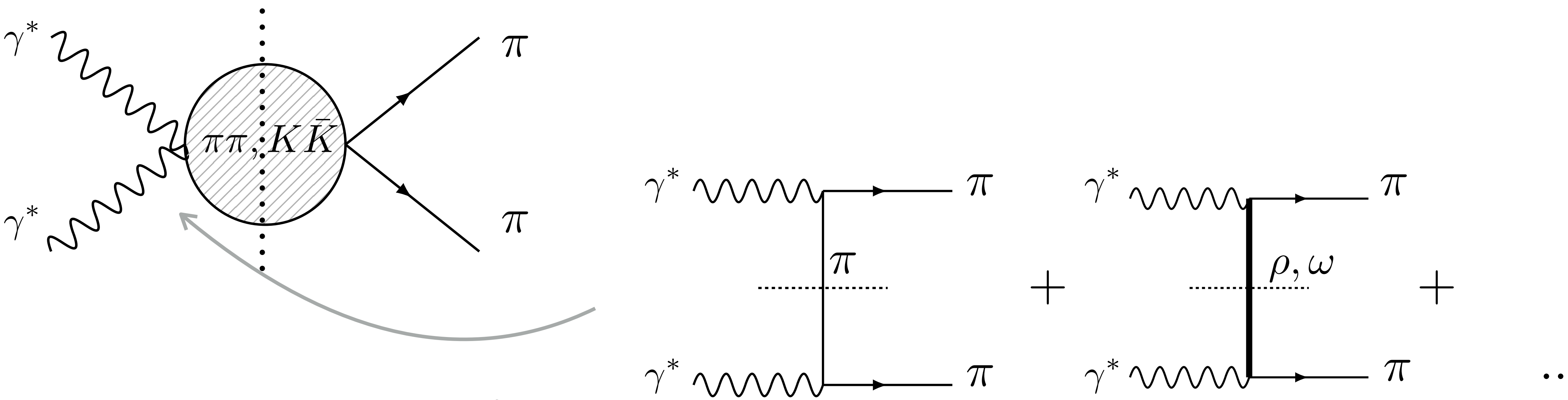


- Input from data: monopole TFF works well for space like region

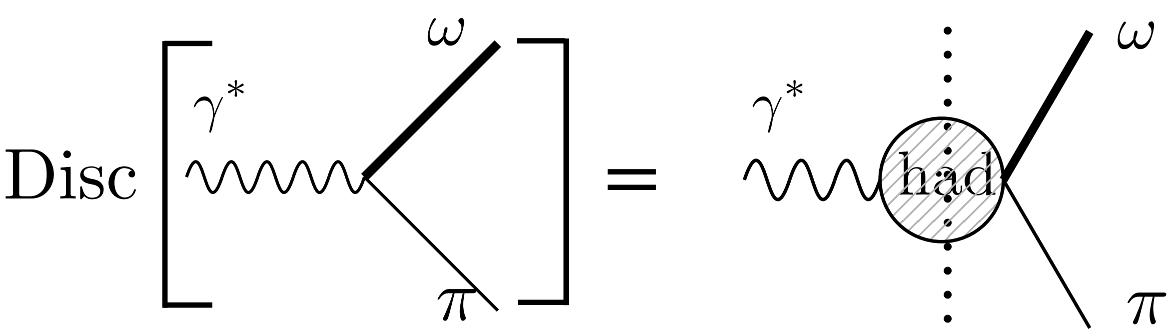


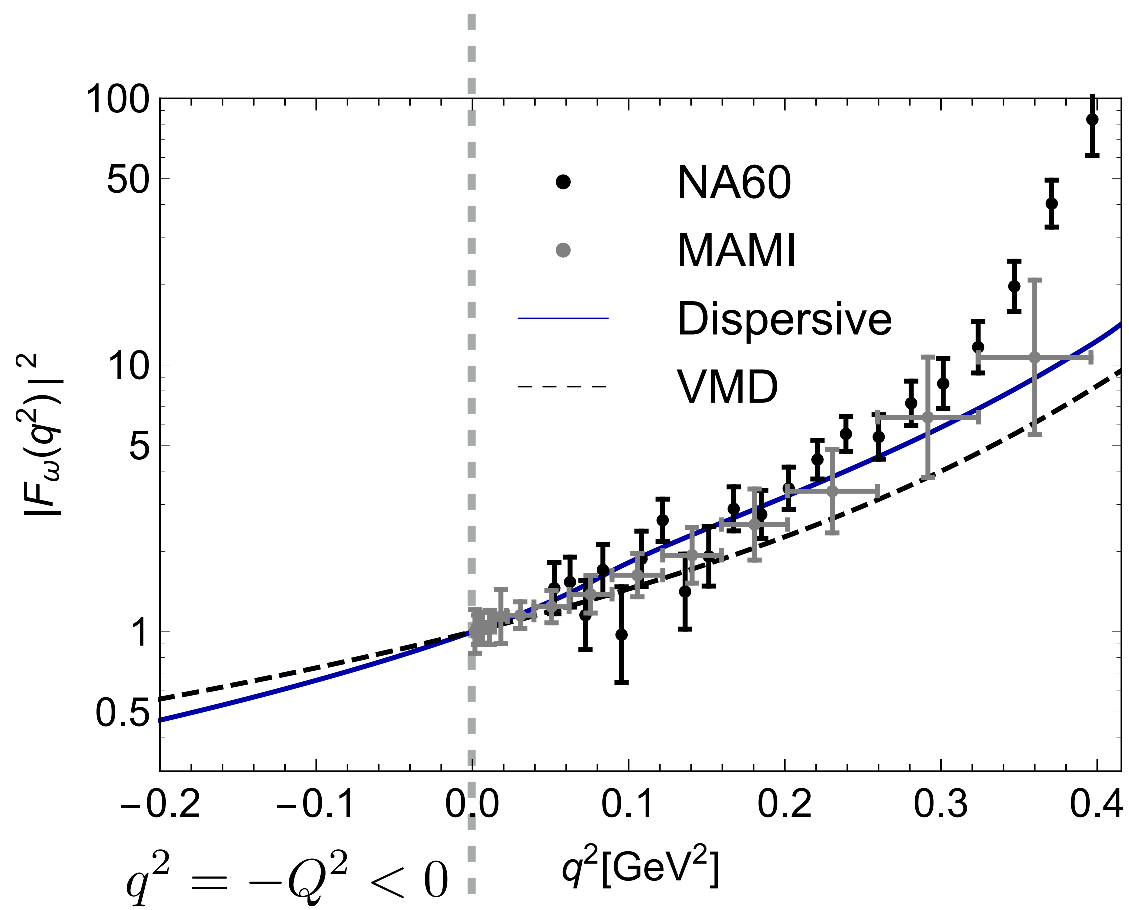
Colangelo et al. (2019)

Left-hand cuts (vector poles)



- Left-hand cuts requires knowledge from $\gamma^*\pi\pi,\ \gamma^*\pi\omega,\ \gamma^*\pi\rho$ transition form factors

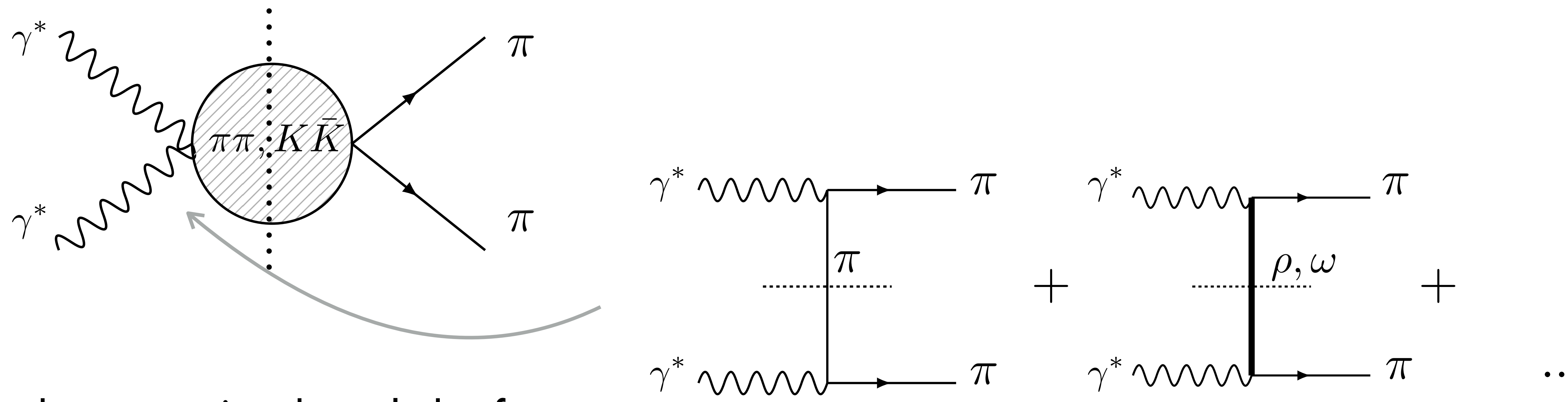




- Input from: $\omega \to 3\pi$

Khuri-Treiman (1960)
Schneider et al. (2012)
I.D. & JPAC (2015)

Left-hand cuts (vector poles)



- Left-hand cuts requires knowledge from $\gamma^* \pi\pi,\ \gamma^* \pi\omega,\ \gamma^* \pi\rho$ transition form factors

- Fitted parameter is the coupling: $g_{V\to\pi\gamma} \simeq C_{\rho^{\pm,0}\to\pi^{\pm,0}\gamma} \simeq \frac{1}{3} C_{\omega\to\pi^0\gamma} \stackrel{\text{PDG}}{=} 0.37(2)\ \text{GeV}^{-1}$

$g_{V\to\pi\gamma} = 0.33\ \text{GeV}^{-1}$

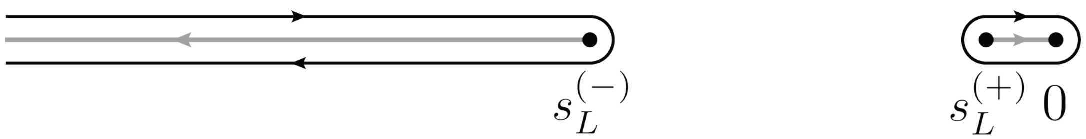
I.D., Vanderhaeghen (2018)

- Left-hand cuts: "anomalous thresholds" for large virtualities $Q_1^2 Q_2^2 > (M_V^2 - m_\pi^2)^2$

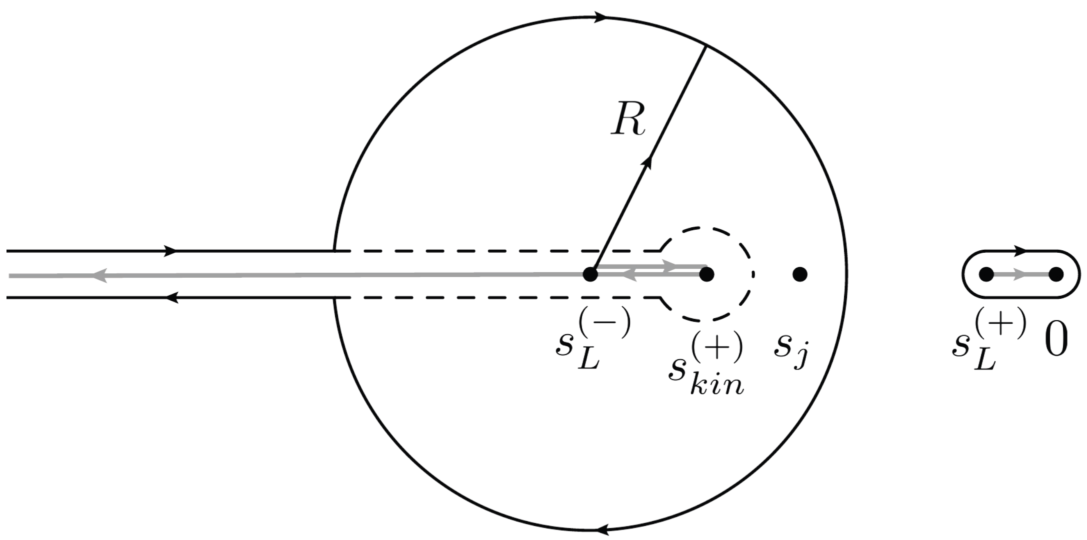
Hoferichter, Stoffer (2019)
Mandelstam (1960)

Left-hand cuts (vector poles)

- Normal case



- Anomaly case requires contour deformation



$$h^V(s) \sim \frac{1}{(s - s_{\text{kin}}^{(+)})^{9/2}} \log\left(\frac{X+1}{X-1}\right) \qquad \text{for J=2}$$

- Dashed curve: cancellations of singular pieces
  It requires a careful numerical implementation

Hoferichter, Stoffer (2019)

- Solid curve: enlarged contour such that one stays away from possible numerical issues related to the anomaly piece

I.D., Deineka, Vanderhaeghen (2019)

  - Easy to implement
  - Independent on the degree of singularity
  - Generalisation to the physical case is straightforward

Kinematic constraints

- Helicity amplitudes

$$H_{\lambda_1,\lambda_2} = \epsilon_\mu(\lambda_1)\epsilon_\nu(\lambda_2) \sum_{n=1}^{5} F_n(s,t)\, L_n^{\mu\nu}$$

Bardeen et al. (1968), Tarrach (1975)
Metz et al. (1998), Colangelo et al. (2015)

where $\lambda_{1,2} = \pm 1, 0$ are photon helicities
(minimal basis for Born subtracted amplitudes)

Low et al. (1954)

- p.w. helicity amplitudes suffer from kinematic constraints

$$h^{(J)}_{\lambda_1\lambda_2} = \int \frac{d\cos\theta}{2}\, d^J_{\lambda_1-\lambda_2,0}(\theta)\, H_{\lambda_1\lambda_2}$$

$$A^{(J)}_n = \frac{1}{(p\,q)^J} \int \frac{d\cos\theta}{2}\, P_J(\theta)\, F_n(s,t)$$ ⟵ object free of kinematic constraints

Lutz et al. (2010, 2014)

- Unconstrained basis for Born subtracted p.w. amplitudes

$$\bar{h}^{(J)}_i \equiv h^{(J)}_i - h^{(J),\mathrm{Born}}_i$$

$$\bar{h}^{(J)}_i = K_{ij}\, \bar{h}^{(J)}_j \qquad j \equiv \lambda_1\lambda_2 = \{++,\, +-,\, +0,\, 0+,\, 00\}$$

K_{ij} is 5×5 matrix

Kinematic constraints

- For s-wave

$$\bar{h}^{(0)}_{++} \pm \bar{h}^{(0)}_{00} \sim (s - s^{(\mp)}_{\text{kin}}), \quad s^{(\pm)}_{\text{kin}} \equiv -(Q_1 \pm Q_2)^2$$

Colangelo et al. (2017)
Pennington (1988), Moussallam (2013)

- For d-wave

$$(s + Q_1^2 + Q_2^2)\,\bar{h}^{(2)}_{+-} + \frac{2\sqrt{2\,s}\,Q_1^2 Q_2^2}{Q_1^2 - Q_2^2}\left(\frac{\bar{h}^{(2)}_{+0}}{Q_2} - \frac{\bar{h}^{(2)}_{0+}}{Q_1}\right) \sim (s - 4m_\pi^2)\left(s - s^{(+)}_{\text{kin}}\right)\left(s - s^{(-)}_{\text{kin}}\right)$$

$$\bar{h}^{(2)}_{+-} + \frac{\sqrt{2\,s}}{Q_1^2 - Q_2^2}\left(Q_2\,\bar{h}^{(2)}_{+0} - Q_1\,\bar{h}^{(2)}_{0+}\right) \sim (s - 4m_\pi^2)\left(s - s^{(+)}_{\text{kin}}\right)\left(s - s^{(-)}_{\text{kin}}\right)$$

$$\sqrt{2}\,\bar{h}^{(2)}_{+-} + \frac{(Q_1^2 + Q_2^2 + s)\sqrt{s}}{Q_1^2 - Q_2^2}\left(\frac{\bar{h}^{(2)}_{+0}}{Q_2} - \frac{\bar{h}^{(2)}_{0+}}{Q_1}\right) \sim (s - 4m_\pi^2)\left(s - s^{(+)}_{\text{kin}}\right)\left(s - s^{(-)}_{\text{kin}}\right)$$

+ 2 more

- Unconstrained basis for Born subtracted p.w. amplitudes

I.D., Deineka, Vanderhaeghen (2019)
cf. also Hoferichter, Stoffer (2019)

$$\bar{h}^{(J)}_i \equiv h^{(J)}_i - h^{(J),\text{Born}}_i$$

$$\bar{h}^{(J)}_i = K_{ij}\,\bar{h}^{(J)}_j \qquad j \equiv \lambda_1\lambda_2 = \{++,\, +-,\, +0,\, 0+,\, 00\}$$

K_{ij} is 5×5 matrix

Dispersion relation

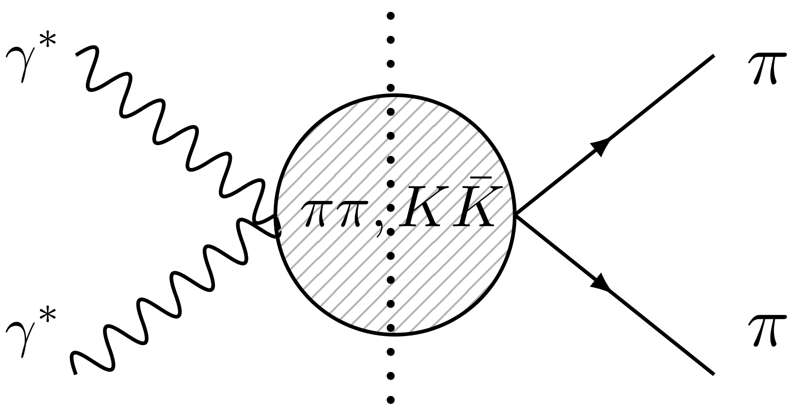
- Unsubtracted dispersion relation for kinematically unconstrained p.w. amplitudes

$$\bar{h}_i^{(J)} = \int_{-\infty}^{0} \frac{ds'}{\pi} \frac{\mathrm{Disc}\,\bar{h}_i^{(J)}(s')}{s'-s} + \int_{4m_\pi^2}^{\infty} \frac{ds'}{\pi} \frac{\mathrm{Disc}\,\bar{h}_i^{(J)}(s')}{s'-s}$$

$$\bar{h}_i^{(J)} \equiv h_i^{(J)} - h_i^{(J),\mathrm{Born}}$$

Garcia-Martin et. al (2010)
Hoferichter et. al. (2011,19)
Dai et al. (2014)
Moussallam (2013)

- Omnès solution of the unitarity relation



$$\mathrm{Disc}\,h_i^{(J)} = h_i^{(J)}\,\rho\,t_{\pi\pi}^{(J)*}$$
$$\mathrm{Disc}\,\Omega^{(J)} = \Omega^{(J)}\,\rho\,t_{\pi\pi}^{(J)*} \qquad |_{s>4m_\pi^2}$$

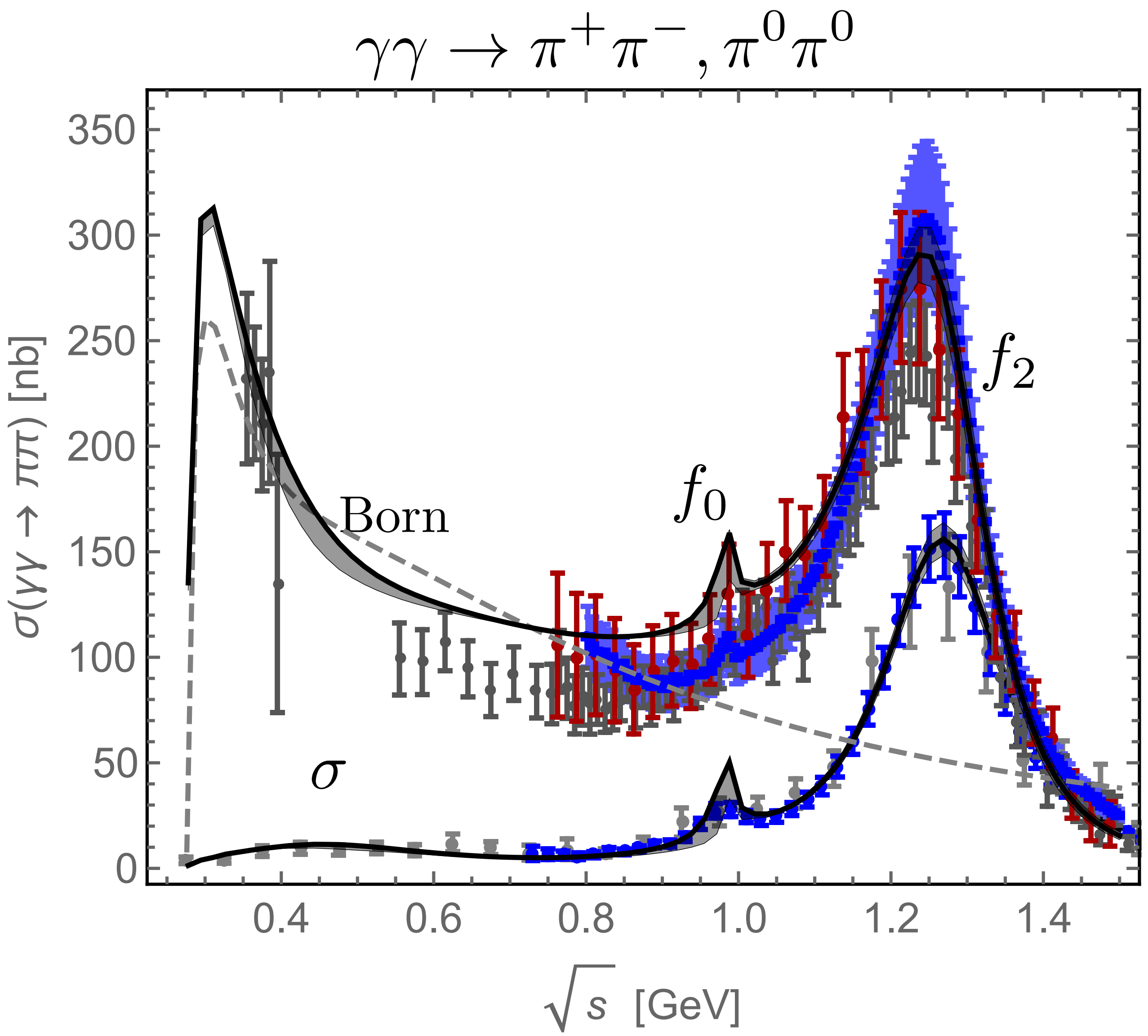
leads to

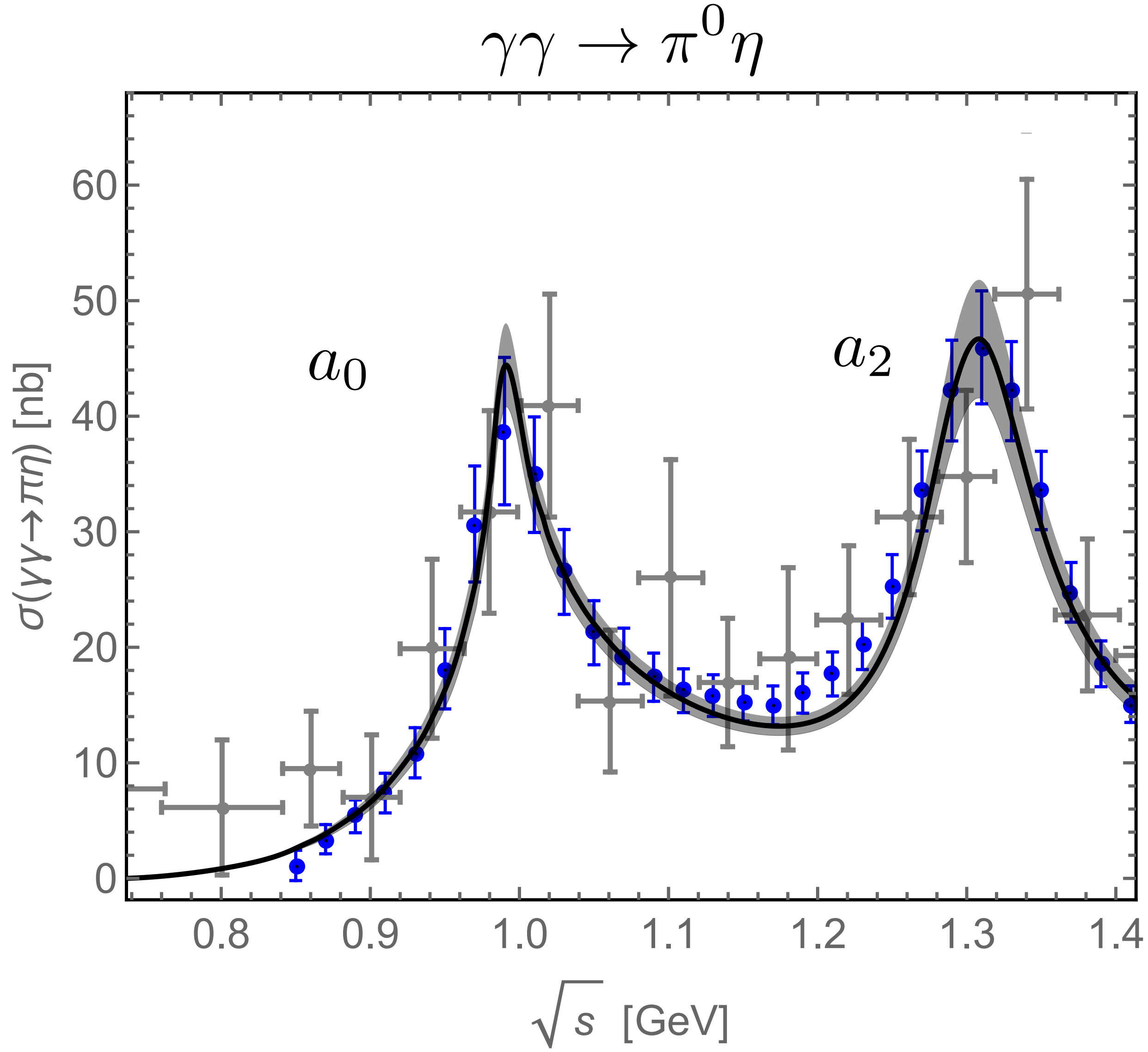
$$\boxed{h_i^{(J)} = h_i^{(J),\mathrm{Born}} + \Omega^{(J)} \left(\int_{-\infty}^{0} \frac{ds'}{\pi} \frac{\mathrm{Disc}(\bar{h}_i^{(J)}(s'))\,\Omega^{(J)}(s')^{-1}}{s'-s} - \int_{4m_\pi^2}^{\infty} \frac{ds'}{\pi} \frac{h_i^{(J),\mathrm{Born}}(s')\,\mathrm{Im}\,\Omega^{(J)}(s')^{-1})}{s'-s} \right)}$$

V-exch

Omnès (1958)
Muskhelishvili (1953)

Results for real photons



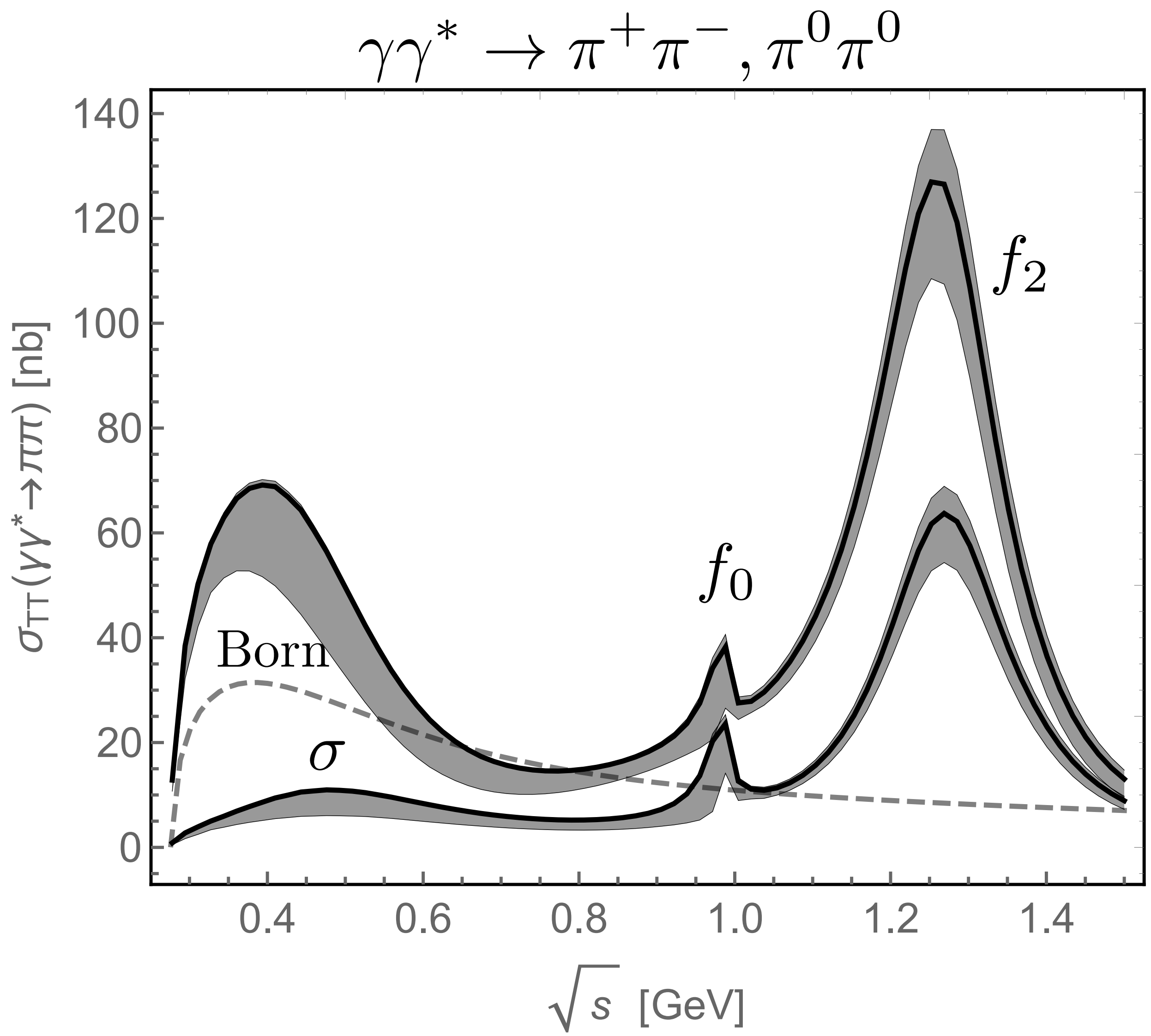


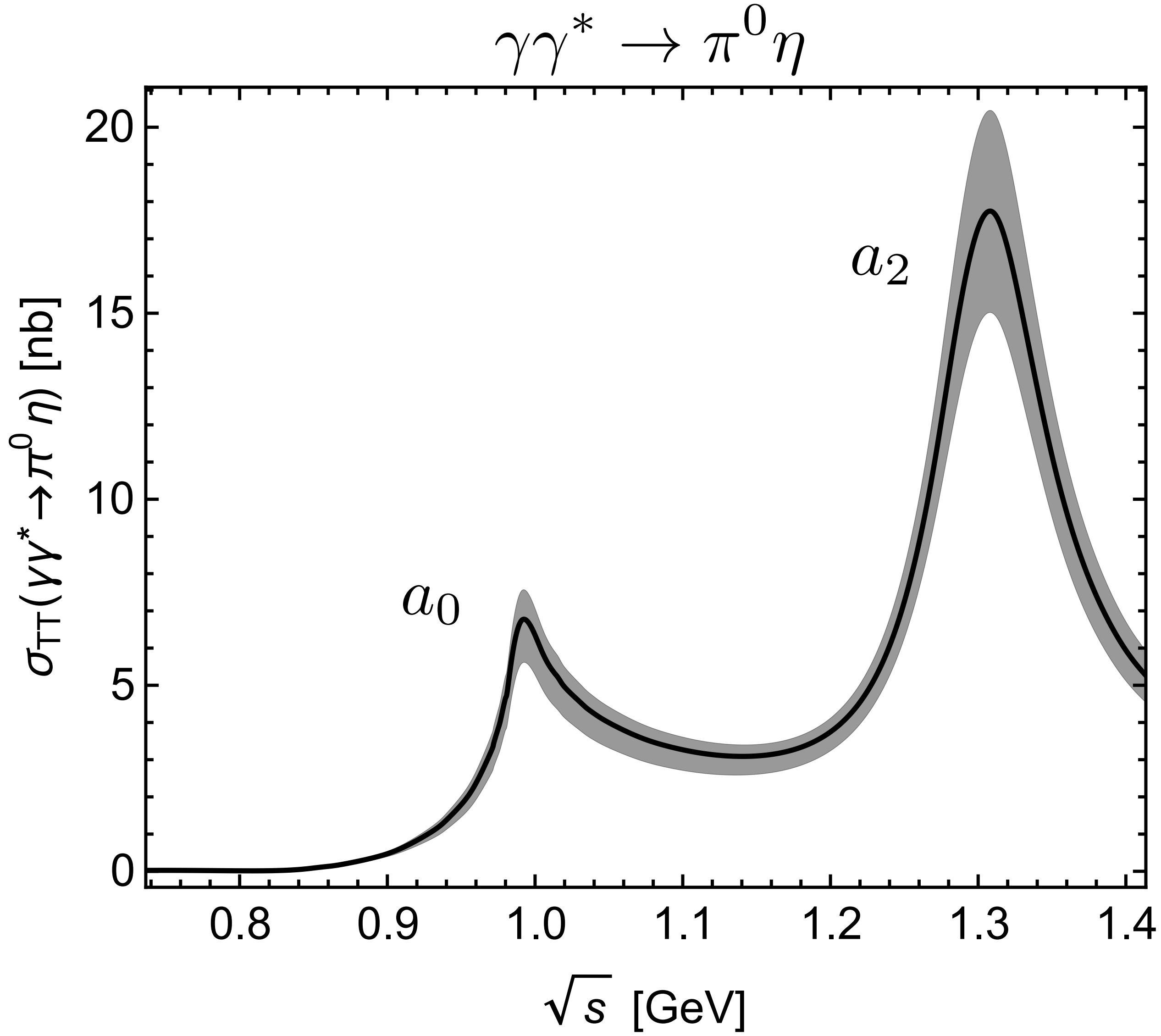
- Coupled-channel dispersive treatment of $f_0(980)$ and $a_0(980)$ is crucial
- $f_2(1270)$ described dispersively through Omnès function
- $a_2(1320)$ described as a Breit Wigner resonance

I.D., Deineka, Vanderhaeghen (2017, 2018)

cf. also Dai et al. (2014)
Hoferichter et. al. (2011,19)
Garcia-Martin et. al (2010)

Results for single virtual photon (Q^2=0.5)

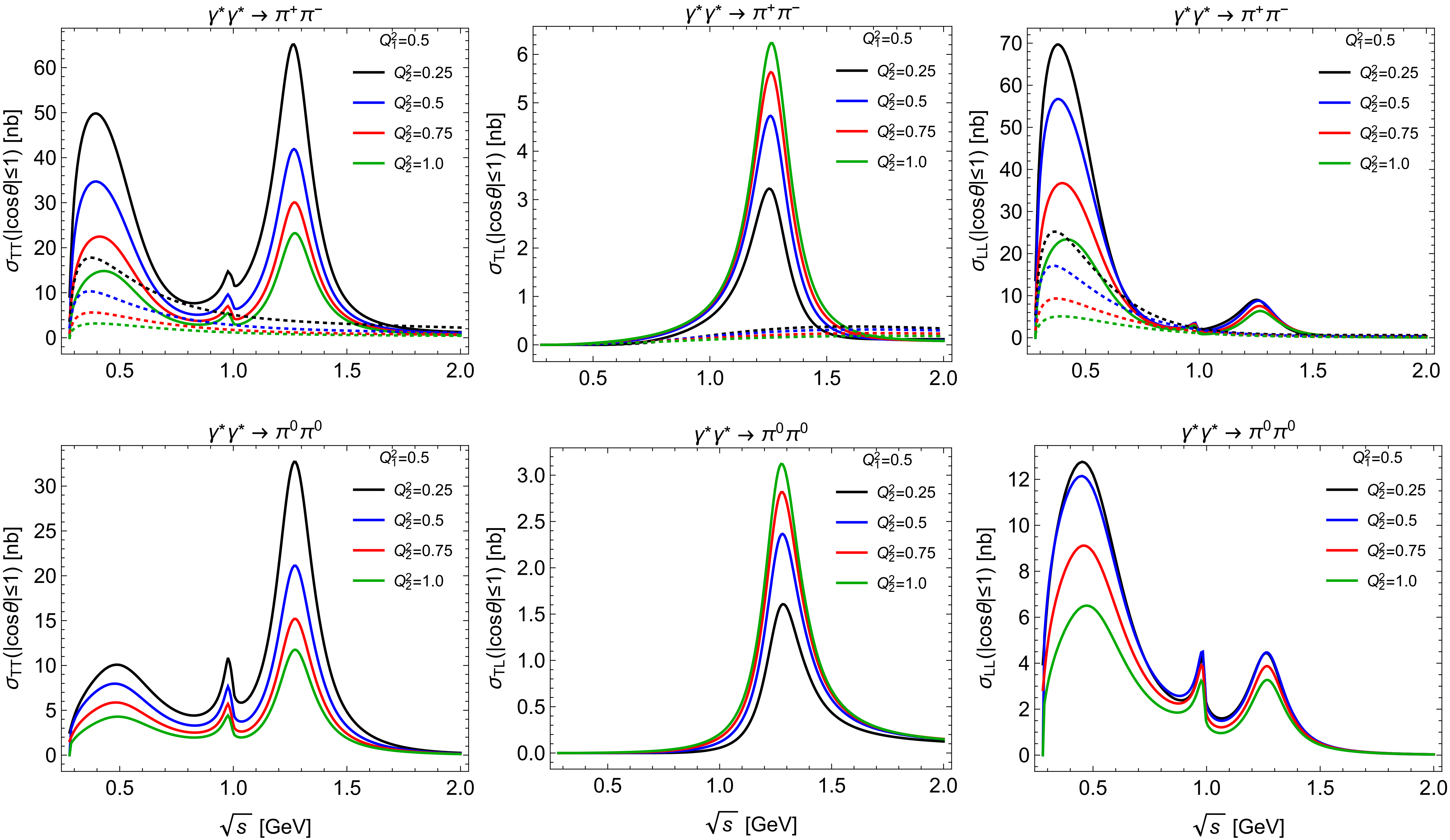




- Single tagged BESIII data for $\pi^+\pi^-$, $\pi^0\pi^0$ in range $0.1\ \mathrm{GeV}^2 < Q^2 < 2\ \mathrm{GeV}^2$ under analysis. It will validate left-hand cuts approximations.

I.D., Deineka, Vanderhaeghen (2018)
cf. also Moussallam (2013)
Hoferichter, Stoffer (2019)

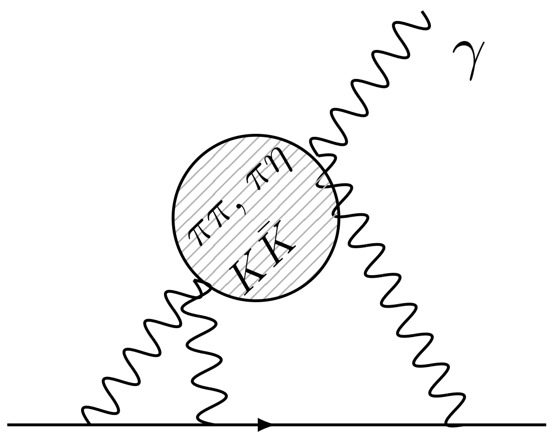
Results for ππ



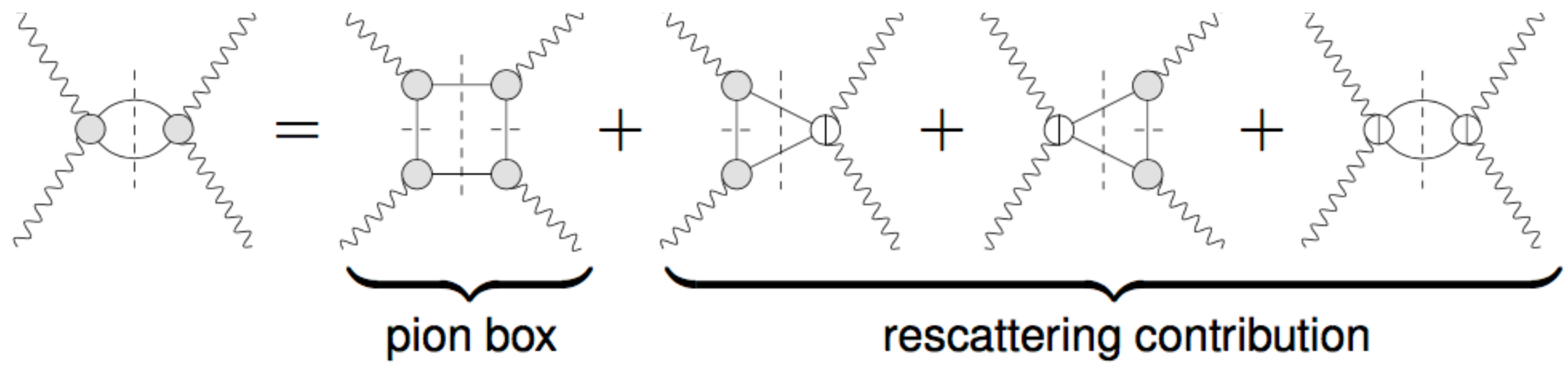
$$\frac{d\sigma_{TT}}{d\cos\theta} \sim |H_{++}|^2 + |H_{+-}|^2, \quad \frac{d\sigma_{TL}}{d\cos\theta} \sim |H_{+0}|^2, \quad \frac{d\sigma_{LL}}{d\cos\theta} \sim |H_{00}|^2$$

I.D., Deineka, Vanderhaeghen (2019)
cf. also Hoferichter, Stoffer (2019)

Multi-meson contribution to (g-2)



- Pioneering dispersive analyses for ππ loop contribution to a_μ

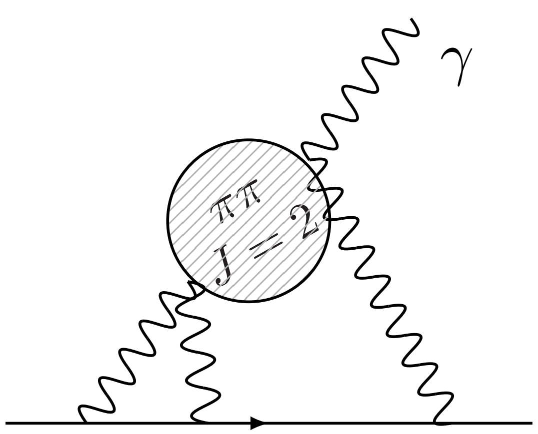


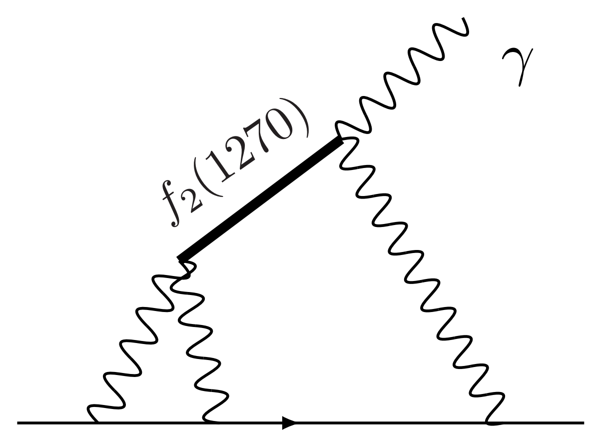
$$a_\mu^{\pi\text{-box}} + a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -2.4(1) \times 10^{-10}$$

$f_0(500)$

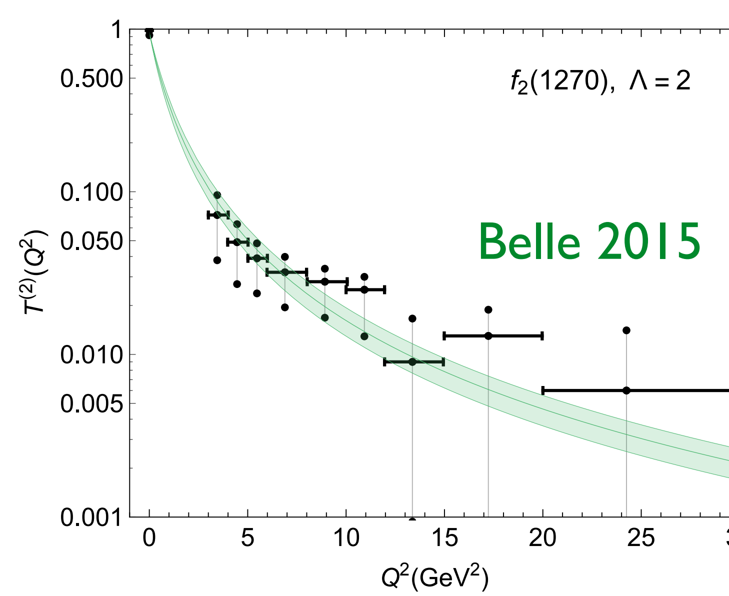
Colangelo et al. (2014-2017)

- Ongoing $f_0(980)$, $a_0(980)$
- One needs to compare $f_2(1270)$ with effective resonance description





$$a_\mu = (0.05 \pm 0.01) \times 10^{-10}$$



Pascalutsa, Pauk, Vanderhaeghen (2012)
I.D., Vanderhaeghen (2016)

Thank you!