meson resonances from lattice QCD

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meson spectroscopy

resonances, scattering, elastic phase-shifts

lattice QCD

discrete spectrum, finite volume, computing the spectrum

elastic scattering

lattice QCD phase-shift results

coupled-channel scattering

mapping the discrete spectrum to the t-matrix

lattice QCD calculation results

the complex energy plane

rigorously determining resonances

I gave similar lectures at a school
"Scattering from the Lattice: Applications to Phenomenology and Beyond"
at HMI Dublin in 2018

slightly more material in those notes
(https://indico.cern.ch/event/690702/)

recent pedagogic review

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Scattering processes and resonances from lattice QCD

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The vast majority of hadrons observed in nature are not stable under the strong interaction; rather they are resonances whose existence is deduced from enhancements in the energy dependence of scattering amplitudes. The study of hadron resonances offers a window into the workings of quantum chromodynamics (QCD) in the low-energy nonperturbative region, and in addition many probes of the limits of the electroweak sector of the standard model consider processes which feature hadron resonances. From a theoretical standpoint, this is a challenging field: the same dynamics that binds quarks and gluons into hadron resonances also controls their decay into lighter hadrons, so a complete approach to QCD is required. Presently, lattice QCD is the only available tool that provides the required nonperturbative evaluation of hadron observables. This article reviews progress in the study of few-hadron reactions in which resonances and bound states appear using lattice QCD techniques. The leading approach is described that takes advantage of the periodic finite spatial volume used in lattice QCD calculations to extract scattering amplitudes from the discrete spectrum of QCD eigenstates in a box. An explanation is given of how from explicit lattice QCD calculations one can rigorously garner information about a variety of resonance properties, including their masses, widths, decay couplings, and form factors. The challenges which currently limit the field are discussed along with the steps being taken to resolve them.

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hadron spectroscopy — ‘rigorously’

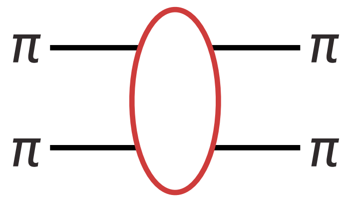
want to study excited hadrons as they really are — rapidly decaying resonances

same dynamics that binds them also causes their decay

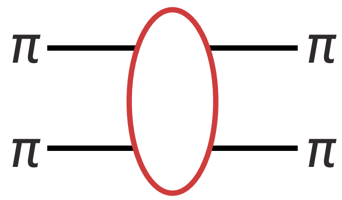
we need to compute **scattering amplitudes** and see if they resonate

start with the simplest case: **elastic scattering** …

e.g.



elastic partial-waves & unitarity



$$\frac{d\sigma}{d\Omega} \propto \left|t(E,\cos\theta)\right|^2$$

elastic scattering amplitude
can be expanded in partial-waves

$$t(E,\cos\theta) = \sum_\ell (2\ell+1)\, t_\ell(E)\, P_\ell(\cos\theta)$$

partial-wave amplitude

resonances appear in
a single partial-wave

ℓ	0	1	2	…
J^P	0^+	1^-	2^+	

conservation of probability
a.k.a **elastic unitarity**

$$\mathrm{Im}\, t_\ell(E) = \rho(E)\left|t_\ell(E)\right|^2 \quad \text{or} \quad \mathrm{Im}\,\frac{1}{t_\ell(E)} = -\rho(E)$$

'phase-space' $\rho(E) = \dfrac{2k(E)}{E}$

c.m. momentum $k(E) = \frac{1}{2}\sqrt{E^2 - 4m^2}$

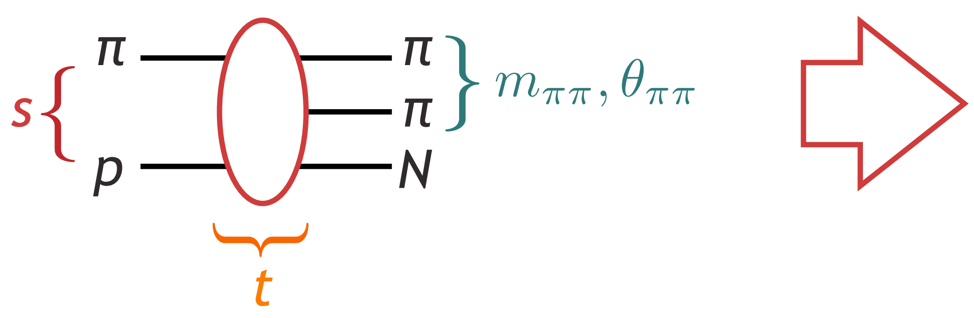
can parameterise elastic scattering
in terms of a single real parameter

$$t_\ell(E) = \frac{1}{\rho(E)}\, e^{i\delta_\ell(E)} \sin\delta_\ell(E)$$

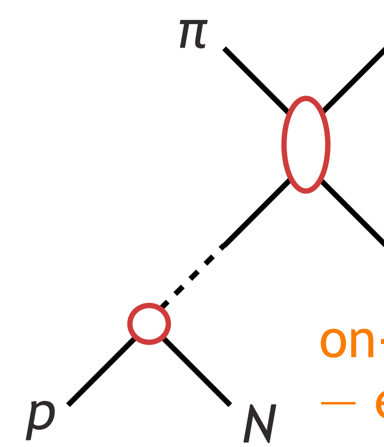
'phase-shift'

the simplest case: $\pi\pi$ elastic scattering

extract from charged pion beams on nucleon targets



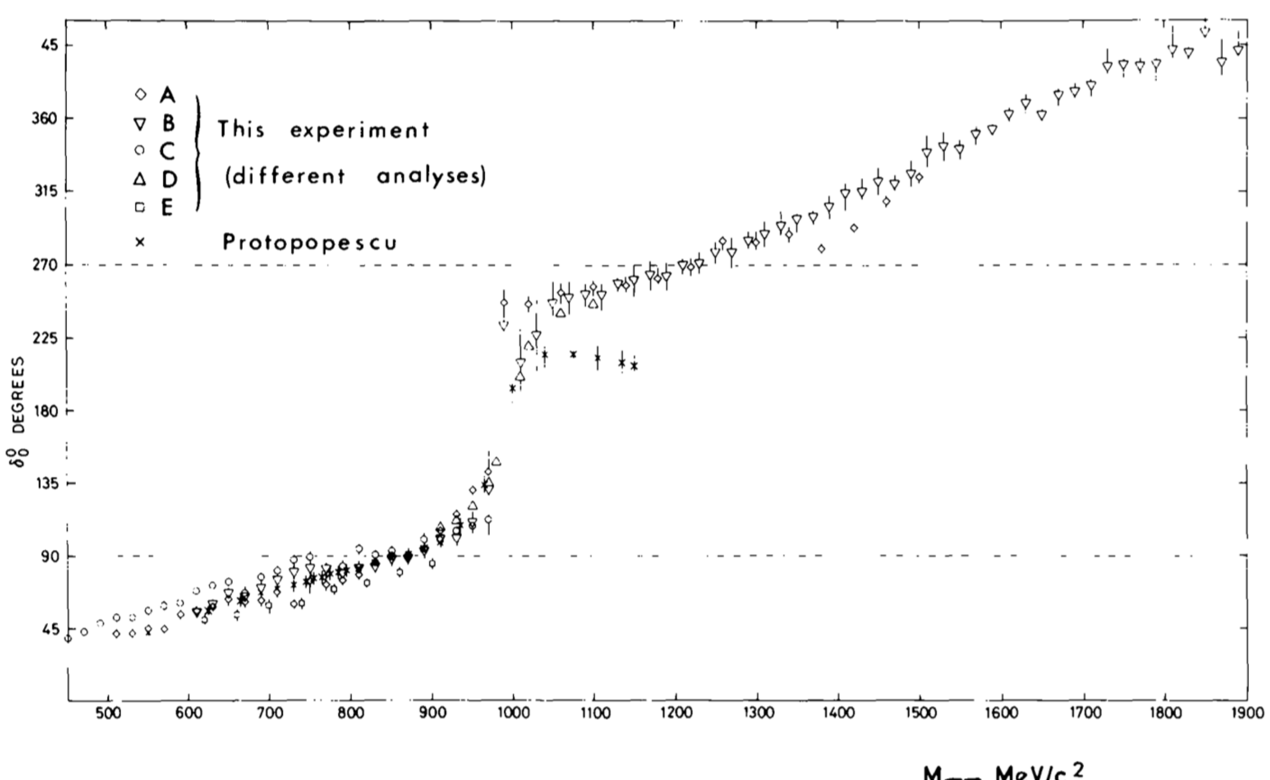
physical scattering
has negative t



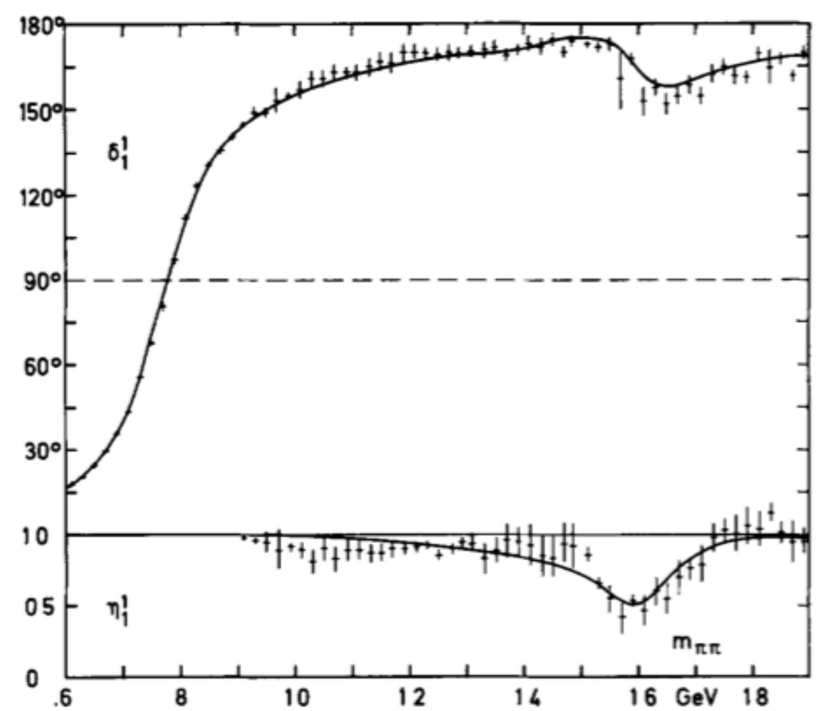
$\pi\pi$ partial-waves
— project $P_\ell(\cos\theta)$

on-shell π exchange
— extrapolate to $t = m_\pi^2$

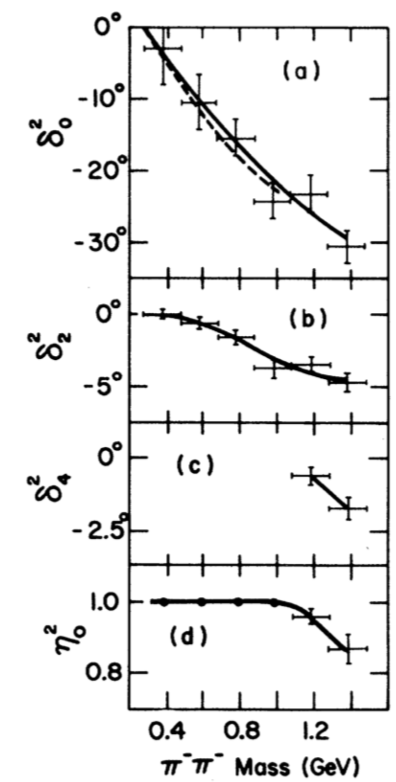
isospin=0	isospin=1	isospin=2



Grayer 1974



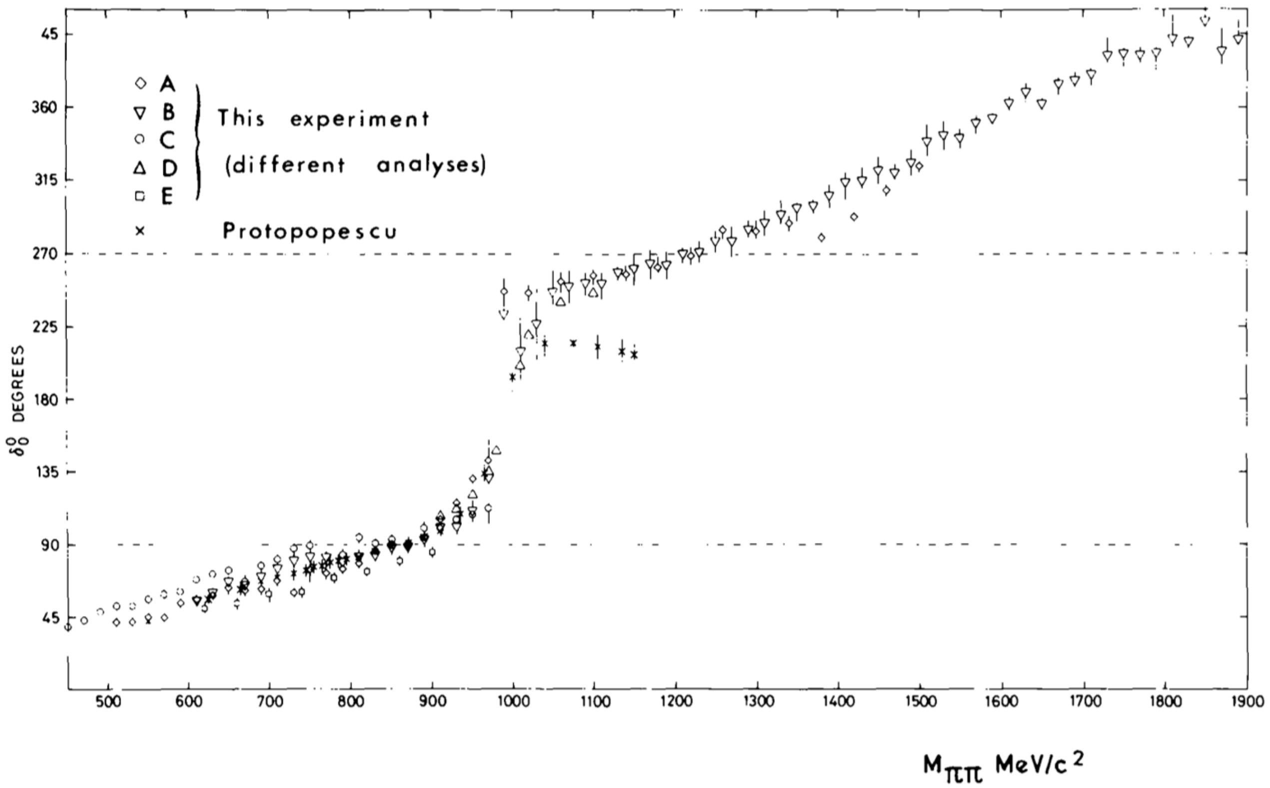
Hyams 1973

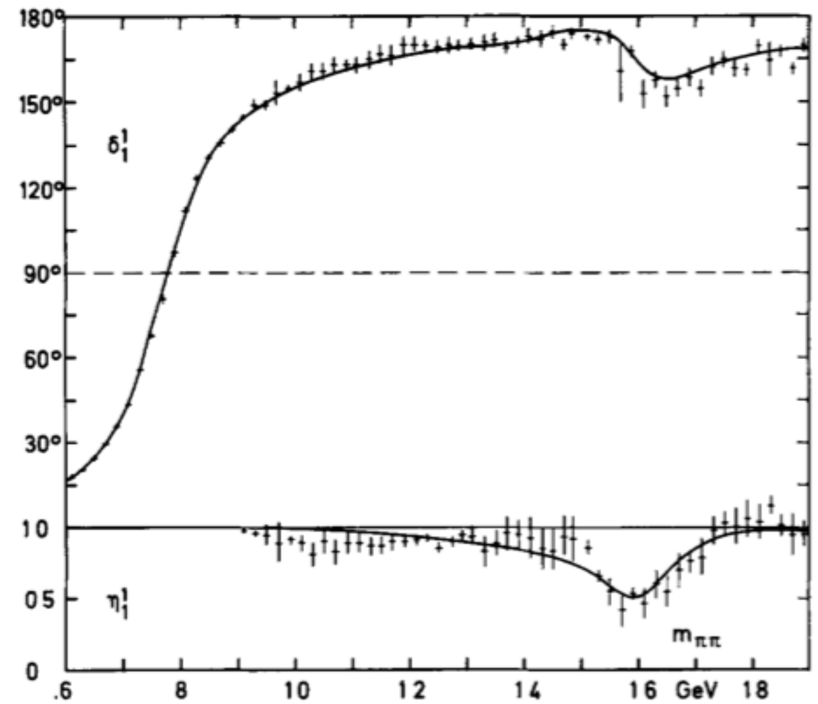


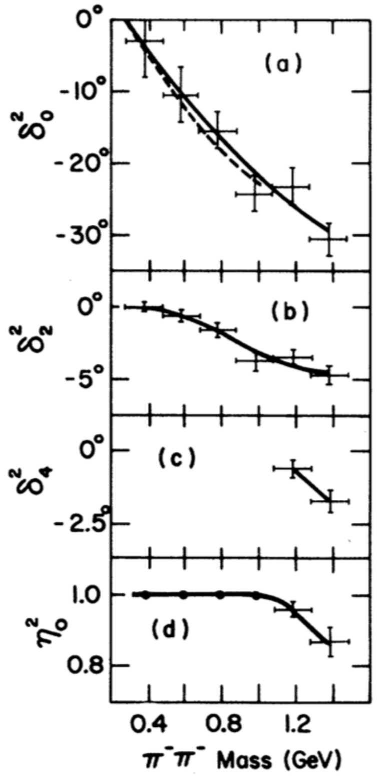
Cohen 1972

the simplest case: $\pi\pi$ elastic scattering

isospin=0	isospin=1	isospin=2







a first target: can a first-principles QCD calculation lead to this kind of behaviour ?

a next target: can we understand these behaviours in terms of resonances ?

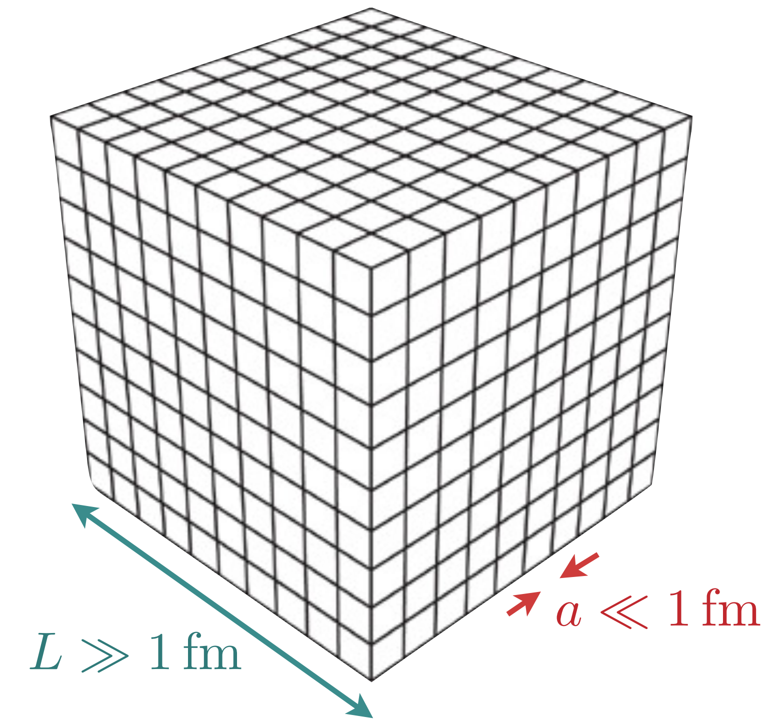
an ultimate target: can we understand the quark-gluon make-up of these resonances ?

lattice QCD

I'm going to assume you're familiar with the basic idea:

discretize the QCD action in Euclidean space-time i.e. $\bar{\psi}_{\mathbf{x}'t'} M_{\mathbf{x}'t',\mathbf{x}t}[U]\, \psi_{\mathbf{x}t}$
integrate out the quark fields
sample gauge field configurations according to a probability $\det M[U]\, e^{-S[U]}$

parameters:
— lattice spacing (just assume fine here)
— lattice volume (very important here!)
— quark masses (might not take physical values)



evaluate **correlation functions** on each configuration in the ensemble

energy eigenstates of the QCD Hamiltonian ?

embedded within two-point correlation functions $C_{ij}(t) = \langle 0 | \mathcal{O}_i(t)\, \mathcal{O}_j^\dagger(0) | 0 \rangle$

e.g. we expect the pion to be a QCD eigenstate with $E = m_\pi$ (in the rest frame)

compute $C_{ij}(t) = \langle 0 | \mathcal{O}_i(t)\, \mathcal{O}_j^\dagger(0) | 0 \rangle$

operator with pion quantum numbers
(color singlet, isospin=1, $J^P = 0^-$)
constructed from quark, gluon fields

$$\mathcal{O}_i(t) = e^{Ht}\, \mathcal{O}_i(0)\, e^{-Ht}$$

$$1 = \sum_{\mathfrak{n}} |\mathfrak{n}\rangle\langle\mathfrak{n}|$$

$$C_{ij}(t) = \sum_{\mathfrak{n}} e^{-E_{\mathfrak{n}} t}\, \langle 0 | \mathcal{O}_i(0) | \mathfrak{n} \rangle\, \langle \mathfrak{n} | \mathcal{O}_j^\dagger(0) | 0 \rangle$$

lowest energy eigenstate will be the pion

examine the time-dependence of the correlation function …

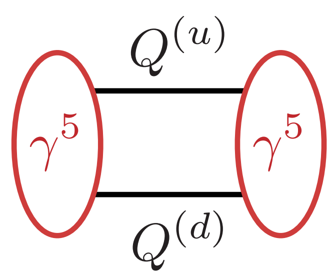
computing the pion mass

e.g. compute $C_{ij}(t) = \langle 0 | \mathcal{O}_i(t)\, \mathcal{O}_j^\dagger(0) | 0 \rangle$ with $\mathcal{O}(t) = \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}}\, \bar{u}_{\mathbf{x},t}\, \gamma^5\, d_{\mathbf{x},t}$

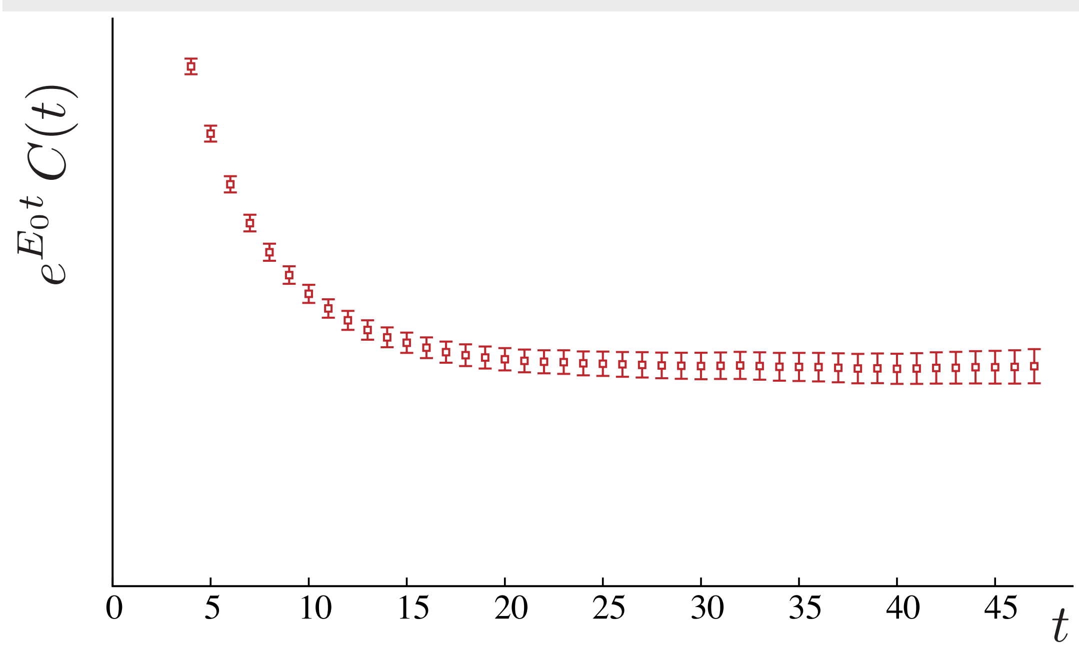
requires evaluation of $\mathrm{tr}\left[Q^{(u)}_{\mathbf{y}0;\mathbf{x}t}\, \gamma^5\, Q^{(d)}_{\mathbf{x}t;\mathbf{y}0}\, \gamma^5 \right]$ averaged over gauge-field configurations

propagator $Q = M^{-1}$

$\bar{\psi}_{\mathbf{x}'t'}\, M_{\mathbf{x}'t',\,\mathbf{x}t}[U]\, \psi_{\mathbf{x}t}$

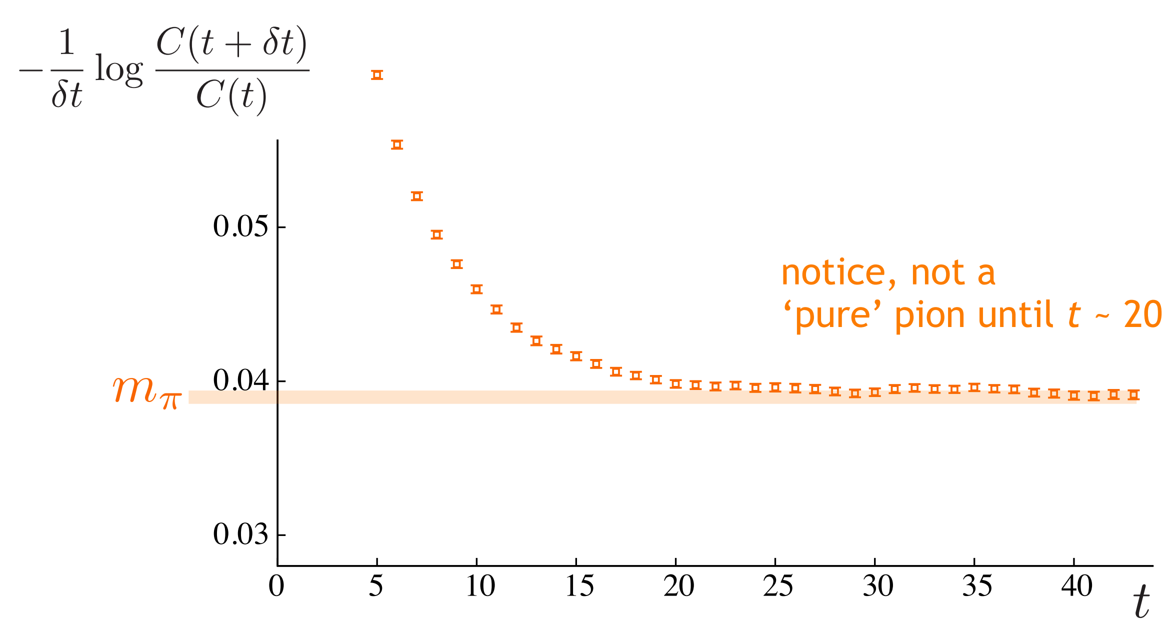


correlation function



$$C(t) = W_0\, e^{-E_0 t} + \ldots$$

effective mass plot



relatively straightforward to determine the 'ground-state' mass ...

what about 'excited' eigenstates ?

they're present in the sum: $C_{ij}(t) = \sum_{\mathfrak{n}} e^{-E_{\mathfrak{n}} t} \langle 0 | \mathcal{O}_i(0) | \mathfrak{n} \rangle \langle \mathfrak{n} | \mathcal{O}_j^\dagger(0) | 0 \rangle$

but why did we assume a **discrete** spectrum of states ? $1 = \sum_{\mathfrak{n}} |\mathfrak{n}\rangle\langle\mathfrak{n}|$

for **scattering**, the spectrum should be **continuous** !

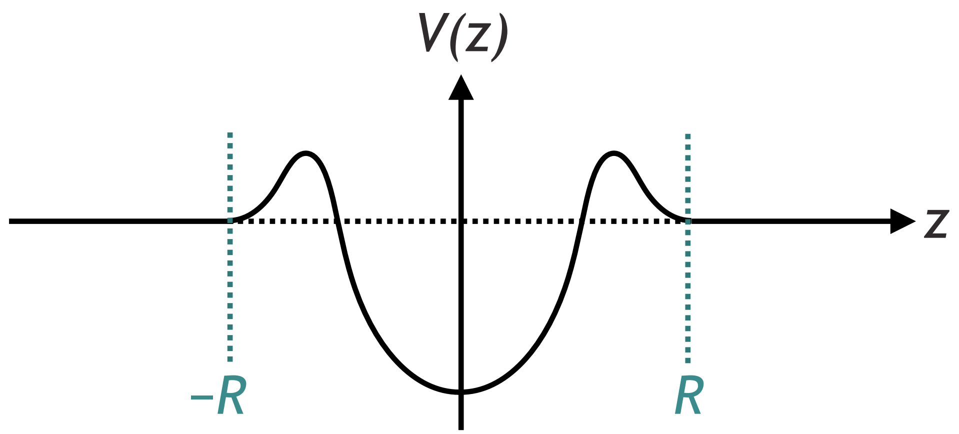
in fact the assumption of a discrete spectrum is correct …

most easily illustrated considering **one-dimensional quantum mechanics**

imagine two identical bosons separated by a distance z interacting through a finite-range potential *V(z)*

solve the Schrödinger equation

$$-\frac{1}{m}\frac{d^2\psi}{dz^2} + V(z)\psi(z) = E\psi(z)$$



scattering

most easily illustrated considering **one-dimensional quantum mechanics**

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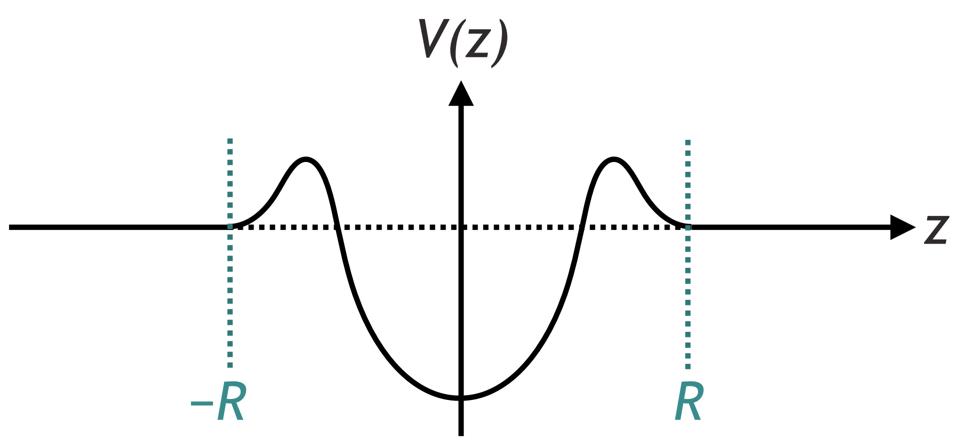

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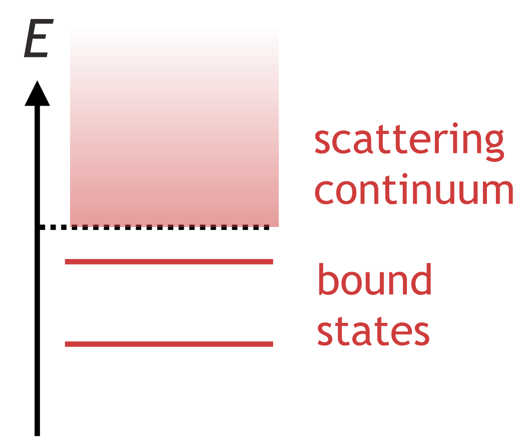
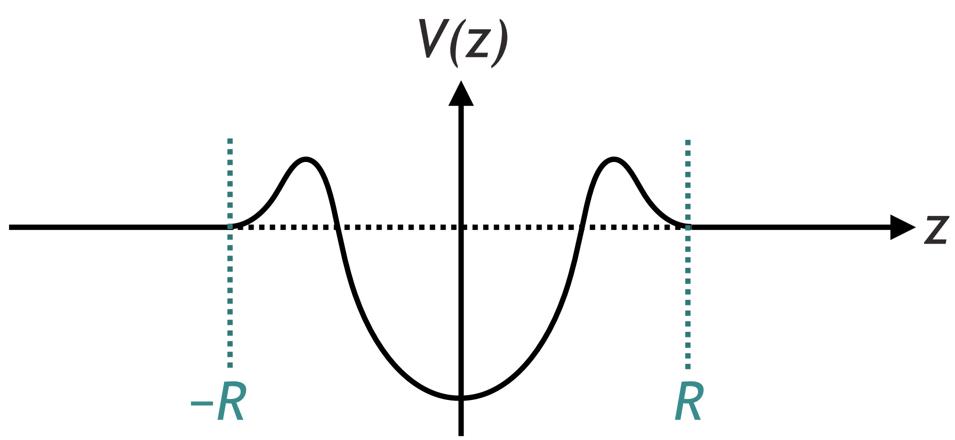


$$\psi(|z| > R) \sim \cos\big(\, p\,|z| + \delta(p)\,\big)$$

most easily illustrated considering **one-dimensional quantum mechanics**

imagine two identical bosons separated by a distance z interacting through a finite-range potential *V(z)*



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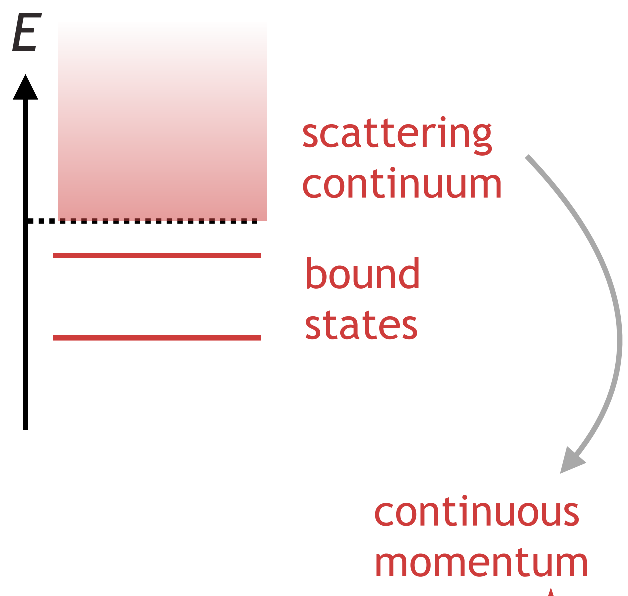
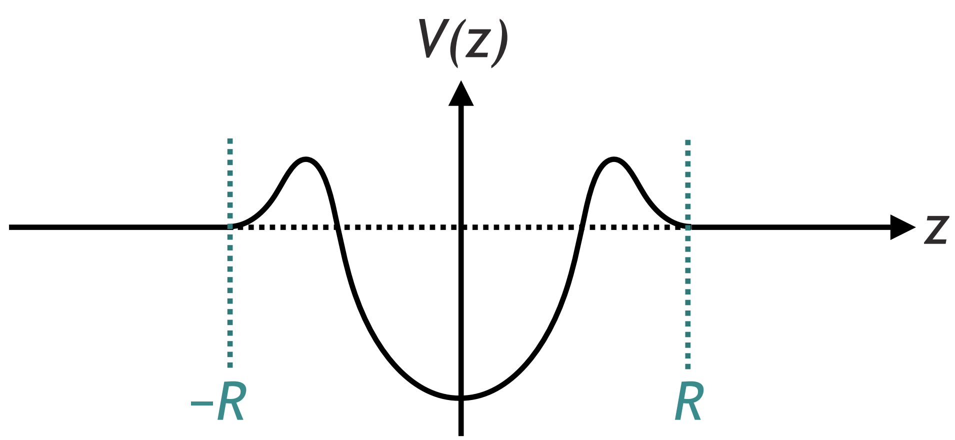


$$\psi(|z| > R) \sim \cos\big(p|z| + \delta(p)\big)$$

most easily illustrated considering **one-dimensional quantum mechanics**

imagine two identical bosons separated by a distance *z* interacting through a finite-range potential *V(z)*



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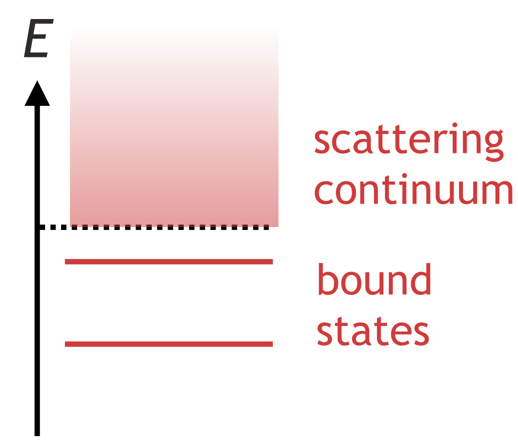
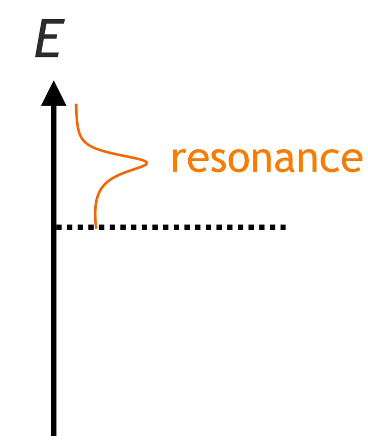
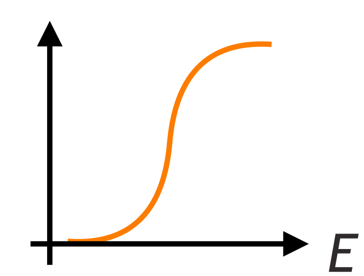




$$\psi(|z| > R) \sim \cos\left(p\,|z| + \delta(p)\right)$$

phase-shift

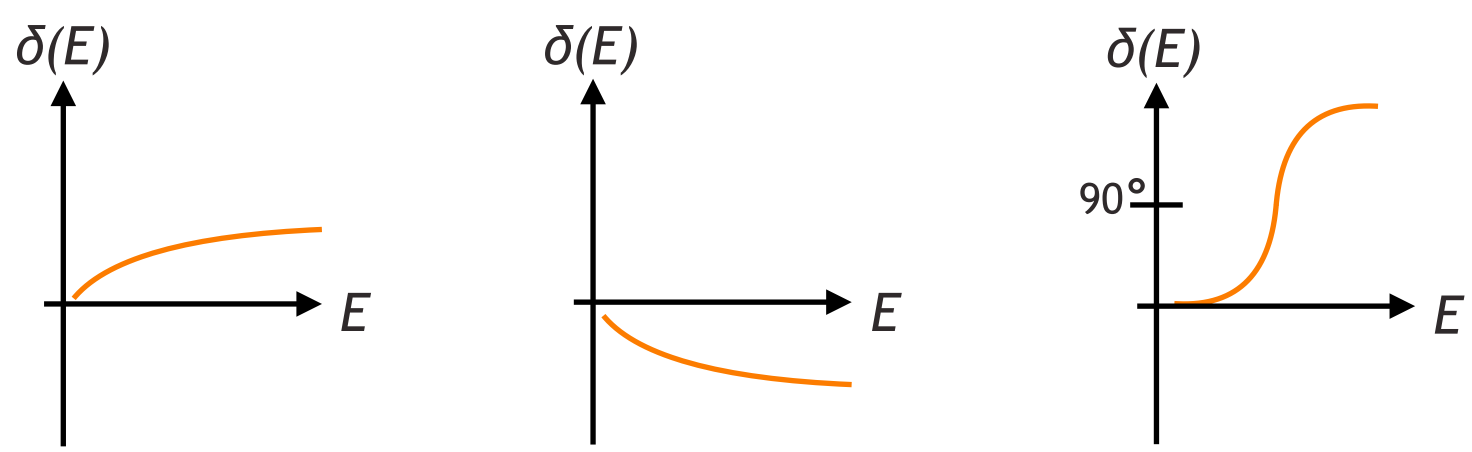


scattering

$$\psi(|z| > R) \sim \cos\left(p\,|z| + \delta(p)\right)$$

phase-shift

e.g.



'weak' attraction

'weak' repulsion

resonance

generally, consider S-matrix

$$|\mathrm{in}\rangle = S\,|\mathrm{out}\rangle$$

elastic scattering

$$S(E) = e^{2i\delta(E)}$$

‘scattering’ in a finite-volume

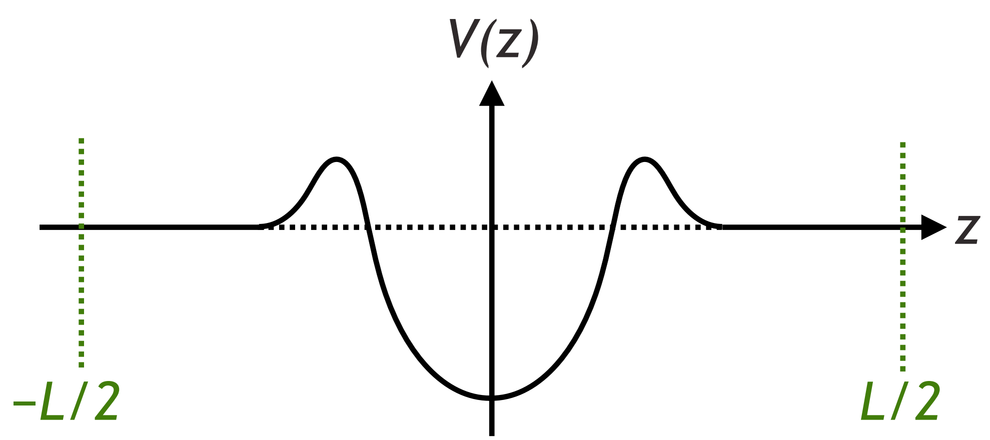
now put the system in a ‘box’ — periodic boundary condition at $z = \pm L/2$

$$\psi(|z| > R) \sim \cos\big(\, p\,|z| + \delta(p)\,\big)$$

$$\psi(L/2) = \psi(-L/2)$$
$$\frac{d\psi}{dz}(L/2) = \frac{d\psi}{dz}(-L/2)$$

momentum quantization condition

$$p = \frac{2\pi}{L} n - \frac{2}{L}\delta(p)$$



3+1 dim quantum field theory result

for elastic scattering in a cube the corresponding relationship is $\cot \delta_\ell(E) = \mathcal{M}_\ell\big(E(L), L\big)$

Lüscher 1986

⋮

many subsequent works
see the RMP for a complete list

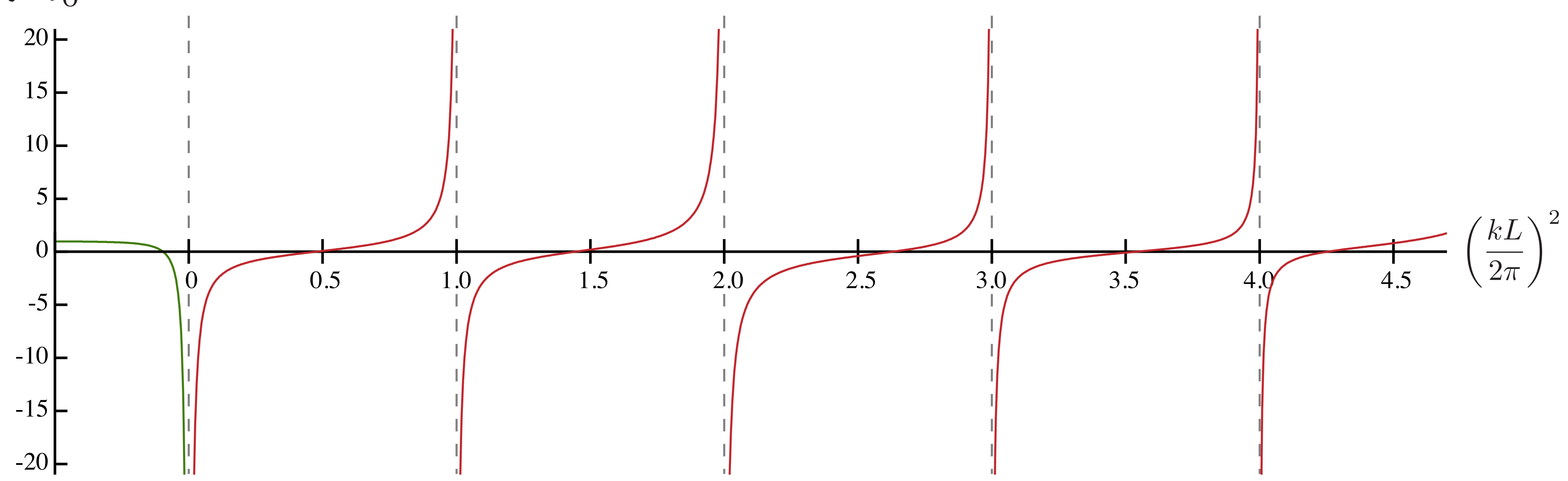
in the simplest case of a single partial wave being non-zero

will present some complications later …

$$\cot \delta_\ell(E) = \boxed{\mathcal{M}_\ell\big(E(L), L\big)}$$

known function expressing the 'kinematics' of the finite-volume

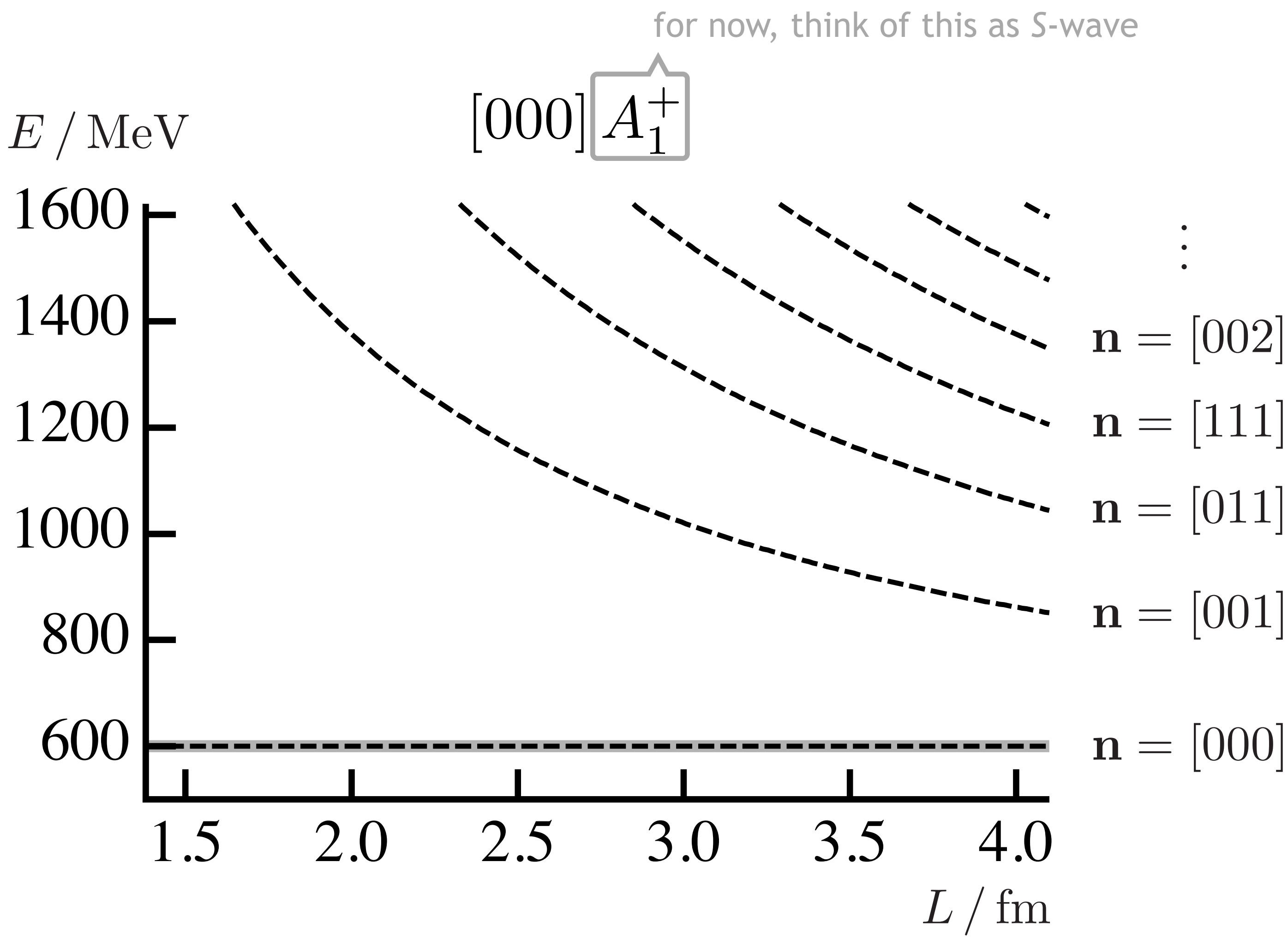
e.g. $\mathcal{M}_0$



so find the intersections of this curve with $\cot \delta(E)$

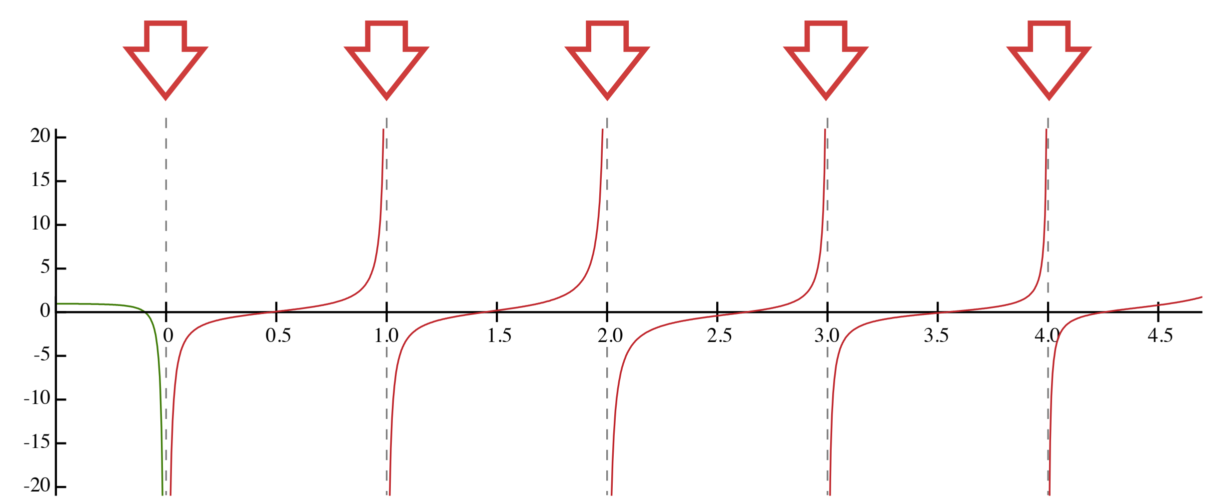
no interaction

scattering particles m = 300 MeV



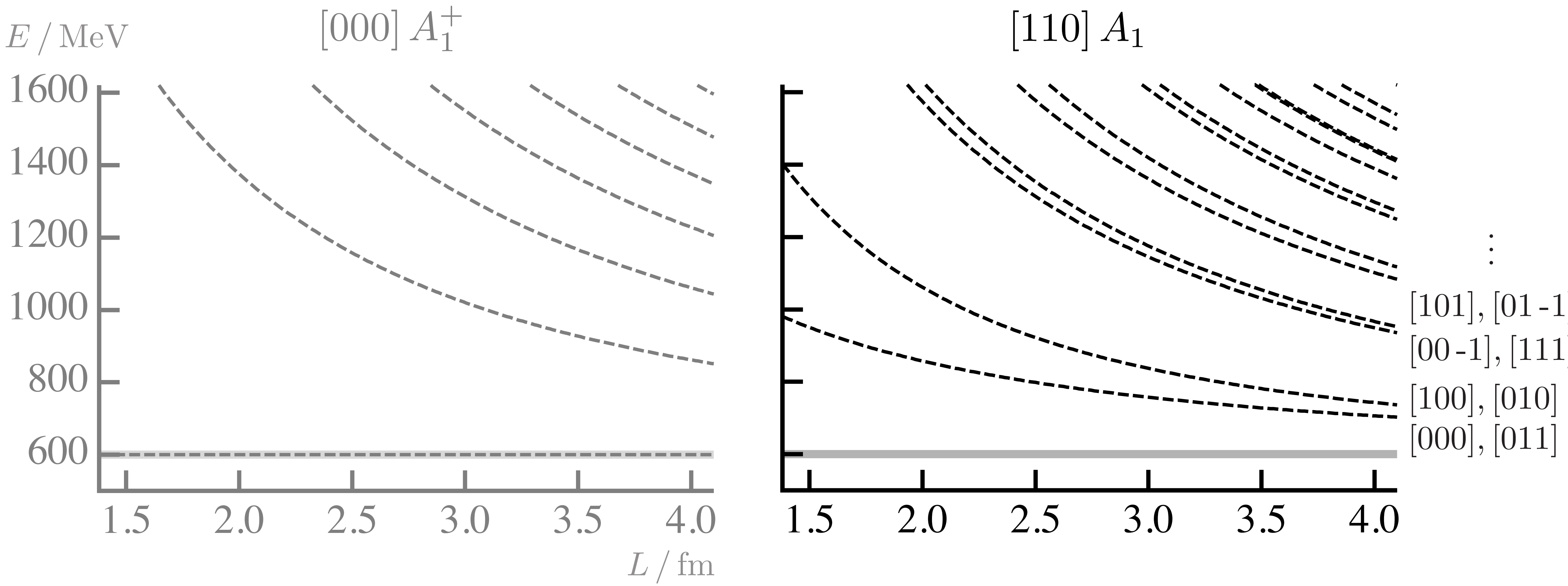
$$E_{\mathrm{ni}} = 2\sqrt{m^2 + \mathbf{p}^2} \qquad \mathbf{p} = \frac{2\pi}{L}\mathbf{n}$$

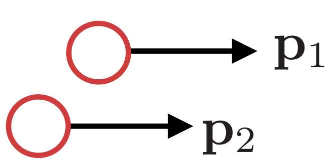
$$\cot\delta(E) = \mathcal{M}\big(E(L), L\big) \qquad \delta \to 0,\ \cot\delta \to \infty$$



no interaction – in a moving frame

$\cot \delta_\ell(E) = \mathcal{M}_\ell^{\mathbf{P}}(E(L), L)$

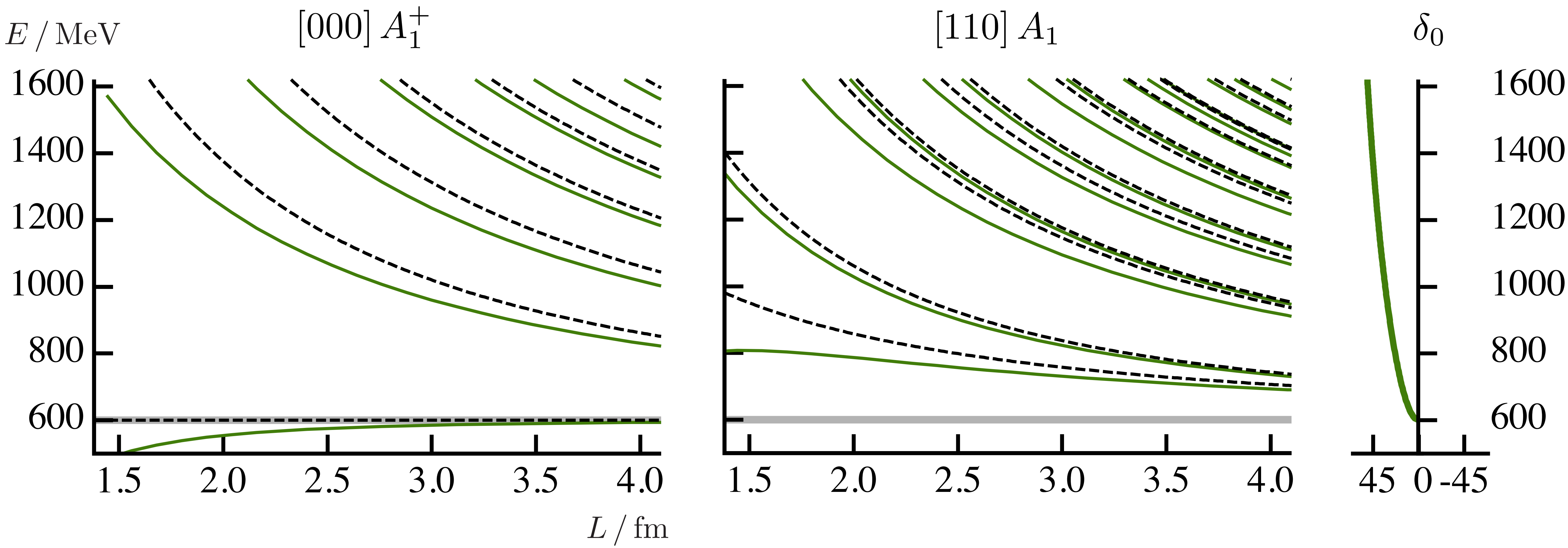




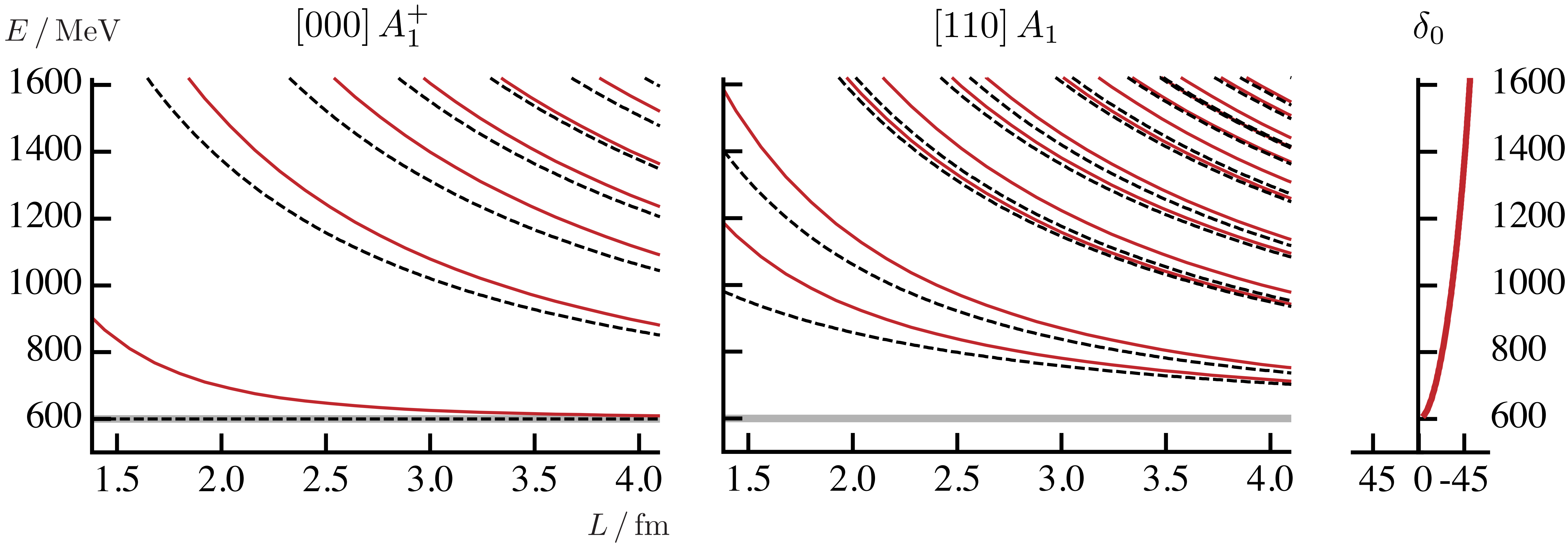
$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 \qquad \mathbf{P} = \frac{2\pi}{L}\mathbf{n}$$

$$E_{\mathrm{ni}} = \sqrt{m^2 + \mathbf{p}_1^2} + \sqrt{m^2 + \mathbf{p}_2^2} \qquad \mathbf{p}_{1,2} = \frac{2\pi}{L}\mathbf{n}_{1,2}$$

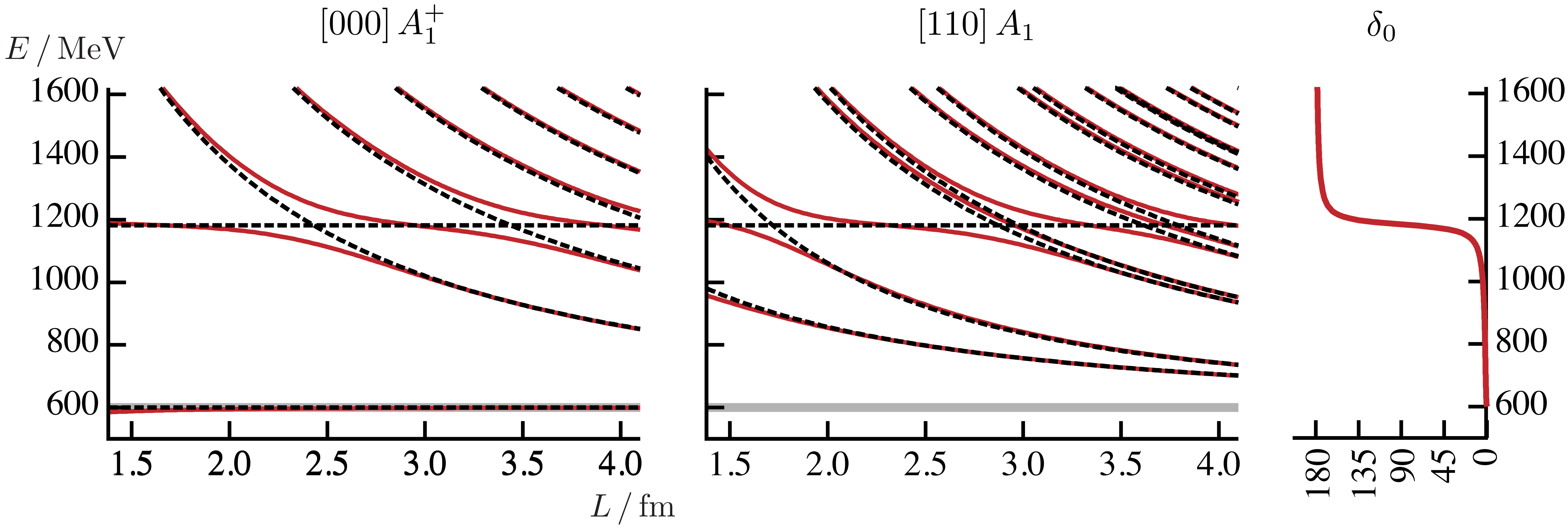
in the cm frame $E_{\mathrm{cm}} = \sqrt{E^2 - \mathbf{P}^2}$


E / MeV
[000] A_1^+
[110] A_1
δ_0
1600
1400
1200
1000
800
600
1.5
2.0
2.5
3.0
3.5
4.0
L / fm
45
0
-45

weak repulsion



an elastic resonance

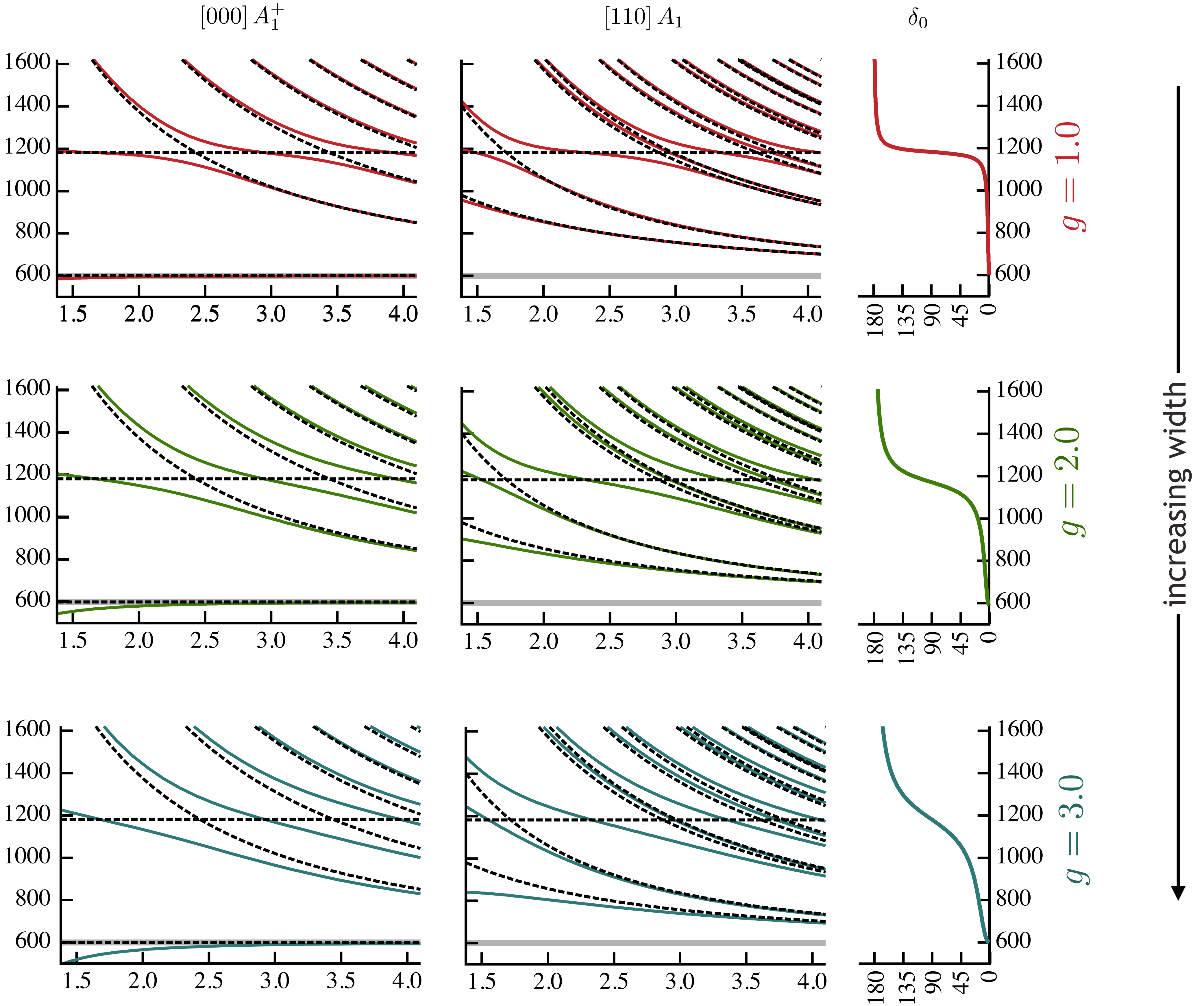


note the avoided level crossings

$$\tan\delta = \frac{E\,\Gamma(E)}{m_R^2 - E^2}$$

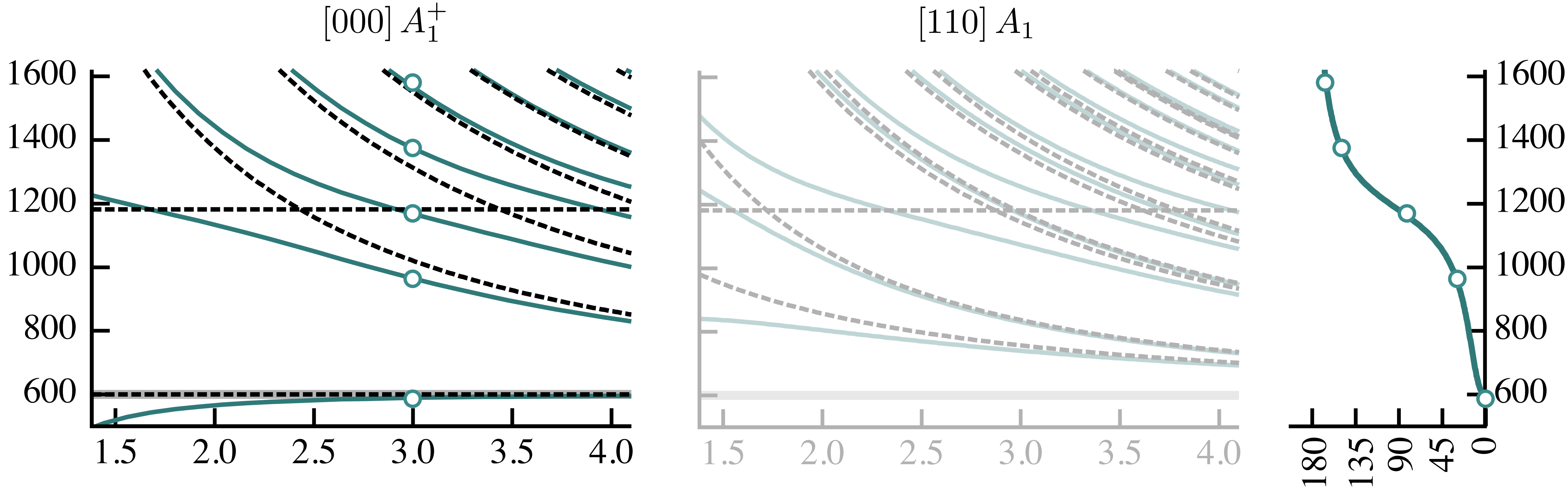
$$\Gamma(E) = \frac{g^2}{6\pi}\frac{m_R^2}{E^2}k(E)$$

elastic resonance

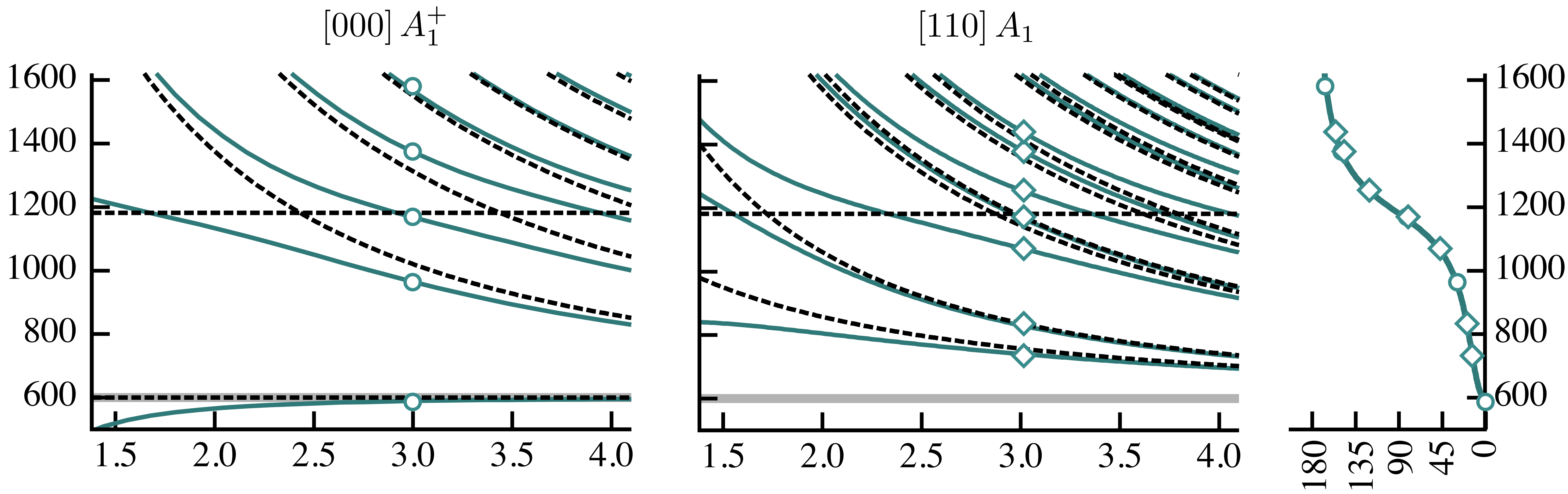


$$\Gamma(E) = \frac{g^2}{6\pi}\frac{m_R^2}{E^2}k(E)$$

an elastic resonance — finite-volume mapping



an elastic resonance — finite-volume mapping



determining the moving-frame spectrum provides much more information

job of lattice QCD: determine the discrete spectrum

we need to reliably determine excited state spectra in multiple volumes / in moving frames

spectrum information is in two-point correlation functions $\langle 0 | \mathcal{O}_i(t)\, \mathcal{O}_j(0) | 0 \rangle = \sum_{\mathfrak{n}} Z_i^{\mathfrak{n}}\, Z_j^{\mathfrak{n}*}\, e^{-E_{\mathfrak{n}} t}$

two-point correlation functions & the excited spectrum

conceptually straightforward, consider a single correlation function

$$\langle 0 | \mathcal{O}_i(t)\, \mathcal{O}_j(0) | 0 \rangle = \sum_{\mathfrak{n}} Z_i^{\mathfrak{n}}\, Z_j^{\mathfrak{n}*}\, e^{-E_{\mathfrak{n}} t}$$

provided the 'overlaps' $Z_i^{\mathfrak{n}} = \langle 0 | \mathcal{O}_i(0) | \mathfrak{n} \rangle$ are non-zero, every state in the spectrum contributes

fit to a sum of exponentials ?

in practice fitting to a sum of exponentials is unreliable:

– overlaps for some states might be very small (operator dependent)
– don't know how many states required in the sum
– (nearly) degenerate states can't be distinguished by *t*-dependence alone
– limited number of timeslices & statistical noise

a more powerful approach makes use of a basis of operators & linear algebra ...

‘variational’ approach

try to build an ‘optimum’ operator for state *n* as a linear superposition in some basis of operators

$$\Omega^\dagger = \sum_i v_i\, \mathcal{O}_i \qquad \Omega^\dagger|0\rangle = |\mathfrak{n}\rangle + \sum_{\mathfrak{m}\neq\mathfrak{n}} \epsilon_{\mathfrak{m}} |\mathfrak{m}\rangle$$

with the ϵ_m as small as possible

corresponding correlation function would be $\langle 0|\Omega(t)\,\Omega^\dagger(0)|0\rangle = e^{-E_{\mathfrak{n}} t} + \sum_{\mathfrak{m}\neq\mathfrak{n}} |\epsilon_{\mathfrak{m}}|^2\, e^{-E_{\mathfrak{m}} t}$

and we want to minimize this, by varying the v_i

$$\langle 0|\Omega(t)\,\Omega^\dagger(0)|0\rangle = \sum_{i,j} v_i^*\, \langle 0|\mathcal{O}_i(t)\,\mathcal{O}_j^\dagger(0)|0\rangle\, v_j = \sum_{i,j} v_i^*\, C_{ij}(t)\, v_j$$

need to avoid the trivial minimum $v_i = 0$ ⇒ constrain normalization $\sum_{i,j} v_i^*\, C_{ij}(t_0)\, v_j = N$

e.g. $N = 1$

implement constraint via a Lagrange multiplier

⇒ minimize $\Lambda = \sum_{i,j} v_i^*\, C_{ij}(t)\, v_j - \lambda\Big[\sum_{i,j} v_i^*\, C_{ij}(t_0)\, v_j - 1\Big]$

which leads to a generalized eigenvalue problem for ***v***

$$\mathbf{C}(t)\,\mathbf{v} = \lambda\,\mathbf{C}(t_0)\,\mathbf{v}$$

n^{th} eigenvalue

$\lambda_{\mathfrak{n}}(t) \sim e^{-E_{\mathfrak{n}}(t-t_0)}$

job of lattice QCD: determine the discrete spectrum

we need to reliably determine excited state spectra

in multiple volumes / in moving frames

spectrum information is in two-point correlation functions $\langle 0 | \mathcal{O}_i(t)\, \mathcal{O}_j(0) | 0 \rangle = \sum_{\mathfrak{n}} Z_i^{\mathfrak{n}}\, Z_j^{\mathfrak{n}*}\, e^{-E_{\mathfrak{n}} t}$

but what operators $\mathcal{O}$ should we consider ?

must be constructed
out of quark/gluon fields

easiest constructions with meson quantum numbers – fermion bilinears $\bar{\psi}\mathbf{\Gamma}\psi$

well motivated by
success of quark model

'looks' like a $q\bar{q}$ system

Wick
contractions

'annihilation'
required for isospin=0

quark propagation from t to t
⇒ matrix inversions on many t

job of lattice QCD: determine the discrete spectrum

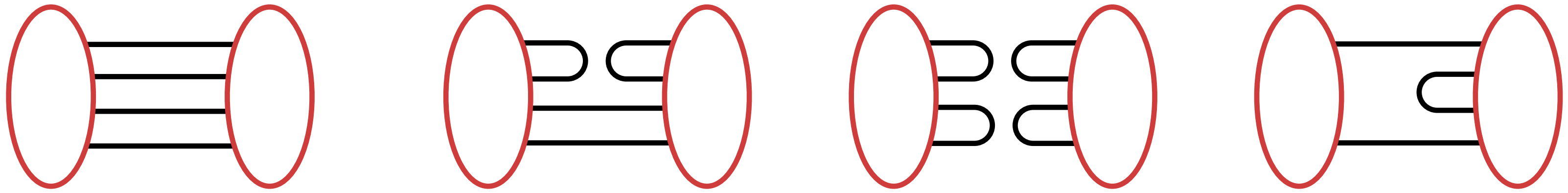
easiest constructions with meson quantum numbers – fermion bilinears $\bar{\psi}\mathbf{\Gamma}\psi$

but can also construct operators with more quark fields

e.g. ‘local’ tetraquark operators $\bar{\psi}_{\mathbf{x}}\bar{\psi}_{\mathbf{x}}\psi_{\mathbf{x}}\psi_{\mathbf{x}}$

e.g. ‘meson-meson’-like operators $\sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}}\,\bar{\psi}_{\mathbf{x}}\mathbf{\Gamma}\psi_{\mathbf{x}}\,\sum_{\mathbf{y}} e^{i\mathbf{q}\cdot\mathbf{y}}\,\bar{\psi}_{\mathbf{y}}\mathbf{\Gamma}'\psi_{\mathbf{y}}$

schematic Wick contractions



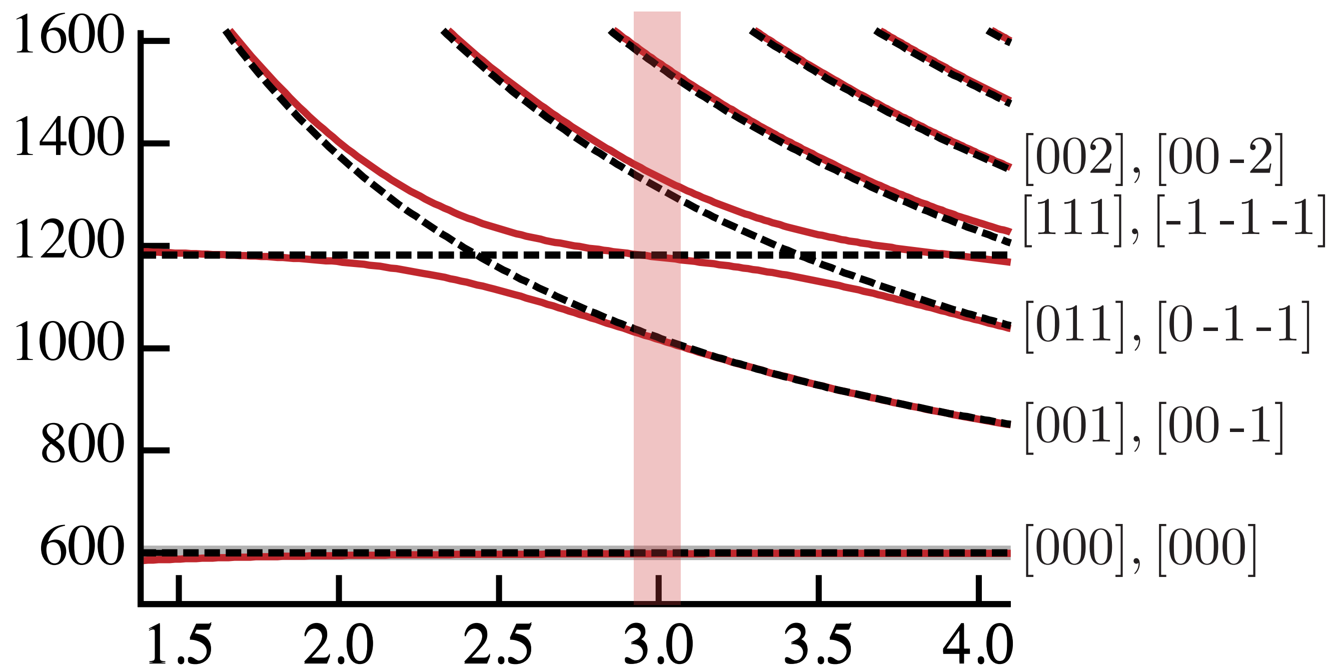
‘annihilation’ generally required

and can clearly include still more quark fields ad infinitum …

… is there some organizing principle
which suggests what operator basis we should use ?

the non-interacting spectrum as an operator basis guide

e.g. narrow resonance (in rest frame)



suppose we want to determine all states up to 1500 MeV on a 3 fm lattice

we might try an operator basis featuring 'meson-meson'-like operators with back-to-back momentum up to [111]

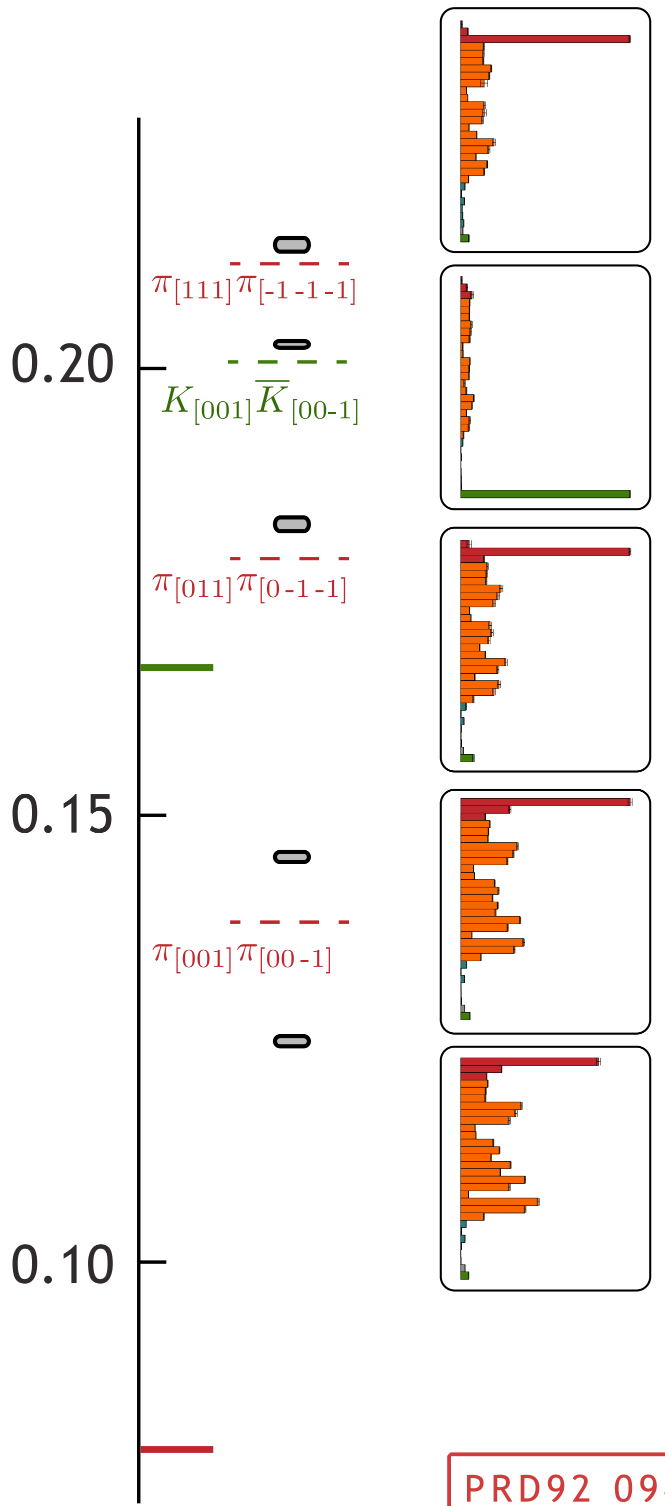
'look like' the expected meson-meson basis states

plus a set of $\bar{\psi}\Gamma\psi$ operators

'look like' a $q\bar{q}$-like basis state

isospin=1 T_1^- irrep spectrum

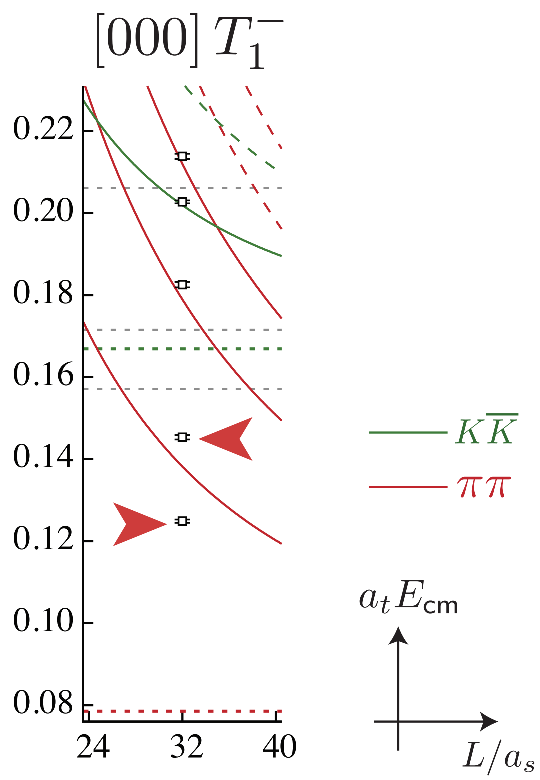
variational analysis of 30×30 correlation matrix: $3\times\pi\pi$, $26\times\bar{\psi}\Gamma\psi$, $1\times K\bar{K}$



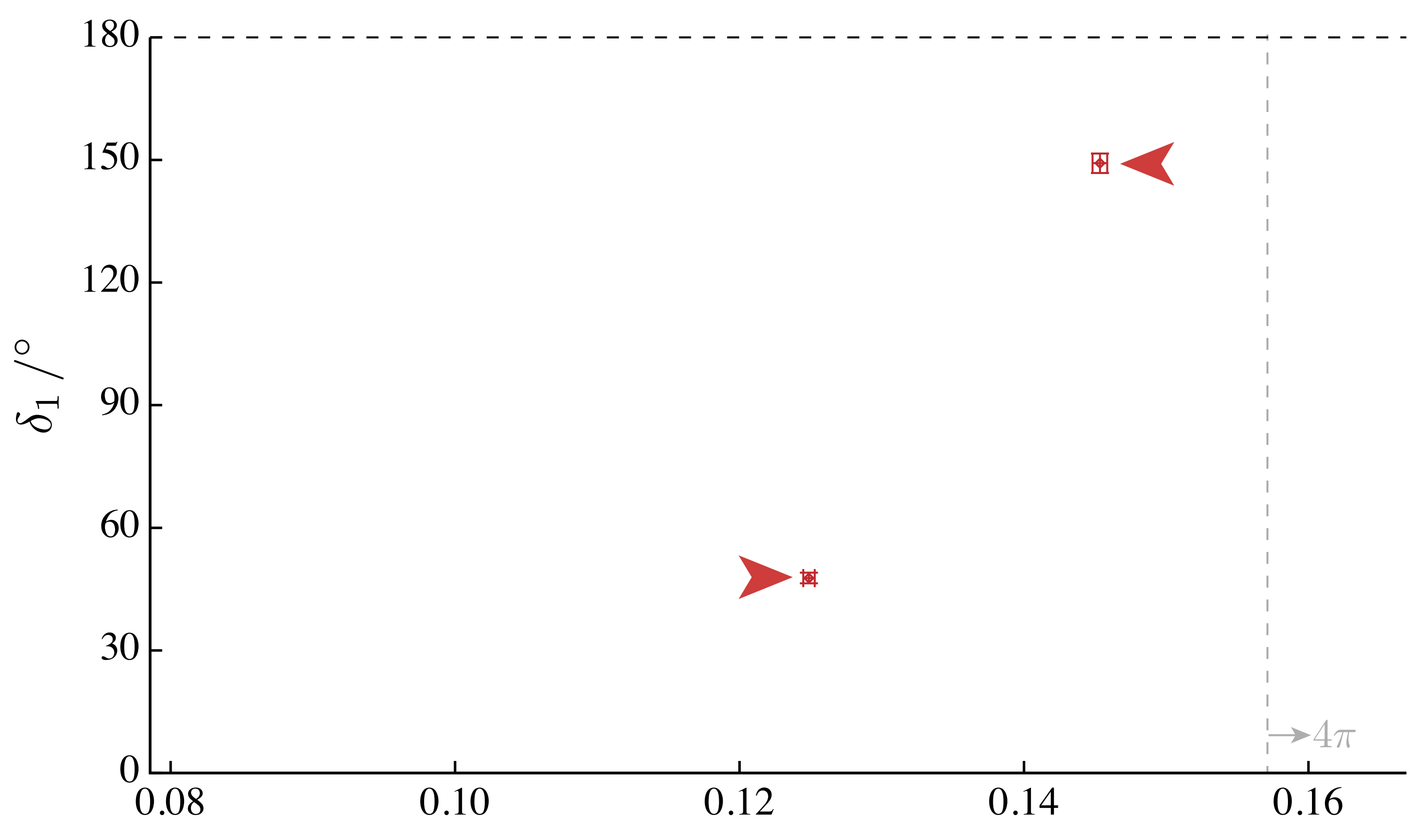
PRD92 094502 (2015)

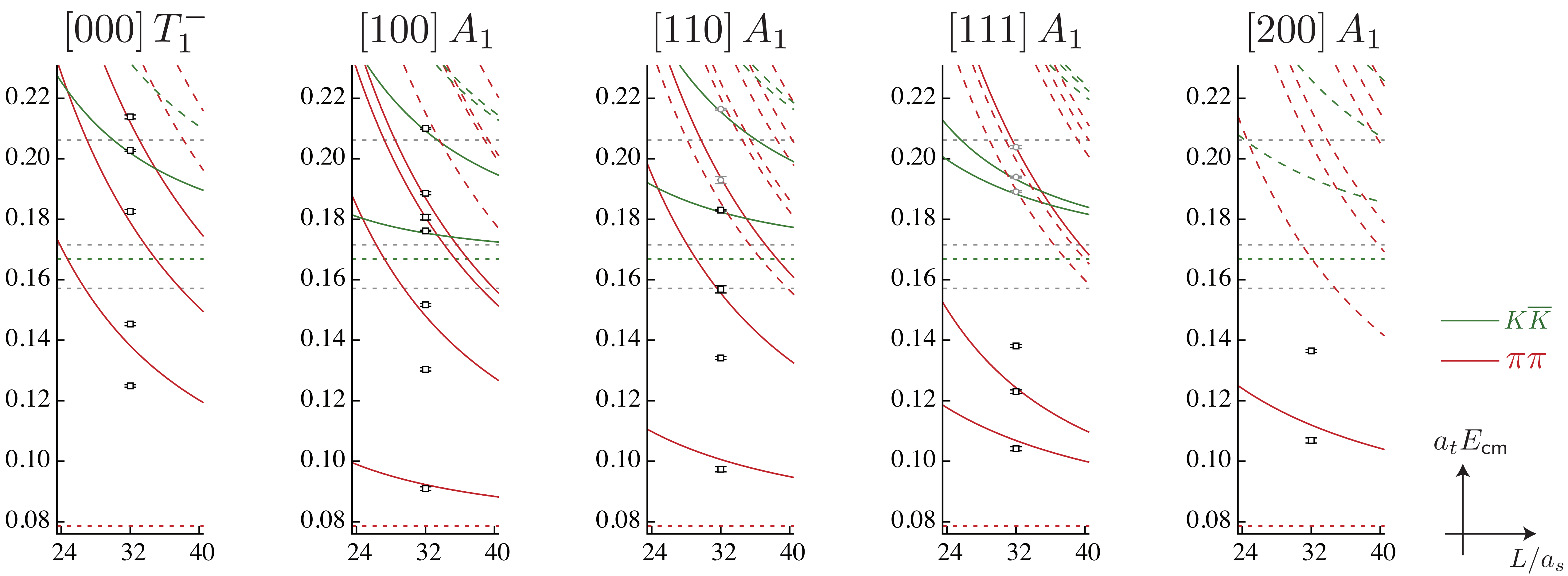
$m_\pi = 0.039$
$m_K = 0.083$

L ~ 3.8 fm

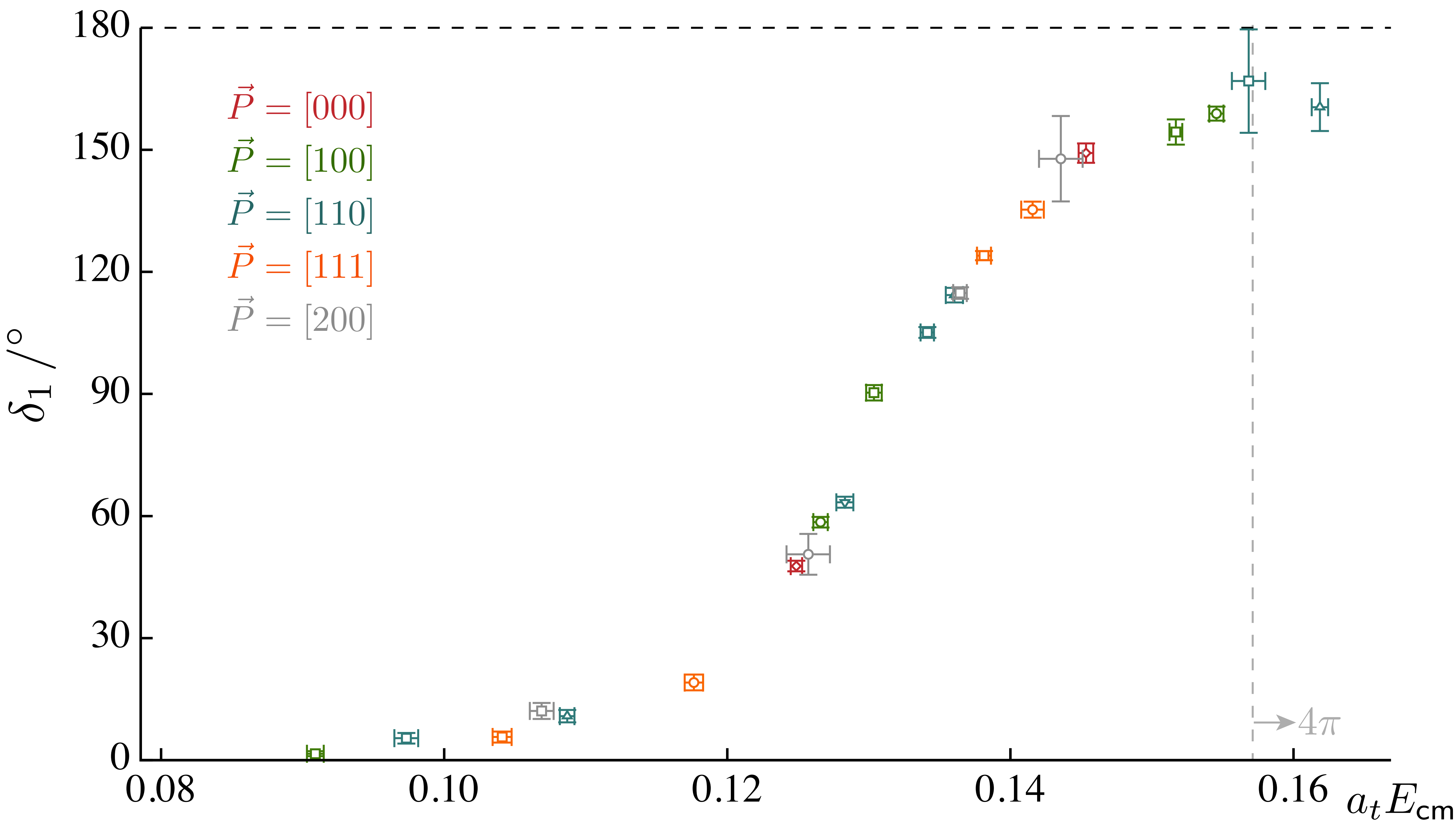


$$t(E) = \frac{1}{\rho(E)} e^{i\delta(E)} \sin\delta(E)$$




[000] T_1^-
[100] A_1
[110] A_1
[111] A_1
[200] A_1
0.22
0.20
0.18
0.16
0.14
0.12
0.10
0.08
24
32
40
$K\overline{K}$
$\pi\pi$
$a_t E_{\mathsf{cm}}$
L/a_s

… giving a phase-shift



some technical stuff — rotational symmetry

a finite cubic lattice has a smaller rotational symmetry group than an infinite continuum

simpler example of the problem: a rotationally symmetric two-dim system $\psi(r,\theta) = R_m(r)\, e^{im\theta}$

now considered on a square grid — minimum rotation is by $\pi/2$

m and $m+4n$ transform the same !

back in 3D — **irreducible representations** of the reduced symmetry group contain multiple spins

cubic symmetry

$\Lambda(\mathrm{dim})$	$A_1(1)$	$T_1(3)$	$T_2(3)$	$E(2)$	$A_2(1)$
J	$\mathbf{0}, 4\ldots$	$\mathbf{1}, 3, 4\ldots$	$\mathbf{2}, 3, 4\ldots$	$\mathbf{2}, 4\ldots$	$\mathbf{3}\ldots$

subduction $\big|\Lambda, \rho\big\rangle = \sum_m S^{\Lambda,\rho}_{J,m} \big|J, m\big\rangle$

for non-zero momentum it's even worse
— in continuum have **little group**, those rotations which don't change ***p***

⇒ label by **helicity**

can subduce helicity states into irreps of the reduced cubic symmetry

PRD85 014507 (2012)

reduction of rotational symmetry is an important feature of the quantization condition too

for elastic scattering, what we previously presented as $\cot\delta_\ell(E) = \mathcal{M}_\ell\big(E(L), L\big)$

should actually be $0 = \det\Big[\cot\delta_\ell\,\delta_{\ell,\ell'}\,\delta_{m,m'} - \mathcal{M}_{\ell m;\ell' m'}\Big]$

which when subduced becomes $0 = \det\Big[\cot\delta_\ell\,\delta_{\ell,\ell'}\,\delta_{n,n'} - \mathcal{M}^{\Lambda}_{\ell n;\ell' n}\Big]$

features **all ℓ** subduced into irrep Λ

n = embedding of ℓ into Λ

what allows us to make progress is that $\delta_\ell(E) \sim k^{2\ell+1}$ at energies not too far from threshold

so higher angular momenta are naturally suppressed

in practice, truncate at some $\ell_{\max}$ ...

some technical stuff — 'meson-meson'-like operators

what actually goes into a '$\pi\pi$'-like operator ?

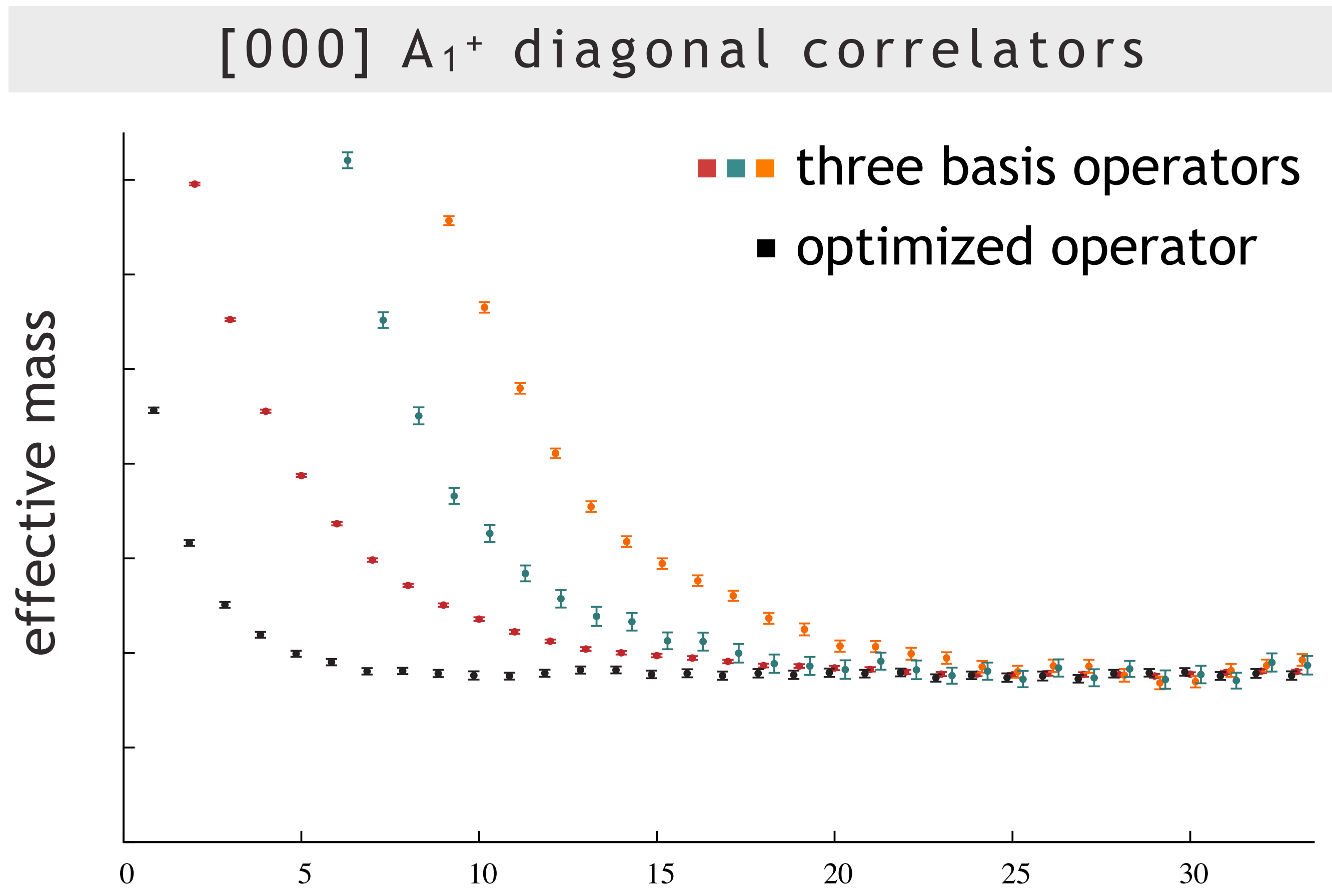
one option for construction is to use products of single-meson operators in lattice irreps

$$\sum_{\substack{\hat{\mathbf{p}}_1,\hat{\mathbf{p}}_2 \\ \mathbf{p}_1+\mathbf{p}_2=\mathbf{P}}} C_{\Lambda_1\otimes\Lambda_2\to\Lambda}(\hat{\mathbf{p}}_1,\hat{\mathbf{p}}_2)\,\pi(\mathbf{p}_1;\Lambda_1)\,\pi(\mathbf{p}_2;\Lambda_2)$$

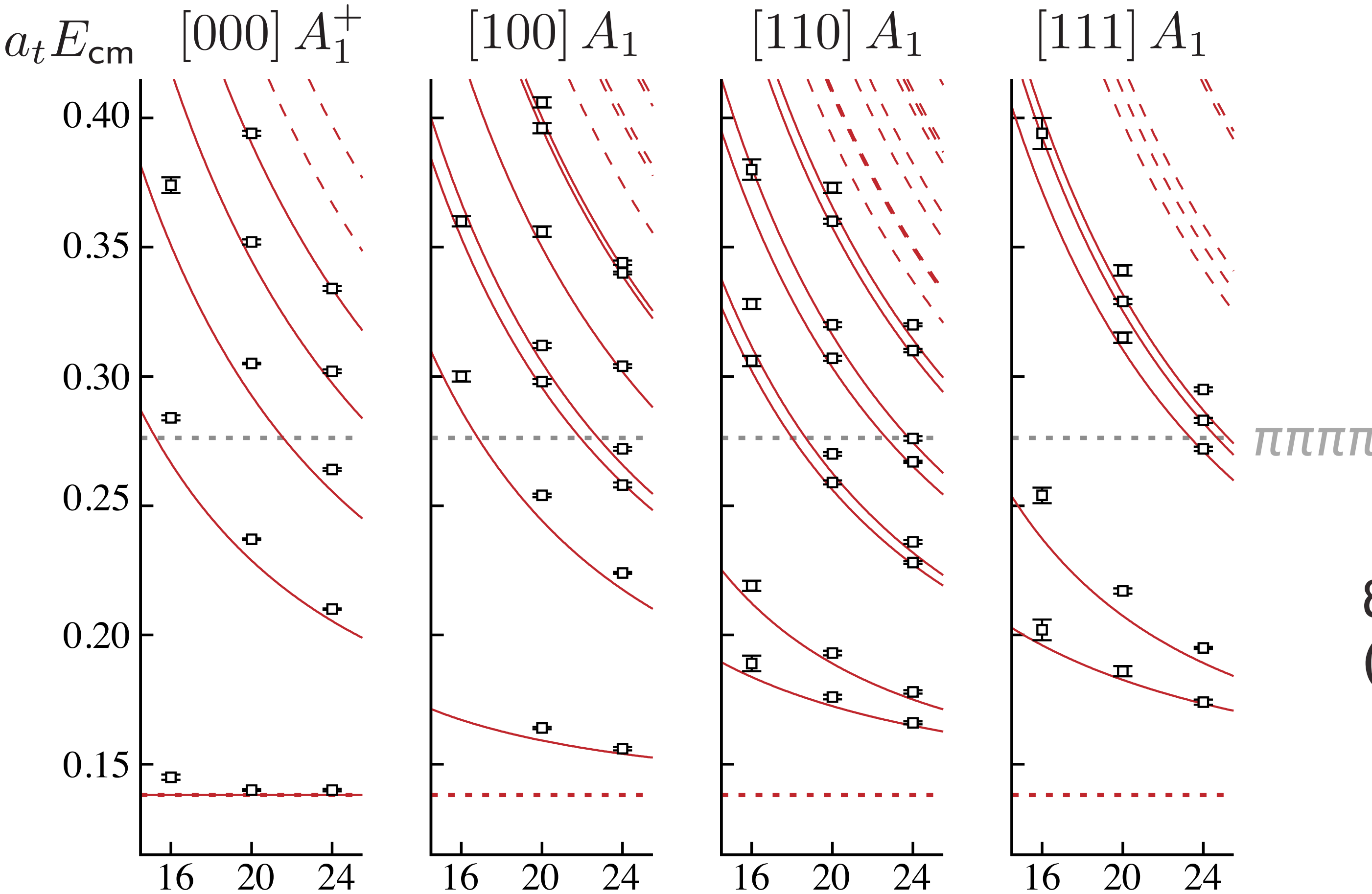
'lattice' Clebsch-Gordan coefficients

some group theory to work them out

then each single-meson operator can be the **variationally optimized** one for that ***p***, *Λ*



optimized operator saturated by the pion by timeslice 7



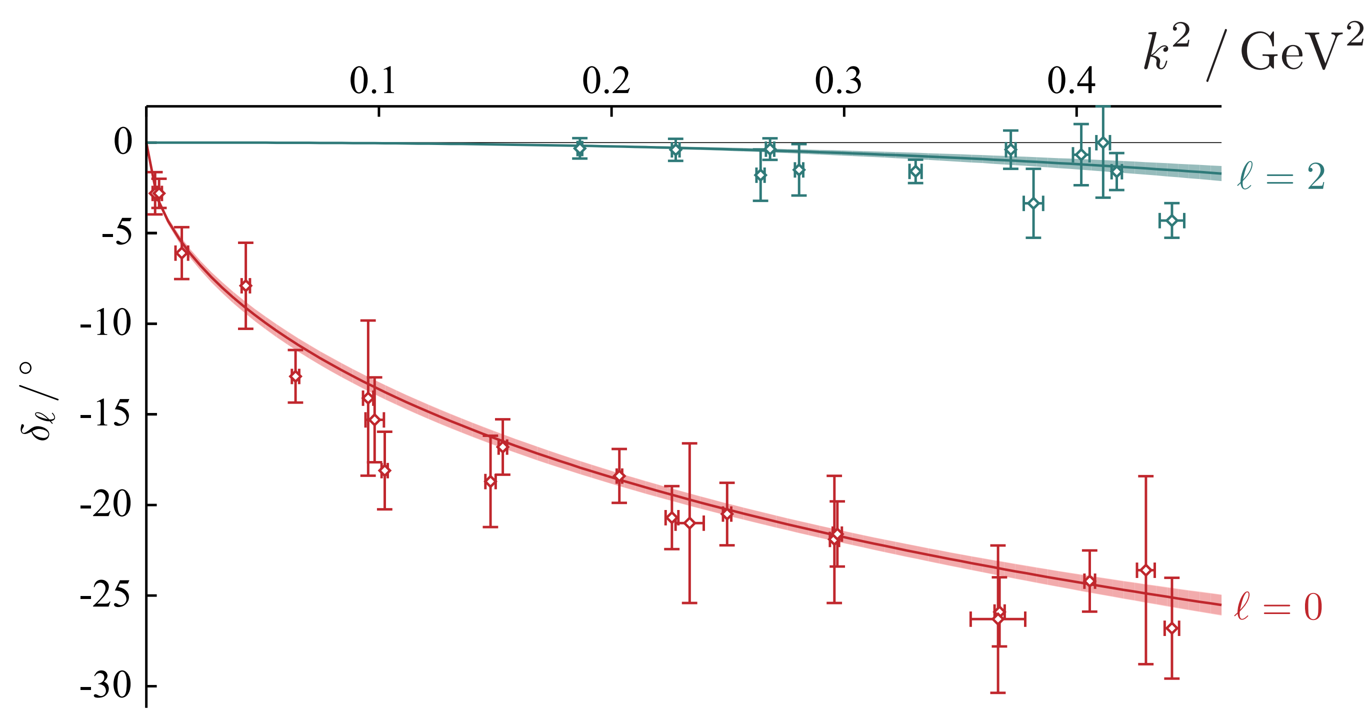
basis of $\pi\pi$ operators only

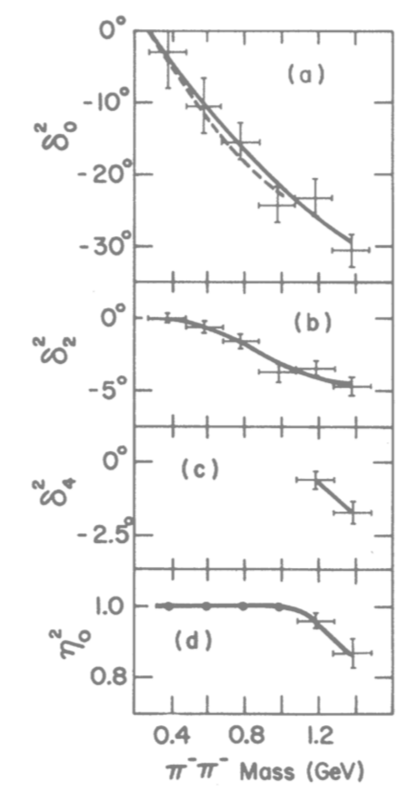
& spectra in irreps with lowest ℓ=2
(not shown here)

effective range expansion

$$k \cot \delta_0 = \frac{1}{a_0} + \frac{1}{2} r_0 k^2 + \ldots$$

$$t \propto \frac{1}{k \cot \delta_0 - ik}$$

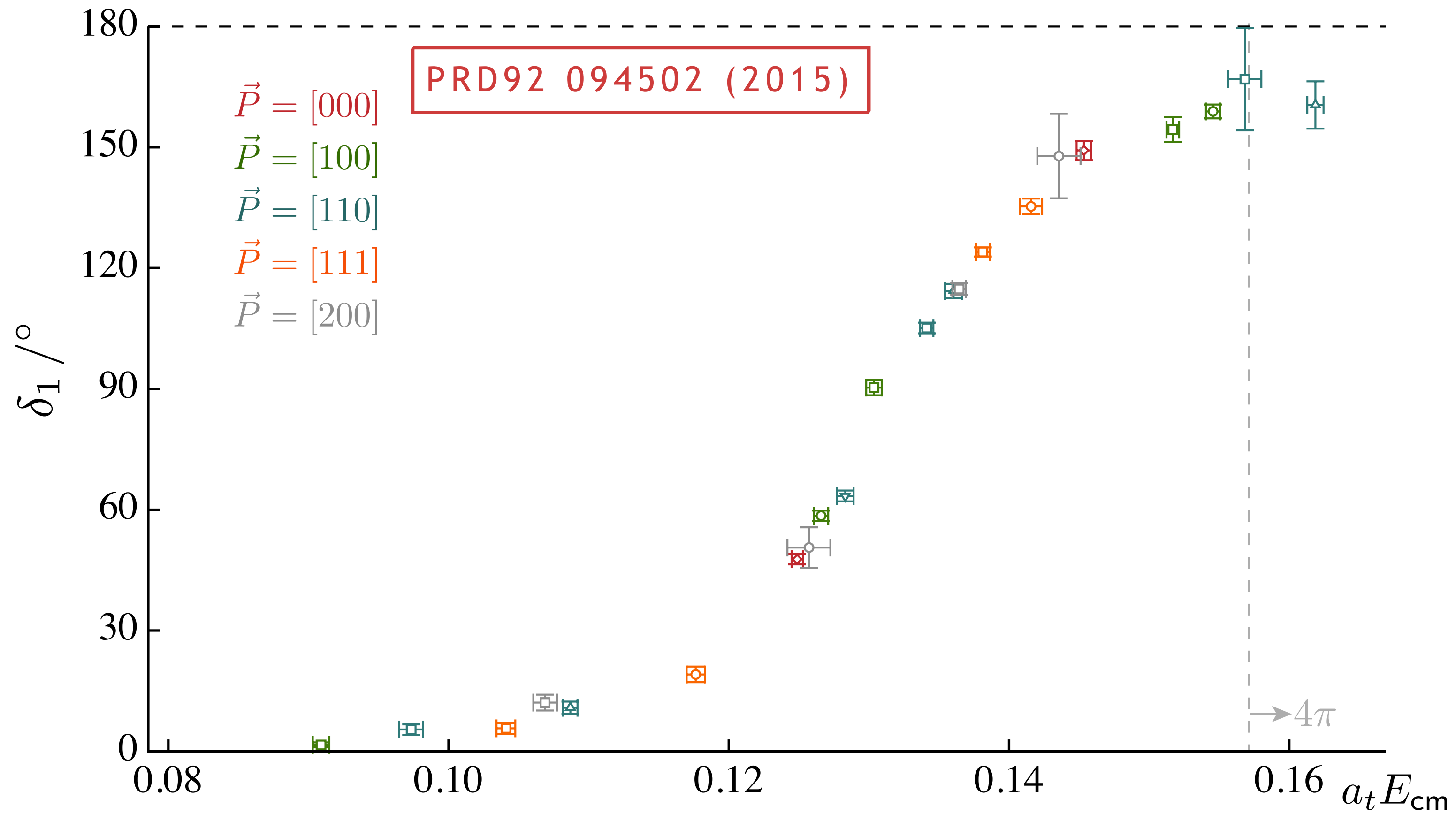




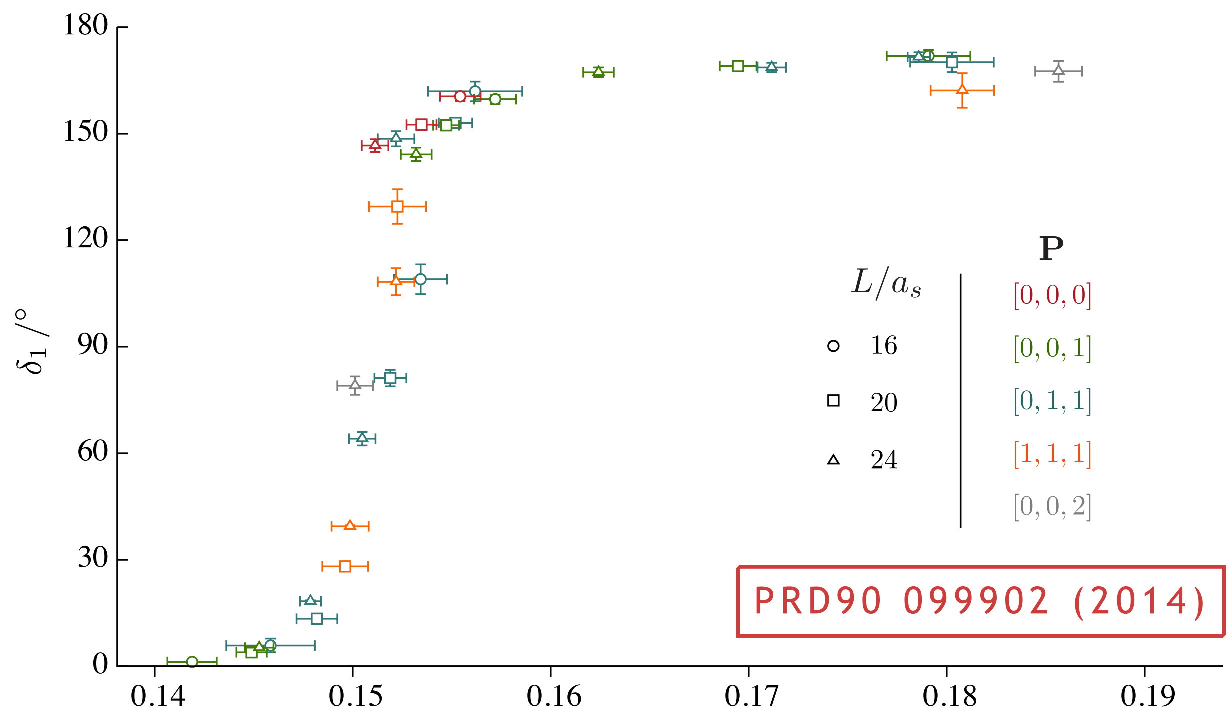
Cohen 1972

$\pi\pi$ isospin=1 — m_π~236 MeV, m_π~391 MeV

you saw this earlier ...

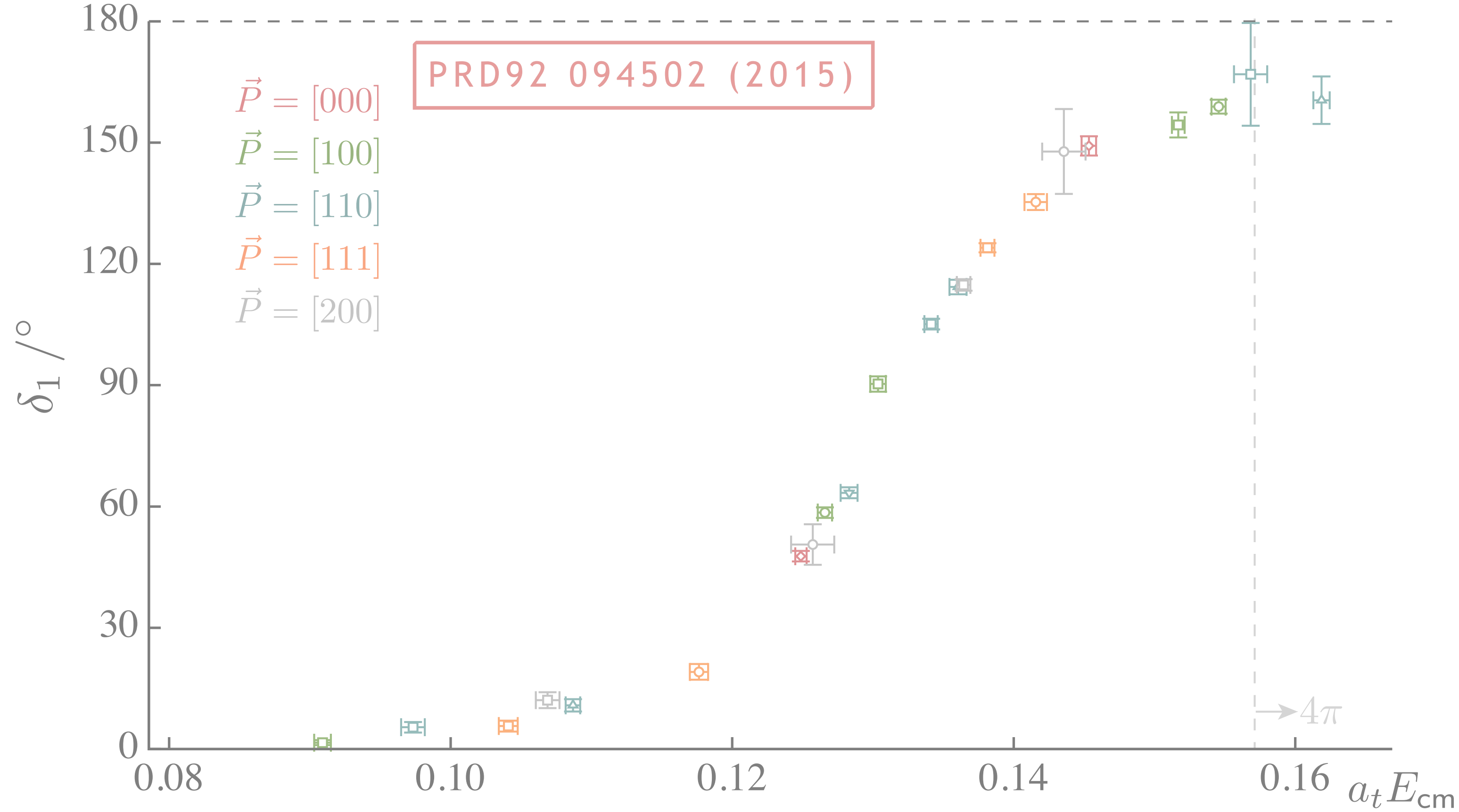


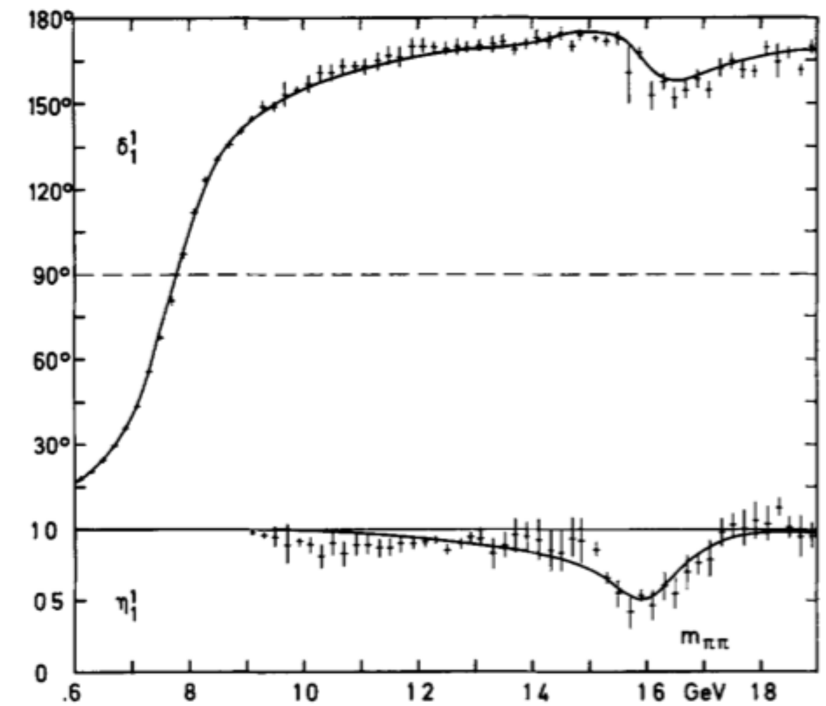
and a similar calculation at a heavier pion mass



$\pi\pi$ isospin=1 — m_π~236 MeV, m_π~391 MeV

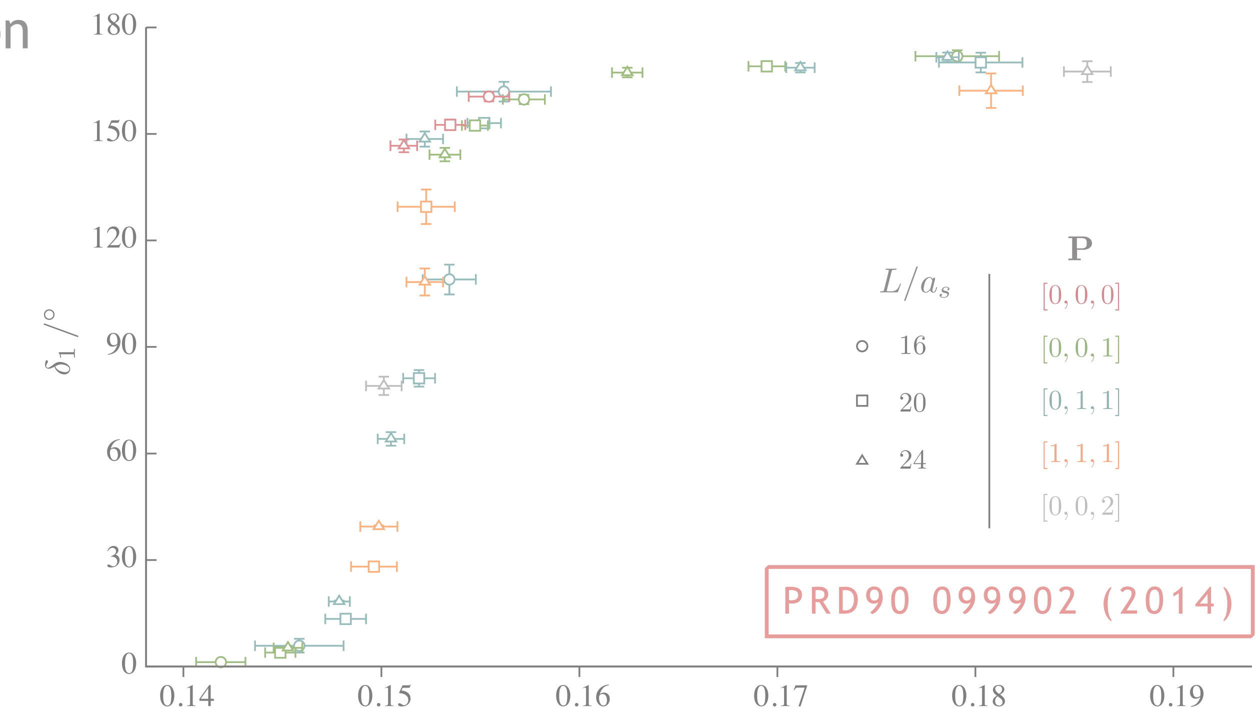
you saw this earlier ...



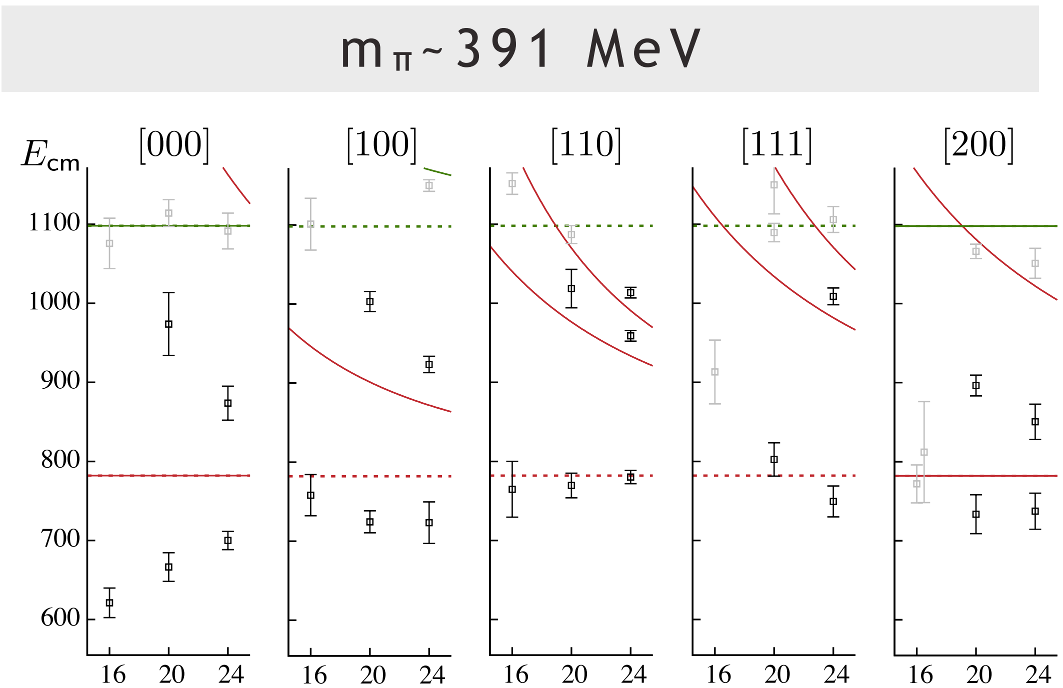


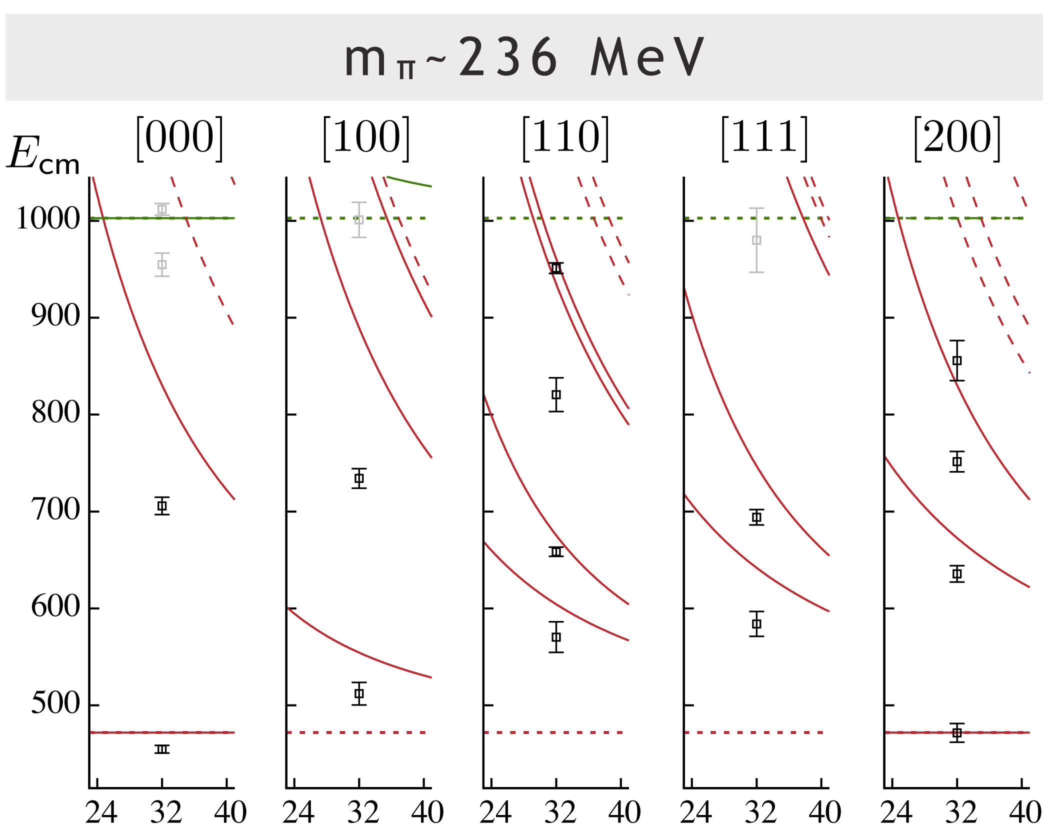
Hyams 1973

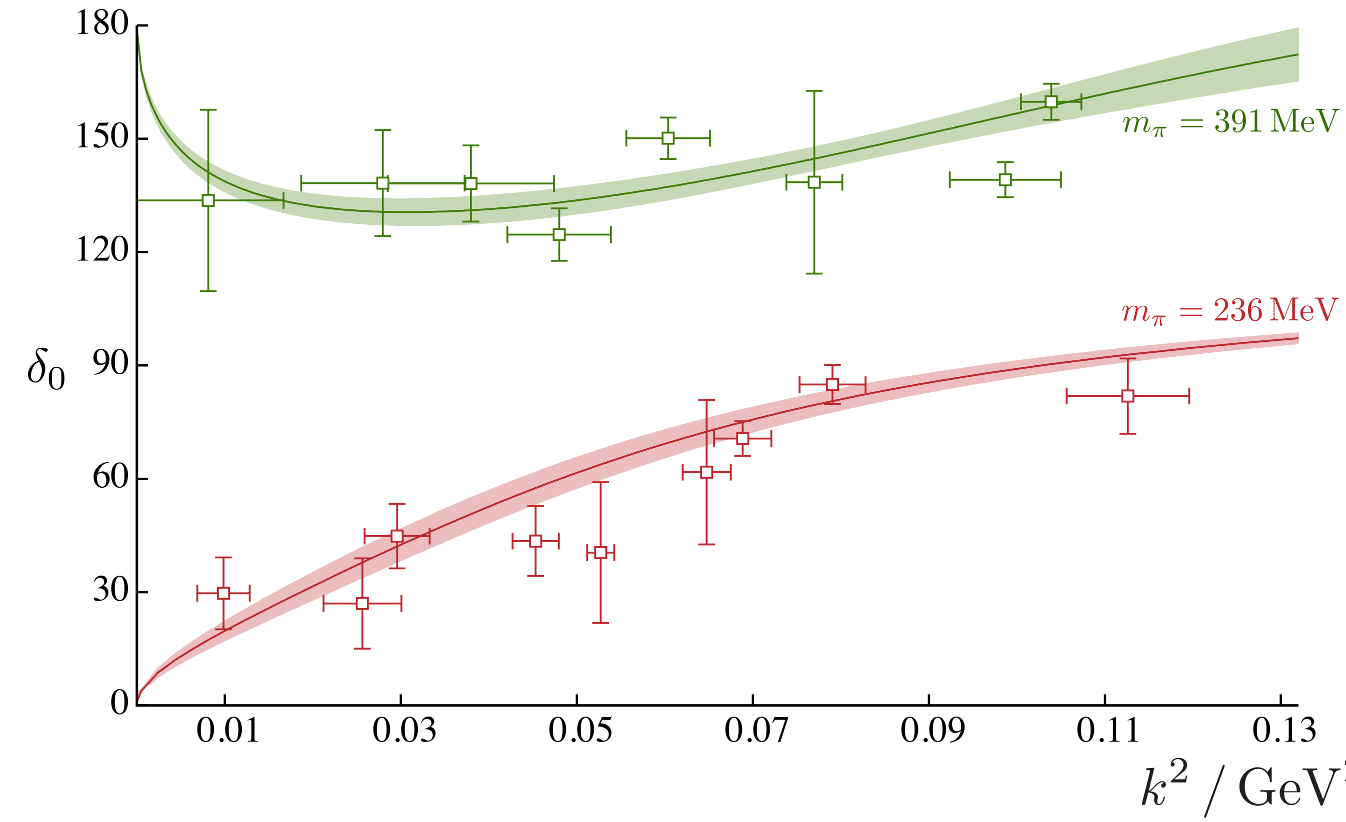
and a similar calculation at a heavier pion mass

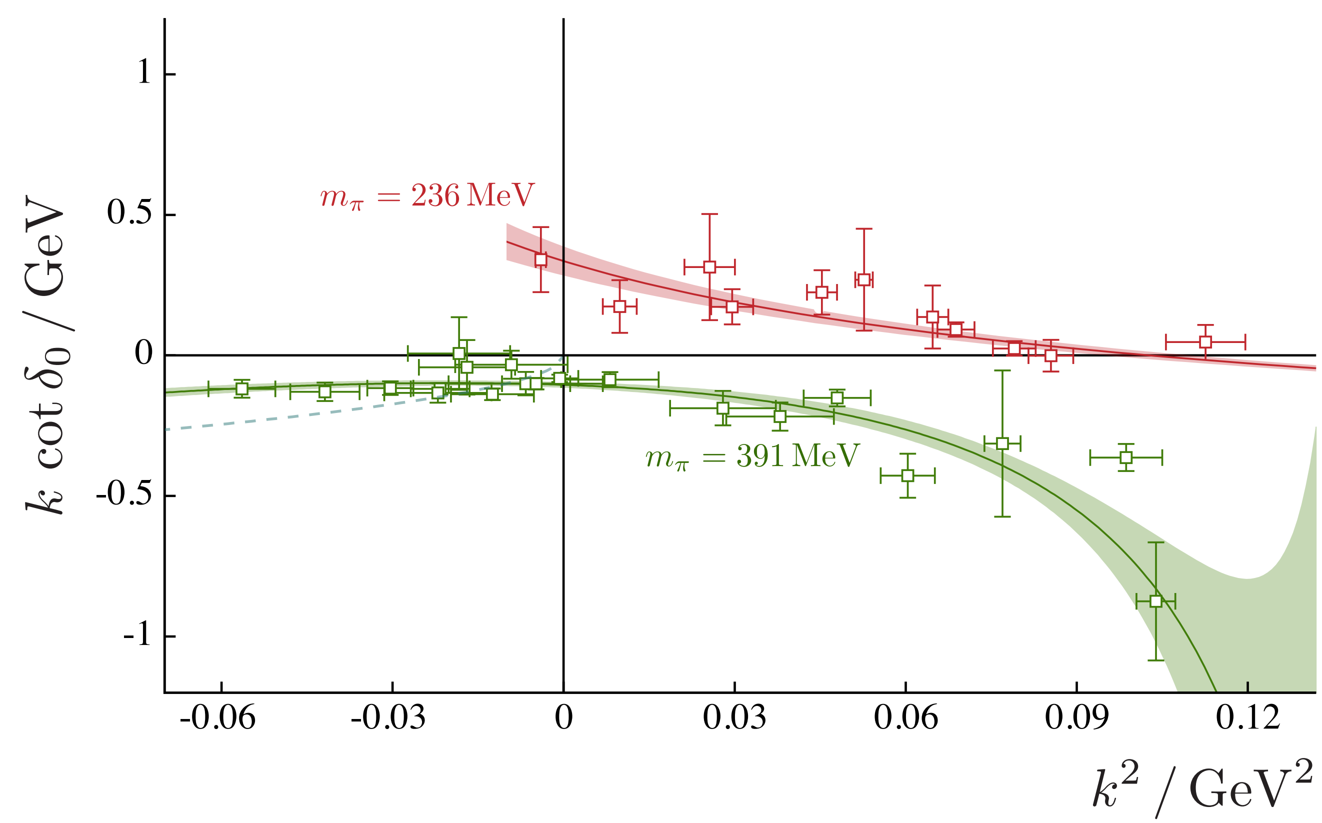


ππ isospin=0

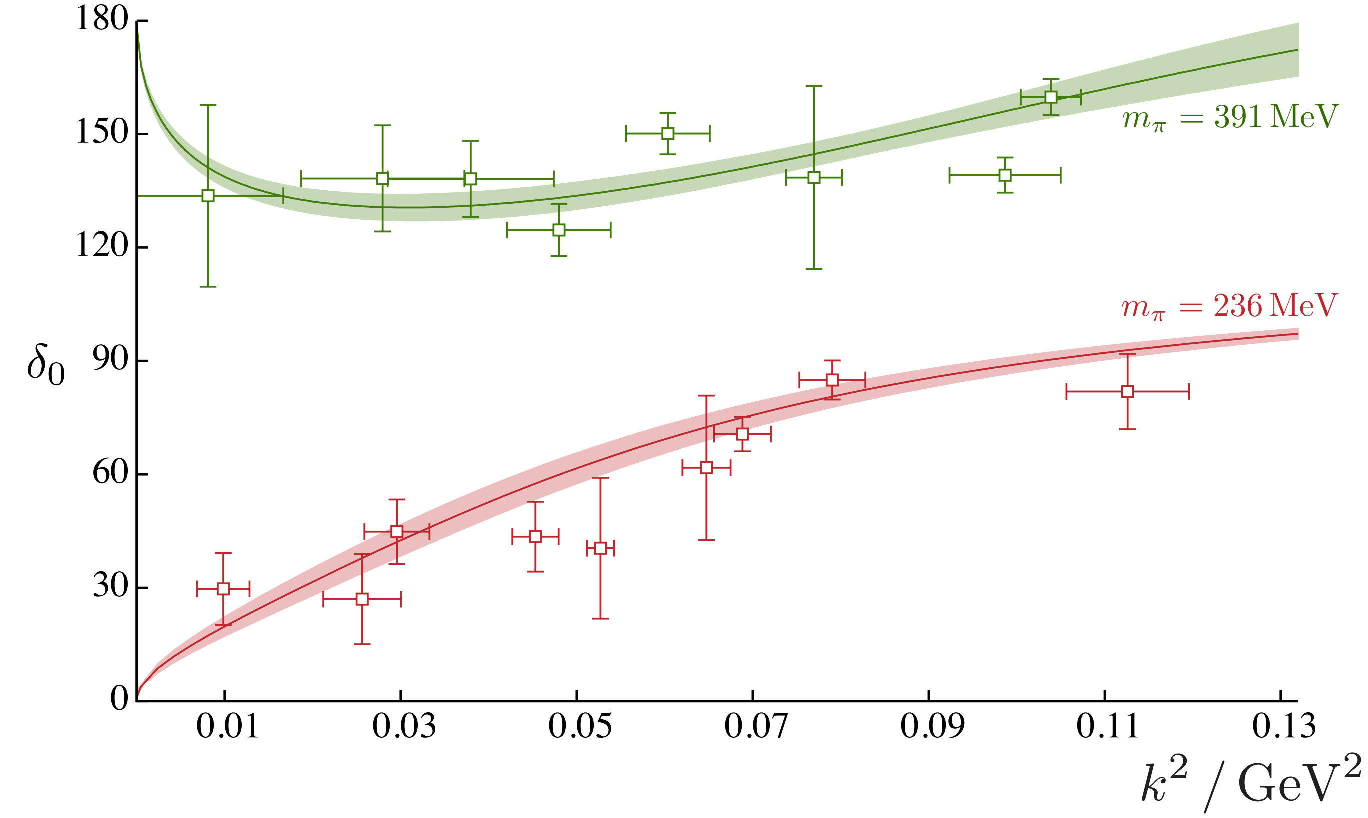

mπ~391 MeV
Ecm
[000]
[100]
[110]
[111]
[200]
1100
1000
900
800
700
600
16
20
24

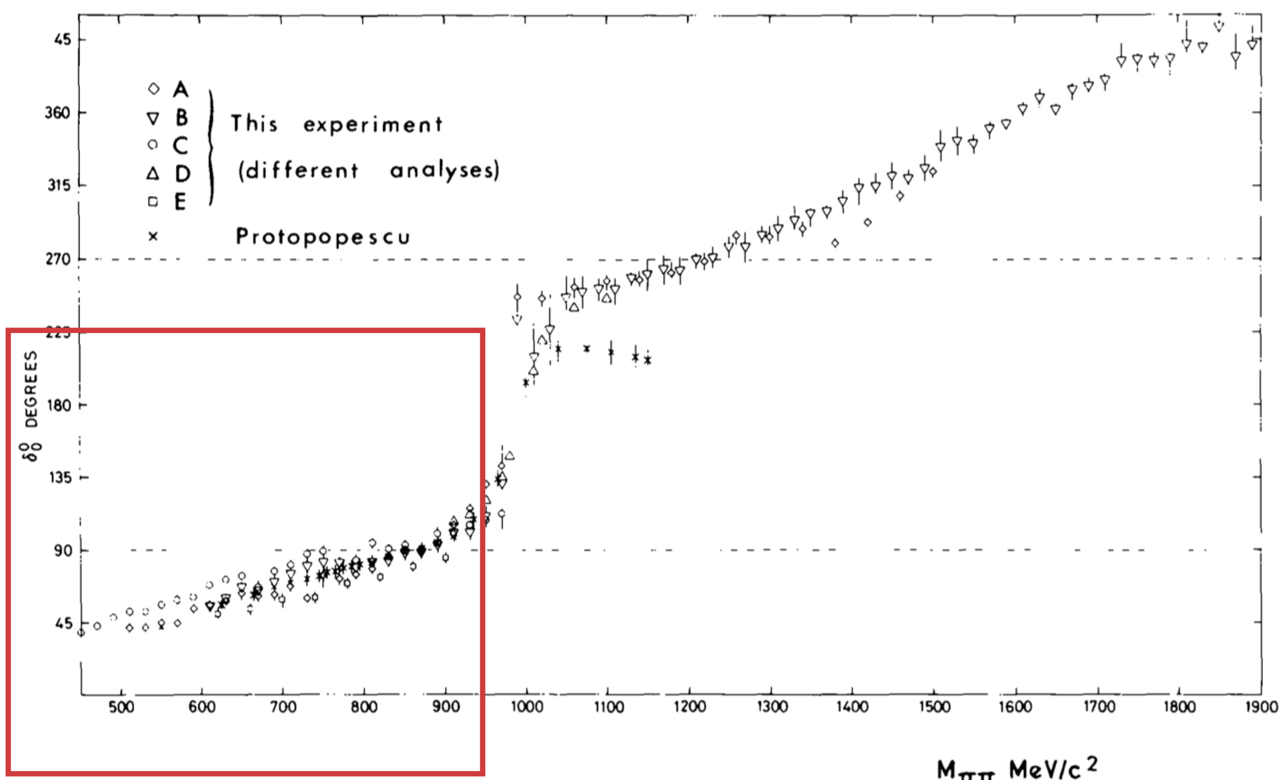

mπ~236 MeV
Ecm
[000]
[100]
[110]
[111]
[200]
1000
900
800
700
600
500
24
32
40


δ_0
$m_\pi = 391$ MeV
$m_\pi = 236$ MeV
180
150
120
90
60
30
0
0.01
0.03
0.05
0.07
0.09
0.11
0.13
k^2 / GeV^2

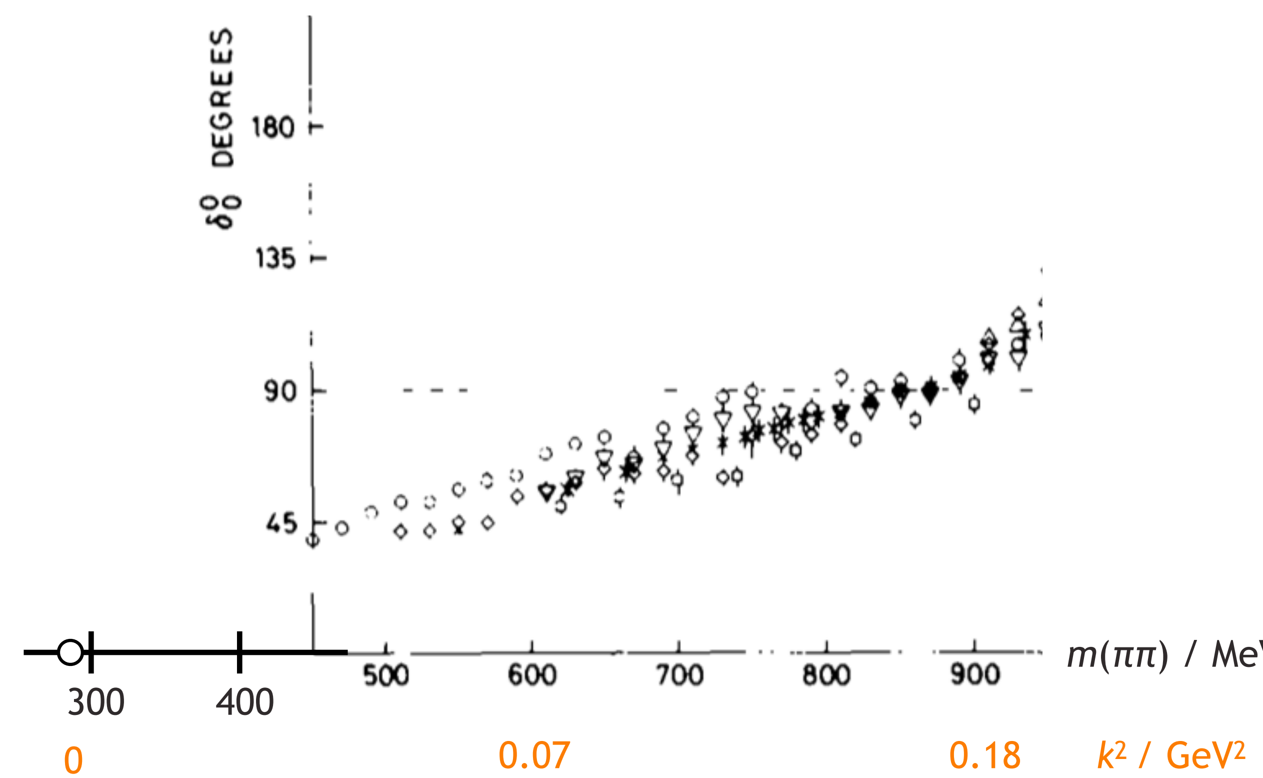

$k \cot \delta_0$ / GeV
$m_\pi = 236$ MeV
$m_\pi = 391$ MeV
1
0.5
0
-0.5
-1
-0.06
-0.03
0
0.03
0.06
0.09
0.12
k^2 / GeV^2

$\pi\pi$ isospin=0





Grayer 1974



coupled-channel scattering

Jozef Dudek

hadron spectrum collaboration
hadspec.org

coupled-channel scattering

evolution from scattering 'in' state to scattering 'out' state given by *S*-matrix elements $S_{ij} = \langle \mathrm{out}, i \,|\, \mathrm{in}, j \rangle$

e.g. in coupled $\pi\pi$, $K\overline{K}$ scattering $\mathbf{S} = \begin{pmatrix} S_{\pi\pi,\pi\pi} & S_{\pi\pi,K\overline{K}} \\ S_{K\overline{K},\pi\pi} & S_{K\overline{K},K\overline{K}} \end{pmatrix}$

more convenient to work with *t*-matrix $\mathbf{S} = \mathbf{1} + 2i\sqrt{\boldsymbol{\rho}} \cdot \mathbf{t} \cdot \sqrt{\boldsymbol{\rho}}$ typically in partial-waves $t^{(\ell)}_{ij}(E)$

in time-reversal invariant theories, ***t*** is symmetric $\Rightarrow \frac{1}{2}N(N+1)$ *complex* numbers at each energy?

conservation of probability, a.k.a. unitarity is an important constraint

$$\mathrm{Im}\, t_{ij} = \sum_k t^*_{ik}\, \rho_k\, t_{kj}$$

sum over channels kinematically open

$$\left(S^\dagger S\right)_{ij} = \sum_k \langle \mathrm{in}, i \,|\, \mathrm{out}, k \rangle \langle \mathrm{out}, k \,|\, \mathrm{in}, j \rangle = \delta_{ij}$$

completeness of outgoing states $1 = \sum_k \left| \mathrm{out}, k \right\rangle \left\langle \mathrm{out}, k \right|$

or

$$\boxed{\mathrm{Im}\left(t^{-1}(E)\right)_{ij} = -\delta_{ij}\, \rho_i(E)\, \Theta(E - E_i^{\mathrm{thr.}})}$$

$\Rightarrow \frac{1}{2}N(N+1)$ ***real*** numbers at each energy

two-channel scattering

a common parameterization uses two phase-shifts, δ_1, δ_2, and an inelastiticity, η

$$S = \begin{pmatrix} \eta\, e^{2i\delta_1} & i\sqrt{1-\eta^2}\, e^{i(\delta_1+\delta_2)} \\ i\sqrt{1-\eta^2}\, e^{i(\delta_1+\delta_2)} & \eta\, e^{2i\delta_2} \end{pmatrix}$$

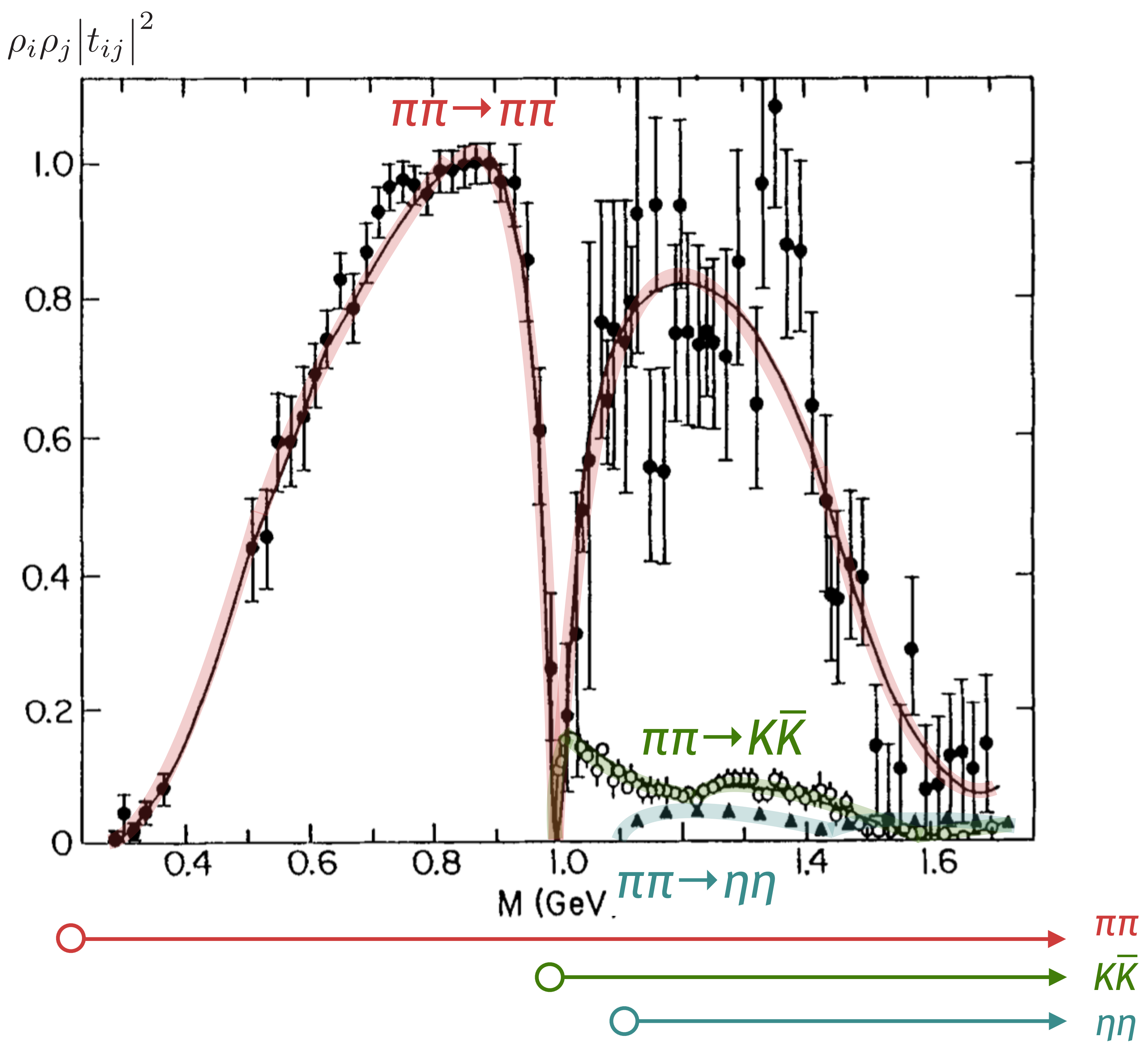
$$t_{11} = \frac{1}{\rho_1} e^{i\delta_1} \left[\tfrac{1}{2}(\eta+1)\sin\delta_1 - \tfrac{i}{2}(\eta-1)\cos\delta_1 \right]$$

elastic form regained if $\eta \to 1$

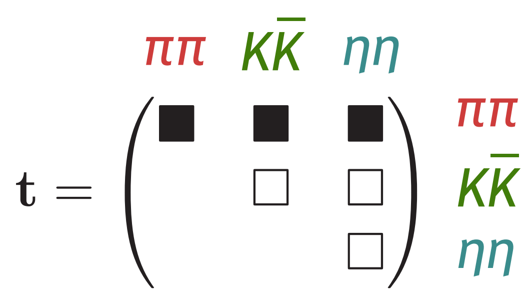
$$\rho_1 \rho_2 \big| t_{12} \big|^2 = 1 - \eta^2$$

channel coupling given by $\eta \neq 1$

$\pi\pi$, $K\bar{K}$, $\eta\eta$ S-wave scattering



experimentally
quite difficult to fill out
the whole matrix



isolating kaon exchange hard
& η beams don't exist

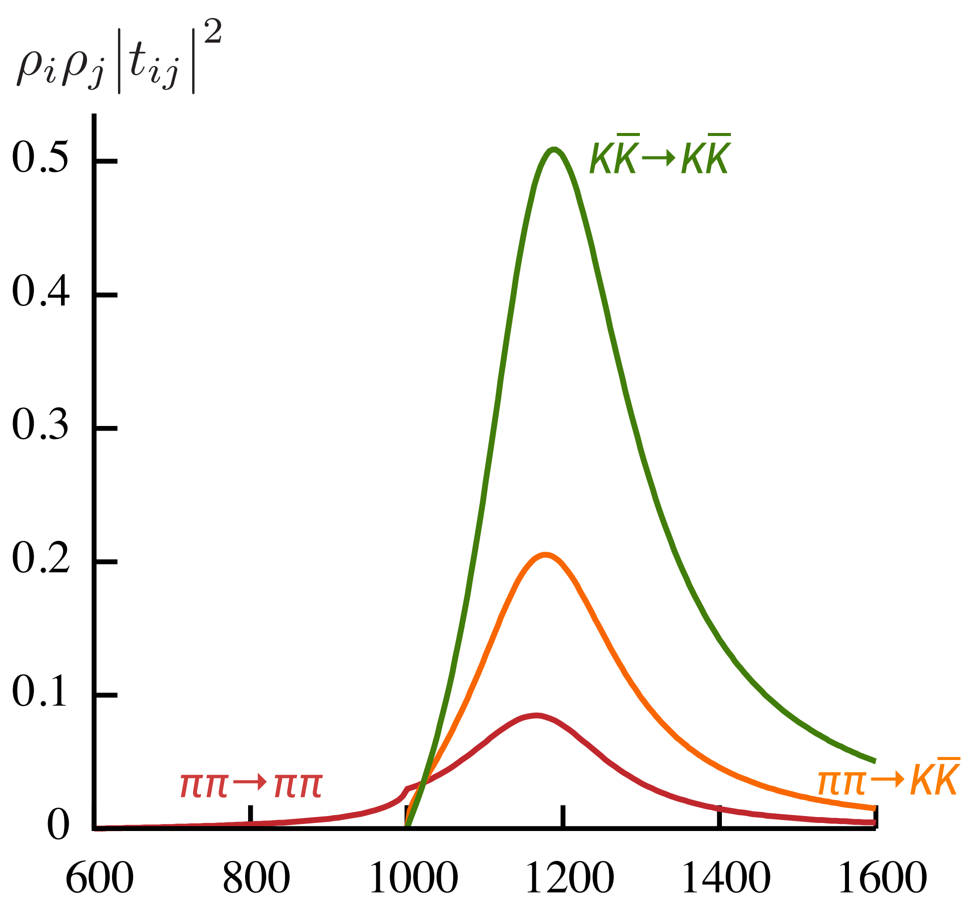
normalization of $\pi\pi \to K\bar{K}$
also slightly uncertain ...

coupled-channel scattering — a simple resonance model

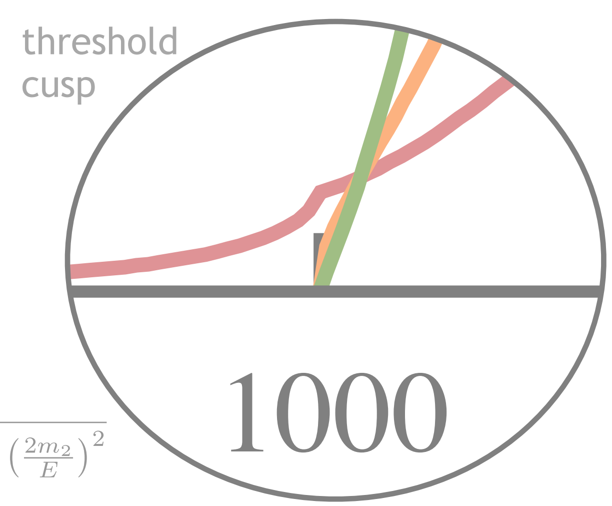
Flatté form — coupled-channel generalisation of Breit-Wigner

m_π = 300 MeV
m_K = 500 MeV

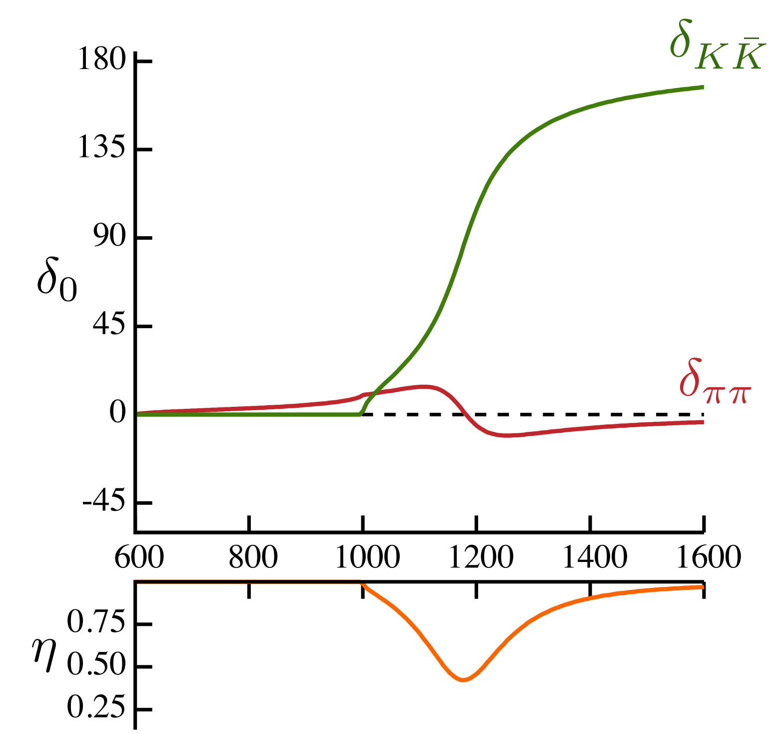
$$t_{ij}(E) = \frac{g_i\, g_j}{m^2 - E^2 - i g_1^2\, \rho_1 - i g_2^2\, \rho_2}$$



m = 1182 MeV
$g_{\pi\pi}$ = 296 MeV
g_{KK} = 592 MeV



$\rho_2(E) = \sqrt{1 - \left(\frac{2m_2}{E}\right)^2}$



coupled-channel scattering in a finite-volume

the quantization condition generalizes to

$$0 = \det\left[\mathbf{1} + i\boldsymbol{\rho}\cdot\mathbf{t}\cdot\left(\mathbf{1}+i\boldsymbol{\mathcal{M}}\right)\right]$$

e.g. in A_1^+ irrep (ℓ = 0, 4 ...)

$$\mathbf{t} = \begin{pmatrix} \begin{pmatrix} t^{(0)}_{11} & t^{(0)}_{12} \\ t^{(0)}_{12} & t^{(0)}_{22} \end{pmatrix} & \mathbf{0} & \cdots \\ \mathbf{0} & \begin{pmatrix} t^{(4)}_{11} & t^{(4)}_{12} \\ t^{(4)}_{12} & t^{(4)}_{22} \end{pmatrix} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

dense in channel space
— infinite-volume dynamics mixes channels

diagonal in angular momentum space
— ℓ good q.n. in infinite-volume

$$\boldsymbol{\mathcal{M}} = \begin{pmatrix} \begin{pmatrix} \mathcal{M}^{A_1^+}_{00}(k_1) & 0 \\ 0 & \mathcal{M}^{A_1^+}_{00}(k_2) \end{pmatrix} & \begin{pmatrix} \mathcal{M}^{A_1^+}_{04}(k_1) & 0 \\ 0 & \mathcal{M}^{A_1^+}_{04}(k_2) \end{pmatrix} & \cdots \\ \begin{pmatrix} \mathcal{M}^{A_1^+}_{40}(k_1) & 0 \\ 0 & \mathcal{M}^{A_1^+}_{40}(k_2) \end{pmatrix} & \begin{pmatrix} \mathcal{M}^{A_1^+}_{44}(k_1) & 0 \\ 0 & \mathcal{M}^{A_1^+}_{44}(k_2) \end{pmatrix} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

diagonal in channel space
— no dynamics

dense in angular momentum
— cubic symmetry lives here

$$k_1 = \tfrac{1}{2}\sqrt{E^2 - 4m_1^2}$$

$$k_2 = \tfrac{1}{2}\sqrt{E^2 - 4m_2^2}$$

coupled-channel scattering in a finite-volume

the quantization condition generalizes to

$$0 = \det\left[\mathbf{1} + i\boldsymbol{\rho}\cdot\mathbf{t}\cdot\left(\mathbf{1} + i\boldsymbol{\mathcal{M}}\right)\right]$$

can also be expressed as $0 = \det\left[\mathbf{t}^{-1} + i\boldsymbol{\rho} - \boldsymbol{\mathcal{M}}\cdot\boldsymbol{\rho}\right)\right]$

which exposes the role of unitarity $\mathrm{Im}\left(t^{-1}(E)\right)_{ij} = -\delta_{ij}\,\rho_i(E)\,\Theta(E - E_i^{\mathrm{thr.}})$

the quantization condition is a single real condition:

the zeroes E=E_n(L) of the function $\det\left[\mathbf{1} + i\boldsymbol{\rho}(E)\cdot\mathbf{t}(E)\cdot\left(\mathbf{1} + i\boldsymbol{\mathcal{M}}(E,L)\right)\right]$

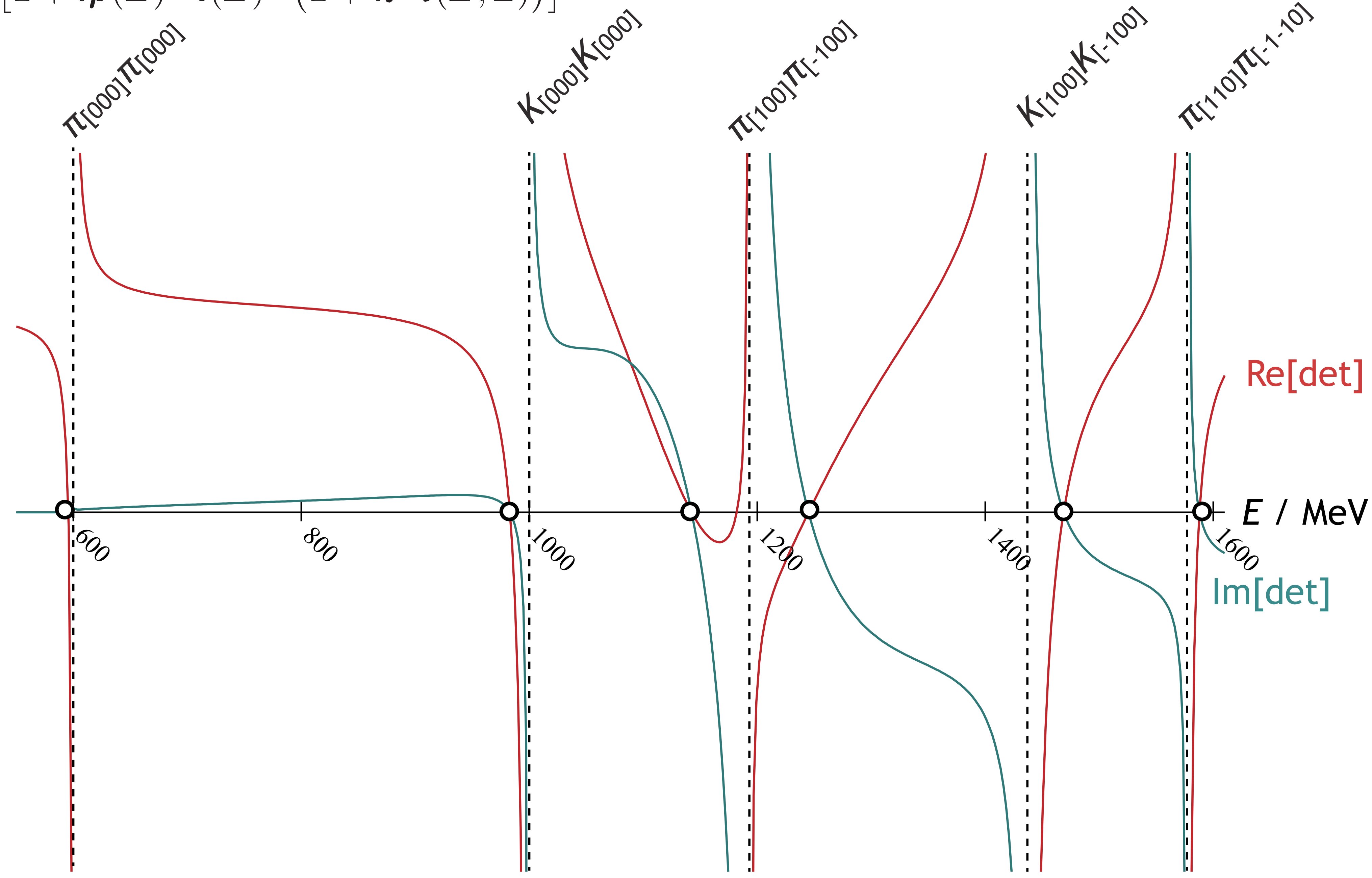
correspond to the spectrum in an $L\times L\times L$ volume

zeroes of the determinant

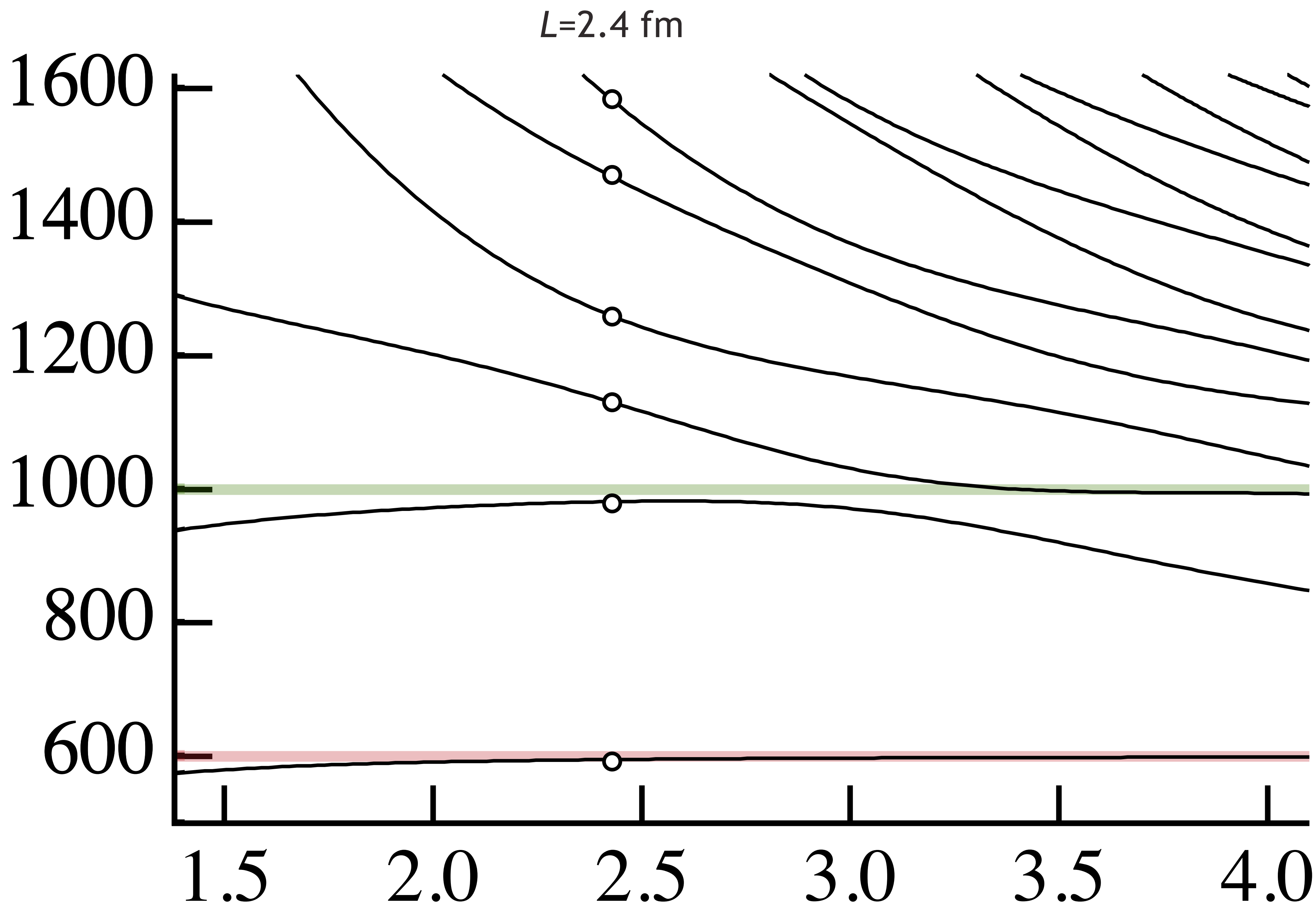
e.g. previously presented two-channel Flatté form — [000] A_1^+ irrep in L=2.4 fm box

m_π = 300 MeV
m_K = 500 MeV

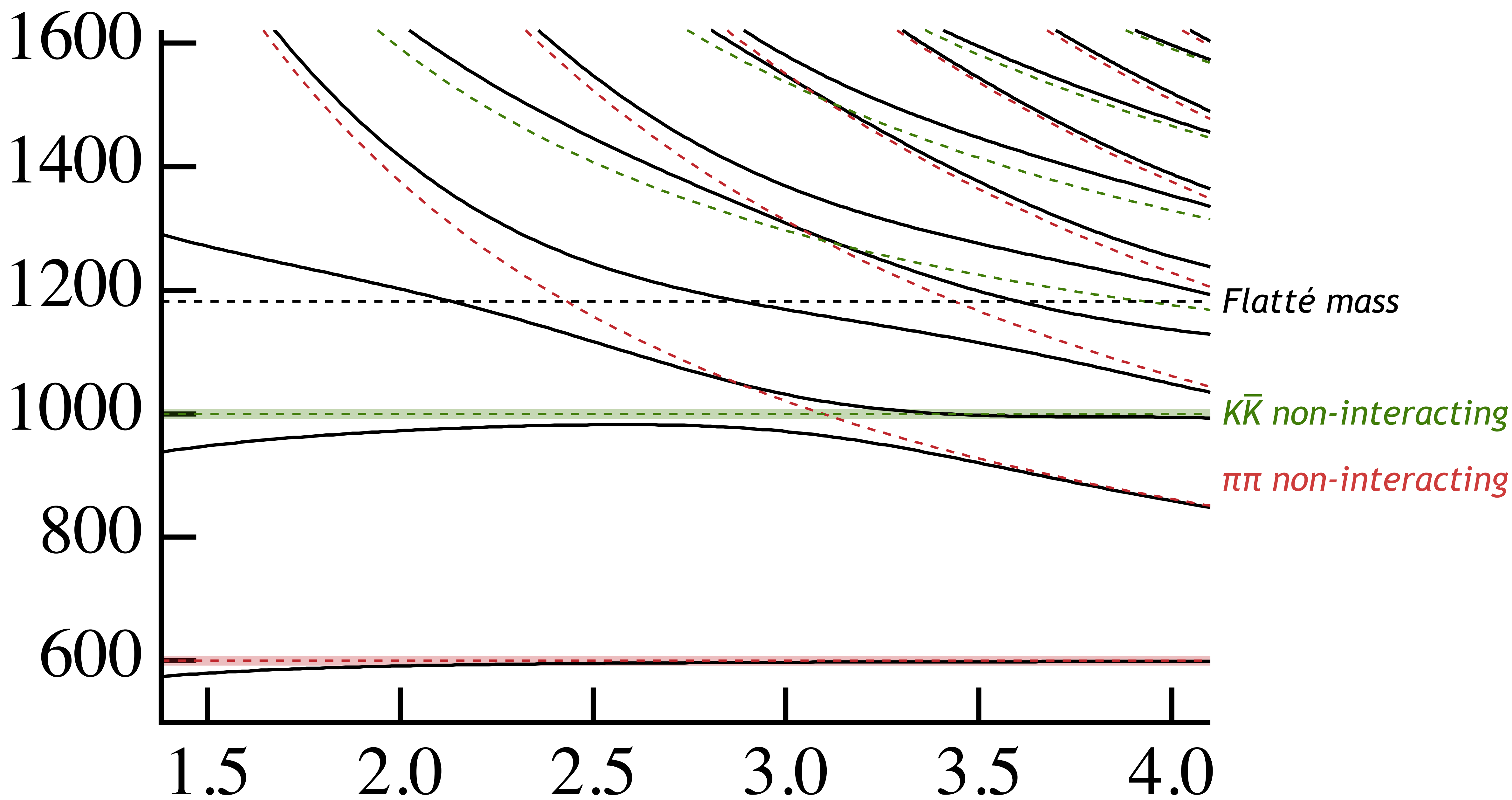
$$\det\left[\mathbf{1} + i\boldsymbol{\rho}(E)\cdot\mathbf{t}(E)\cdot\left(\mathbf{1}+i\boldsymbol{\mathcal{M}}(E,L)\right)\right]$$



zeroes of the determinant

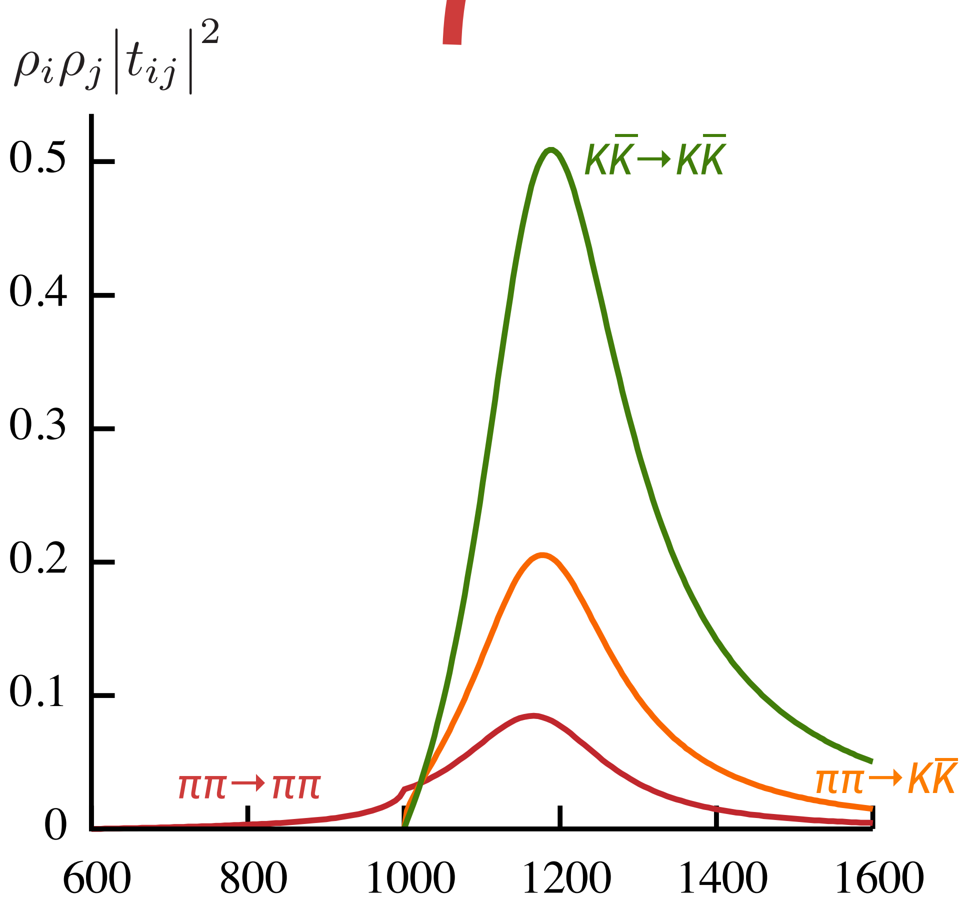


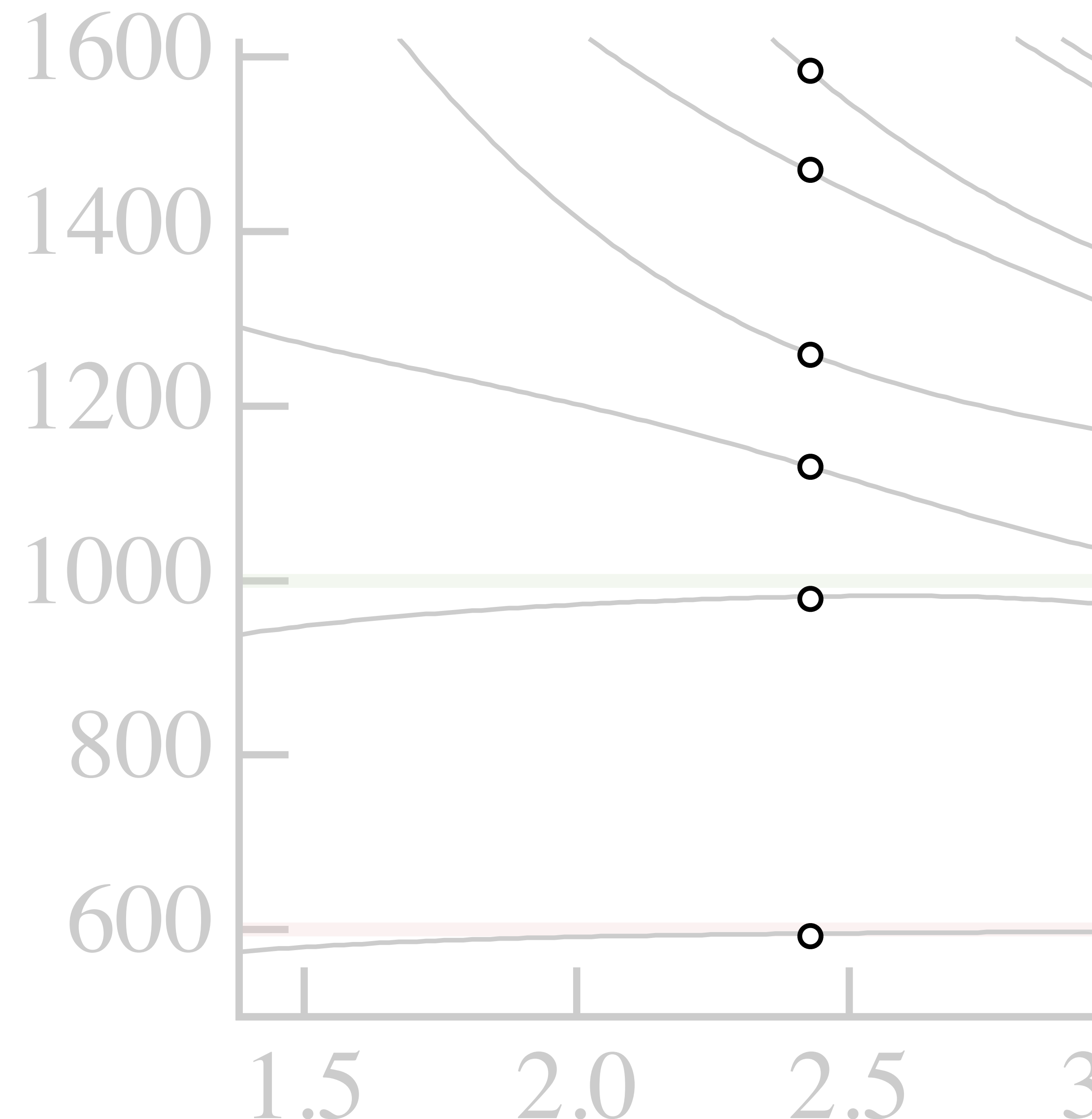
finite-volume spectrum



finite-volume approach

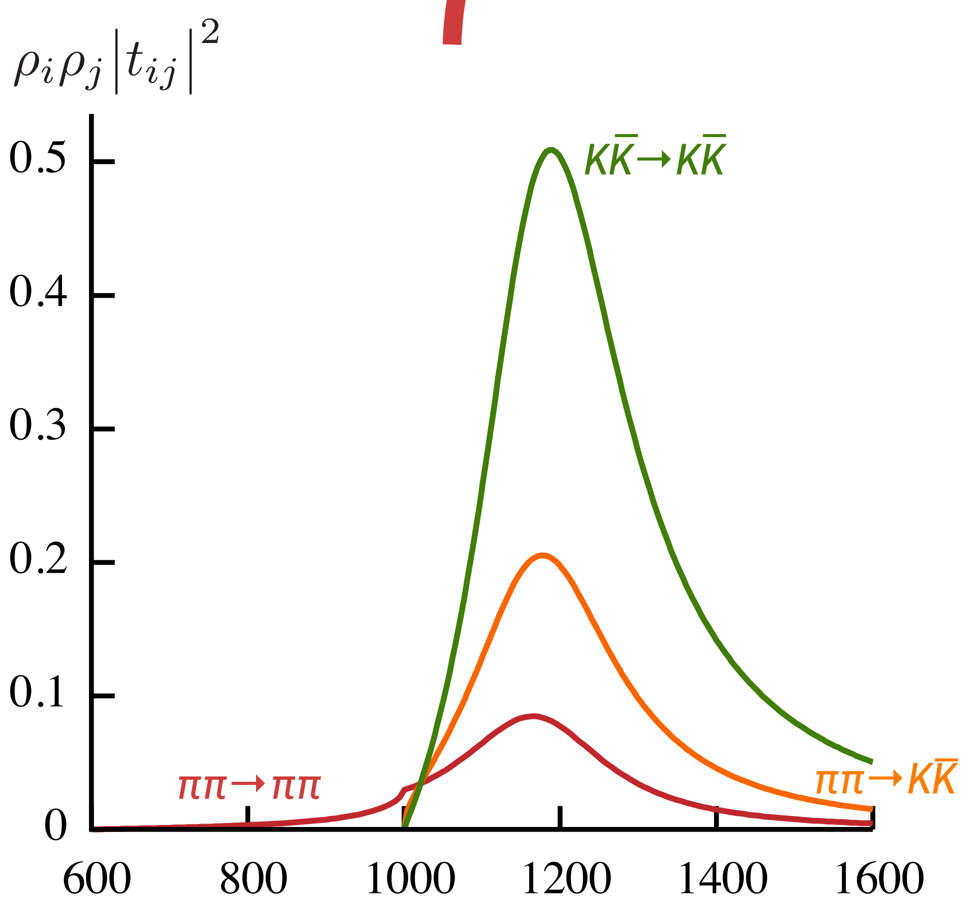
$$0 = \det\left[\mathbf{1} + i\boldsymbol{\rho}\cdot\mathbf{t}\cdot\left(\mathbf{1}+i\boldsymbol{\mathcal{M}}\right)\right]$$

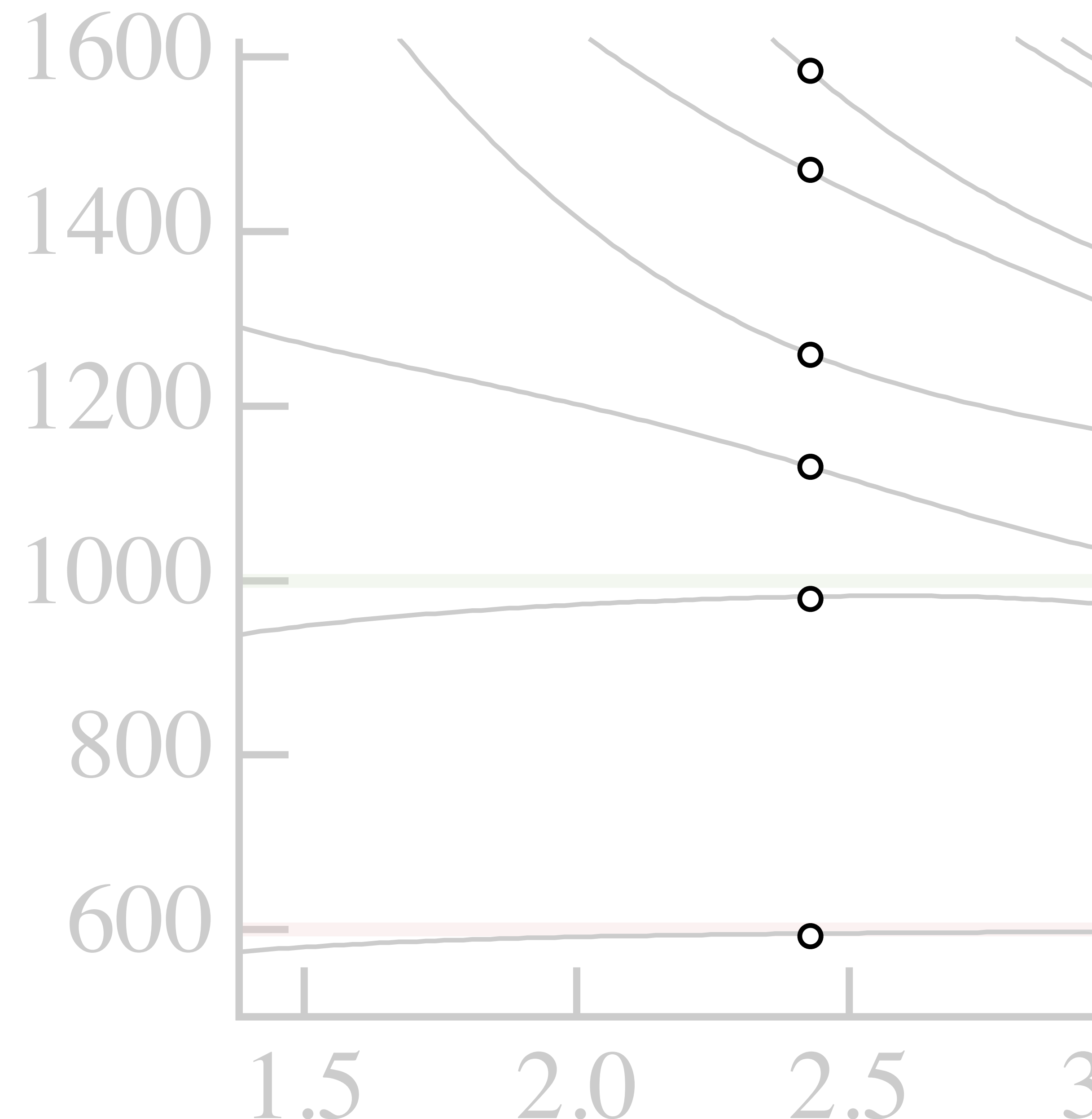




finite-volume approach

$$0 = \det\left[\mathbf{1} + i\boldsymbol{\rho}\cdot\mathbf{t}\cdot\left(\mathbf{1} + i\boldsymbol{\mathcal{M}}\right)\right]$$

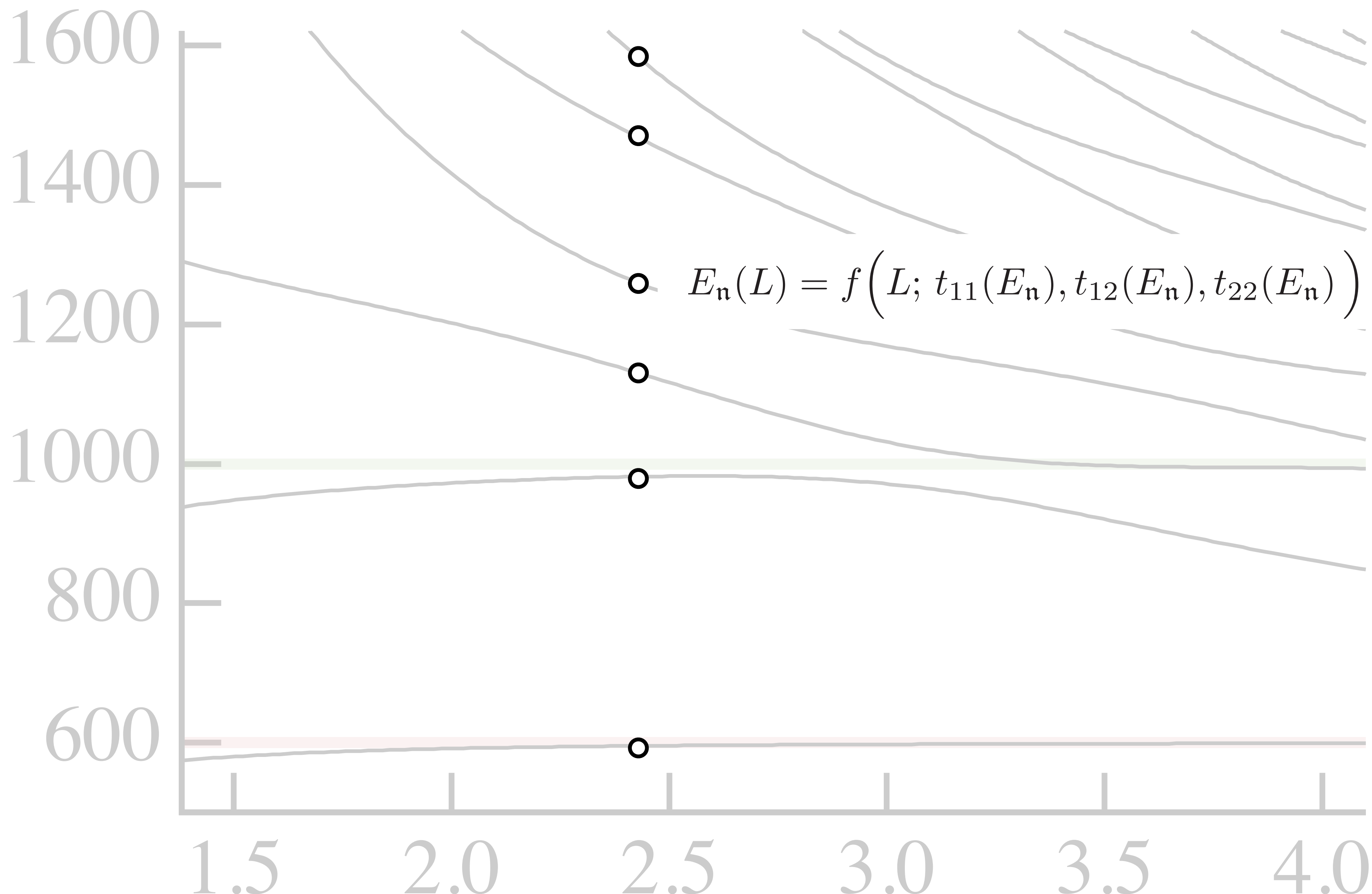




but in a lattice QCD calculation
we have the inverse problem ...

?

position of each energy level depends upon all elements of the ***t***-matrix



$$0 = \det\big[\mathbf{1} + i\boldsymbol{\rho}\cdot\mathbf{t}\cdot\big(\mathbf{1} + i\boldsymbol{\mathcal{M}}\big)\big]$$

at $E = E_n(L)$
is one equation in three unknowns ...

parameterizing the *t*-matrix

a solution is to propose that different energies are not unrelated – parameterize $\mathbf{t}\big(E;\{a_i\}\big)$

then can use many energy levels to constrain the parameters by minimising a χ^2

$$\chi^2\big(\{a_i\}\big) = \sum_{\mathfrak{n},\mathfrak{n}'} \Big(E^{\text{lat.}}_{\mathfrak{n}} - E^{\text{par.}}_{\mathfrak{n}}(L;\{a_i\})\Big)\, \mathbb{C}^{-1}_{\mathfrak{n},\mathfrak{n}'} \Big(E^{\text{lat.}}_{\mathfrak{n}'} - E^{\text{par.}}_{\mathfrak{n}'}(L;\{a_i\})\Big)$$

inverse data covariance

energy levels solving
$0 = \det\big[\mathbf{1} + i\boldsymbol{\rho}\cdot\mathbf{t}\cdot\big(\mathbf{1}+i\boldsymbol{\mathcal{M}}\big)\big]$
for $\mathbf{t}\big(E;\{a_i\}\big)$

parameterizing the *t*-matrix

a solution is to propose that different energies are not unrelated – parameterize $\mathbf{t}\big(E;\{a_i\}\big)$

need to ensure multi-channel unitarity $\mathrm{Im}\,\big(t^{-1}(E)\big)_{ij} = -\delta_{ij}\,\rho_i(E)\,\Theta(E-E_i^{\mathrm{thr.}})$

– ***K*-matrix approach**

$\mathbf{t}^{-1}(E) = \mathbf{K}^{-1}(E) + \mathbf{I}(E)$ with $\mathrm{Im}\,\big(I(E)\big)_{ij} = -\delta_{ij}\,\rho_i(E)$

simplest choice has $\mathrm{Re}\,\mathbf{I}(E)=0$

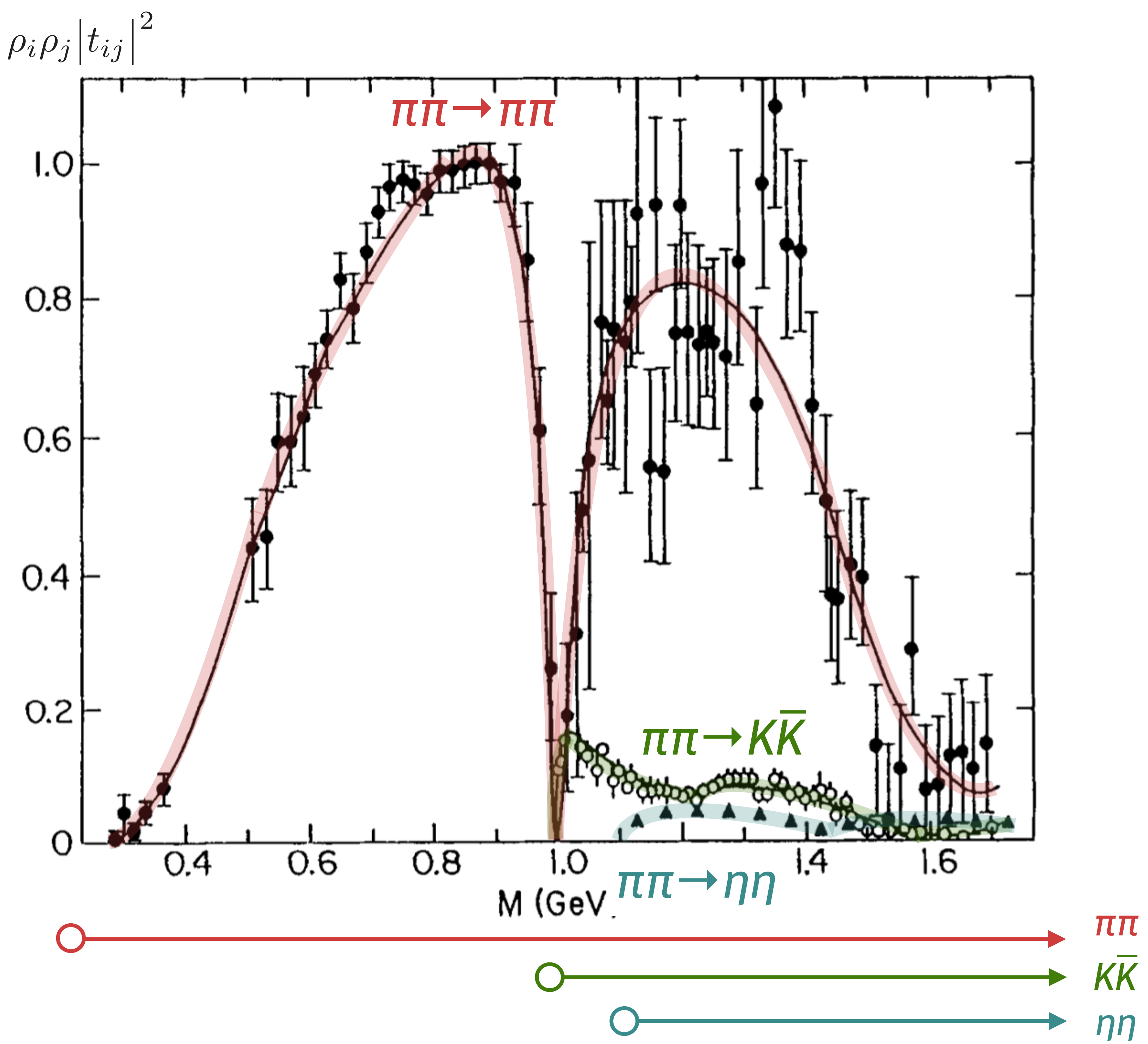
a more sophisticated approach = "Chew-Mandelstam" phase-space

***K*(*E*)** should be a real symmetric matrix

for reasons you'll see later, better to parameterize in terms of $s = E^2$

e.g. $K_{ij} = \dfrac{g_i\,g_j}{m^2-s}$ gives the Flatté form

$\pi\pi$, $K\bar{K}$, $\eta\eta$ S-wave scattering

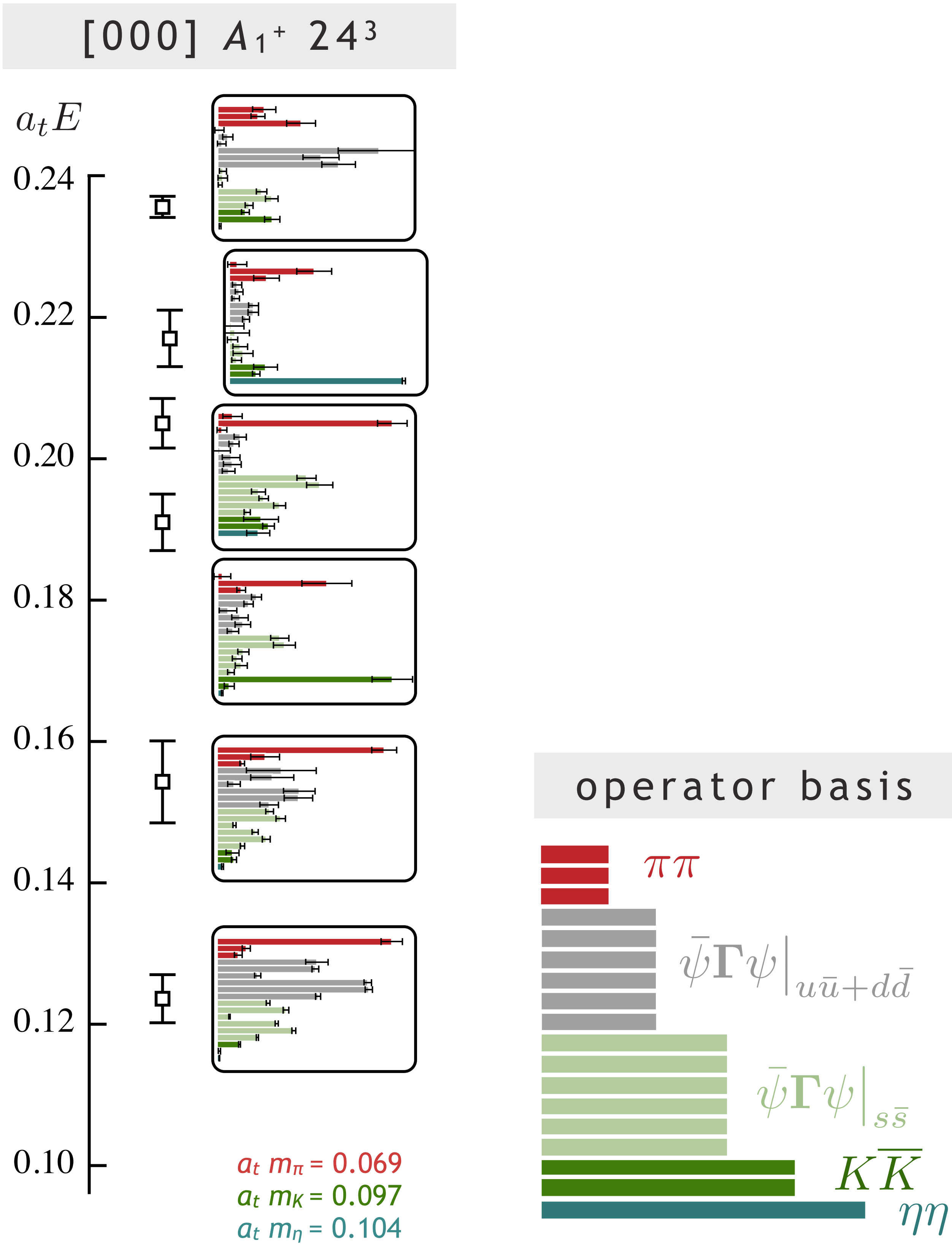


explore this non-trivial system ...

... at a higher quark mass ...

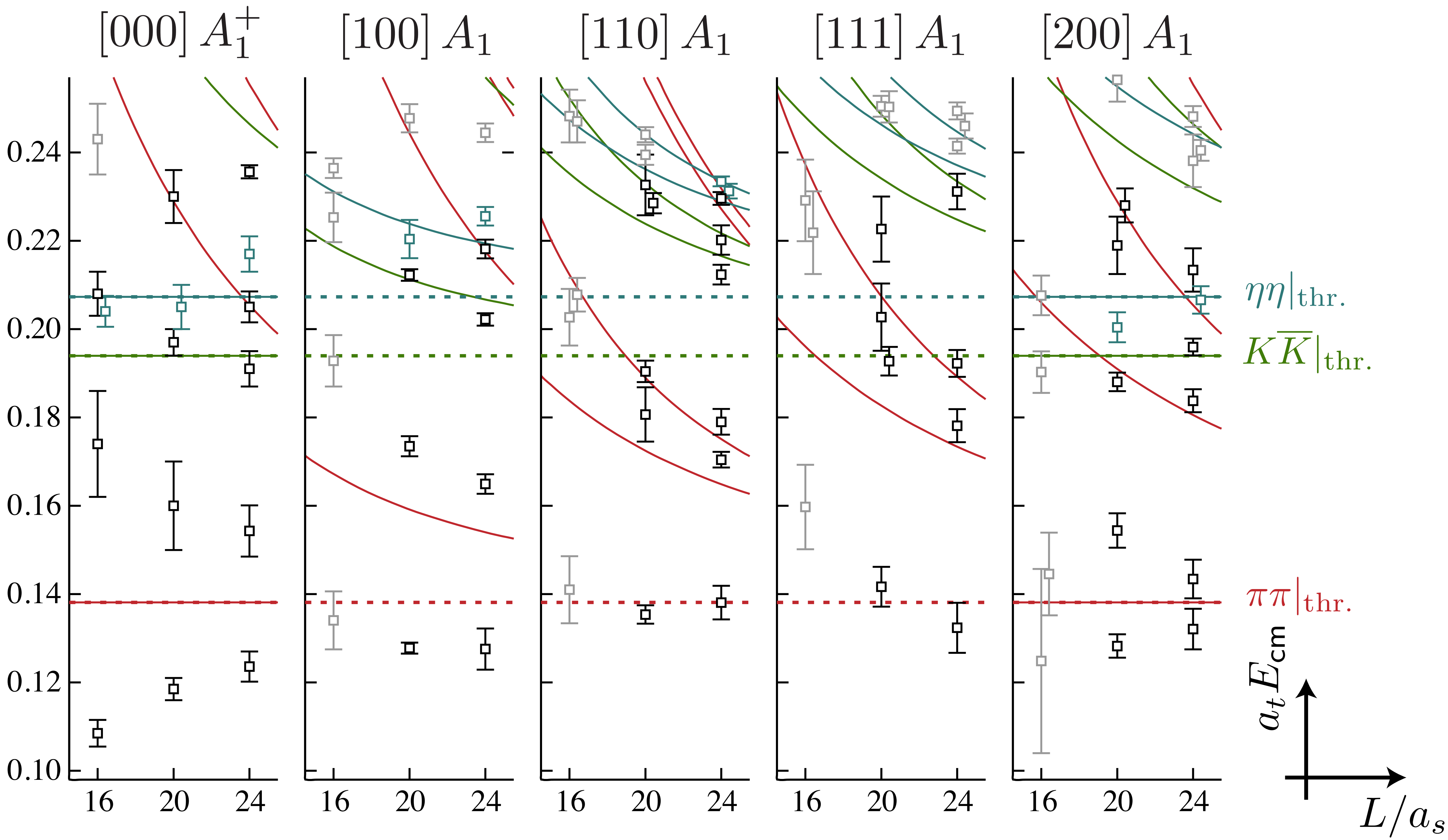
$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

PRD 97 054513 (2018)



$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

PRD 97 054513 (2018)



what t-matrix gives these spectra ?

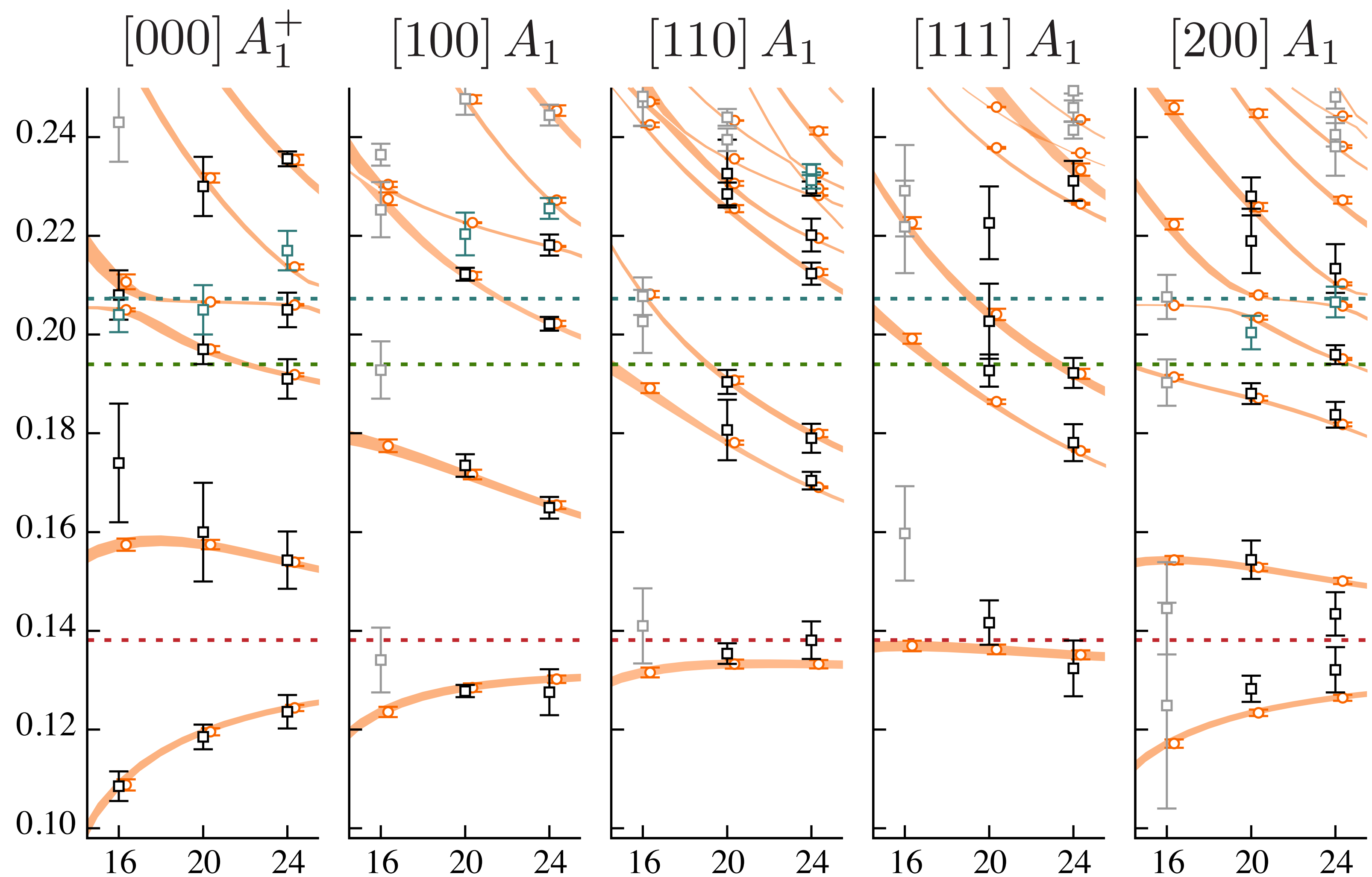
$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

not obvious what amplitude parameterization likely to describe the spectra well — try many …

$$\text{e.g.}\quad \mathbf{K}^{-1}(s) = \begin{pmatrix} a+b\,s & c+d\,s & e \\ c+d\,s & f & g \\ e & g & h \end{pmatrix}$$

{ *a* … *h* } are free parameters

best fit to lattice spectra



$$\frac{\chi^2}{N_{\mathrm{dof}}} = \frac{44.0}{57-8} = 0.90$$

with Chew-Mandelstam phase-space

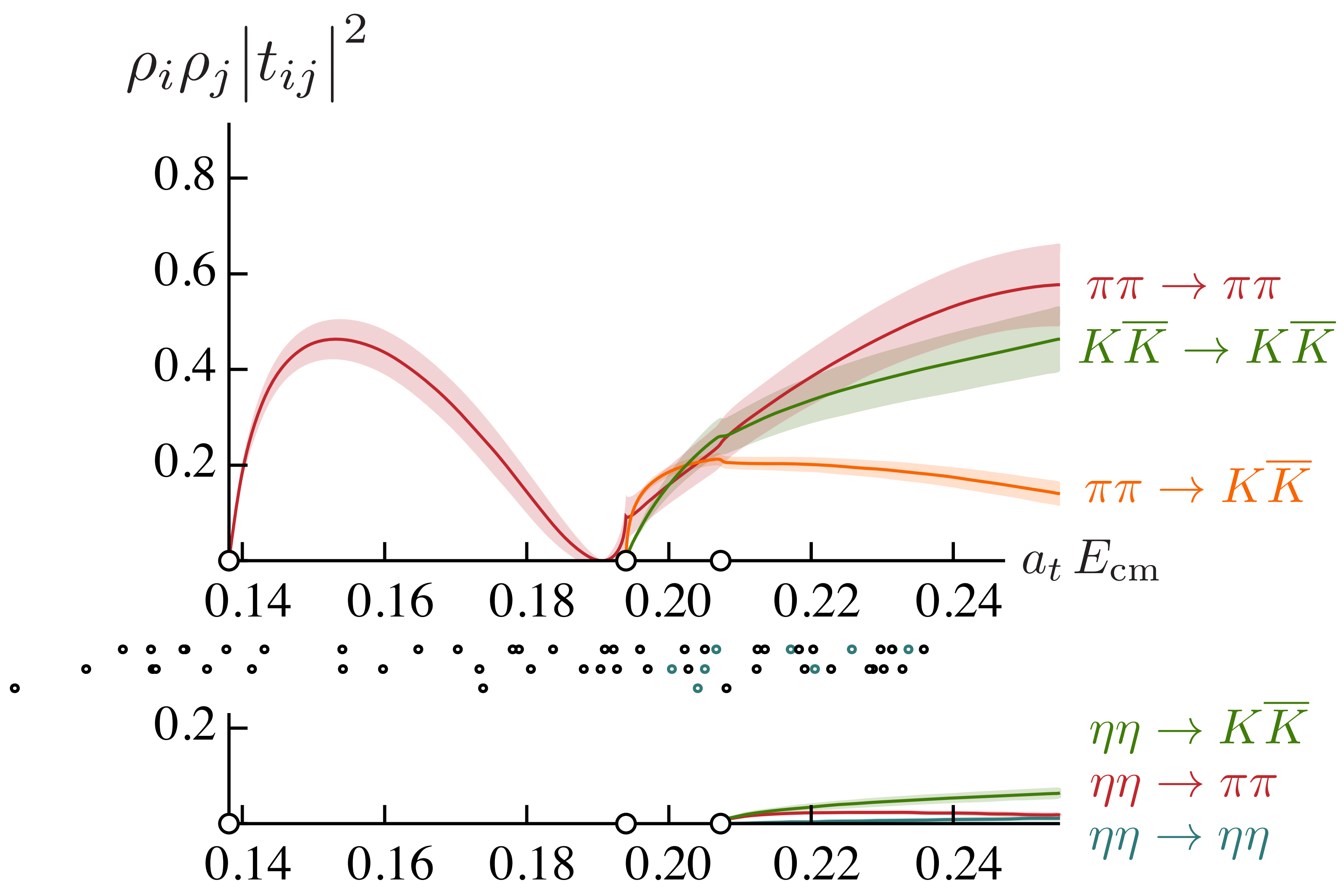
$$I(s) = -\frac{\rho(s)}{\pi}\log\left[\frac{\rho(s)-1}{\rho(s)+1}\right]$$

$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

e.g. $$\mathbf{K}^{-1}(s) = \begin{pmatrix} a + b\,s & c + d\,s & e \\ c + d\,s & f & g \\ e & g & h \end{pmatrix}$$

{ *a* … *h* } are free parameters

S-wave amplitudes

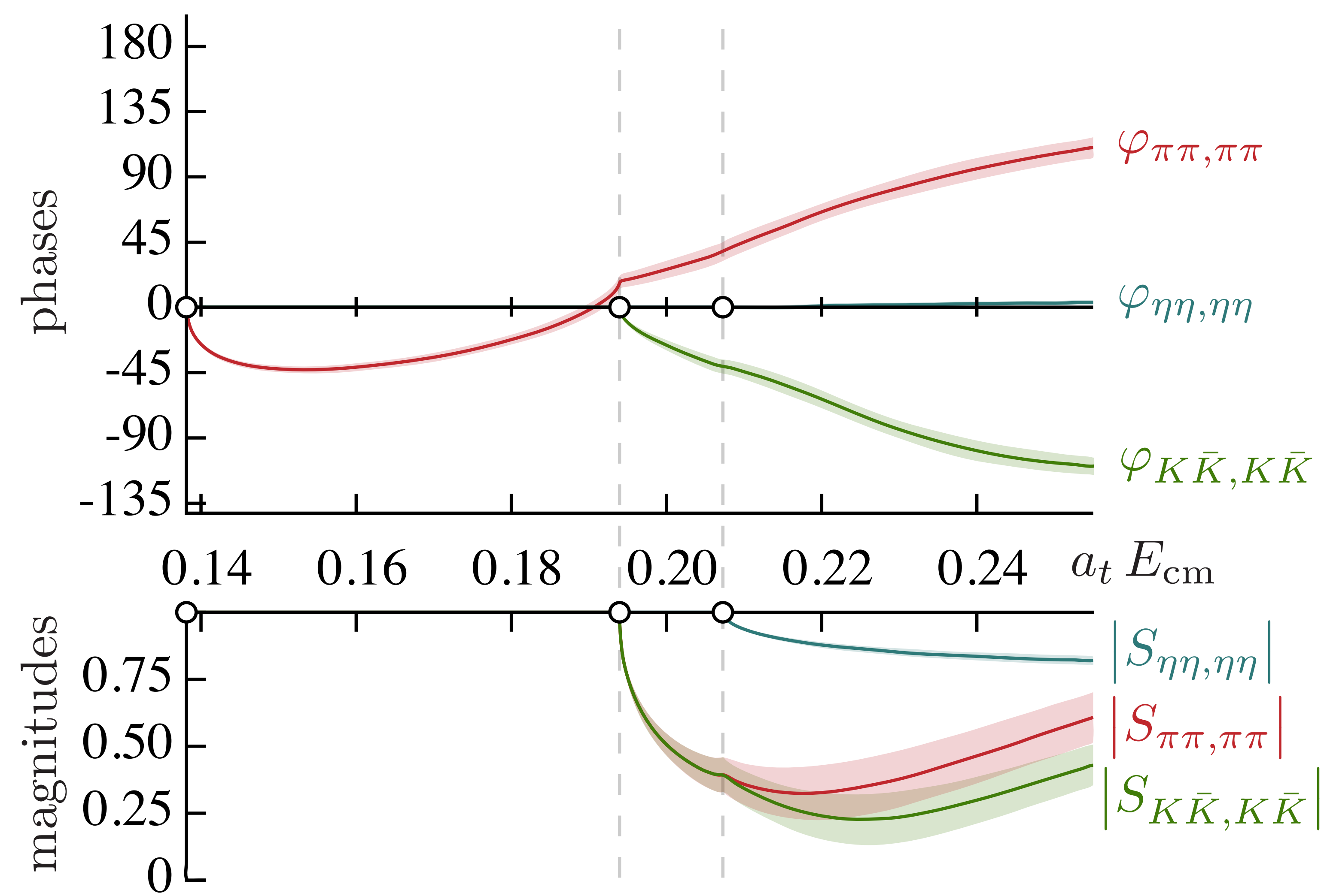


$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

e.g. $$\mathbf{K}^{-1}(s) = \begin{pmatrix} a + b\,s & c + d\,s & e \\ c + d\,s & f & g \\ e & g & h \end{pmatrix}$$

{ a … h } are free parameters

S-wave S-matrix diagonals

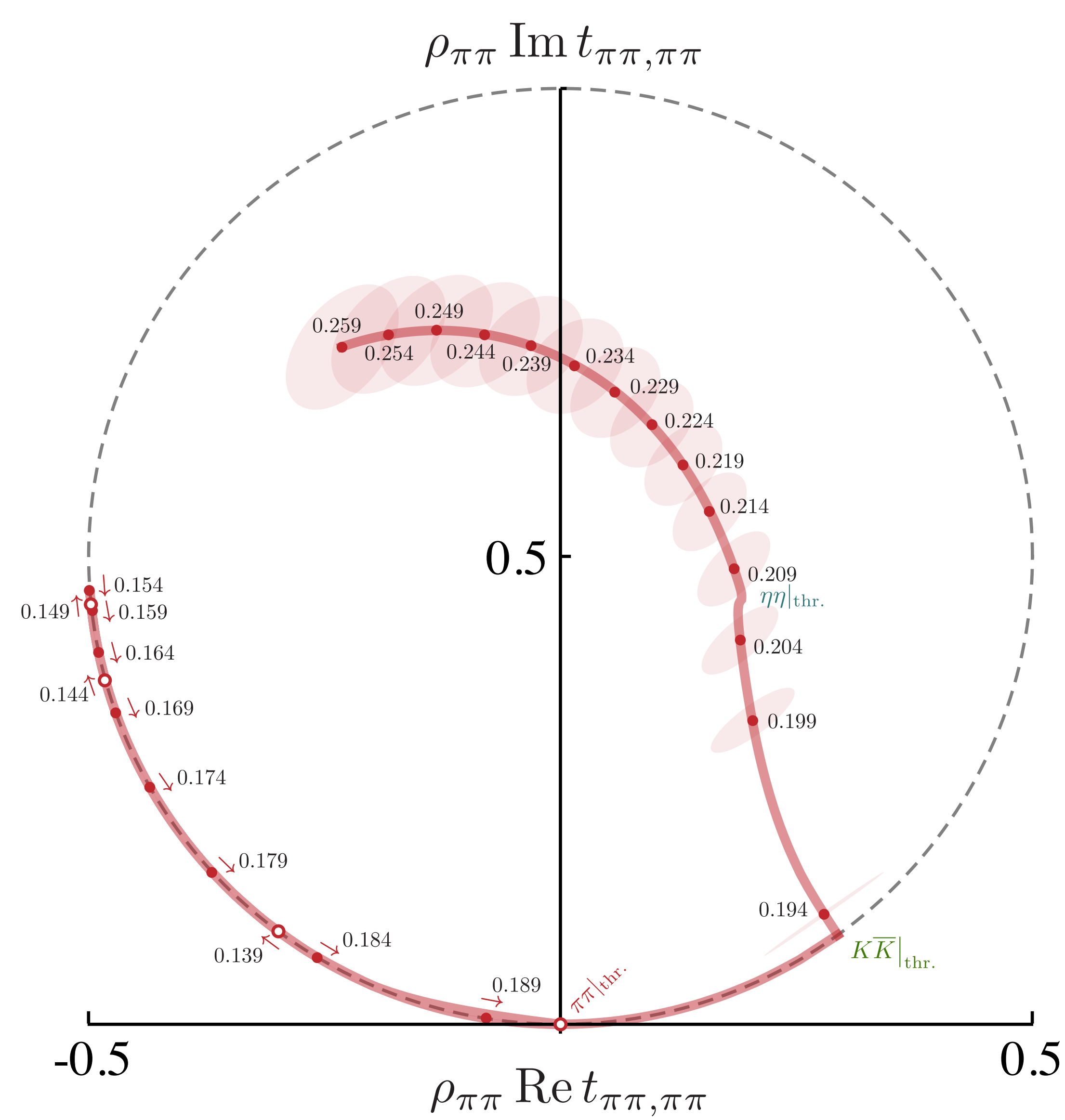


$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

e.g. $$\mathbf{K}^{-1}(s) = \begin{pmatrix} a + b\,s & c + d\,s & e \\ c + d\,s & f & g \\ e & g & h \end{pmatrix}$$

{ a … h } are free parameters

$\pi\pi \rightarrow \pi\pi$ Argand diagram

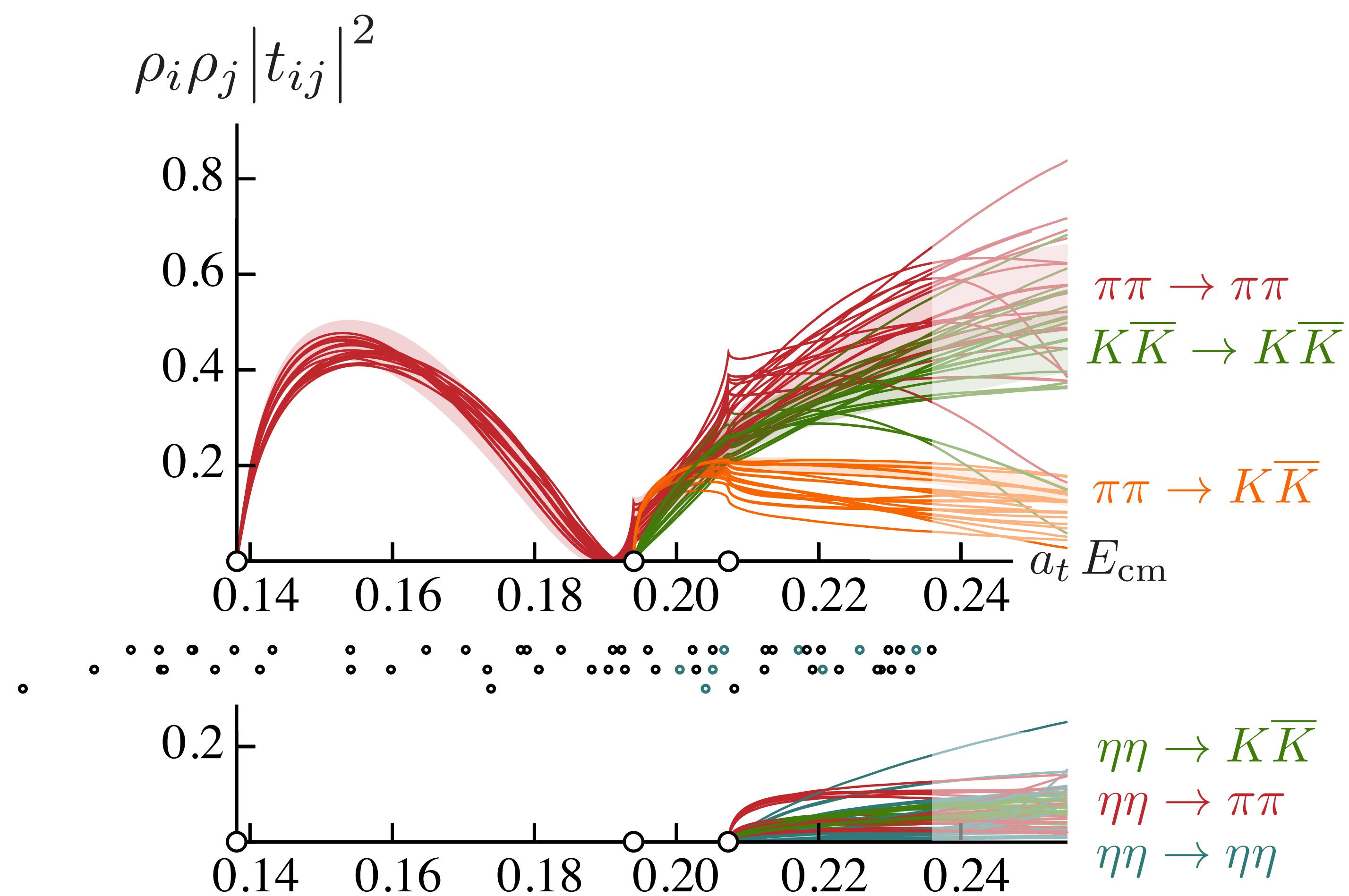


not obvious what amplitude parameterization likely to describe the spectra well — **try many …**

K^{-1} as matrix of polynomials,
K as matrix of polynomials,
K as pole plus matrix of polynomials,
simple versus Chew-Mandelstam phase-space …

keep choices that can describe spectra with good χ^2

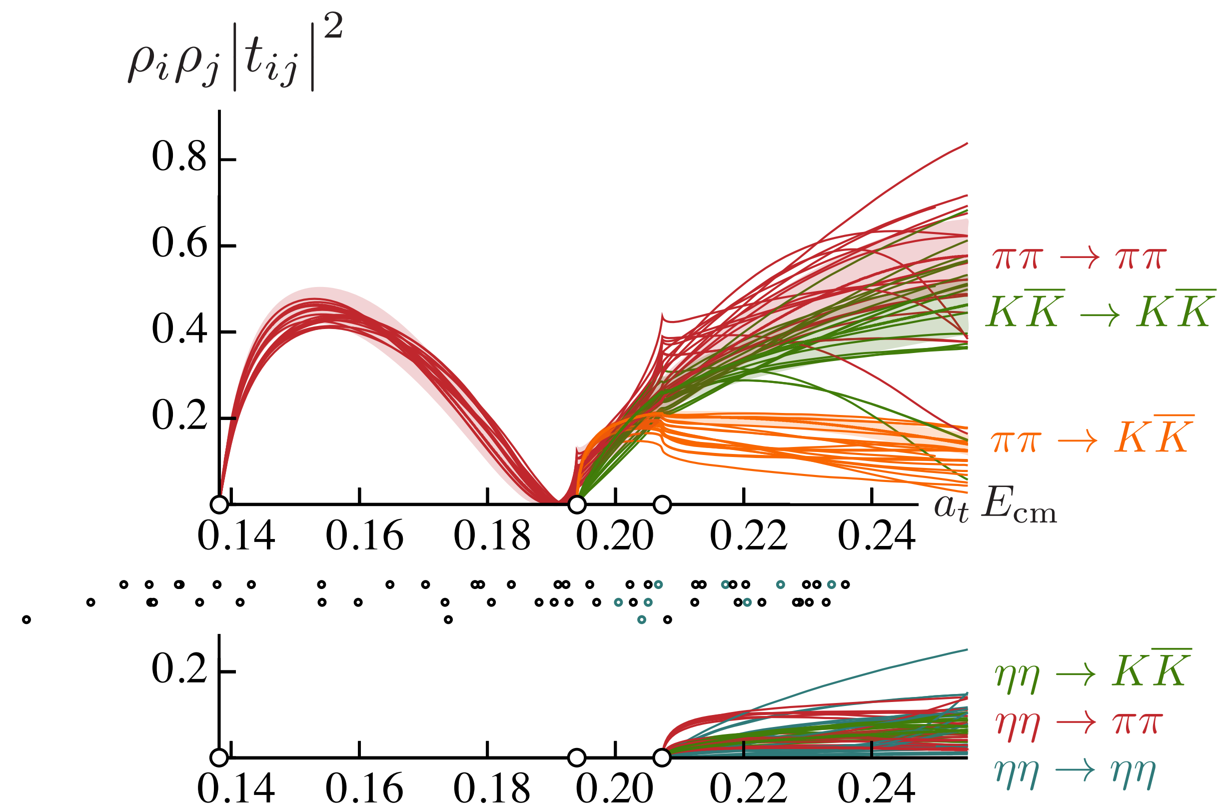
variation with parameterization



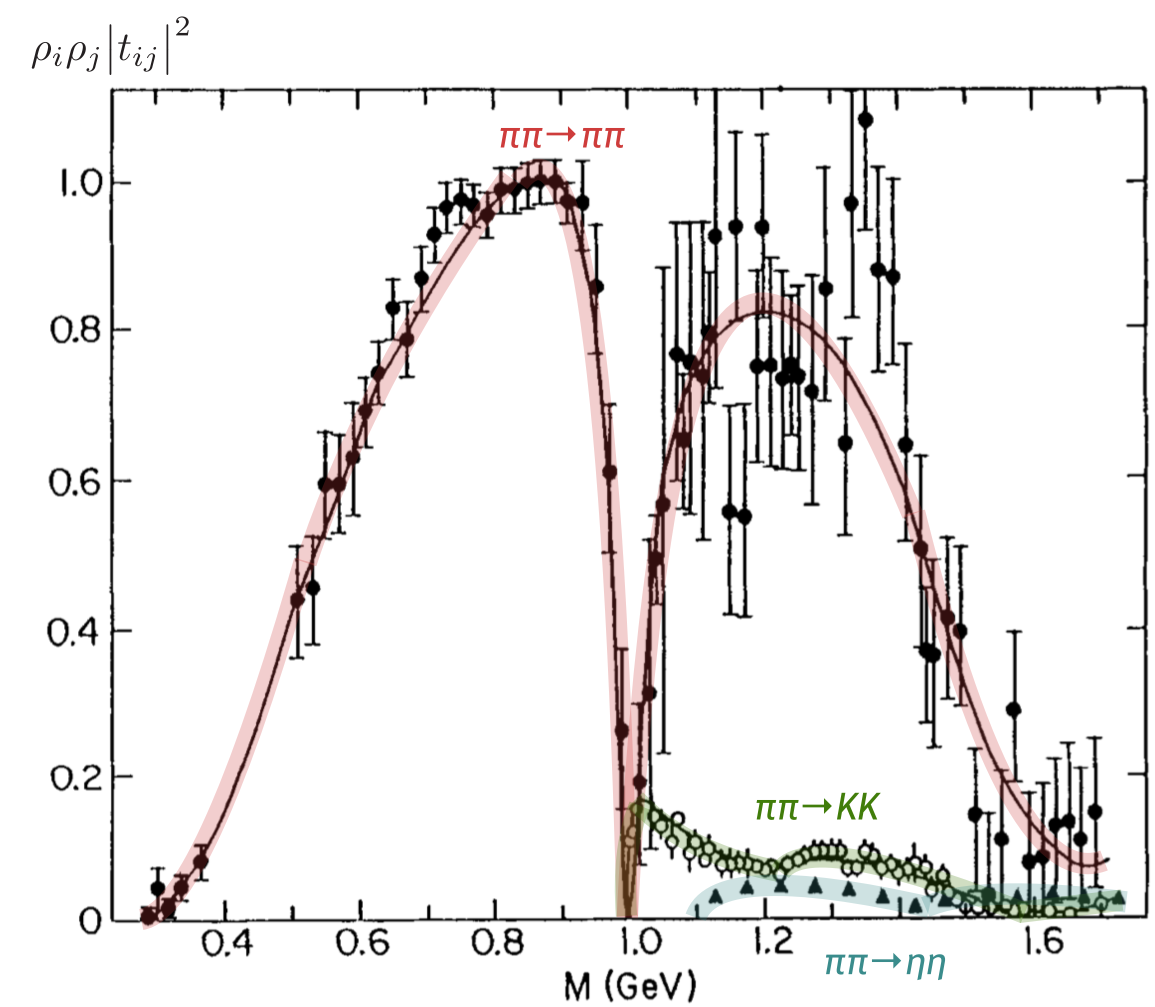
$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

PRD 97 054513 (2018)

scattering amplitude 'prediction'



'analogous' experimental data

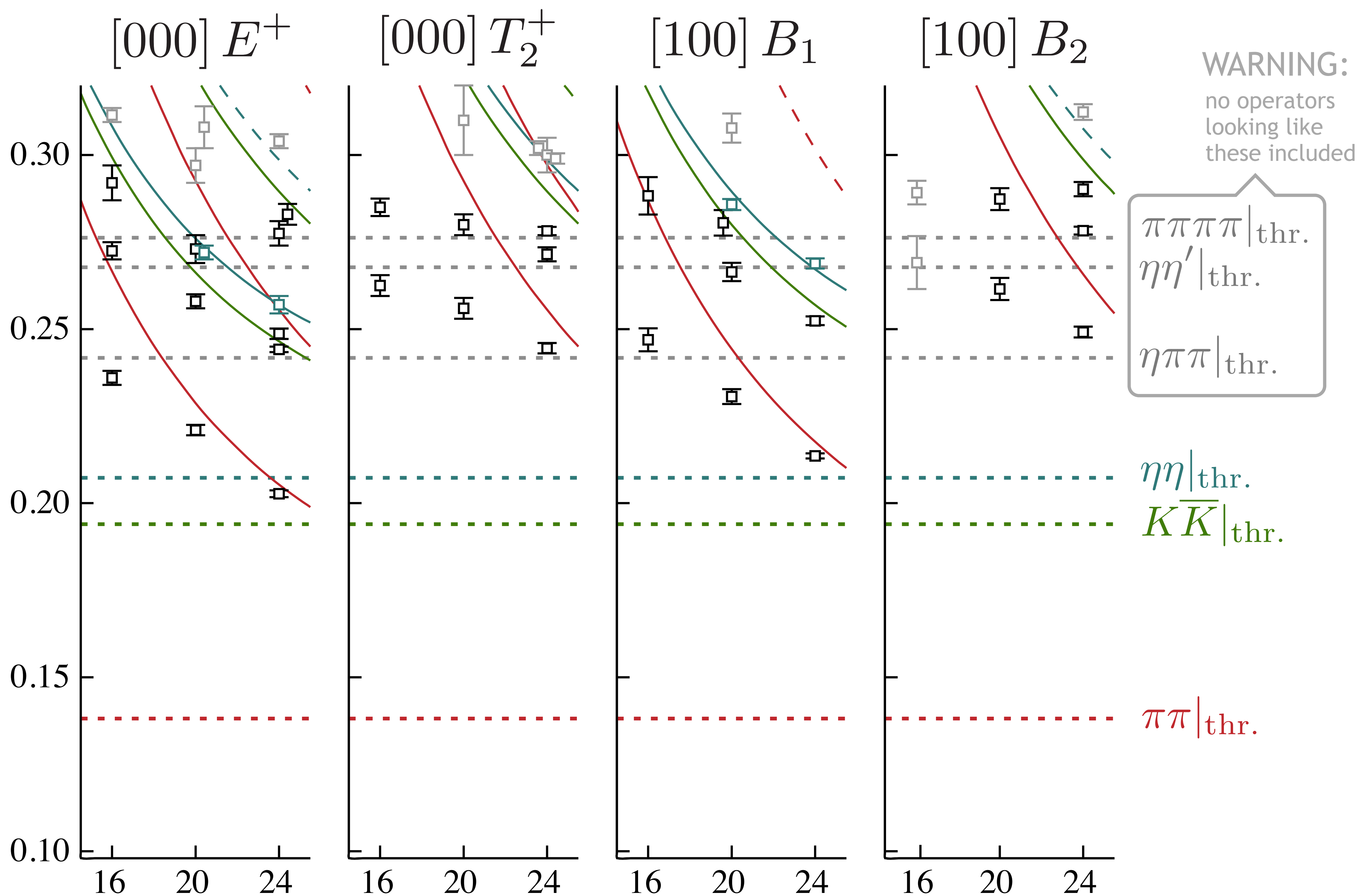


... but what do we do with this ?

... is this strange energy dependence due to resonances ?

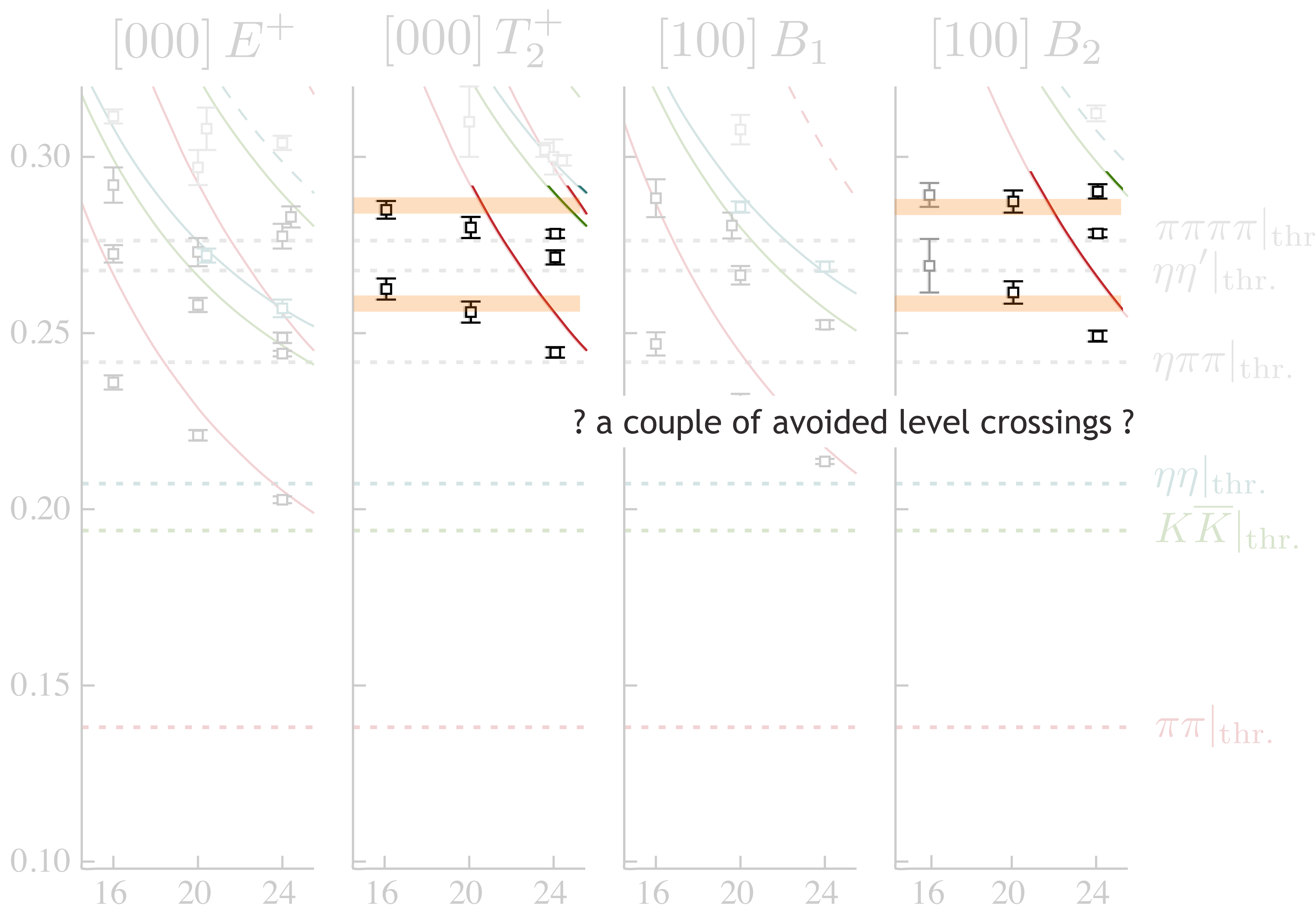
$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

also computed spectra for irreps with lowest subduced spin J=2



$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

also computed spectra for irreps with lowest subduced spin J=2



? a couple of avoided level crossings ?

$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

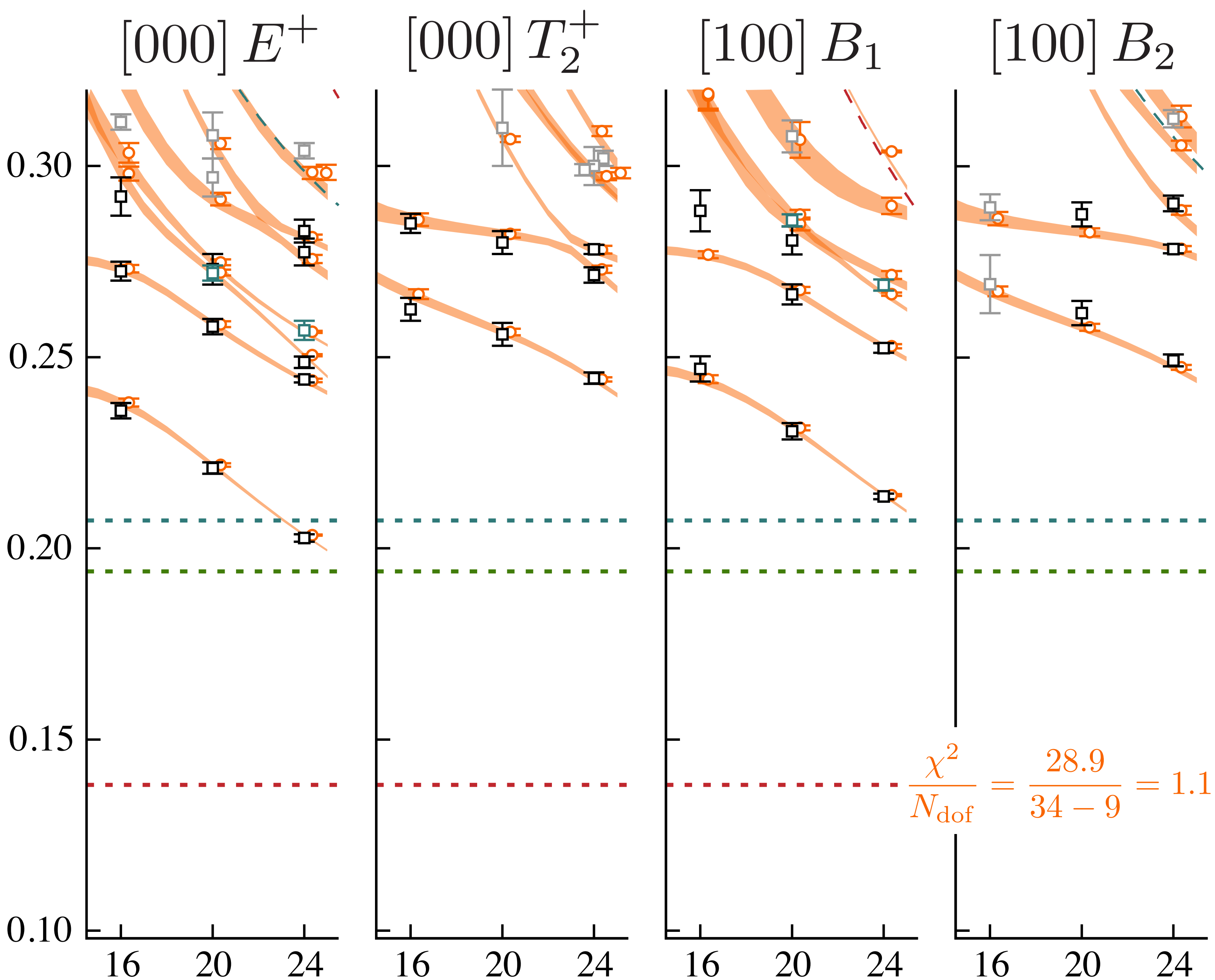
PRD 97 054513 (2018)

e.g. parameterize coupled *D*-wave ***t***-matrix with

$$K_{ij}(s) = \frac{g_i^{(1)} g_j^{(1)}}{m_1^2 - s} + \frac{g_i^{(2)} g_j^{(2)}}{m_2^2 - s} + \gamma_{ij} \qquad \gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma_{\eta\eta,\eta\eta} \end{pmatrix}$$

and the simple phase-space

best fit to lattice spectra



$$\frac{\chi^2}{N_{\mathrm{dof}}} = \frac{28.9}{34-9} = 1.15$$

$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

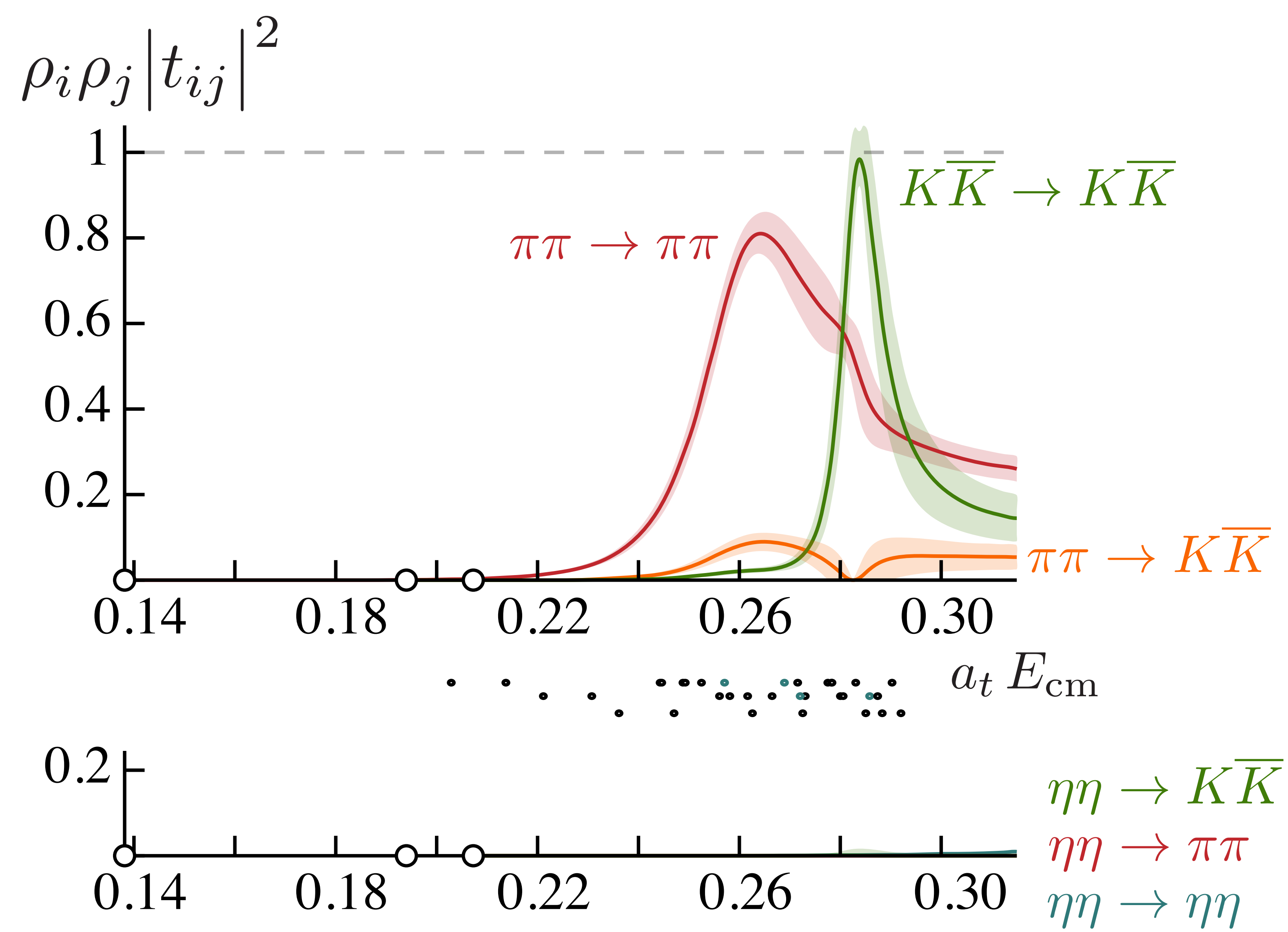
PRD 97 054513 (2018)

e.g. parameterize coupled *D*-wave ***t***-matrix with

$$K_{ij}(s) = \frac{g_i^{(1)} g_j^{(1)}}{m_1^2 - s} + \frac{g_i^{(2)} g_j^{(2)}}{m_2^2 - s} + \gamma_{ij} \qquad \gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma_{\eta\eta,\eta\eta} \end{pmatrix}$$

and the simple phase-space

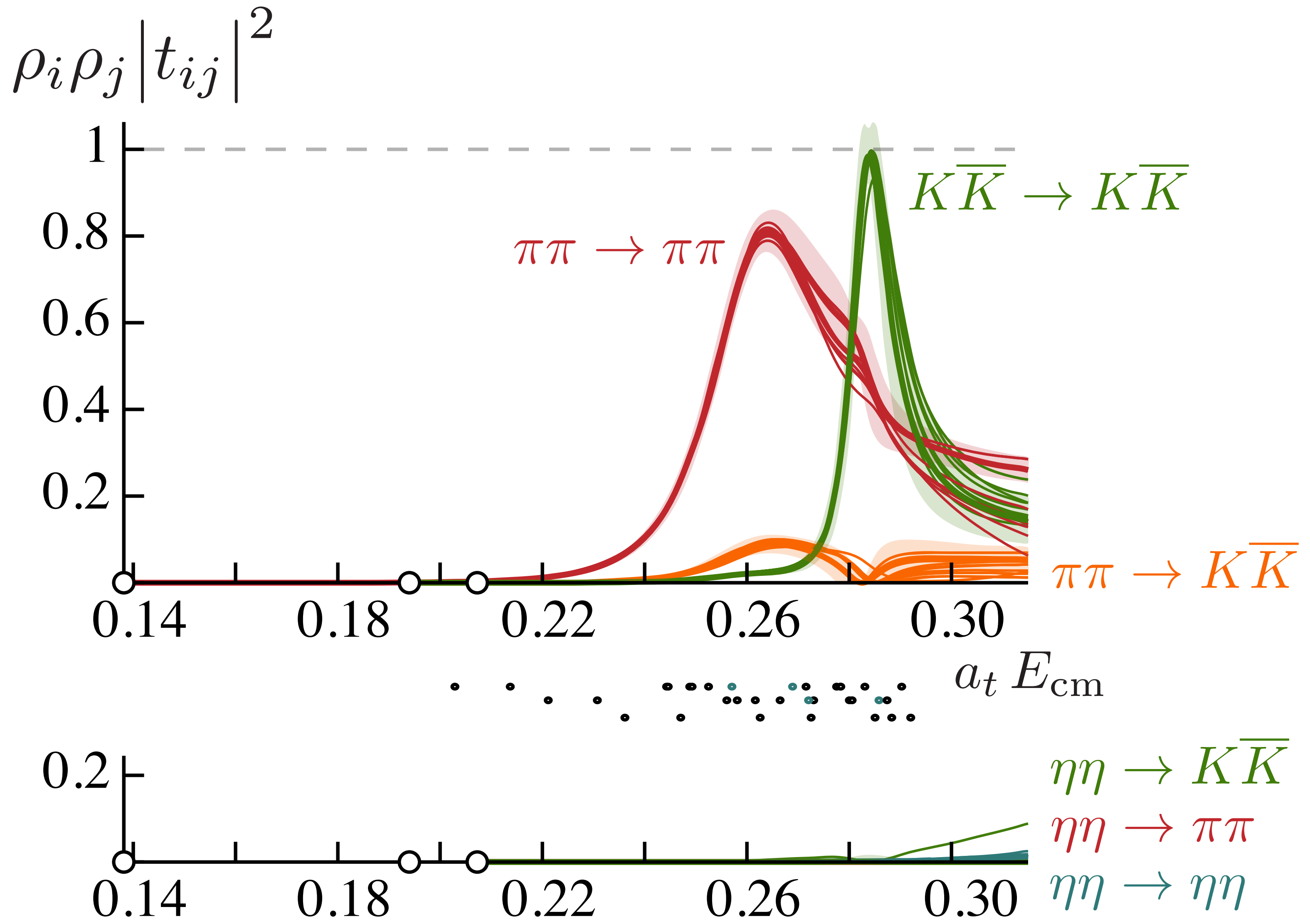
D-wave amplitudes



$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

… and varying the particular choice of parameterization …

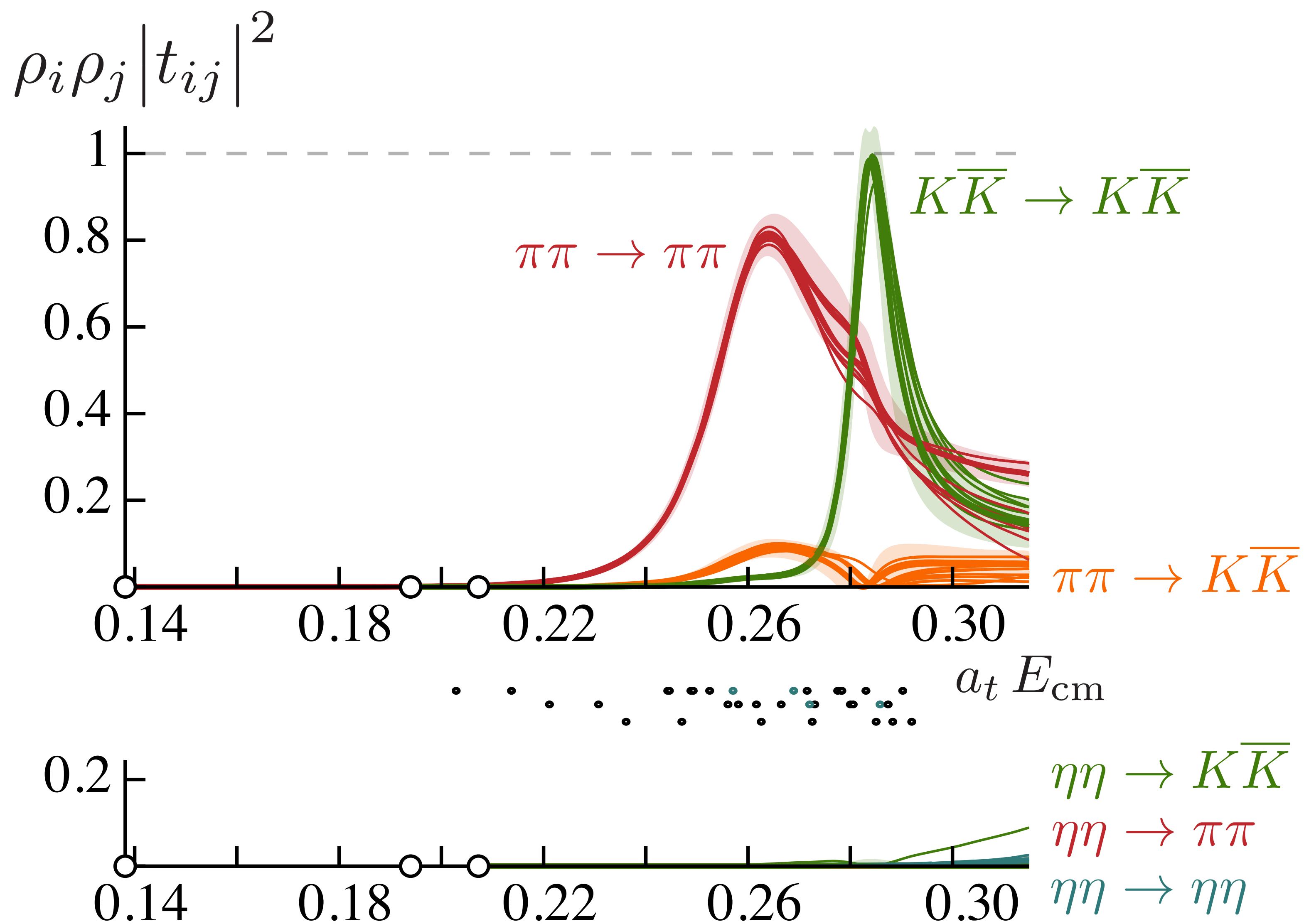
D-wave amplitudes



$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

PRD 97 054513 (2018)

D-wave amplitudes



'looks like' two resonances

– lighter one has larger width, big coupling to $\pi\pi$

– heavier one has smaller width, big coupling to $K\bar{K}$

... there must be a more rigorous way to know the resonance content ?

rigorous resonance determination ?

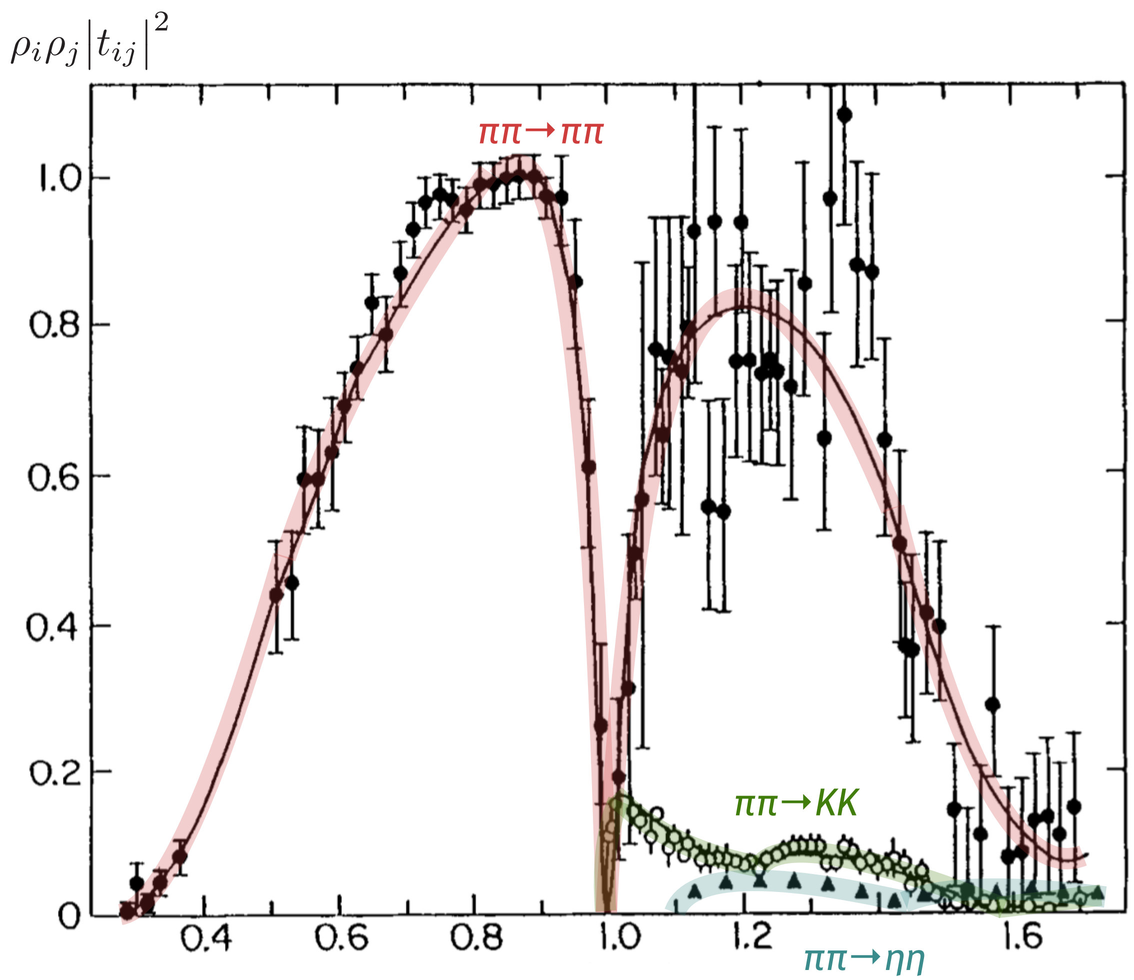
Jozef Dudek

hadron spectrum collaboration
hadspec.org

William & Mary
Chartered 1693

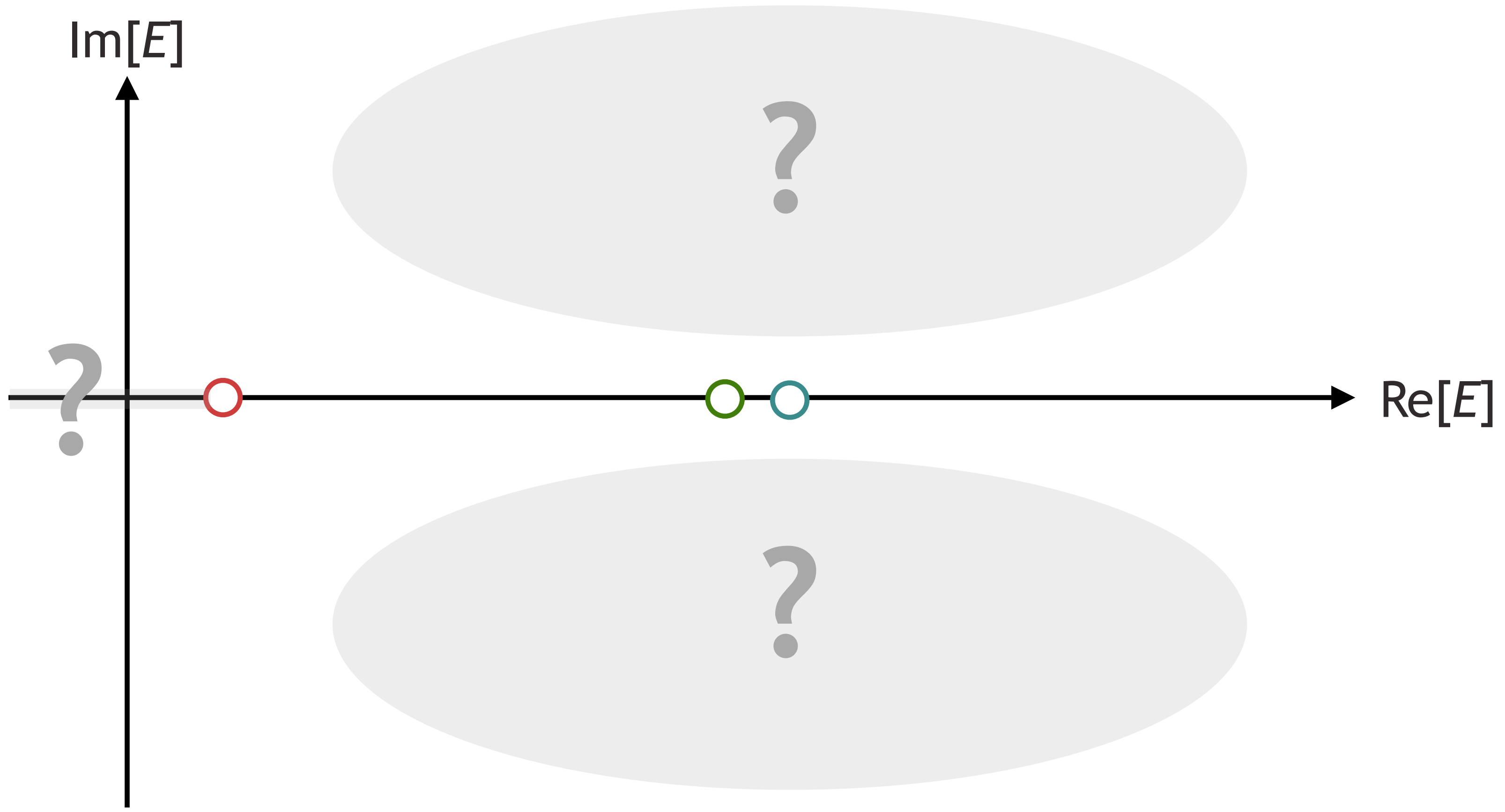
Jefferson Lab
Thomas Jefferson National Accelerator Facility

scattering amplitudes are measured for real energies above threshold



and we've seen that lattice calculations can lead to something similar

does it make sense to consider how the amplitude behaves 'elsewhere'
— below threshold ?
— for complex values of E ?



complex variable theory tells us that
singularities (poles, cuts)
'control' the behaviour of functions

— what singularities can our amplitudes have ?

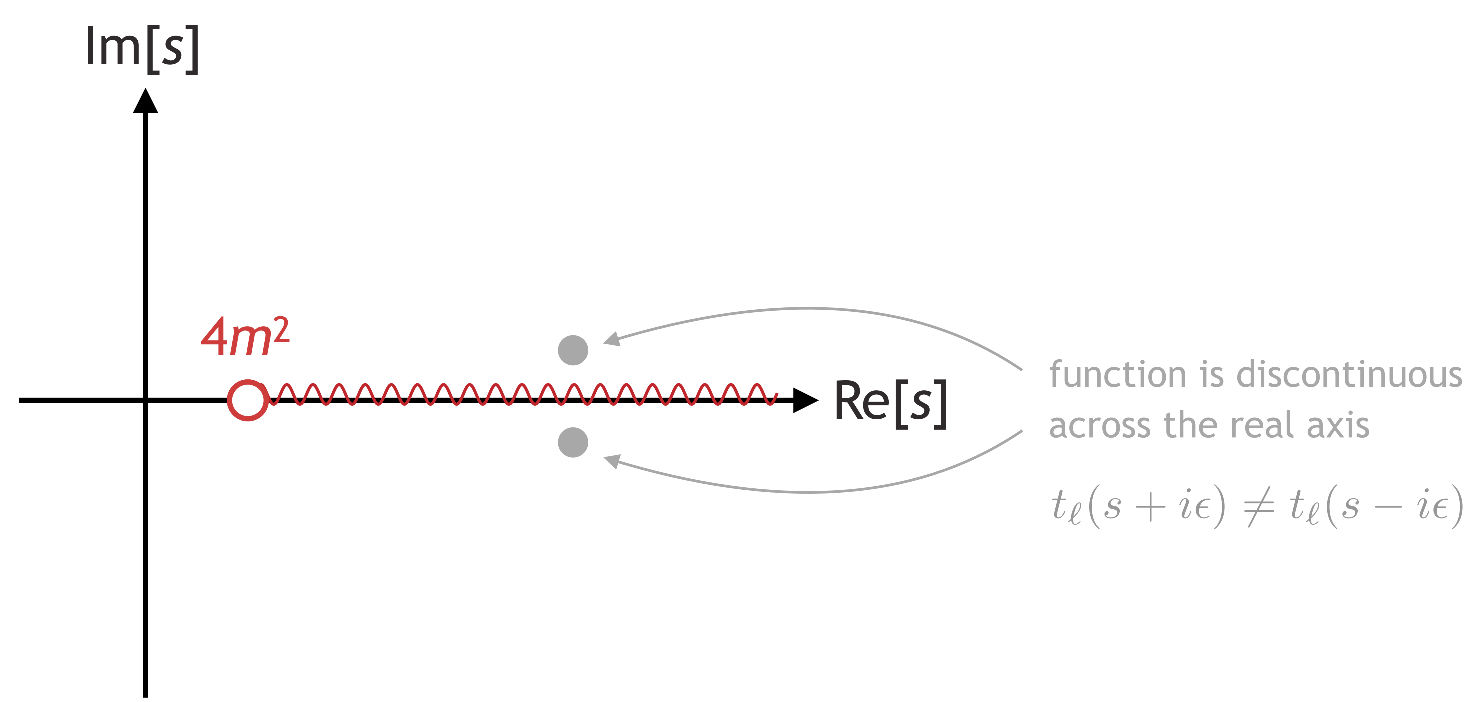
the unitarity cut

unitarity gives us one guaranteed singularity — a branch cut starting at threshold

e.g. elastic partial-wave case: $\mathrm{Im}\, t_\ell(s) = \rho(s)\, \big|t_\ell(s)\big|^2\, \Theta(s-4m^2)$

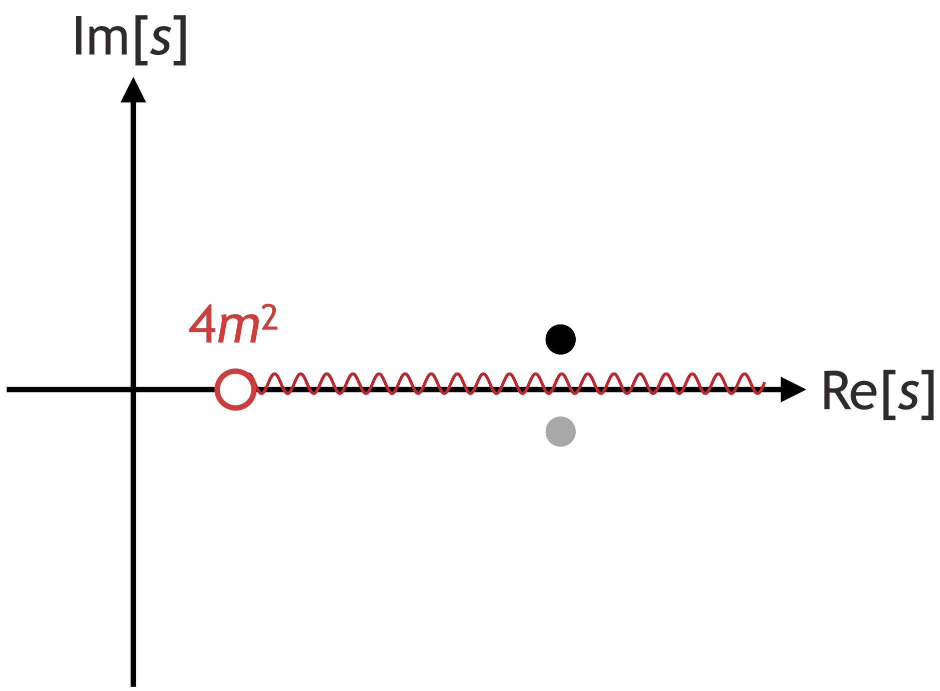
$$\rho(s) = \frac{2k(s)}{\sqrt{s}} = \frac{\sqrt{s-4m^2}}{\sqrt{s}}$$

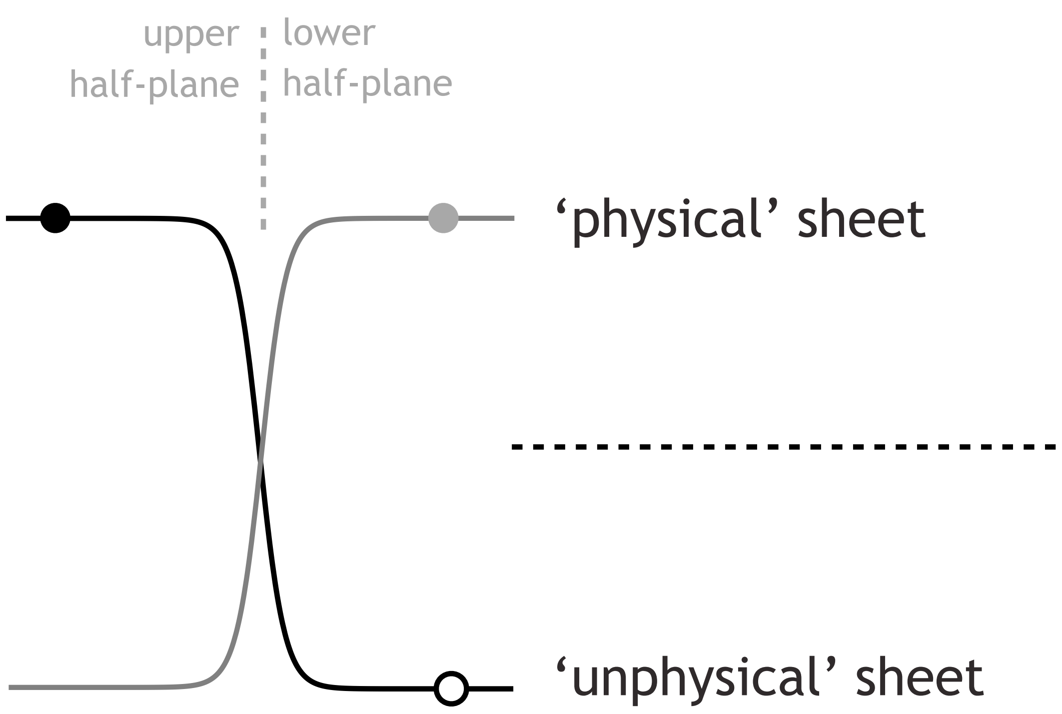
square root branch cut



has an immediate consequence
— the complex plane must be **multi-sheeted**

Riemann sheet structure





sheets can be characterised by the sign of Im[k]

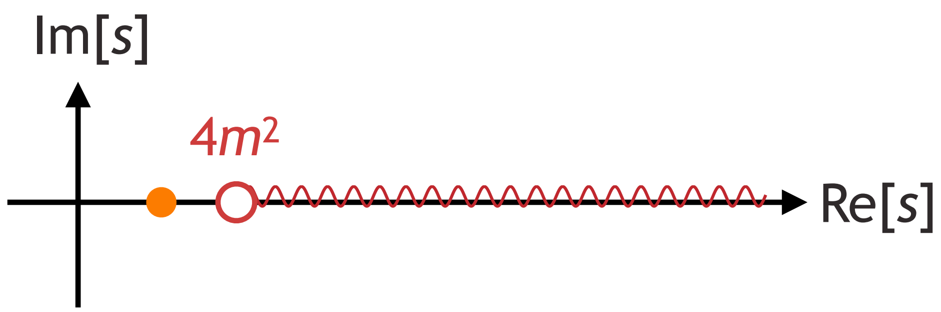
physical sheet = sheet I = Im[k] > 0

unphysical sheet = sheet II = Im[k] < 0

pole singularities ?

scattering amplitudes can have pole singularities only in certain locations

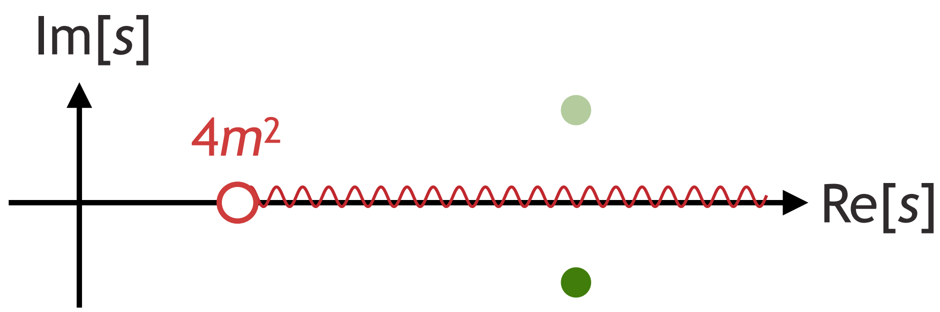
real energy axis, below threshold on **physical sheet**



corresponds to a **stable bound-state**

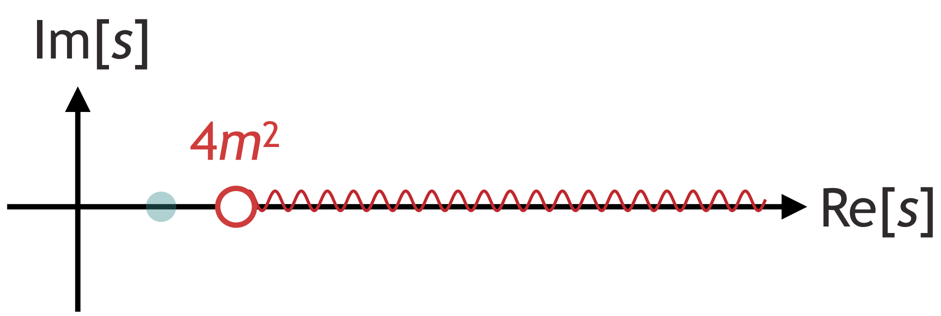
off the real axis, on the **unphysical sheet**

(in complex conjugate pairs)



corresponds to a **resonance**

real energy axis, below threshold on **unphysical sheet**

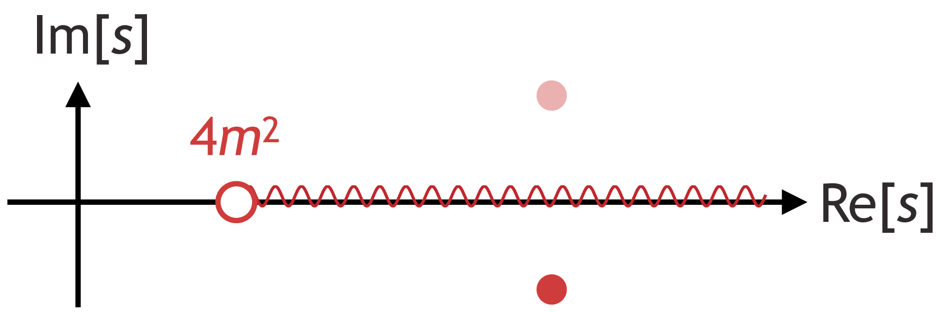


corresponds to a **virtual bound-state**

pole singularities ?

scattering amplitudes can have pole singularities only in certain locations

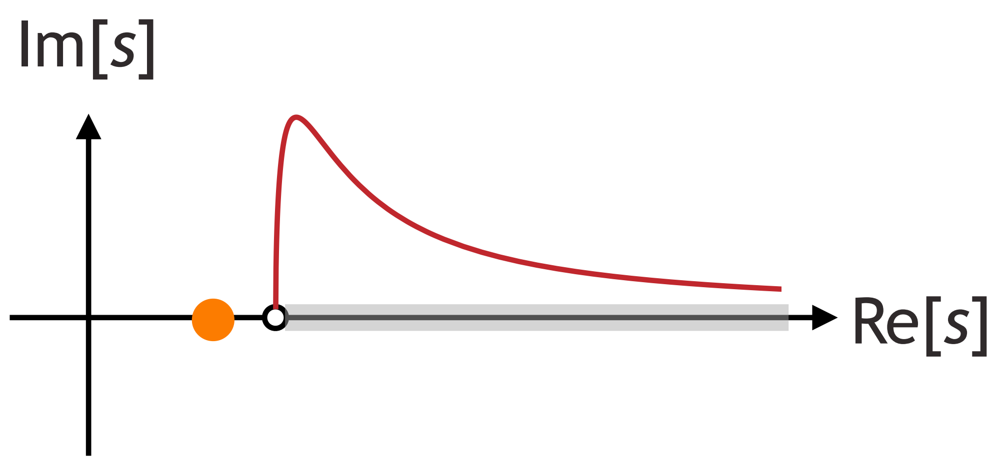
not allowed: poles off the real axis
on the **physical sheet**



would violate **causality**

a bound-state pole

will strongly enhance scattering at threshold

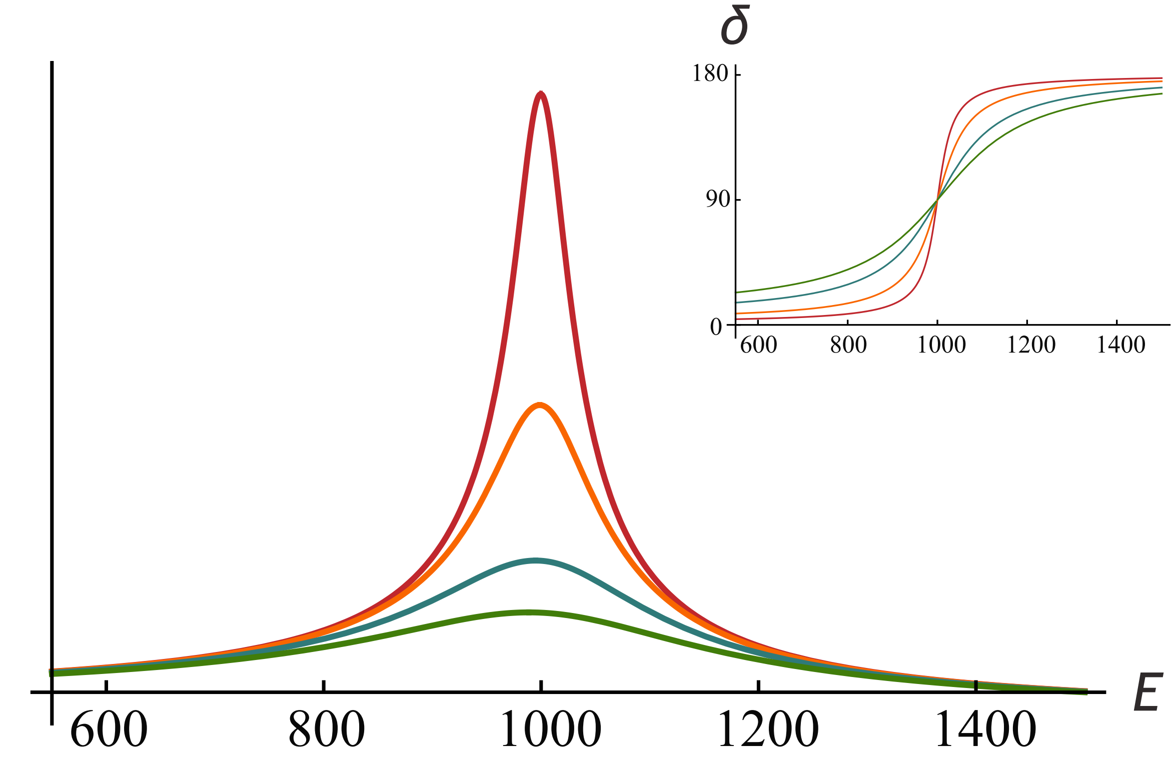


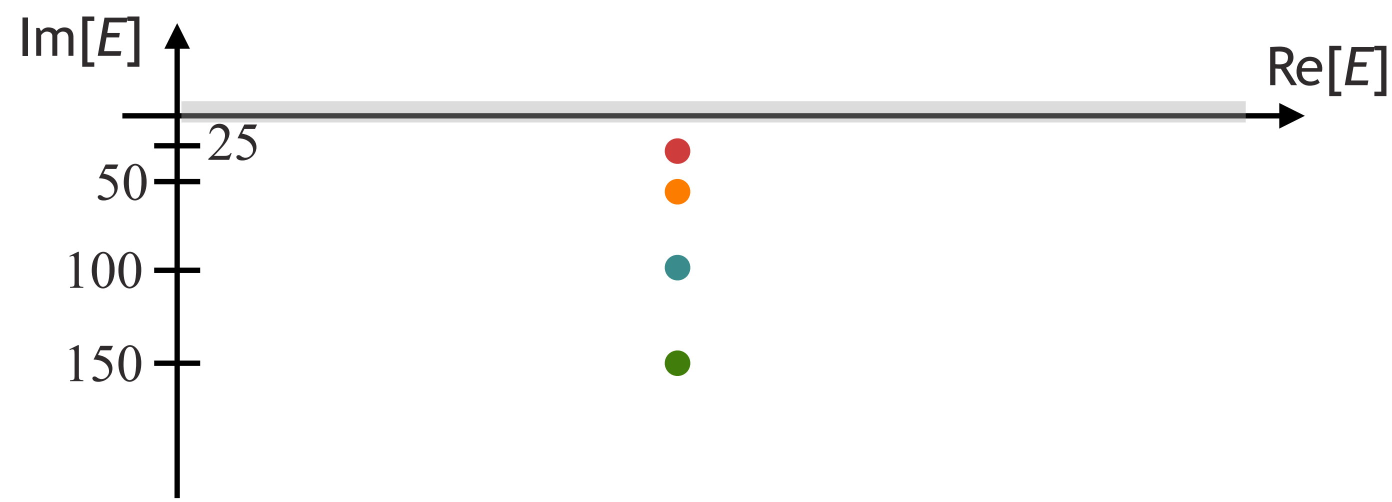
famous example is the deuteron at *NN* threshold

pole on the unphysical sheet

an **isolated** pole on the unphysical sheet will produce a bump on the real axis

– the classic resonance signature





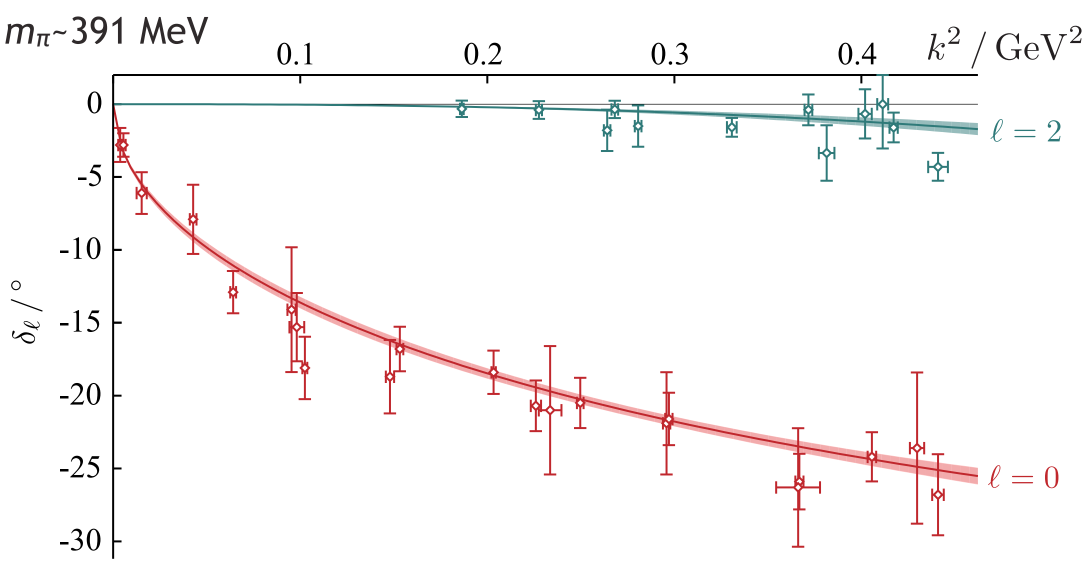
close to the pole

$$t_\ell(s) \sim \frac{1}{s_0 - s}$$

$$s_0 = \left(m - i\tfrac{1}{2}\Gamma\right)^2$$

singularity structure from lattice calculations — elastic

$\pi\pi$ isospin=2



PRD86 034031 (2012)

$$k \cot \delta_0 = \frac{1}{a_0} + \ldots$$

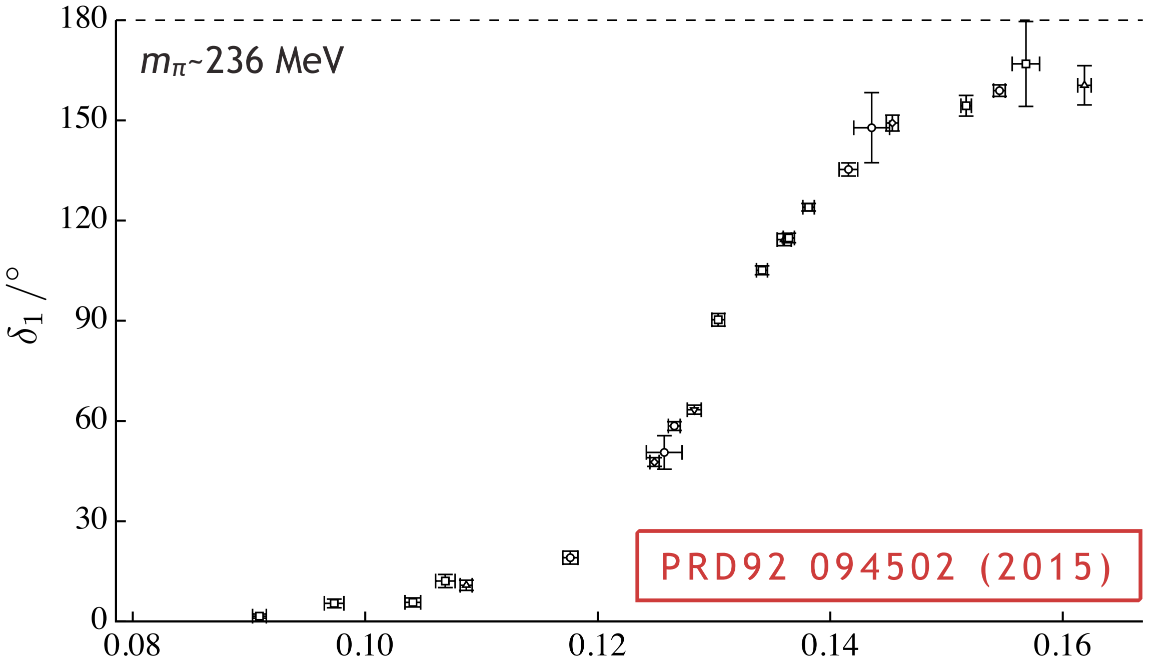
$$m_\pi a_0 = -0.285(6)$$

$$s_0 \approx -45\, m_\pi^2$$

no nearby poles

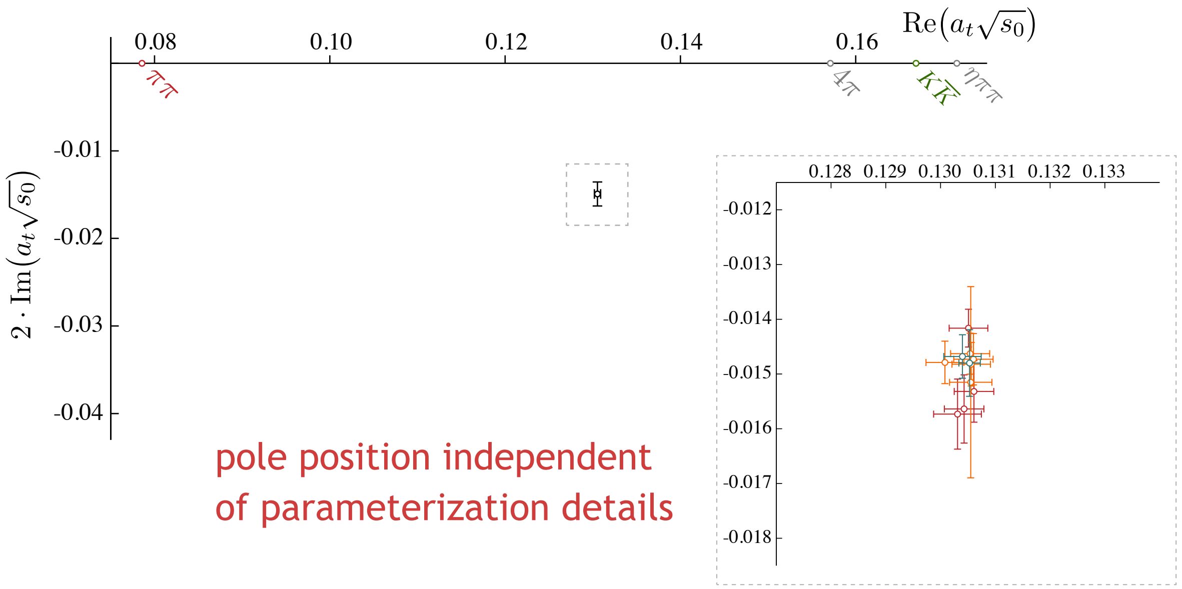
weak and repulsive interaction

$\pi\pi$ isospin=1



a single isolated pole

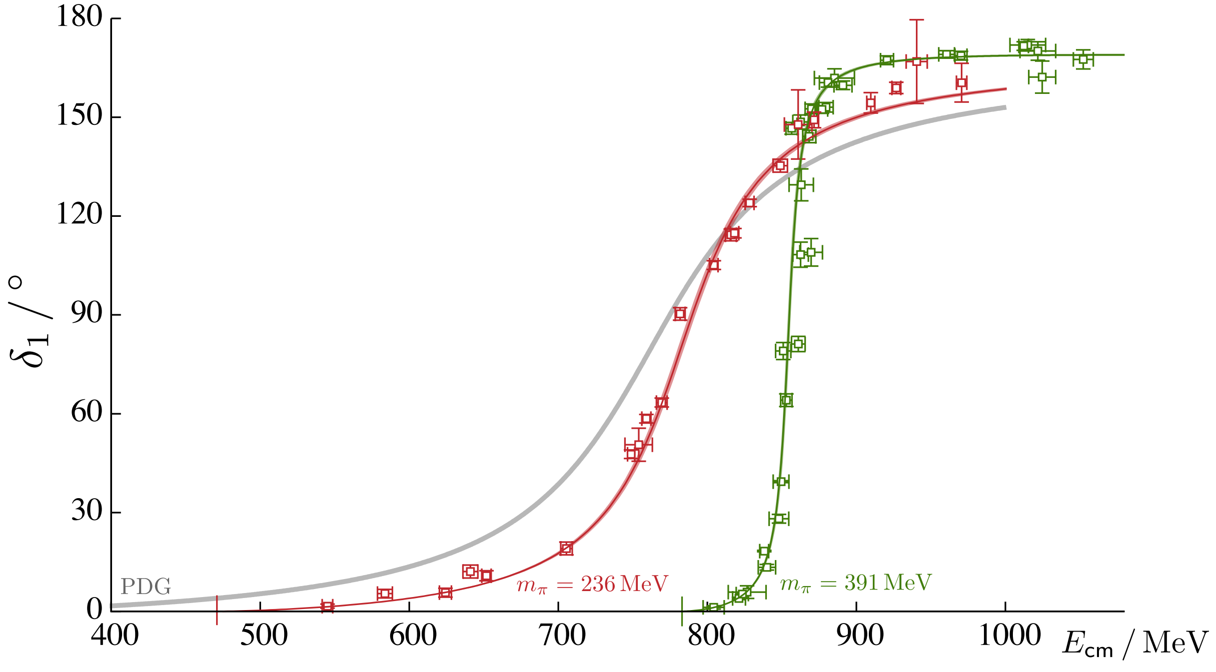
a narrow resonance



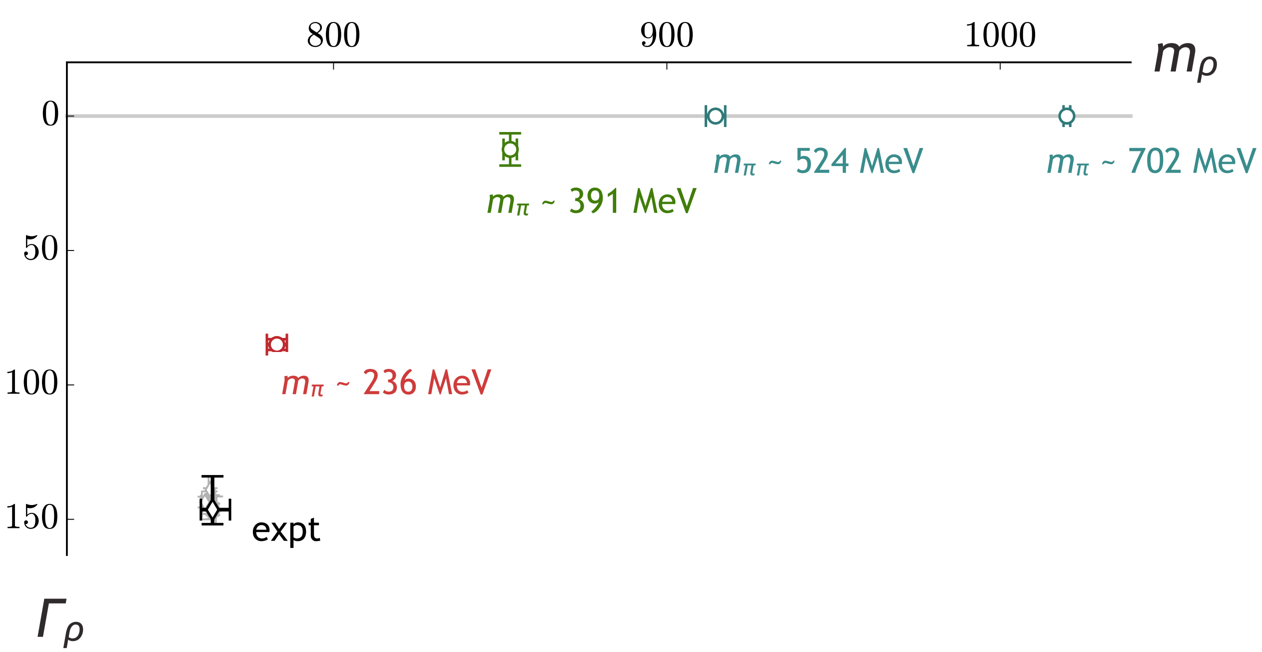
pole position independent of parameterization details

singularity structure from lattice calculations — elastic

$\pi\pi$ isospin=1

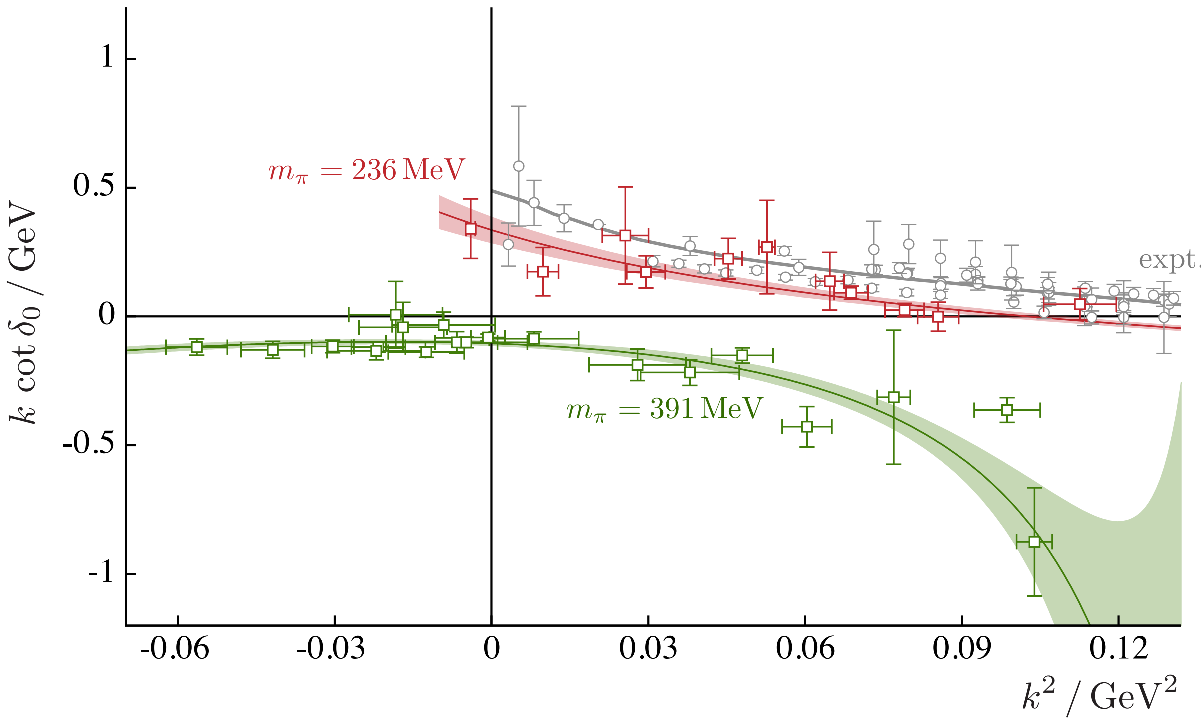


evolution with changing quark mass



singularity structure from lattice calculations — elastic

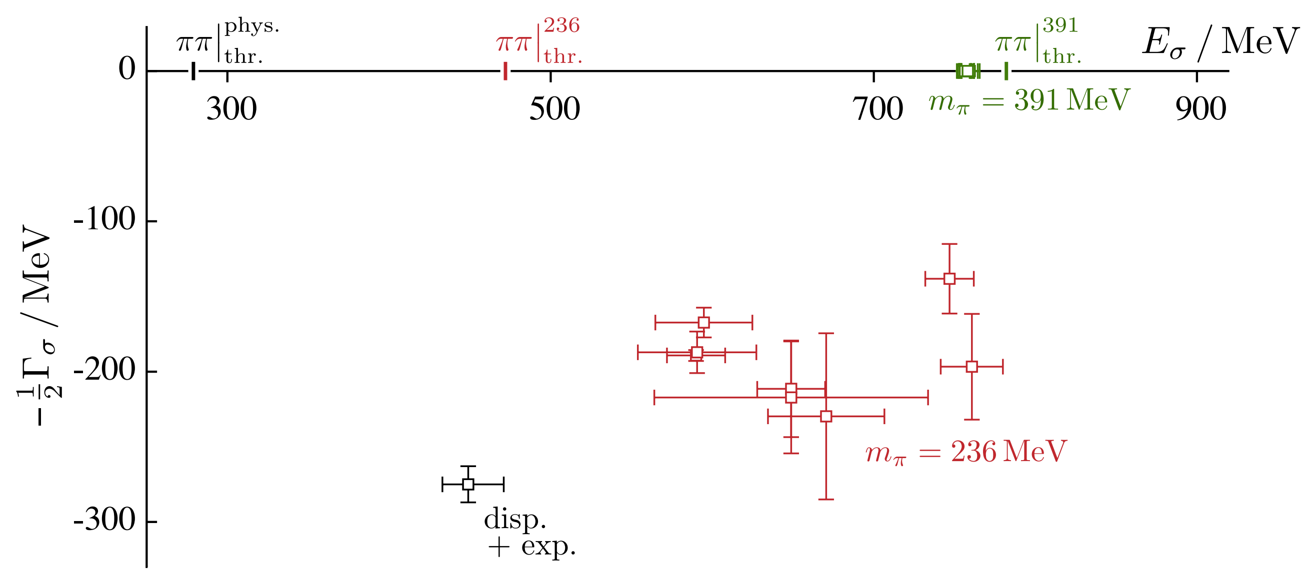
ππ isospin=0



PRL118 022002 (2017)

m_π~391 MeV — a bound-state pole

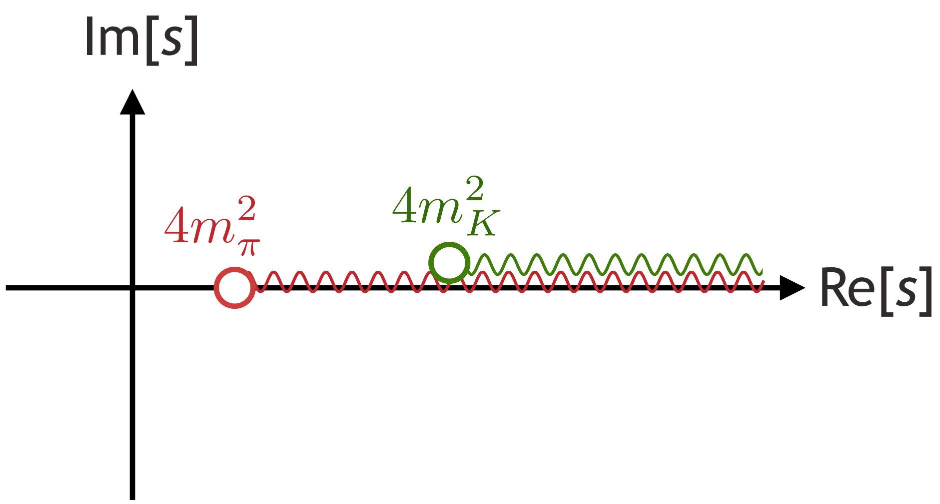
m_π~236 MeV — a resonance pole



coupled-channels

for each new channel, each sheet splits in two ⇒ 2^N sheets for N channels

e.g. two channels ($\pi\pi$, $K\bar{K}$)



coupled-channels

for each new channel, each sheet splits in two ⇒ 2^N sheets for N channels

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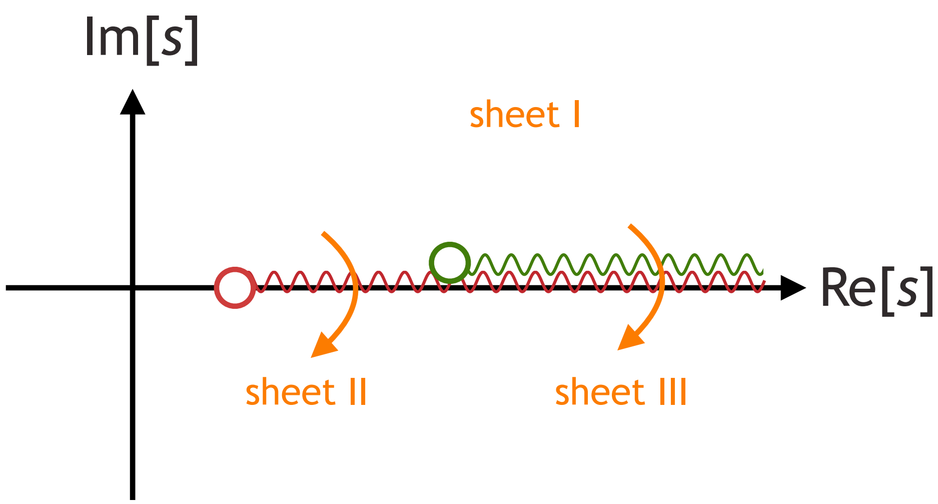
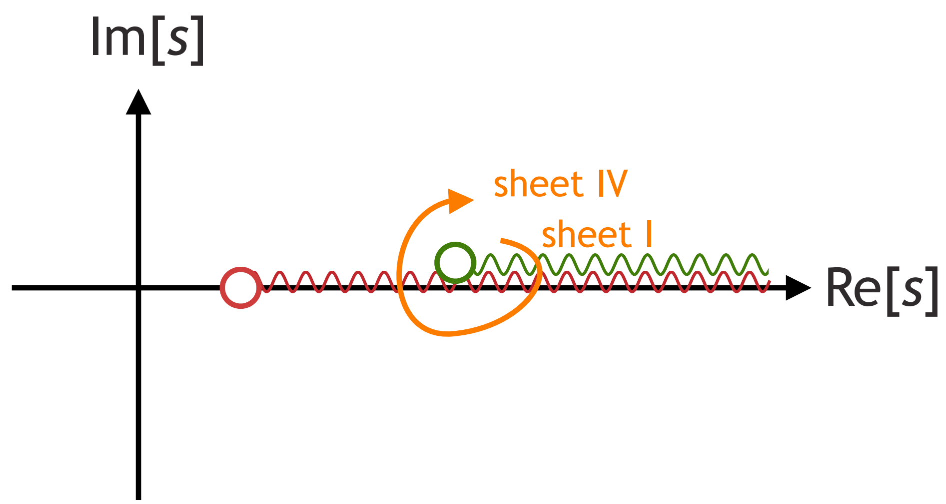


	Im[$k_{\pi\pi}$]	Im[k_{KK}]
sheet I	+	+
sheet II	−	+
sheet III	−	−
sheet IV	+	−

coupled-channels

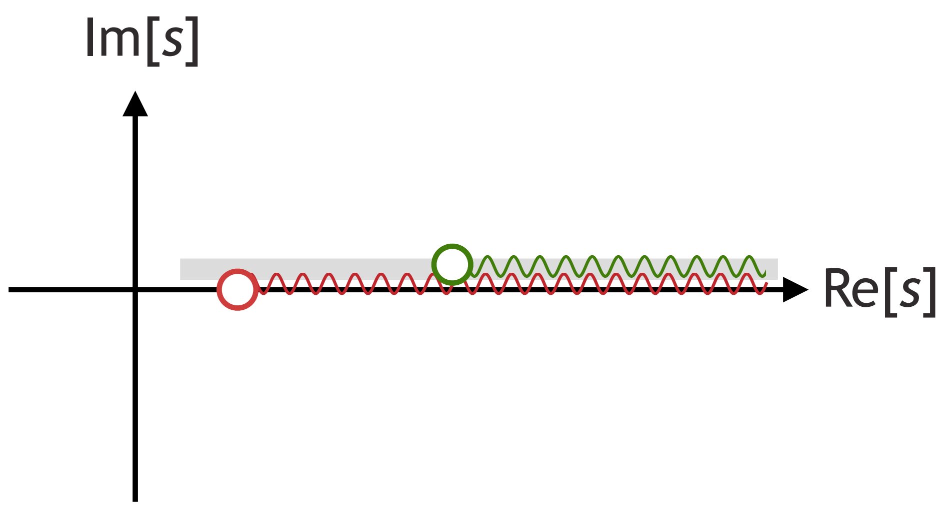
for each new channel, each sheet splits in two ⇒ 2^N sheets for N channels

e.g. two channels

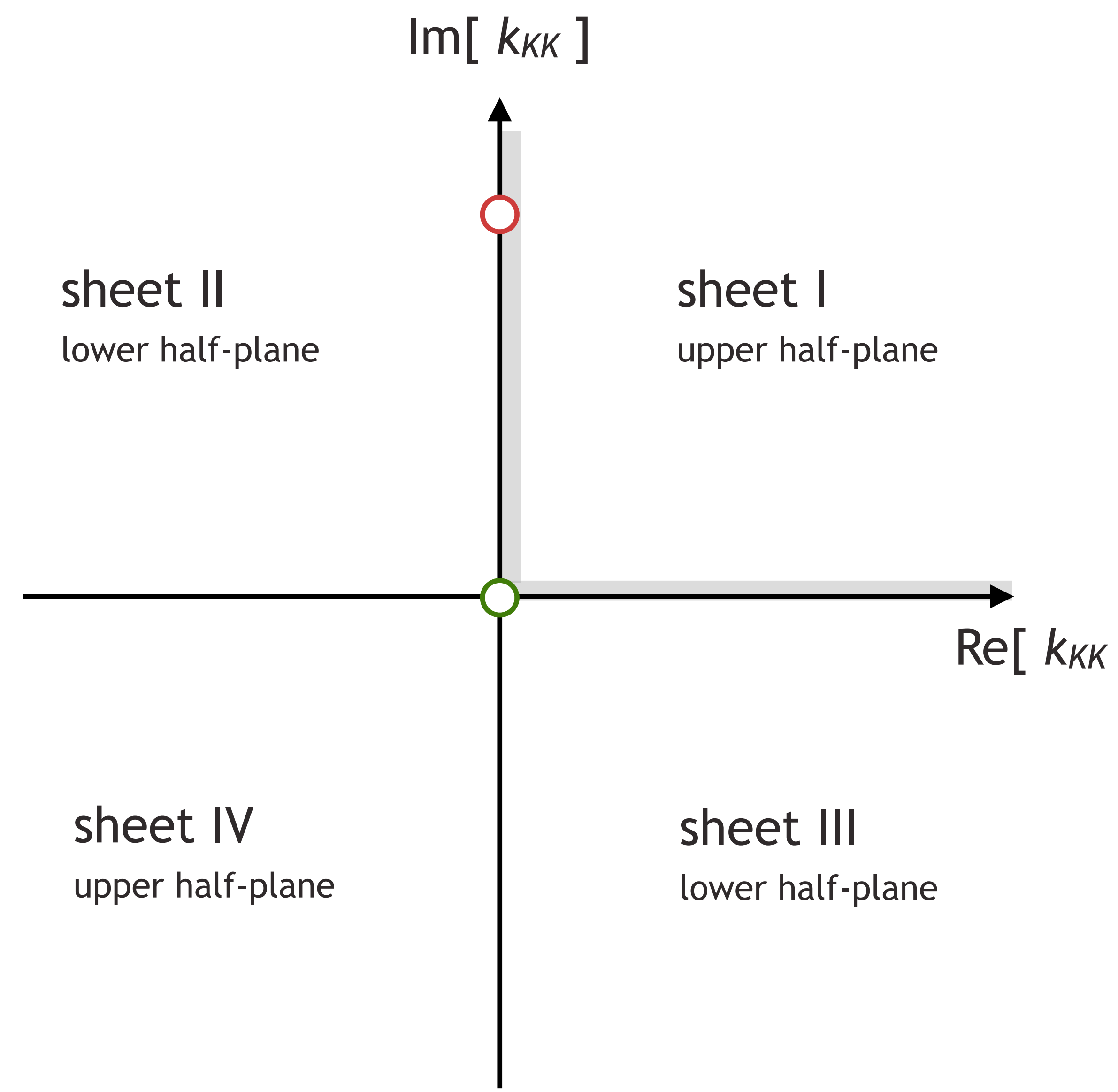


	Im[$k_{\pi\pi}$]	Im[k_{KK}]
sheet I	+	+
sheet II	−	+
sheet III	−	−
sheet IV	+	−

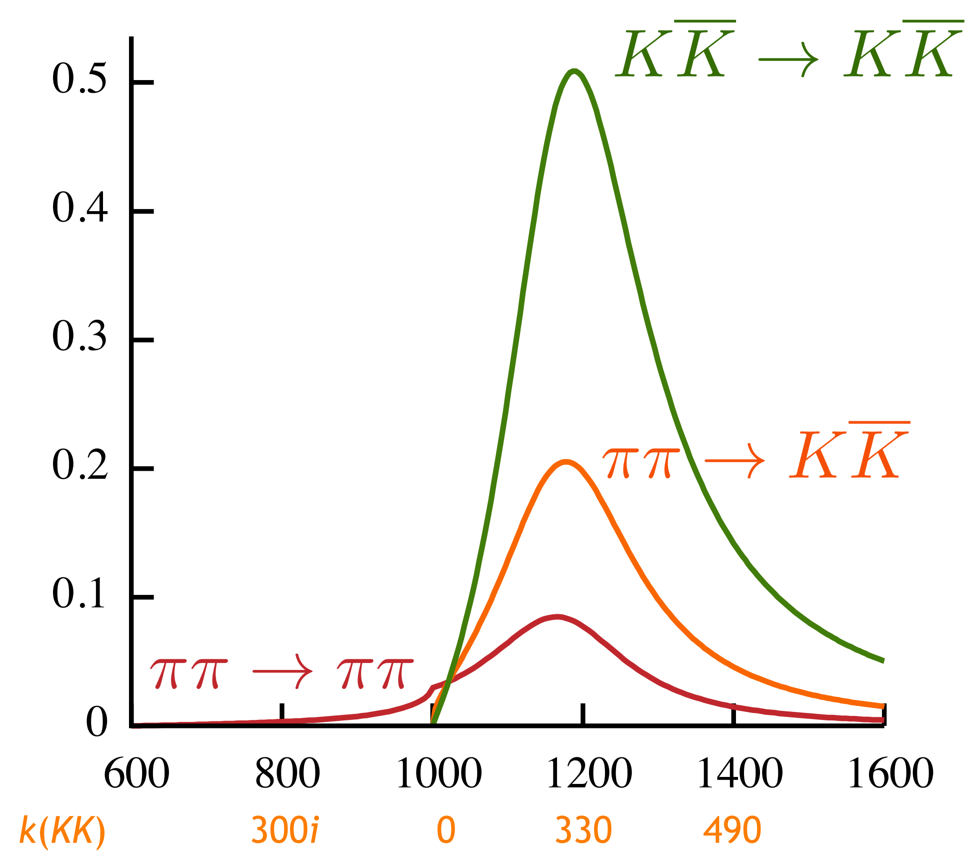
coupled-channels

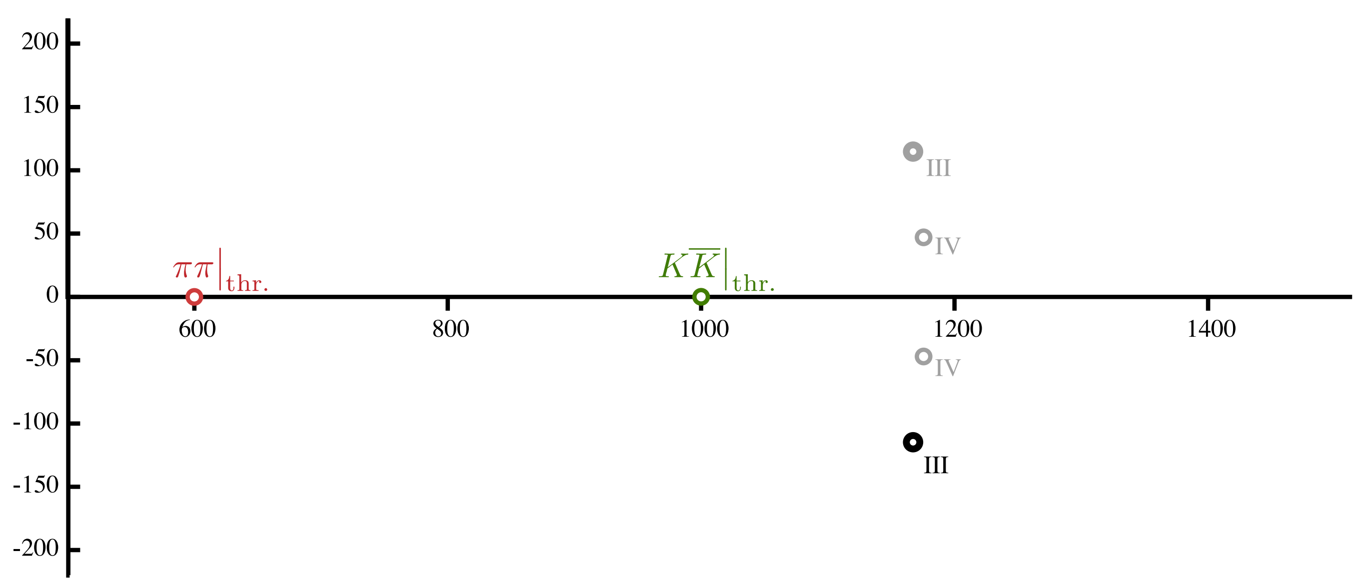


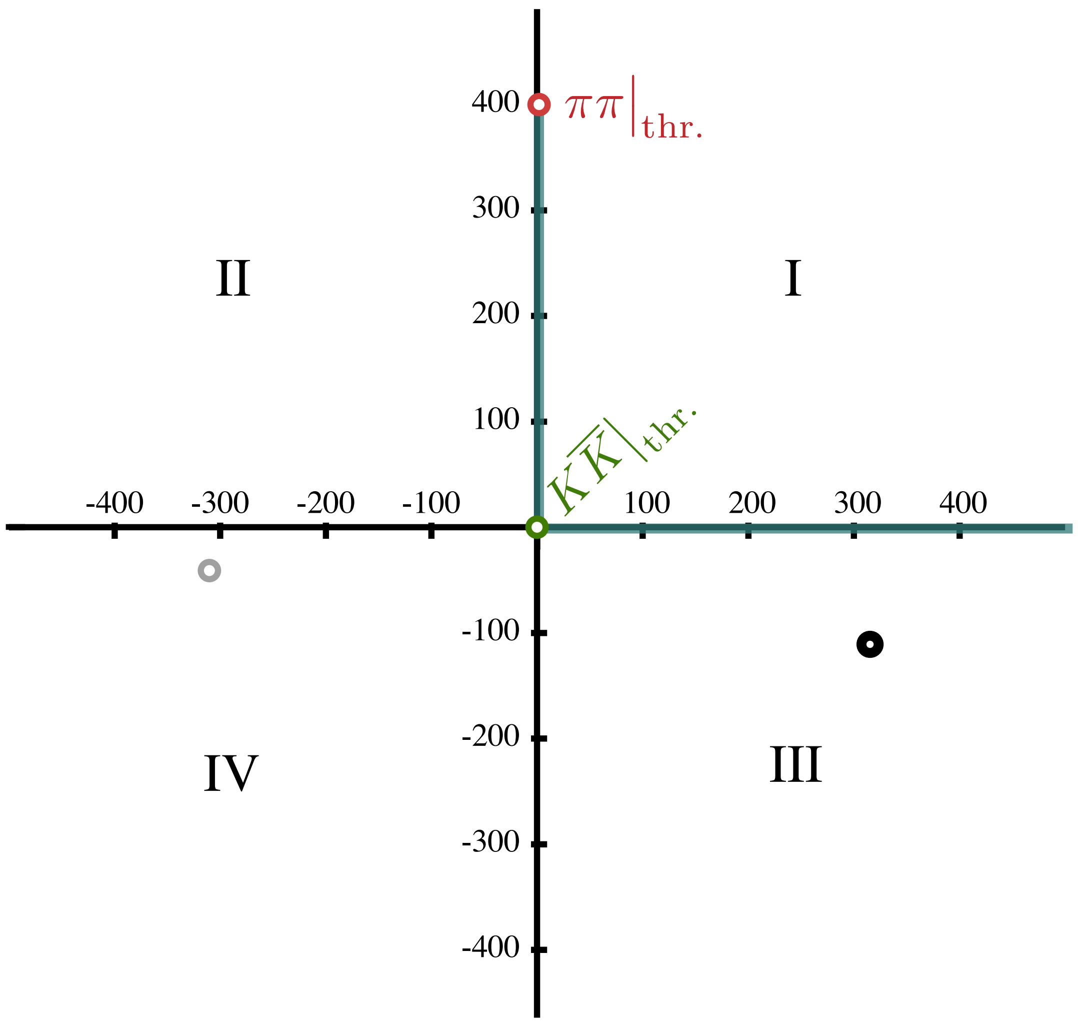
	Im[$k_{\pi\pi}$]	Im[k_{KK}]
sheet I	+	+
sheet II	−	+
sheet III	−	−
sheet IV	+	−



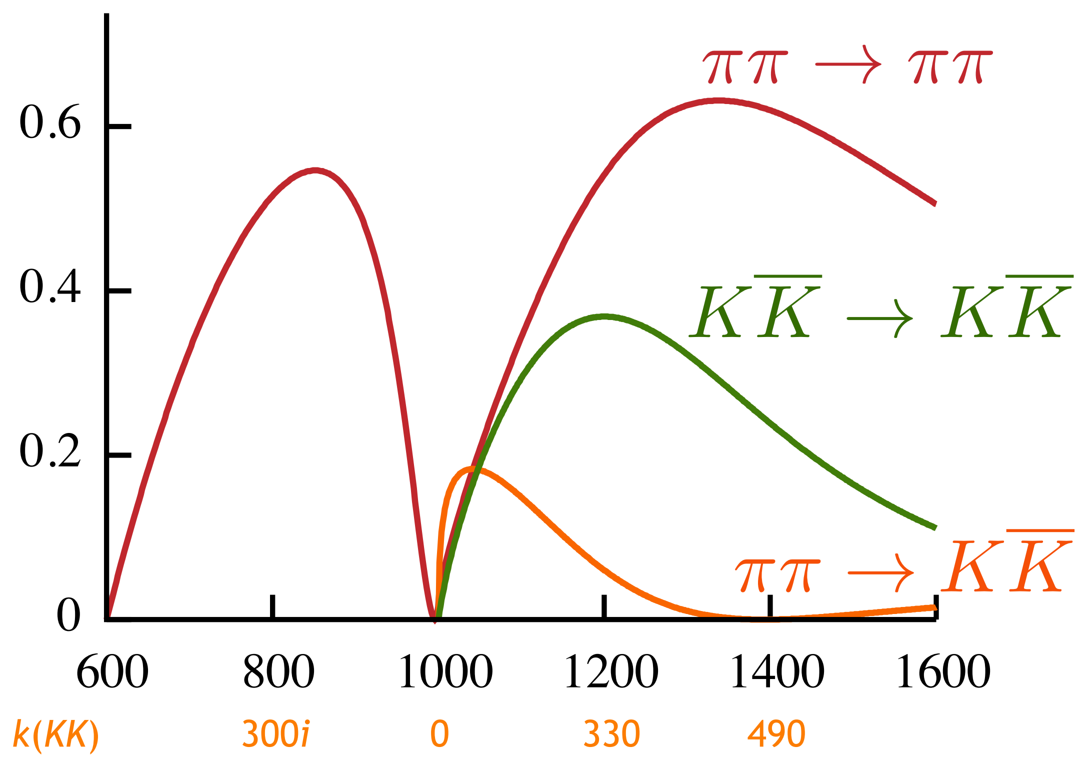
two-channel Flatté amplitude





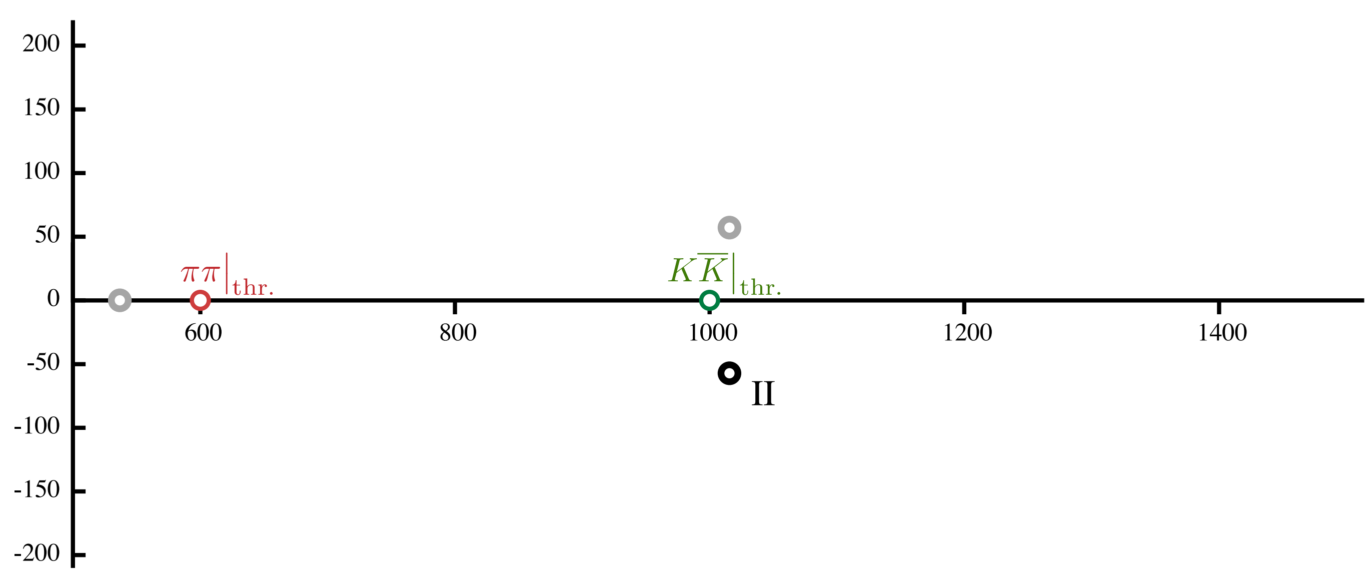


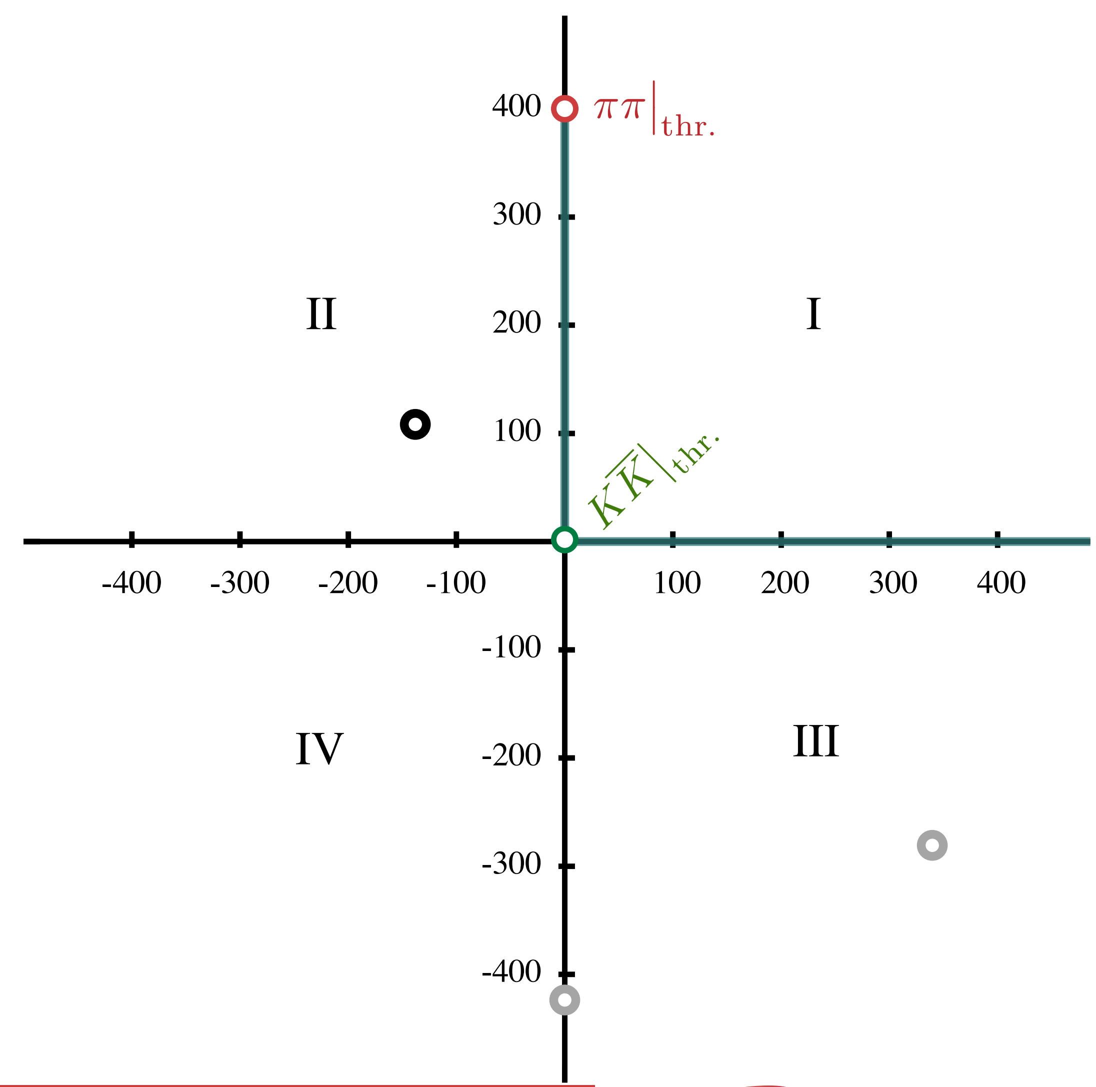
a less obviously resonant amplitude



$$\mathbf{K}^{-1}(s) = \begin{pmatrix} a & b + c\,s \\ b + c\,s & d + e\,s \end{pmatrix}$$

with Chew-Mandelstam phase-space





information from the pole

near the complex pole, s_0

$$t_{ij}(s \sim s_0) \sim \frac{c_i\, c_j}{s_0 - s}$$

pole **position** can be interpreted as **mass** and **width**

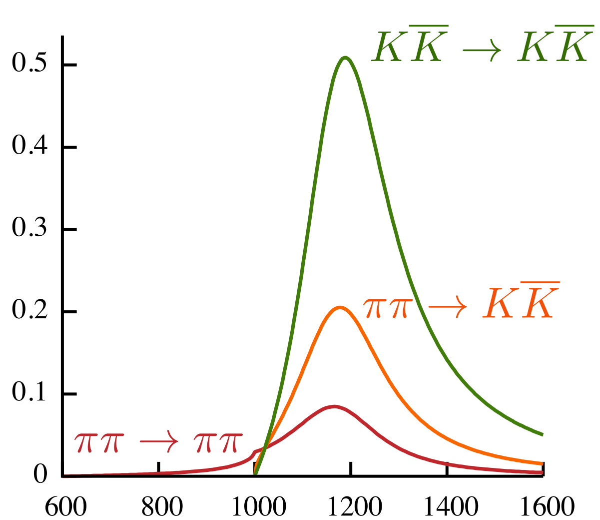
$$s_0 = \left(m_R \pm i\tfrac{1}{2}\Gamma_R\right)^2$$

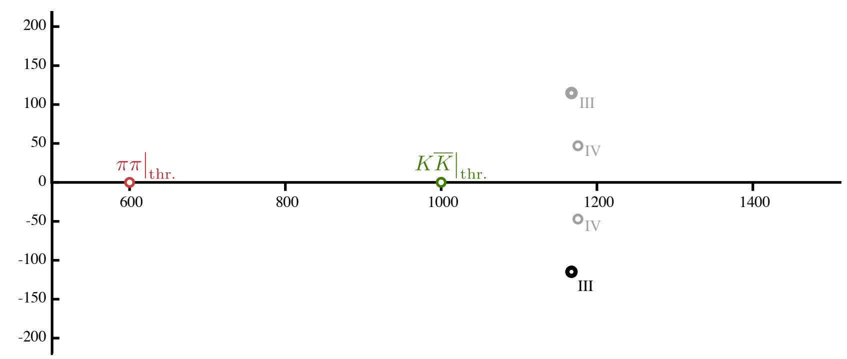
pole **residue** factorizes into a product of resonance **couplings** to the various decay channels

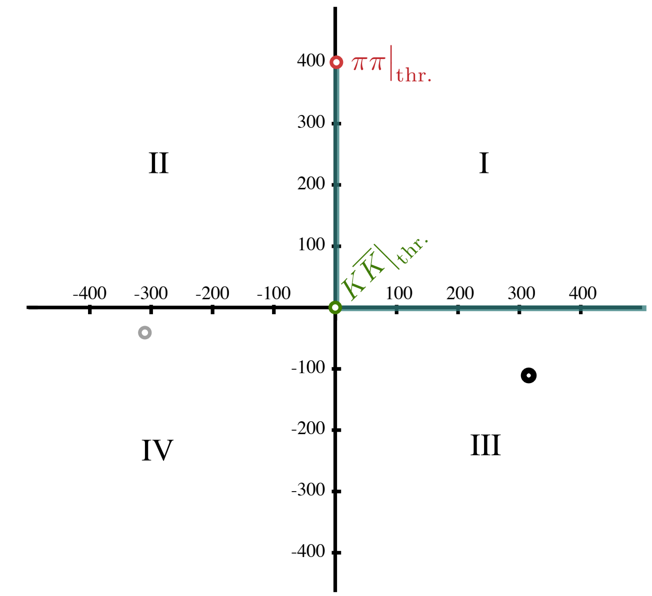
$$c_{\pi\pi},\, c_{K\bar{K}},\, \ldots$$

as we've seen a single resonance can be responsible for poles on more than one sheet

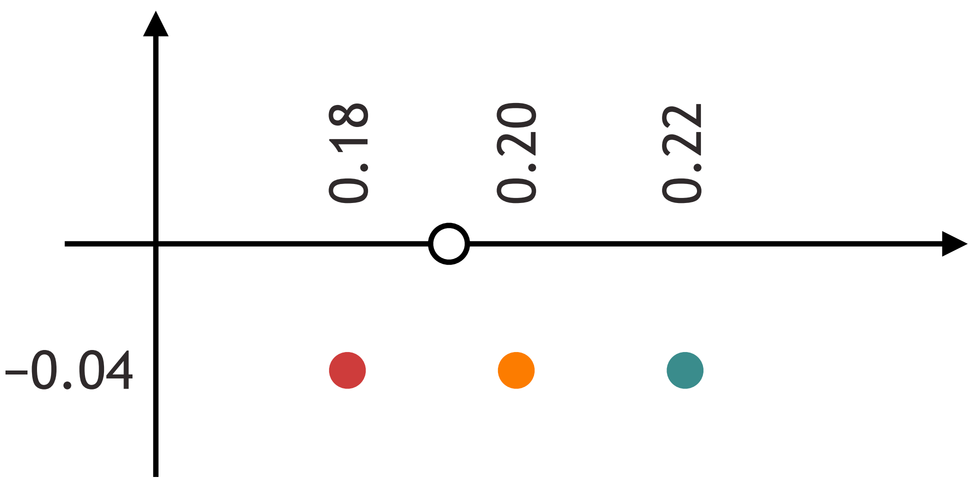
— often only one is close enough to physical scattering to have a large effect

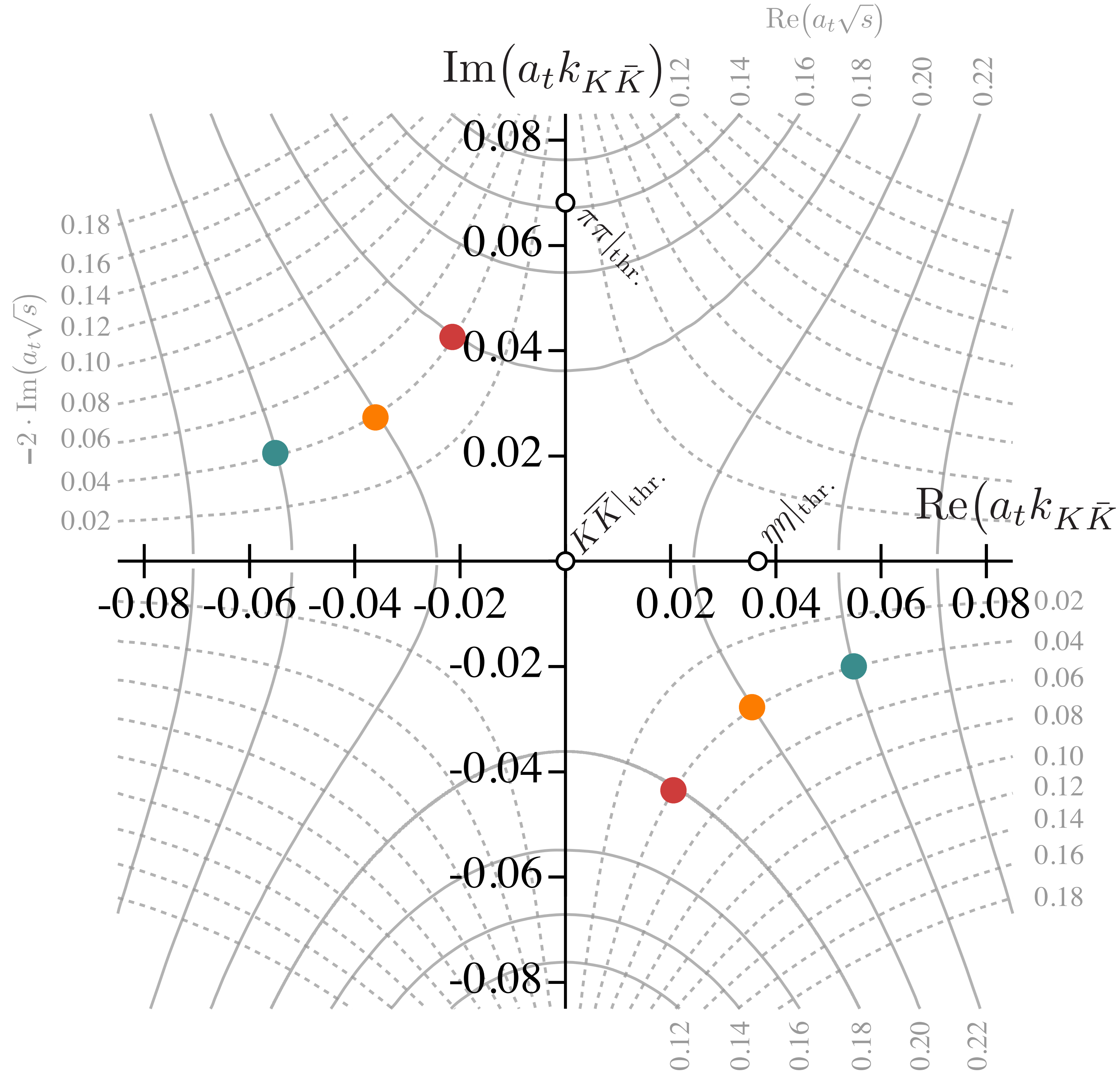






slight complication — mapping of ‘distances’ is rather non-uniform

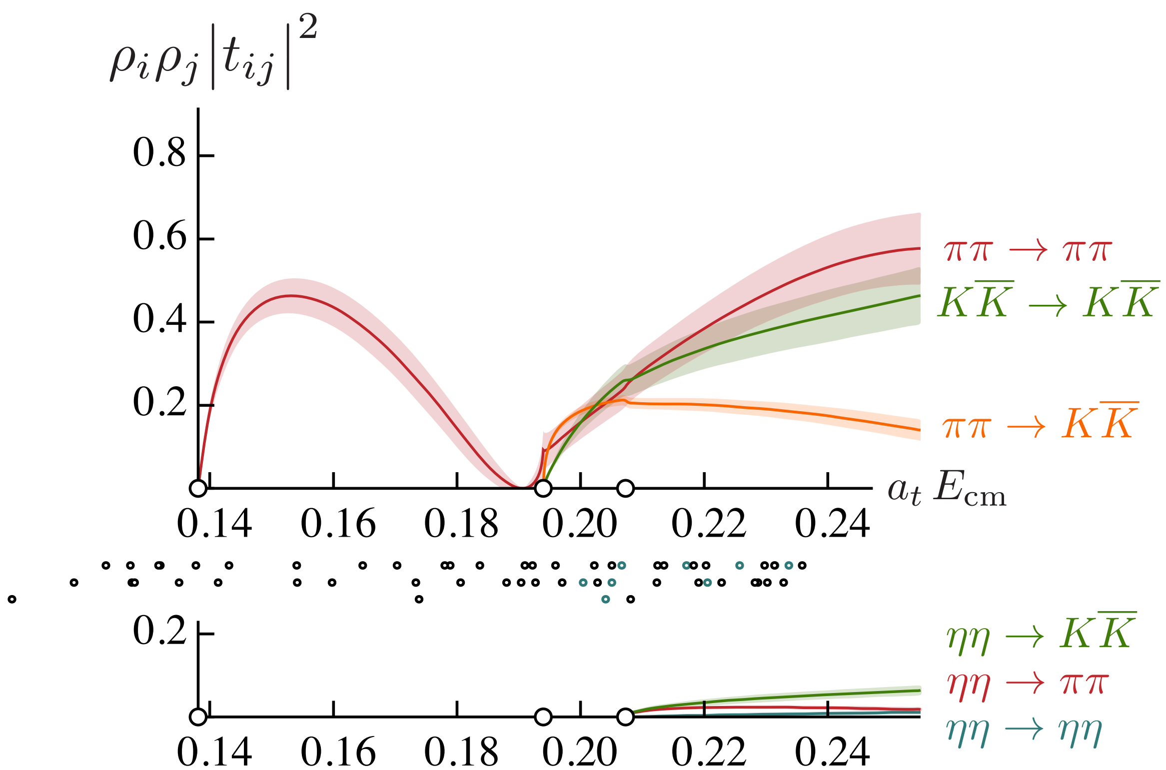




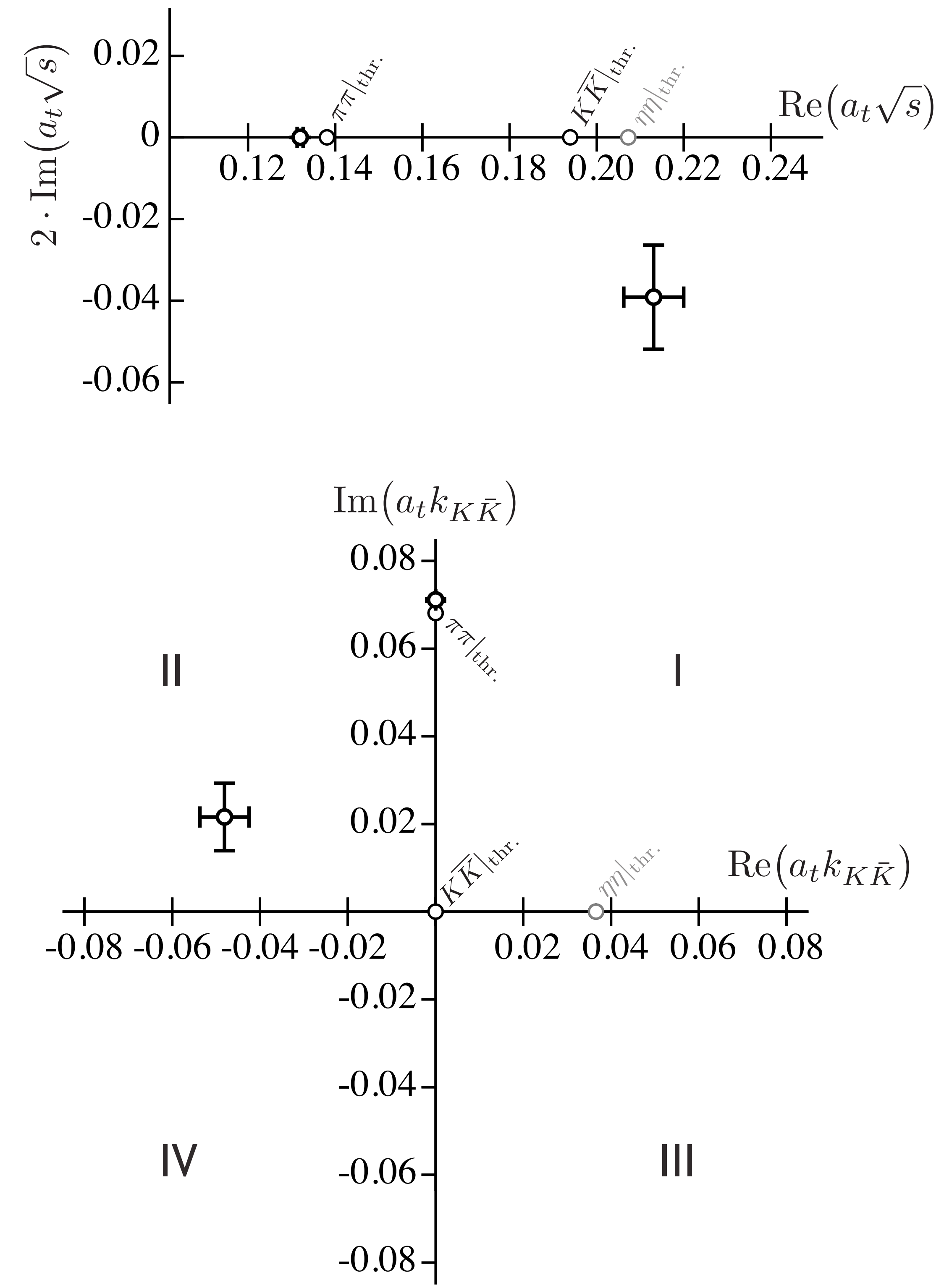
should really be using the *s*-plane rather than the √*s* plane

$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

S-wave amplitudes



pole singularities



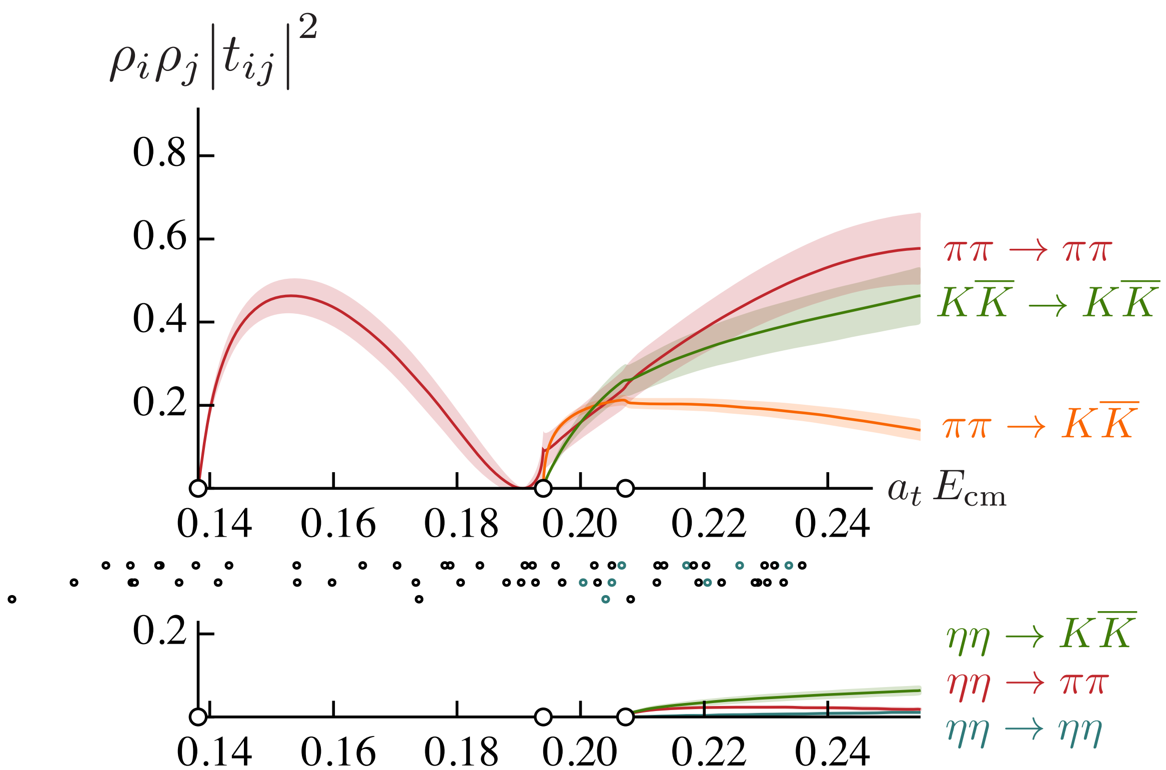
$$\mathbf{K}^{-1}(s) = \begin{pmatrix} a + b\,s & c + d\,s & e \\ c + d\,s & f & g \\ e & g & h \end{pmatrix}$$

with Chew-Mandelstam phase-space

$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

PRD 97 054513 (2018)

S-wave amplitudes



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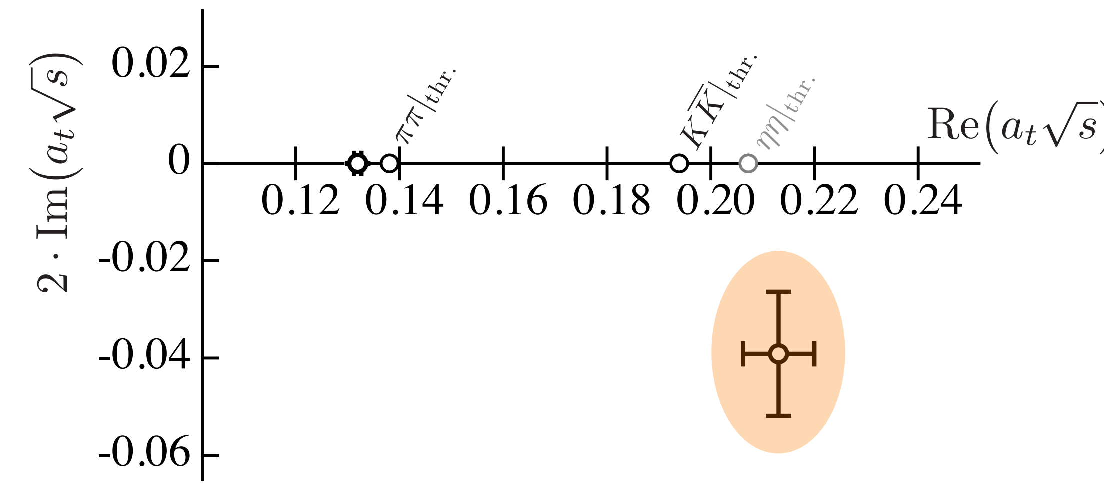
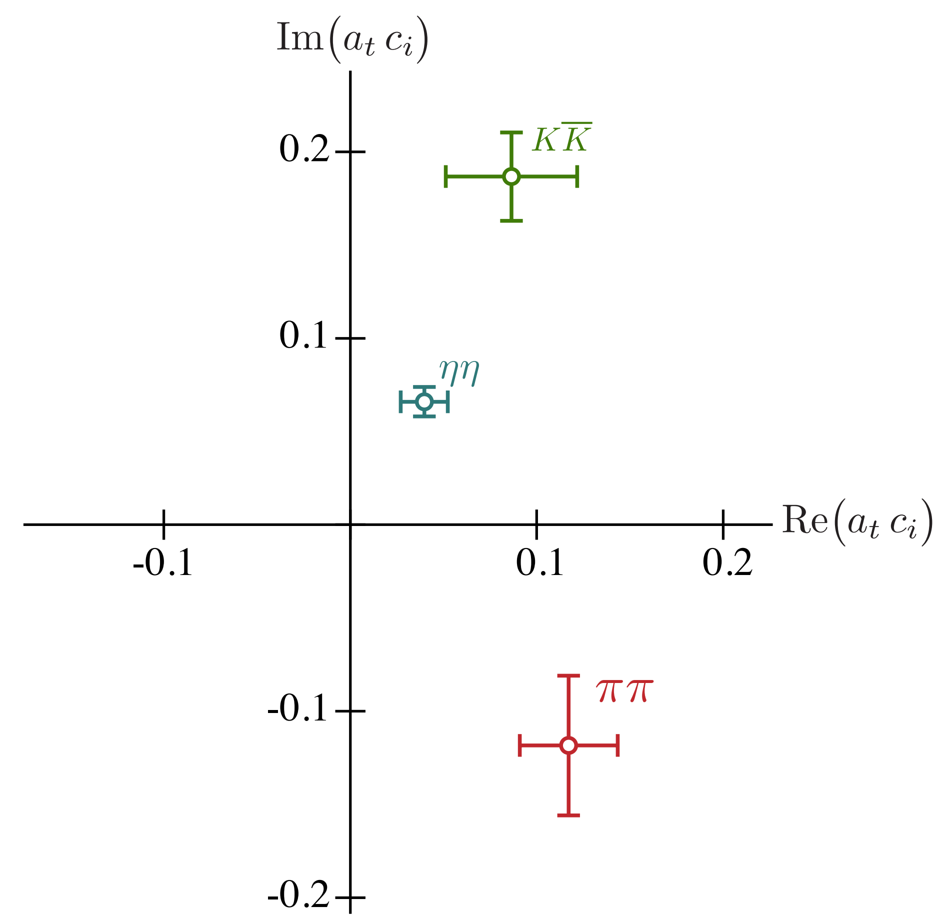


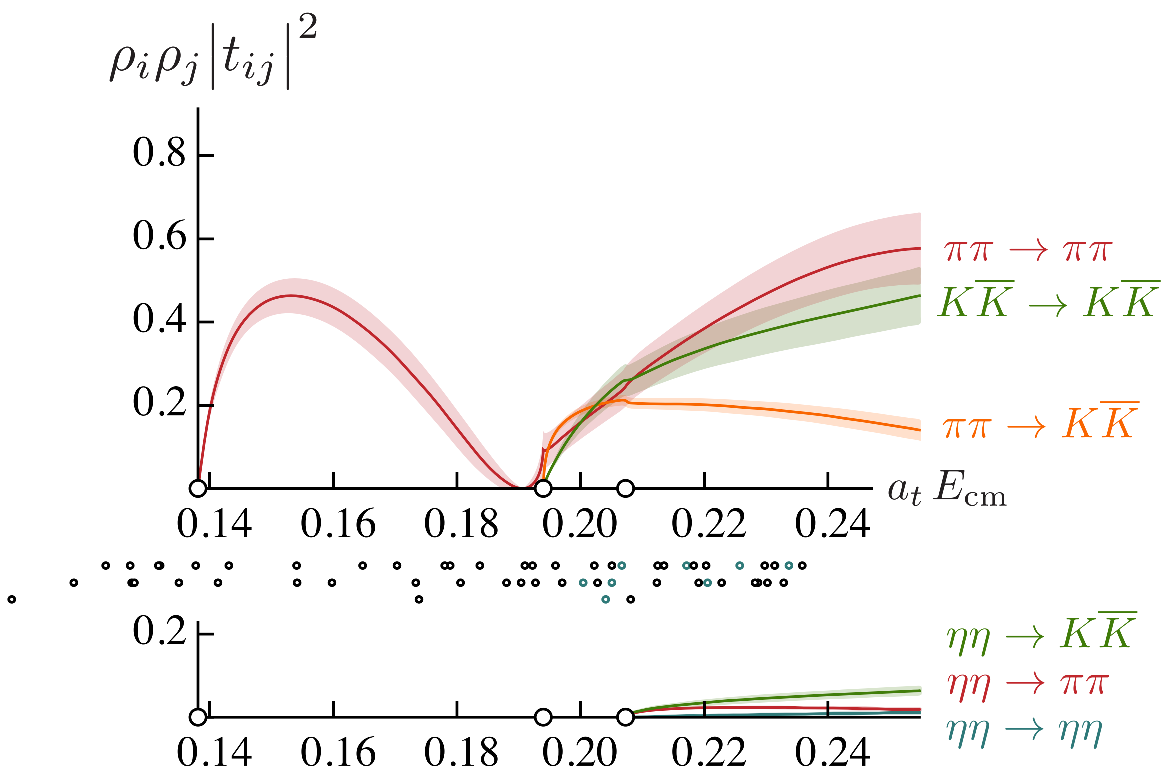
sheet II pole couplings



$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

PRD 97 054513 (2018)

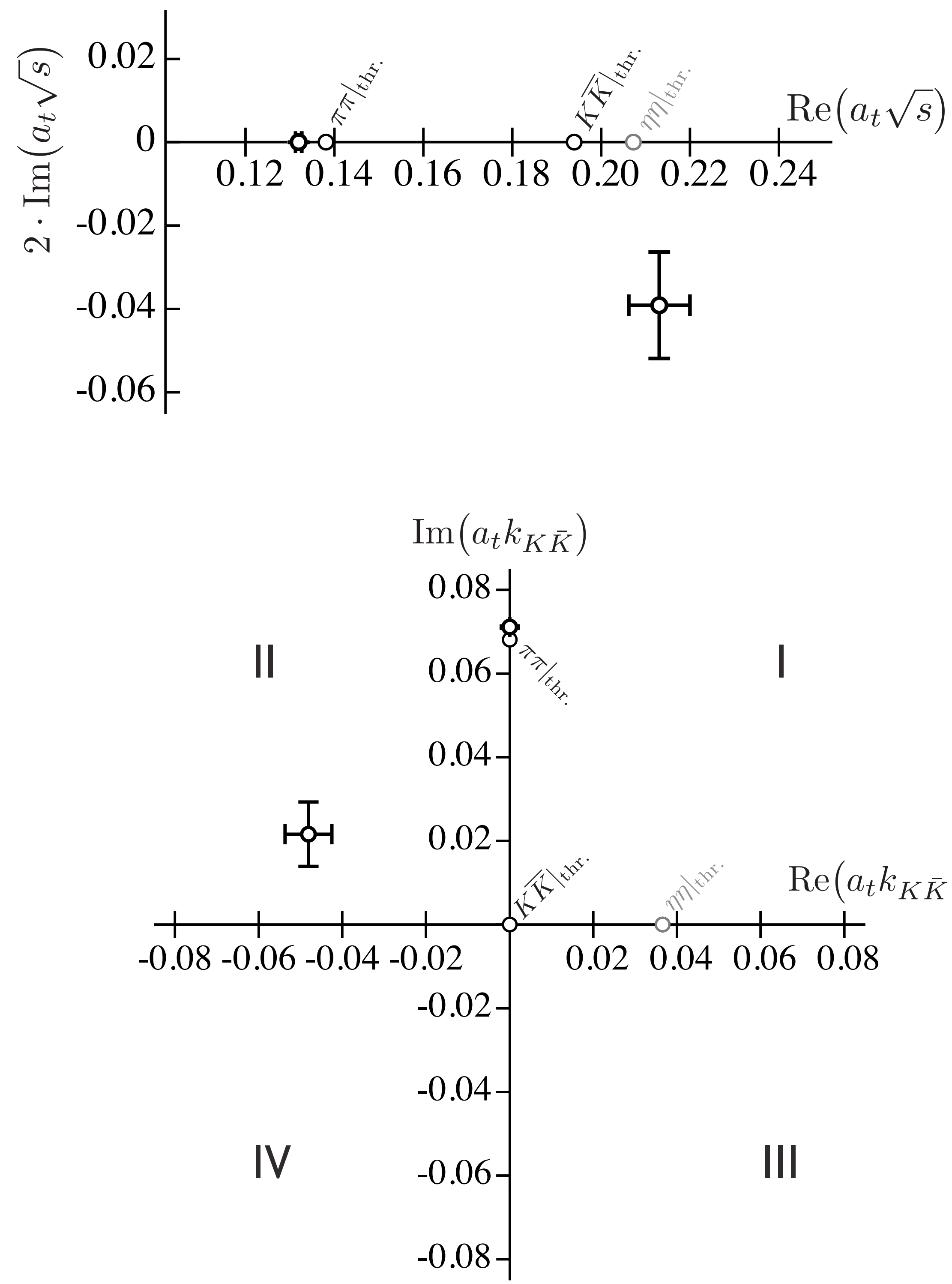
S-wave amplitudes



$$\mathbf{K}^{-1}(s) = \begin{pmatrix} a + b\,s & c + d\,s & e \\ c + d\,s & f & g \\ e & g & h \end{pmatrix}$$

with Chew-Mandelstam phase-space

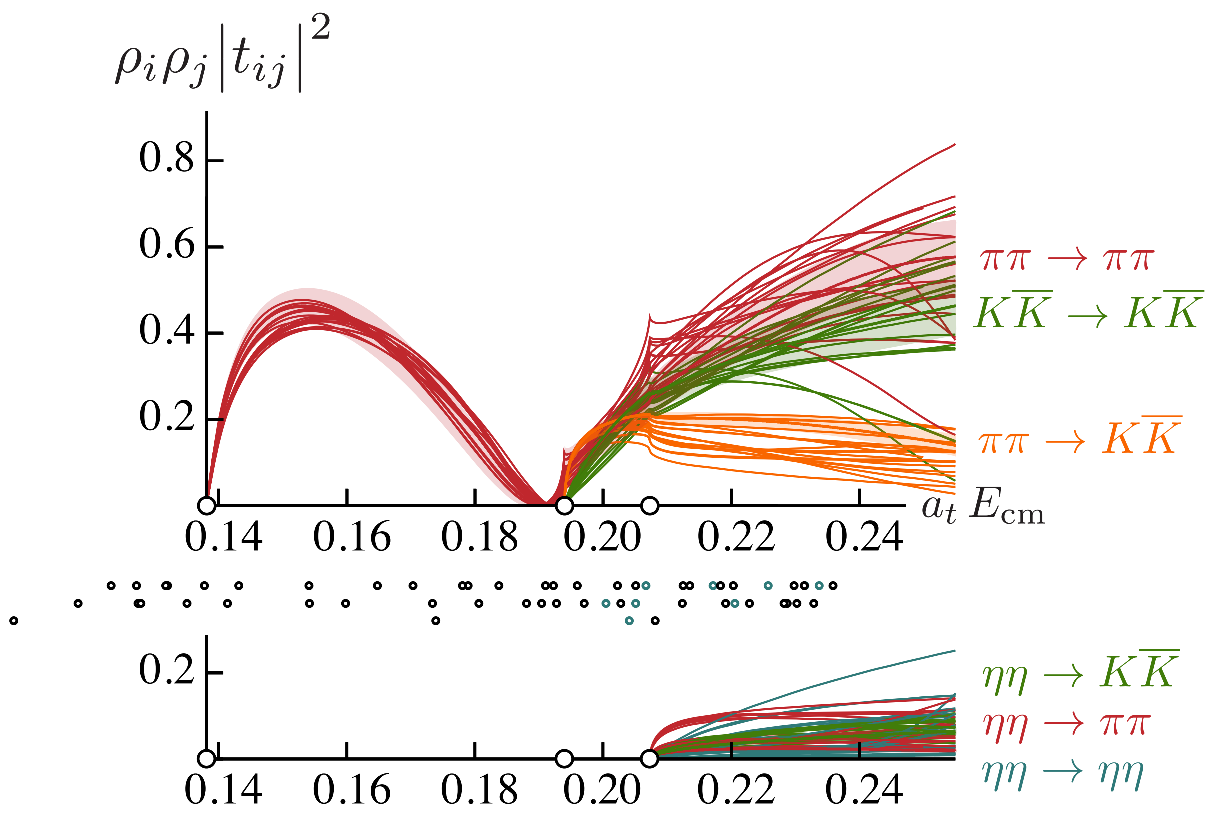
pole singularities



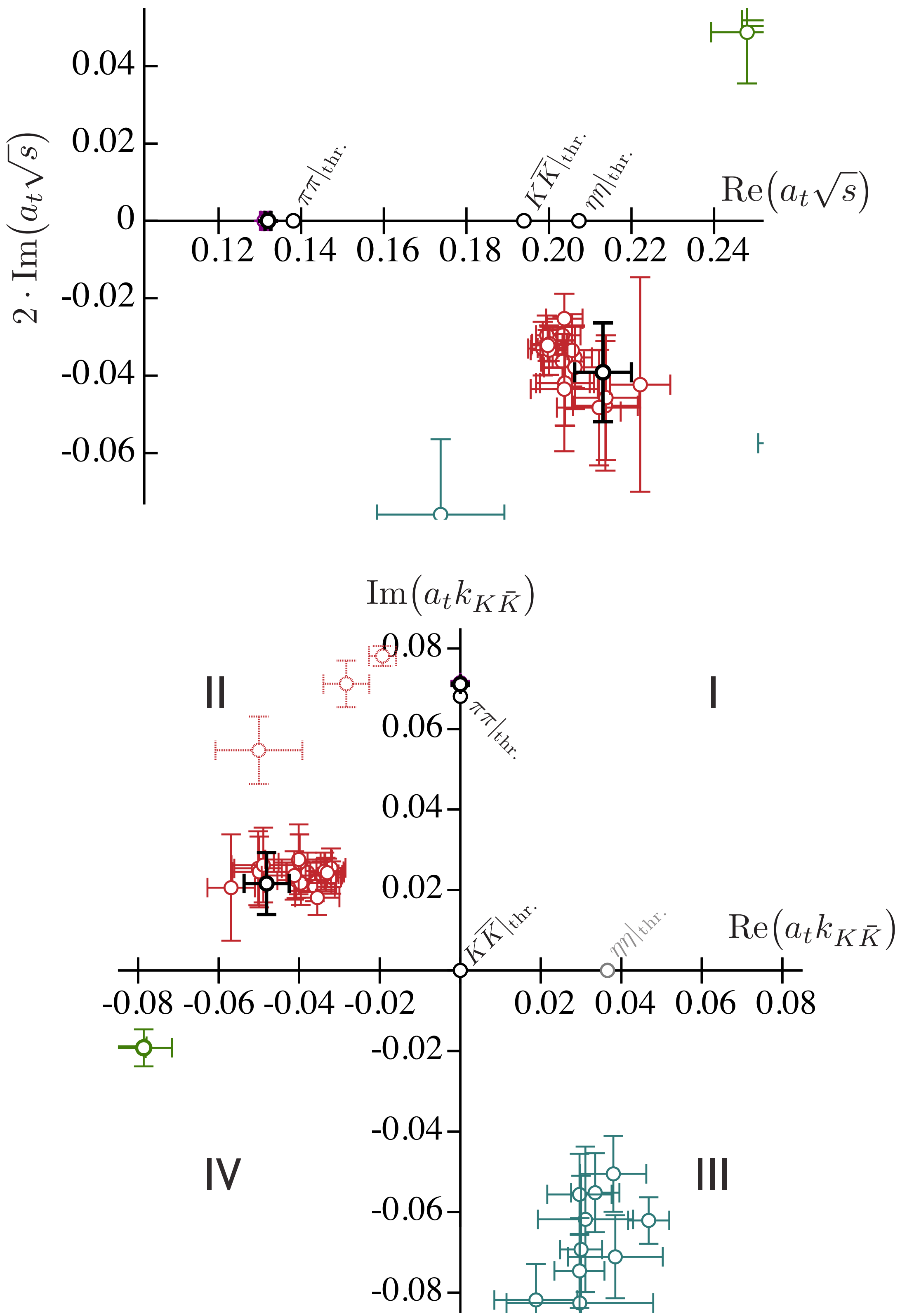
$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

PRD 97 054513 (2018)

S-wave amplitudes



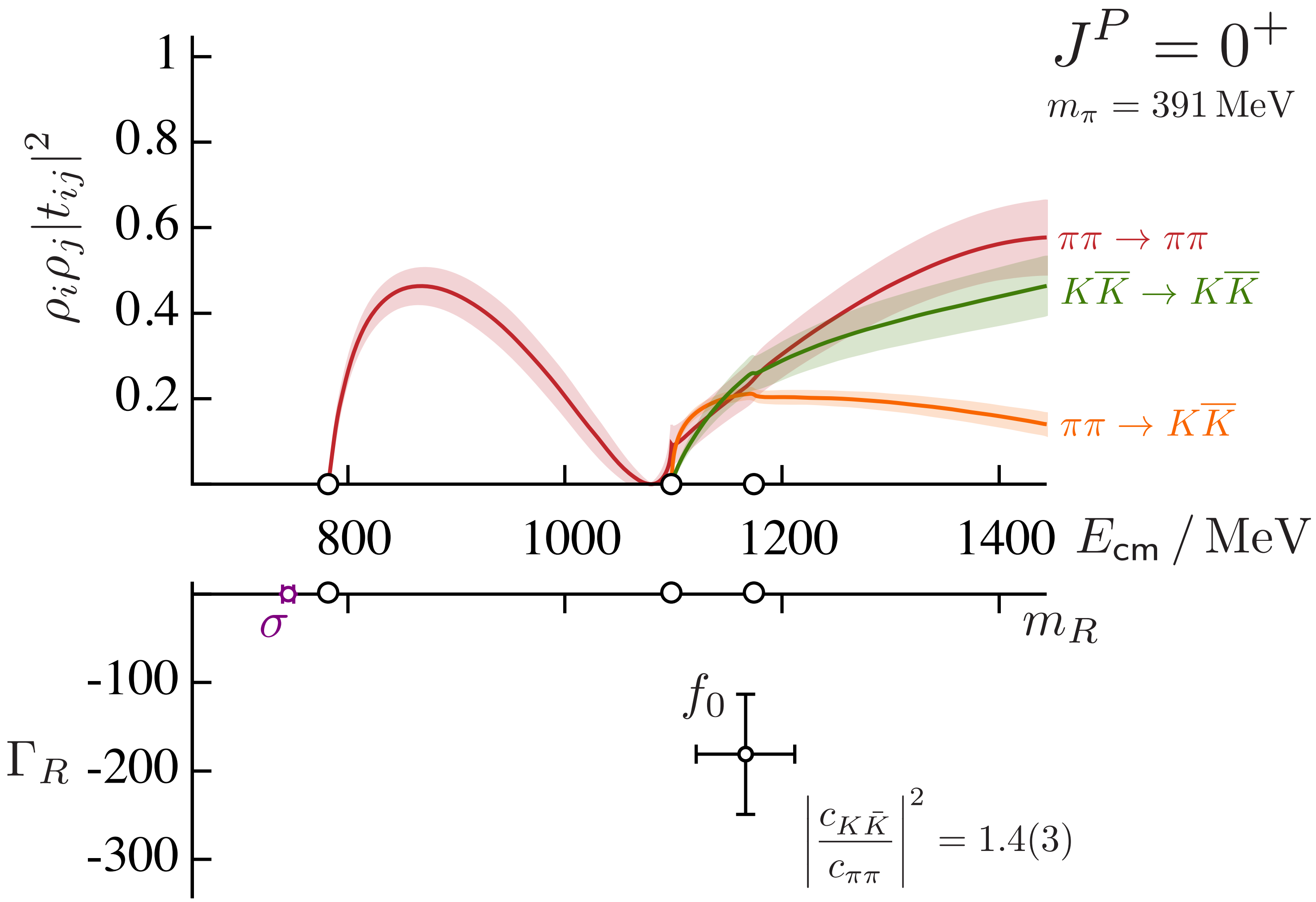
pole singularities



$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

summary, including spread over parameterizations in pole uncertainty

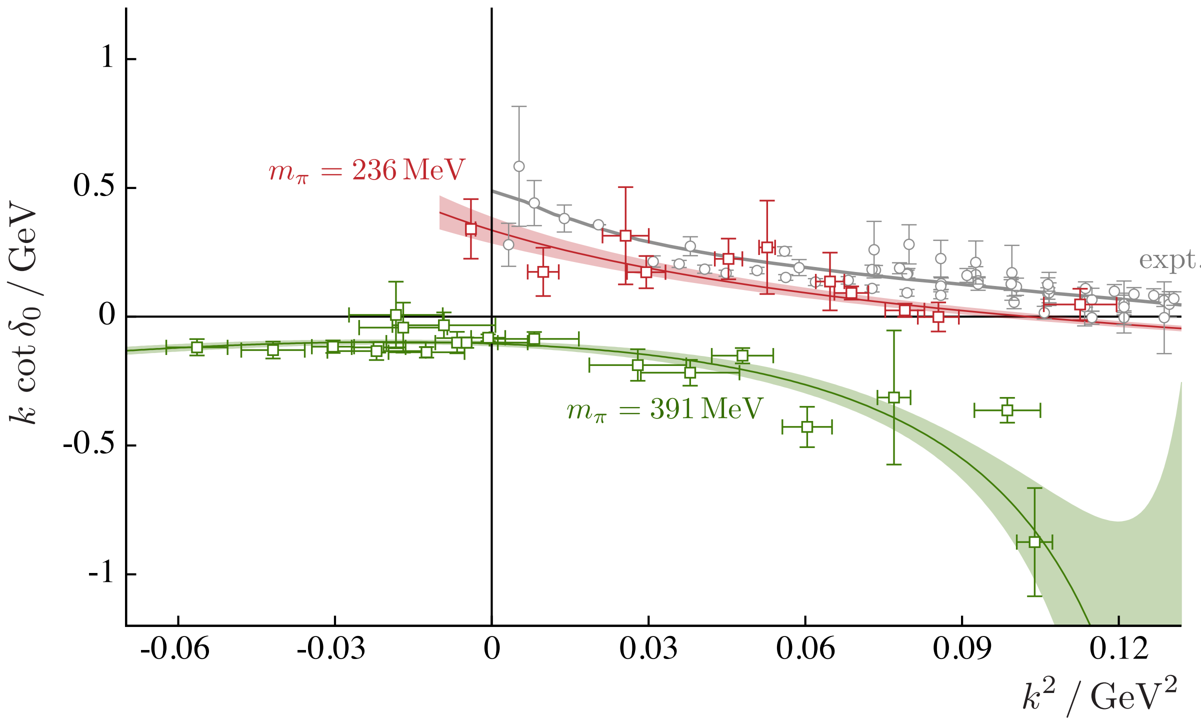
S-wave amplitudes & poles



the f_0("980")

singularity structure from lattice calculations — elastic

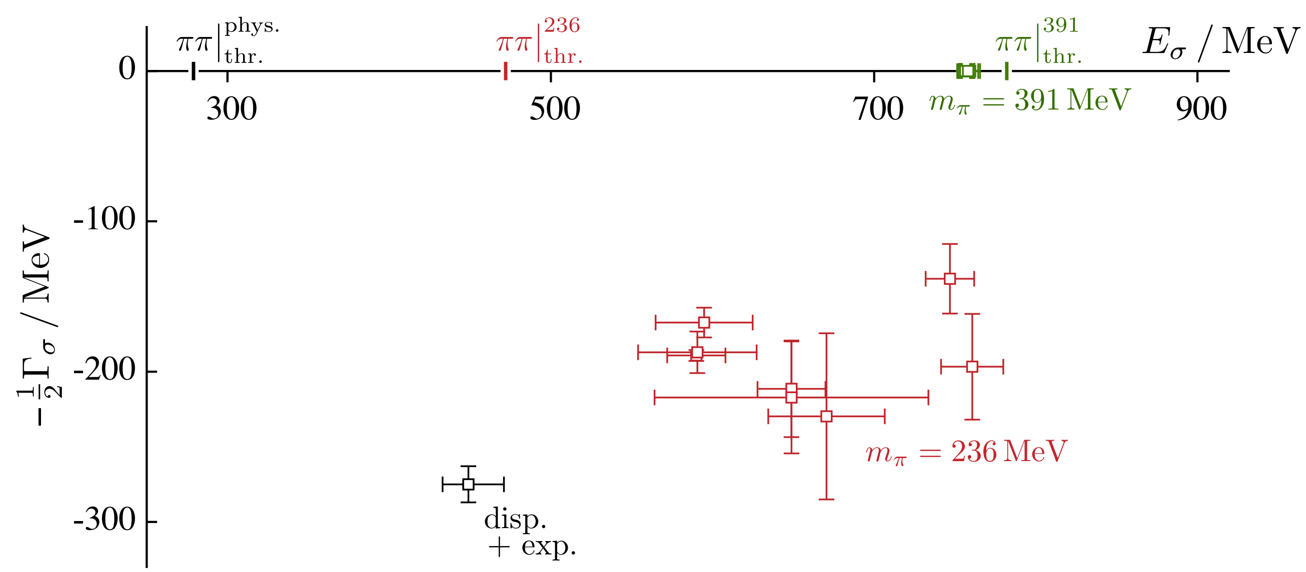
$\pi\pi$ isospin=0

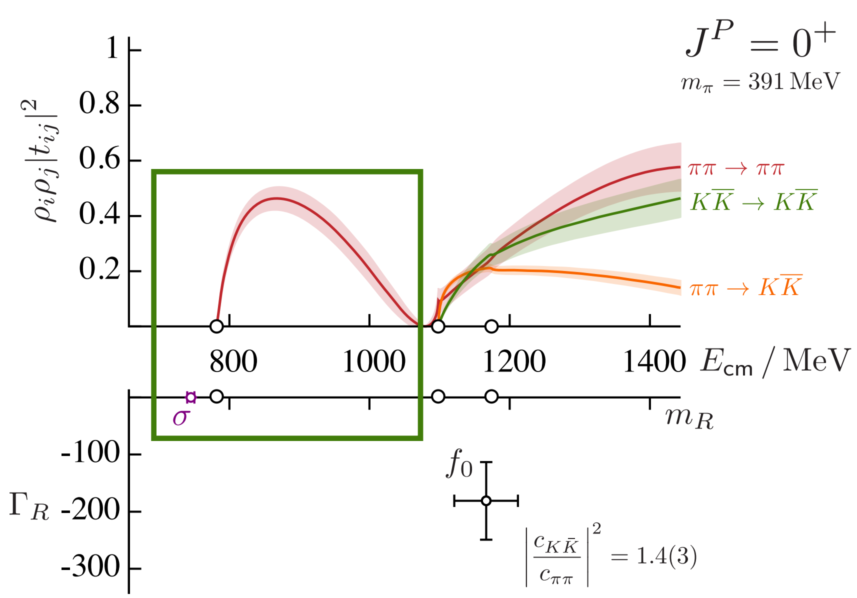


PRL118 022002 (2017)

m_π~391 MeV — a bound-state pole

m_π~236 MeV — a resonance pole



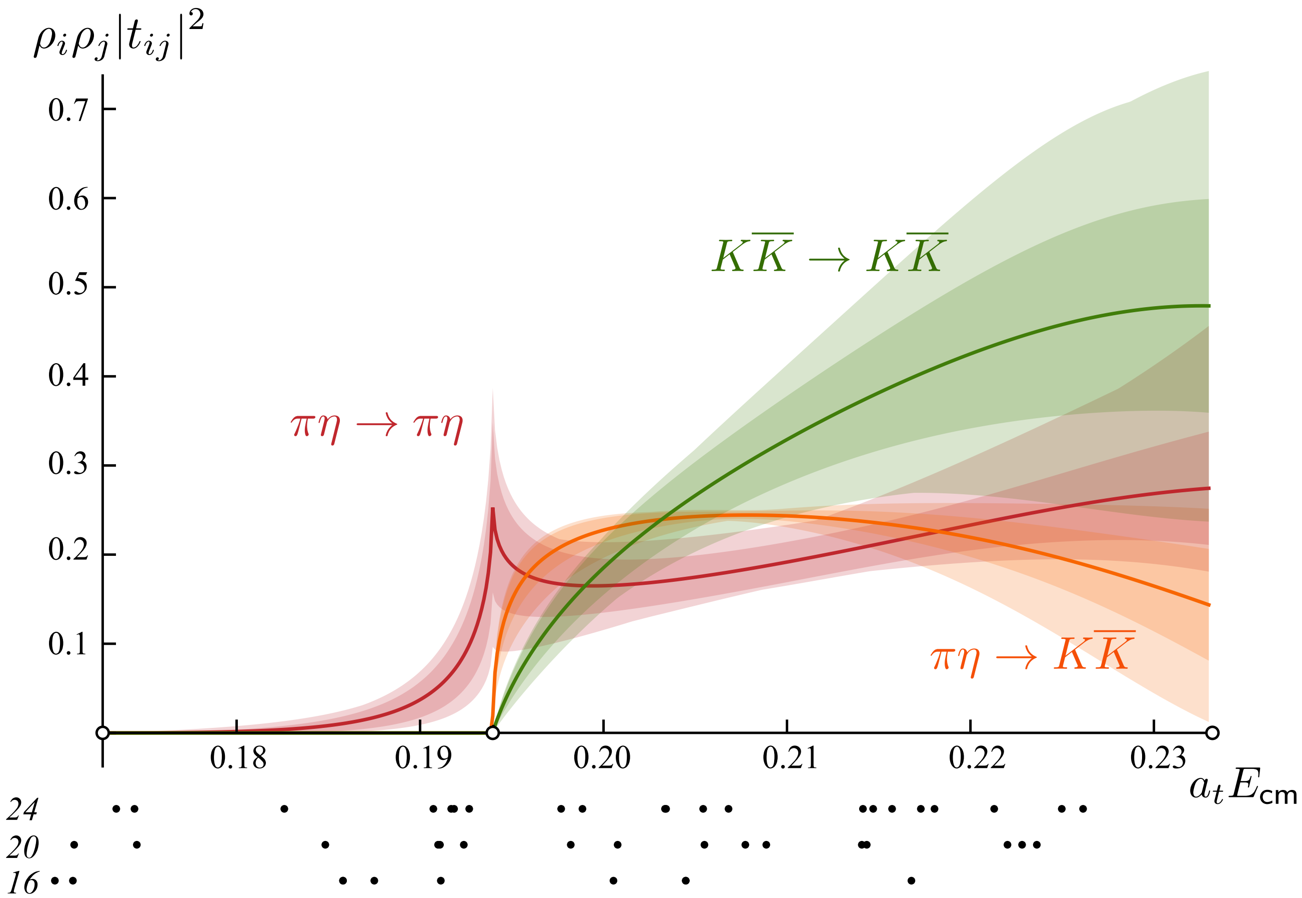


$\pi\eta$, $K\bar{K}$ scattering with m_π~391 MeV

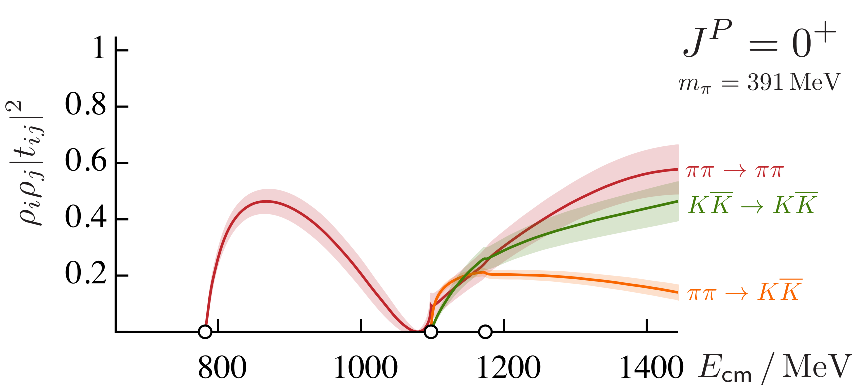
PRD 93 094506 (2016)

similar calculation in isospin=1, *G*-parity negative channel

S-wave amplitudes

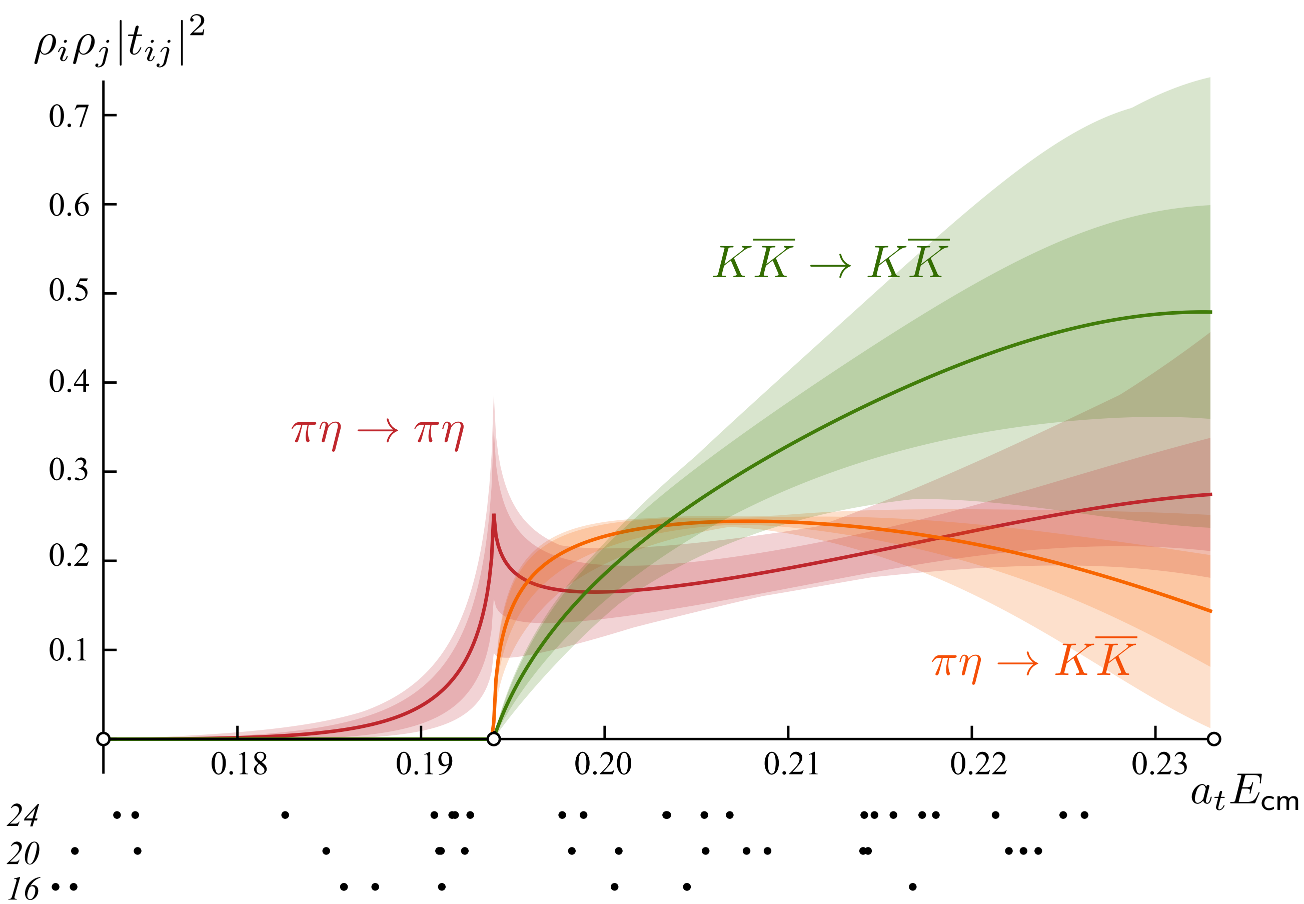


looks very different to isospin=0 case shown before

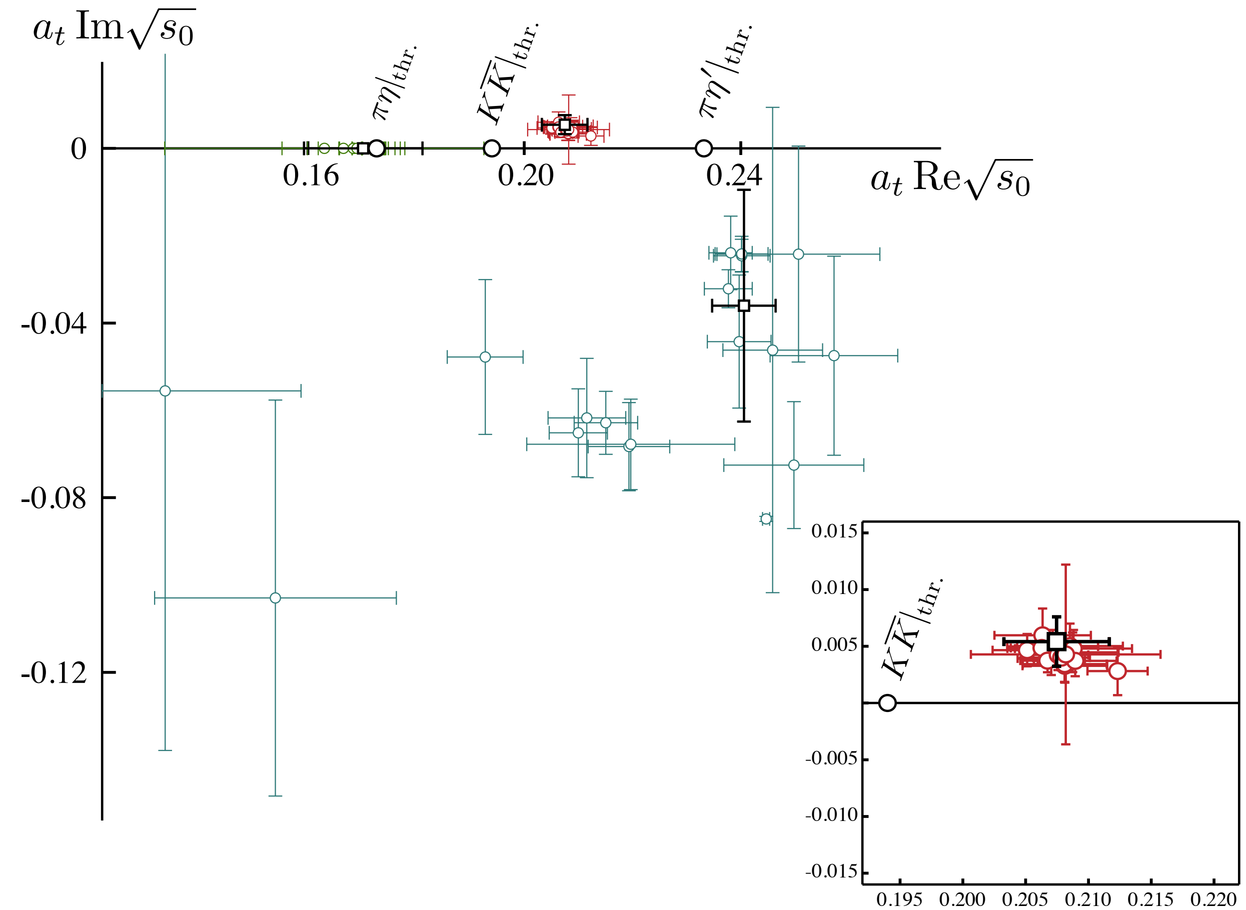


$\pi\eta$, $K\bar{K}$ scattering with m_π~391 MeV

S-wave amplitudes

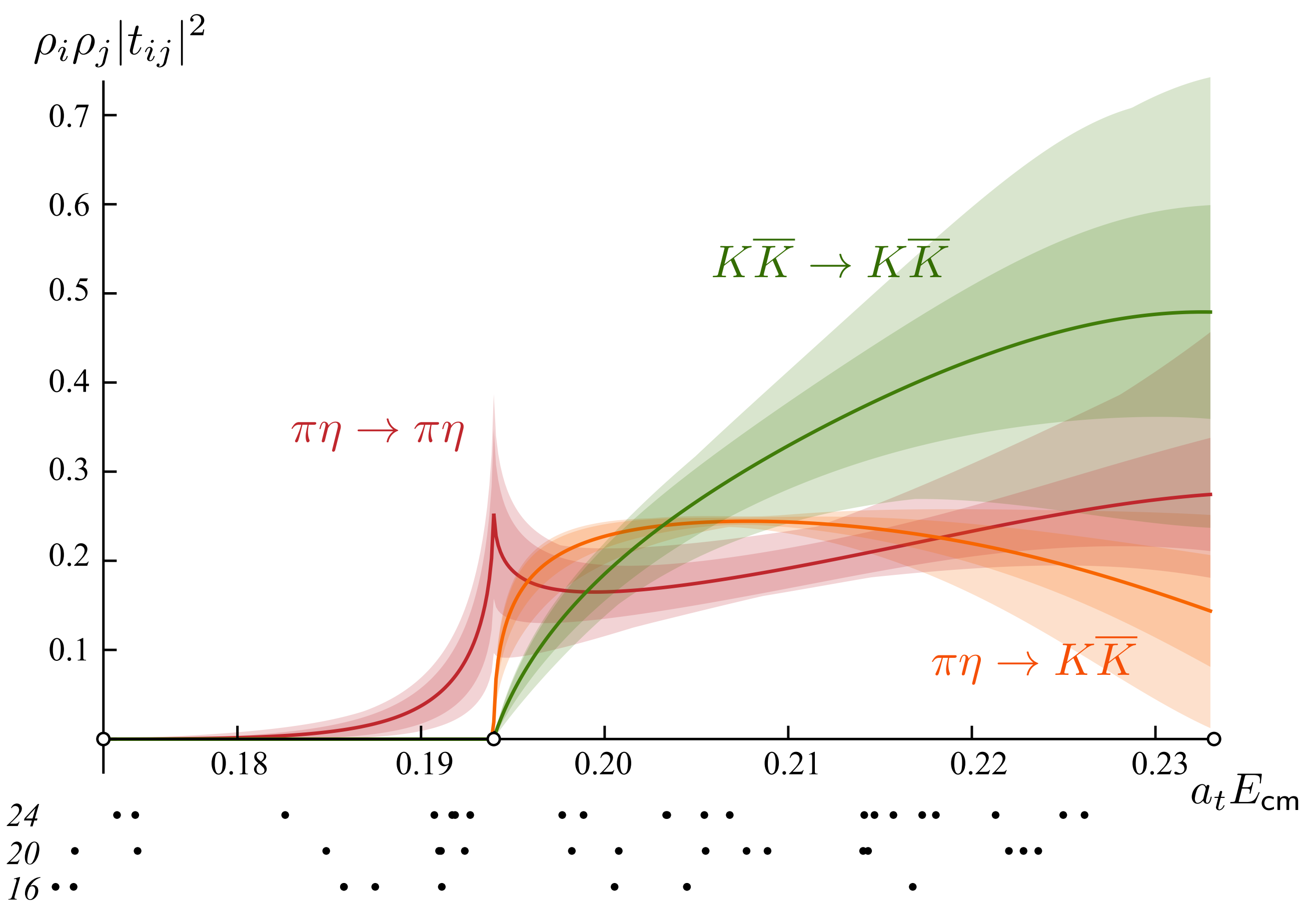


pole singularities

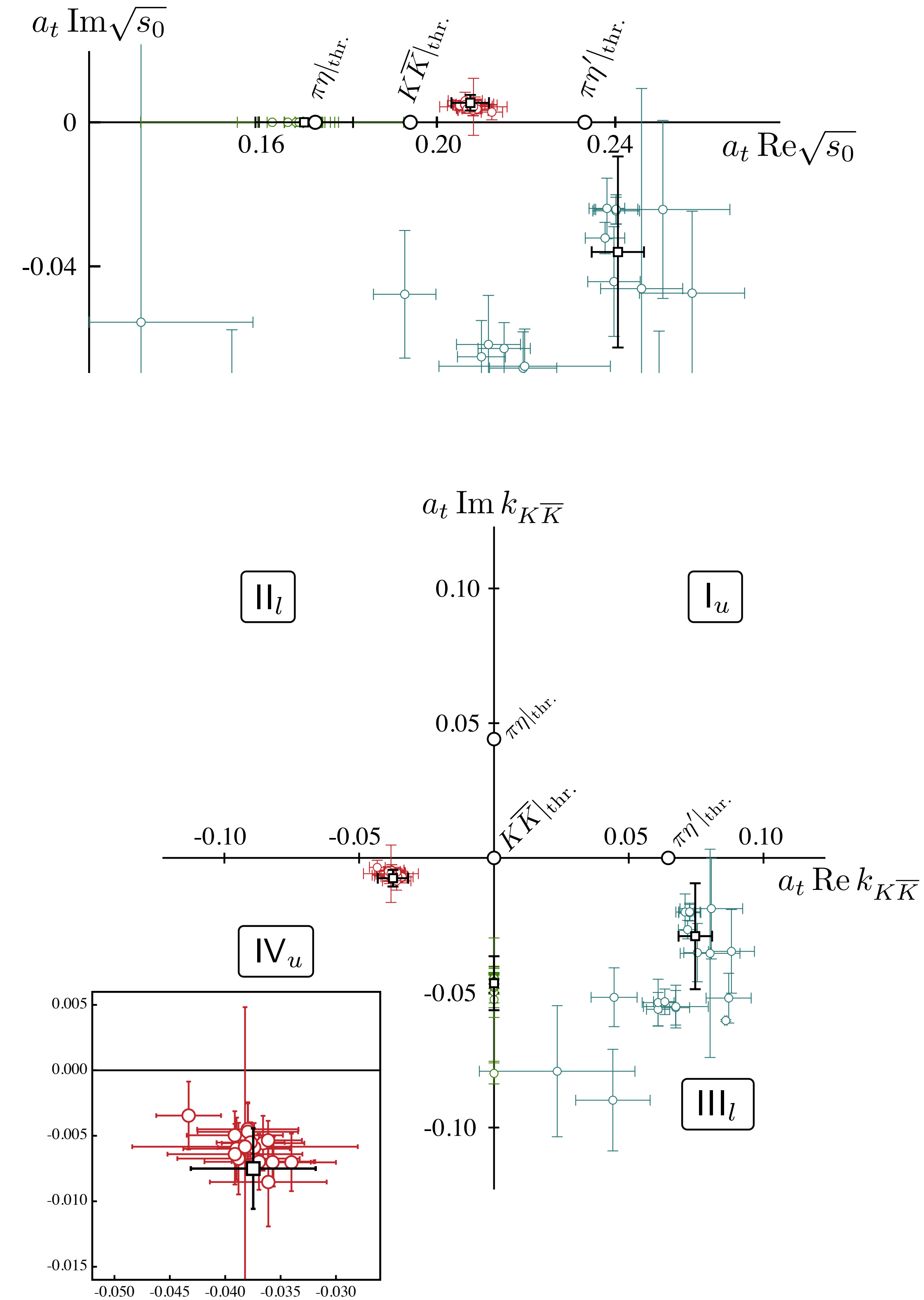


$\pi\eta$, $K\bar{K}$ scattering with m_π~391 MeV

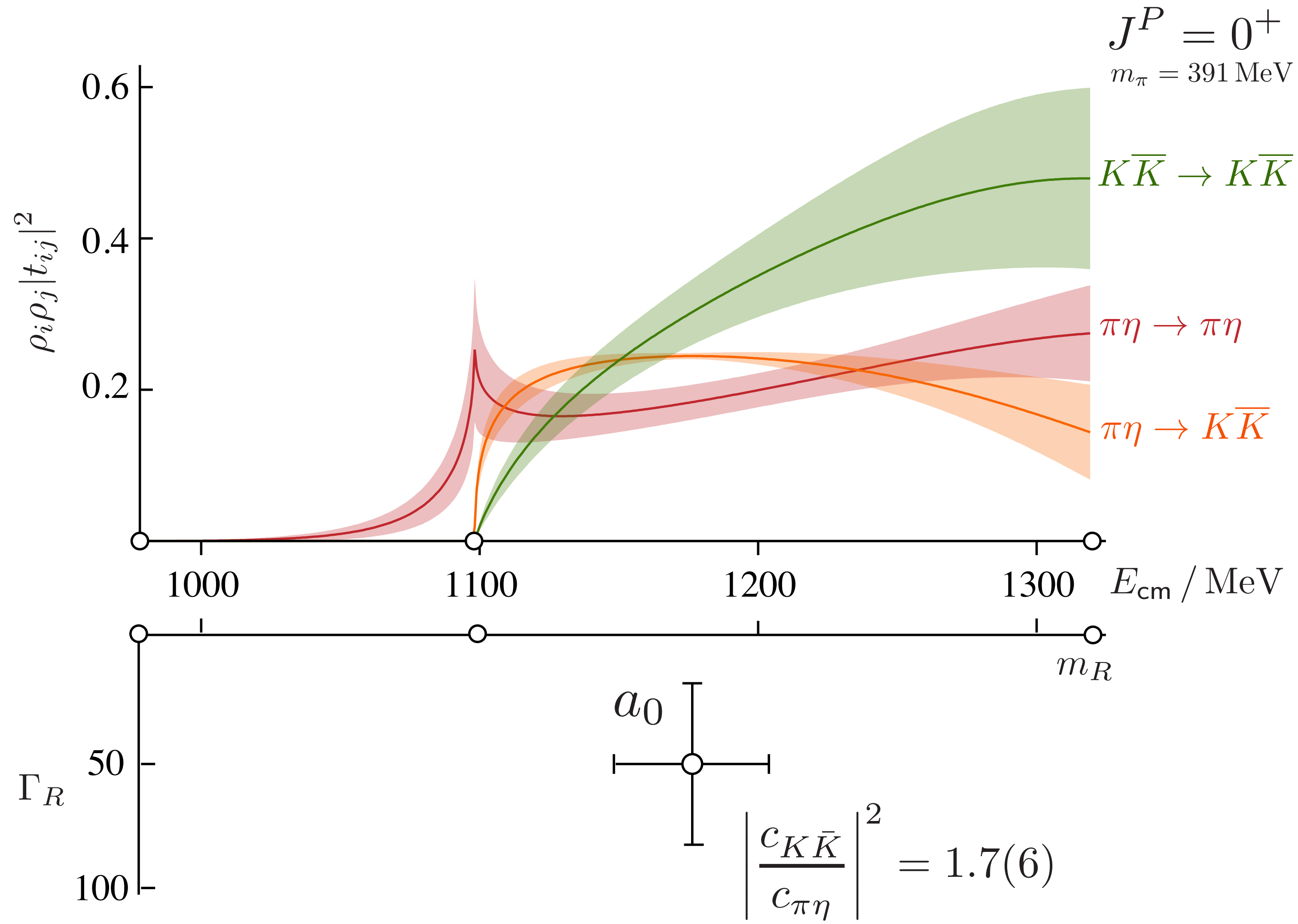
S-wave amplitudes



pole singularities



S-wave amplitudes & poles



the a_0("980")

masses similar

$$m_R(f_0) = 1166(45)\,\mathrm{MeV},$$
$$m_R(a_0) = 1177(27)\,\mathrm{MeV},$$

widths a little different

$$\Gamma_R(f_0) = 181(68)\,\mathrm{MeV},$$
$$\Gamma_R(a_0) = 49(33)\,\mathrm{MeV}.$$

but channel couplings quite similar ?

$$\big|c(a_0 \to K\overline{K})\big| \approx \big|c(f_0 \to K\overline{K})\big| \quad \sim 850\,\mathrm{MeV}$$
$$\big|c(a_0 \to \pi\eta)\big| \approx \big|c(f_0 \to \pi\pi)\big| \quad \sim 700\,\mathrm{MeV}.$$

main difference is the **larger phase-space for $\pi\pi$ compared to $\pi\eta$**

can explore the effect using the simple Flatté amplitude

Flatté denominator $D(s) = m_0^2 - s - i g_1^2\, \rho_1(s) - i g_2^2\, \rho_2(s)$

has zeros at

$$\sqrt{s_0} \approx m_0 \pm \frac{i}{2}\frac{g_2^2\,\rho_2}{m_0}\left[\left(\frac{g_1}{g_2}\right)^2 \frac{\rho_1}{\rho_2} - 1\right] \quad \text{on sheet II, if } \left(\frac{g_1}{g_2}\right)^2 \frac{\rho_1}{\rho_2} > 1, \text{ or,}$$

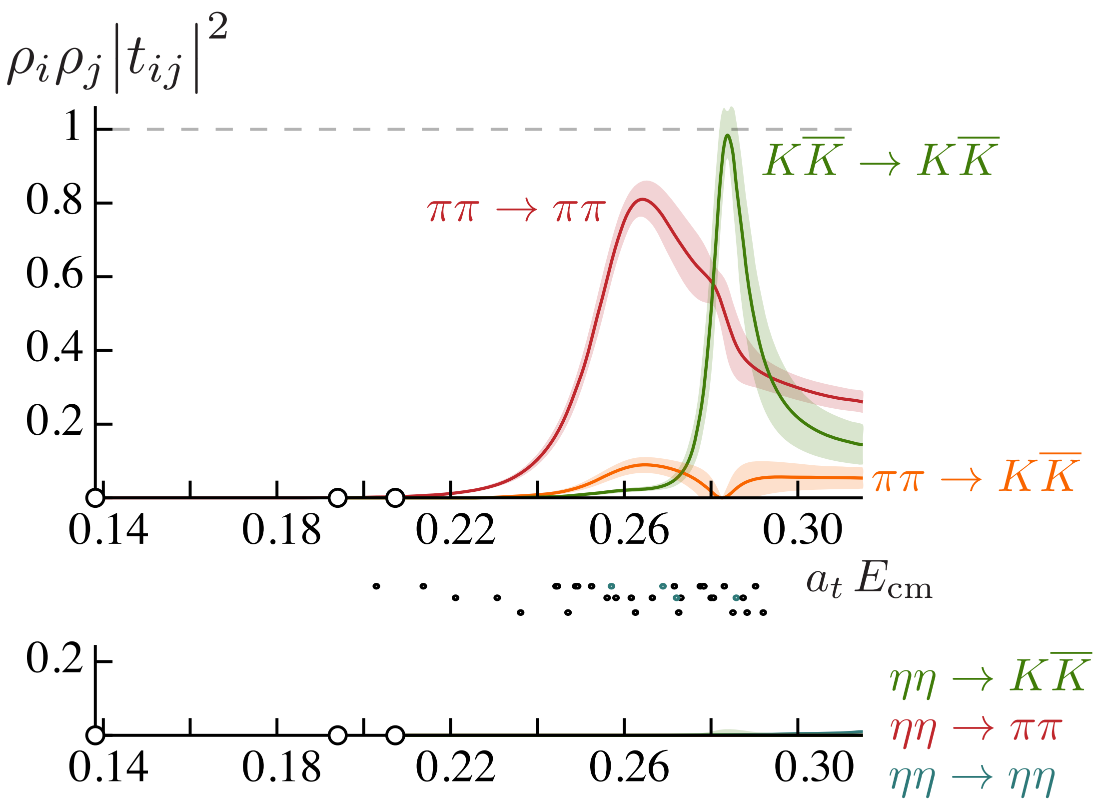
$$\sqrt{s_0} \approx m_0 \pm \frac{i}{2}\frac{g_2^2\,\rho_2}{m_0}\left[1 - \left(\frac{g_1}{g_2}\right)^2 \frac{\rho_1}{\rho_2}\right] \quad \text{on sheet IV, if } \left(\frac{g_1}{g_2}\right)^2 \frac{\rho_1}{\rho_2} < 1, \text{ and,}$$

$$\sqrt{s_0} \approx m_0 \pm \frac{i}{2}\frac{g_2^2\,\rho_2}{m_0}\left[1 + \left(\frac{g_1}{g_2}\right)^2 \frac{\rho_1}{\rho_2}\right] \quad \text{on sheet III, in all cases,}$$

$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

PRD 97 054513 (2018)

D-wave amplitudes

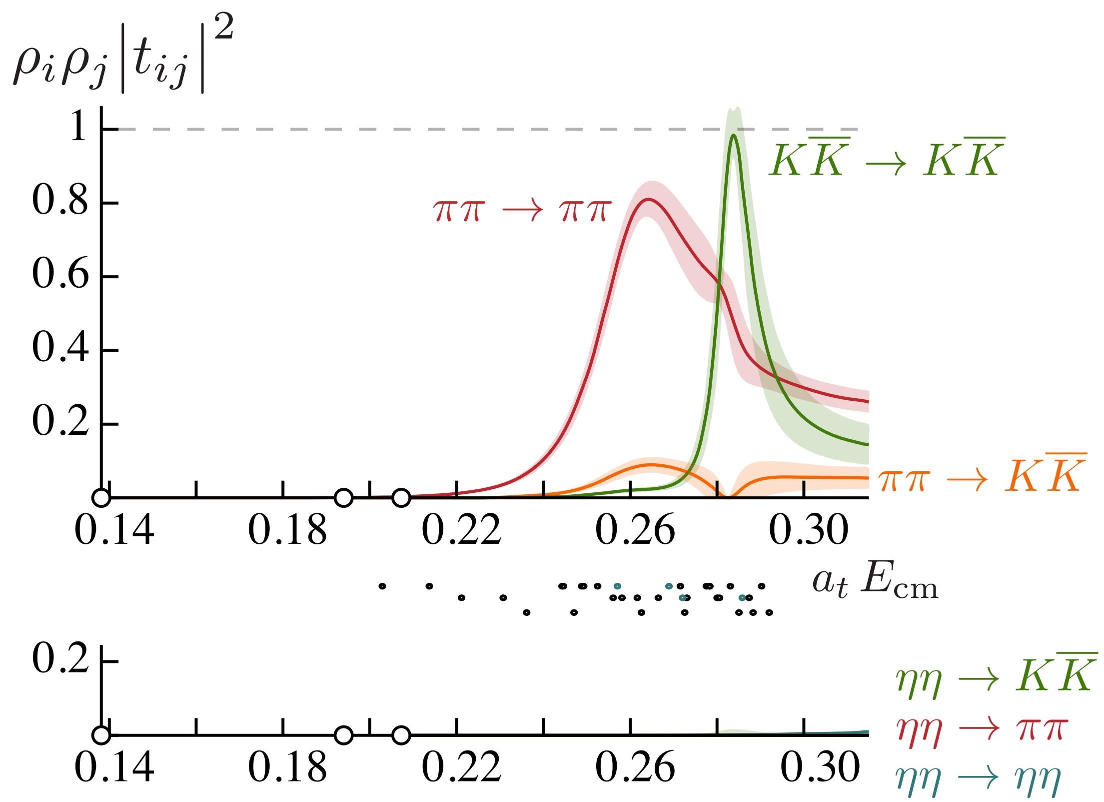


bumps are in the three channel region ⇒ **8 sheets !**

won't burden you with the sheet details here …

$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

D-wave amplitudes



bumps are in the three channel region ⇒ **8 sheets !**

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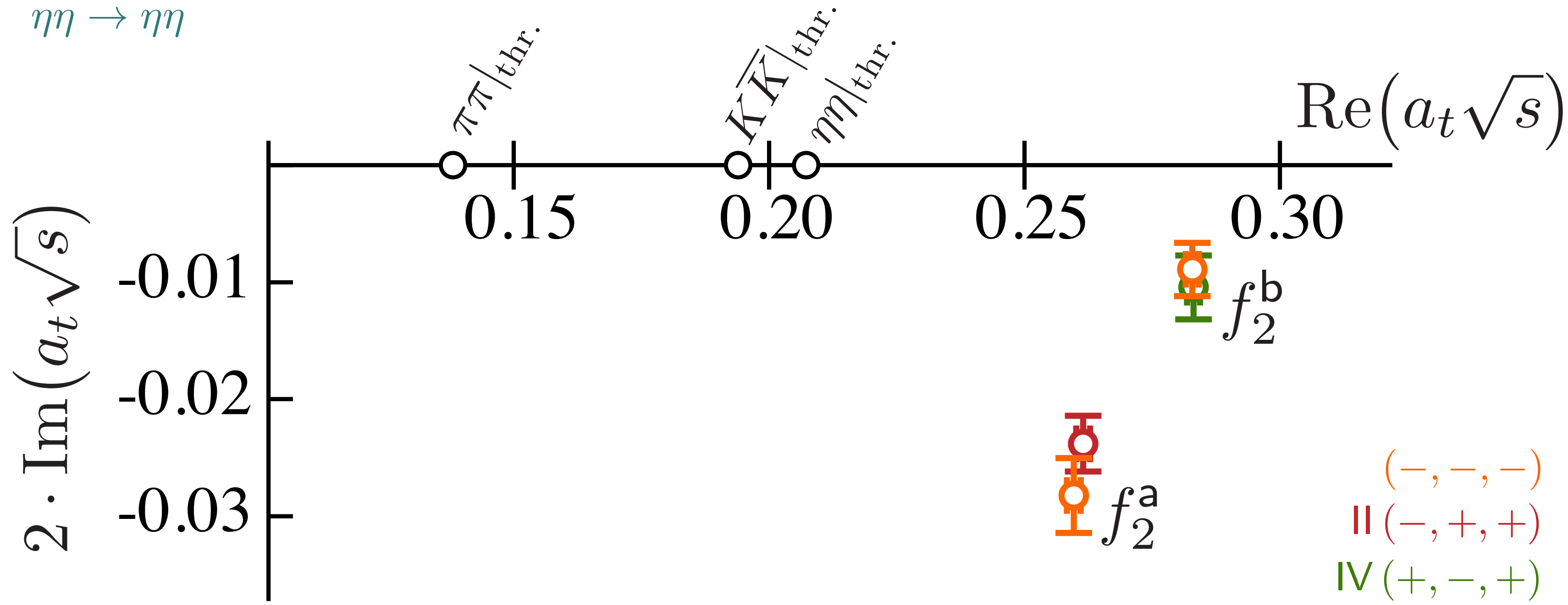
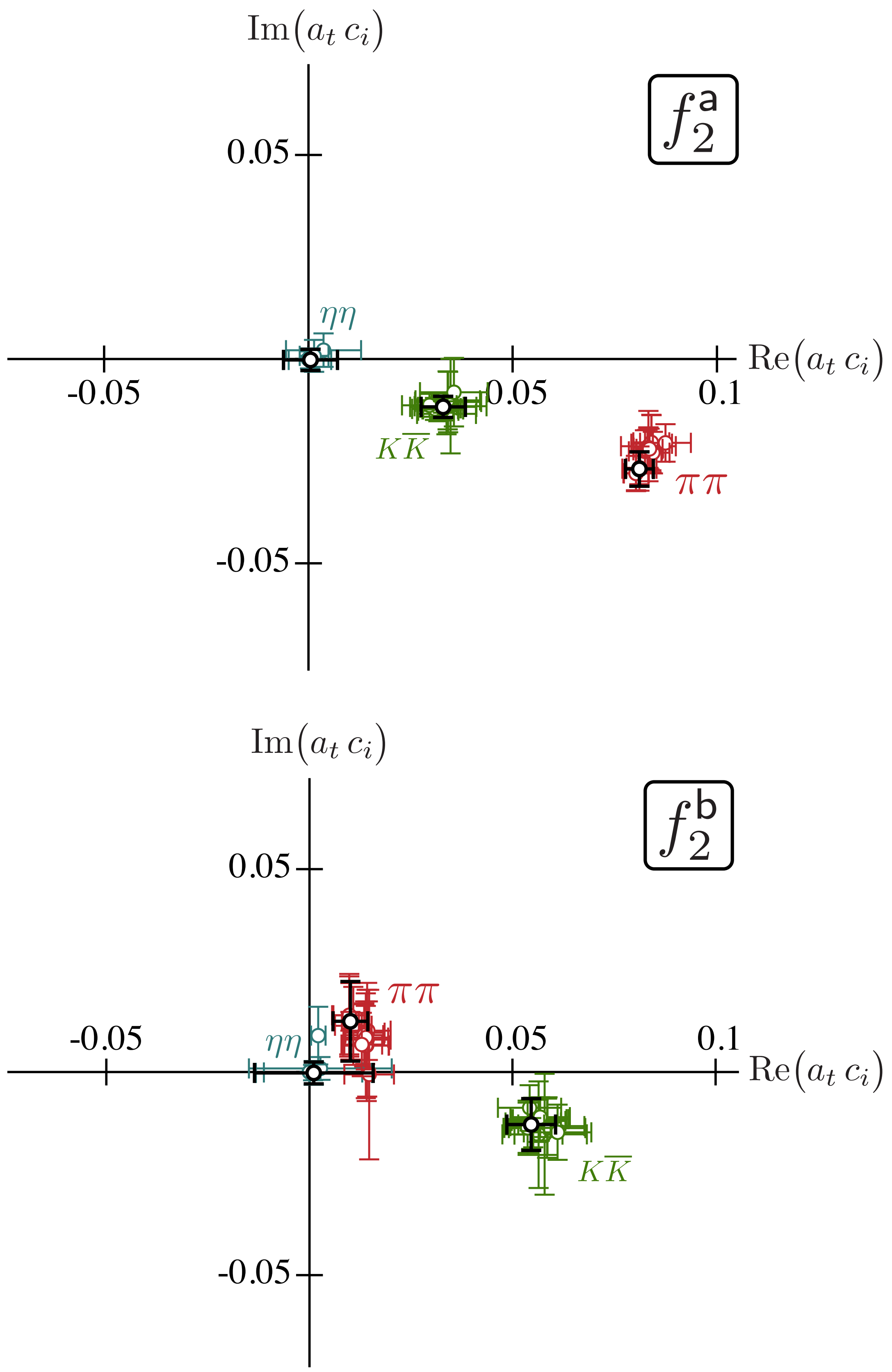


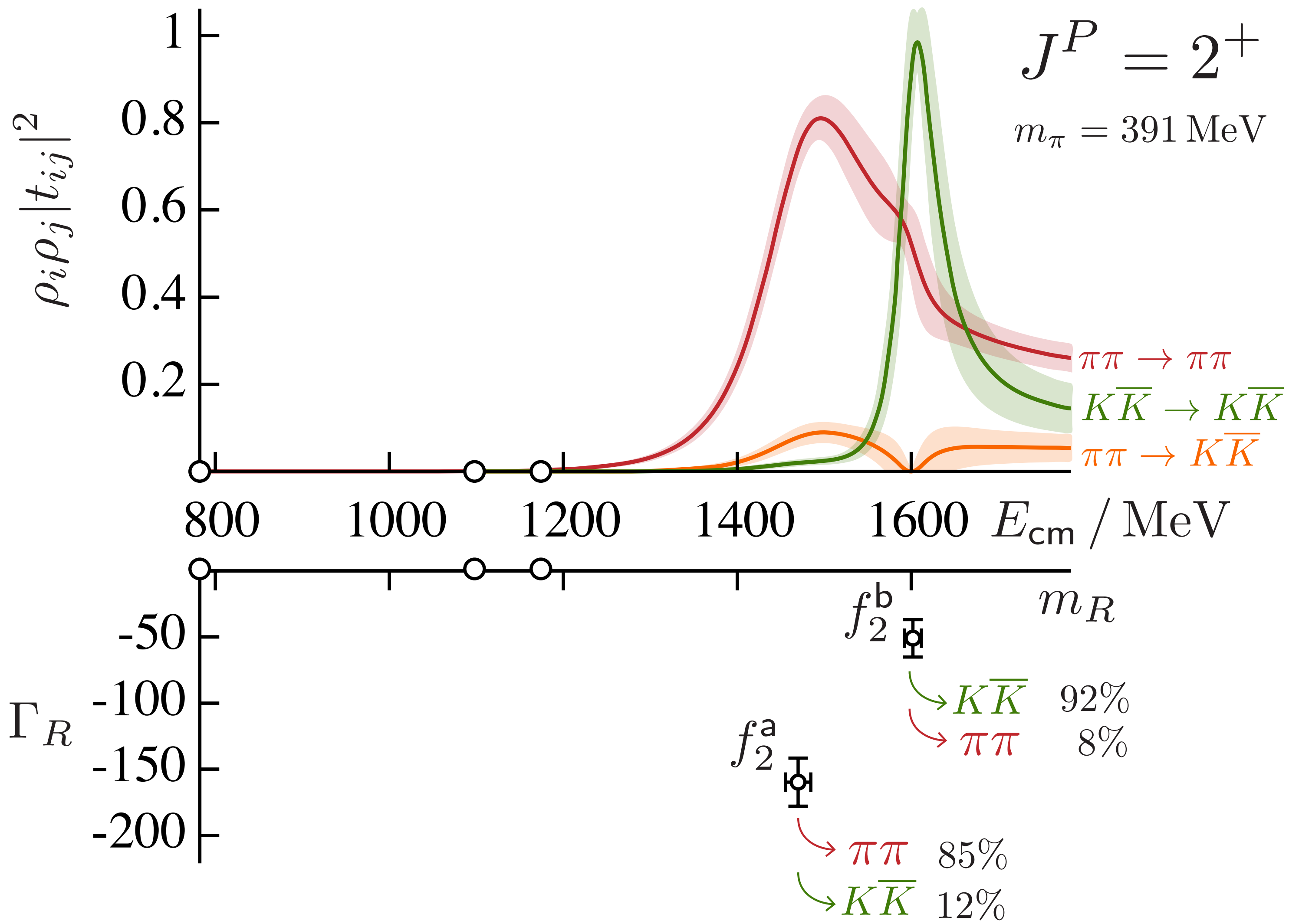
(–,–,–) is 'closest' sheet to physical scattering above all three thresholds

couplings at the poles



ππ, $K\bar{K}$, ηη scattering with m_π~391 MeV

D-wave amplitudes & poles



pdg summary

$f_2(1270)$ $I^G(J^{PC}) = 0^+(2^{++})$

Mass $m = 1275.5 \pm 0.8$ MeV
Full width $\Gamma = 186.7^{+2.2}_{-2.5}$ MeV (S = 1.4)

$f_2(1270)$ DECAY MODES	Fraction (Γ_i/Γ)	Scale factor/ Confidence level	p (MeV/c)
$\pi\pi$	$(84.2\ ^{+2.9}_{-0.9})$ %	S=1.1	623
$\pi^+\pi^-2\pi^0$	$(7.7\ ^{+1.1}_{-3.2})$ %	S=1.2	563
$K\bar{K}$	$(4.6\ ^{+0.5}_{-0.4})$ %	S=2.7	404
$2\pi^+2\pi^-$	(2.8 ± 0.4) %	S=1.2	560
$\eta\eta$	$(4.0 \pm 0.8) \times 10^{-3}$	S=2.1	326
$4\pi^0$	$(3.0 \pm 1.0) \times 10^{-3}$		565

$f_2'(1525)$ $I^G(J^{PC}) = 0^+(2^{++})$

Mass $m = 1525 \pm 5$ MeV [l]
Full width $\Gamma = 73^{+6}_{-5}$ MeV [l]

$f_2'(1525)$ DECAY MODES	Fraction (Γ_i/Γ)	p (MeV/c)
$K\bar{K}$	(88.7 ± 2.2) %	581
$\eta\eta$	(10.4 ± 2.2) %	530
$\pi\pi$	$(8.2 \pm 1.5) \times 10^{-3}$	750

except at very low quark masses, the ω is a stable meson

the non-zero spin of the ω introduces new features, e.g. $J^P = 1^+$ in **two** partial-waves $\begin{pmatrix} {}^3S_1 \\ {}^3D_1 \end{pmatrix}$

expect a ***b*₁ resonance**

$b_1(1235)$ $I^G(J^{PC}) = 1^+(1^{+-})$

Mass $m = 1229.5 \pm 3.2$ MeV (S = 1.6)
Full width $\Gamma = 142 \pm 9$ MeV (S = 1.2)

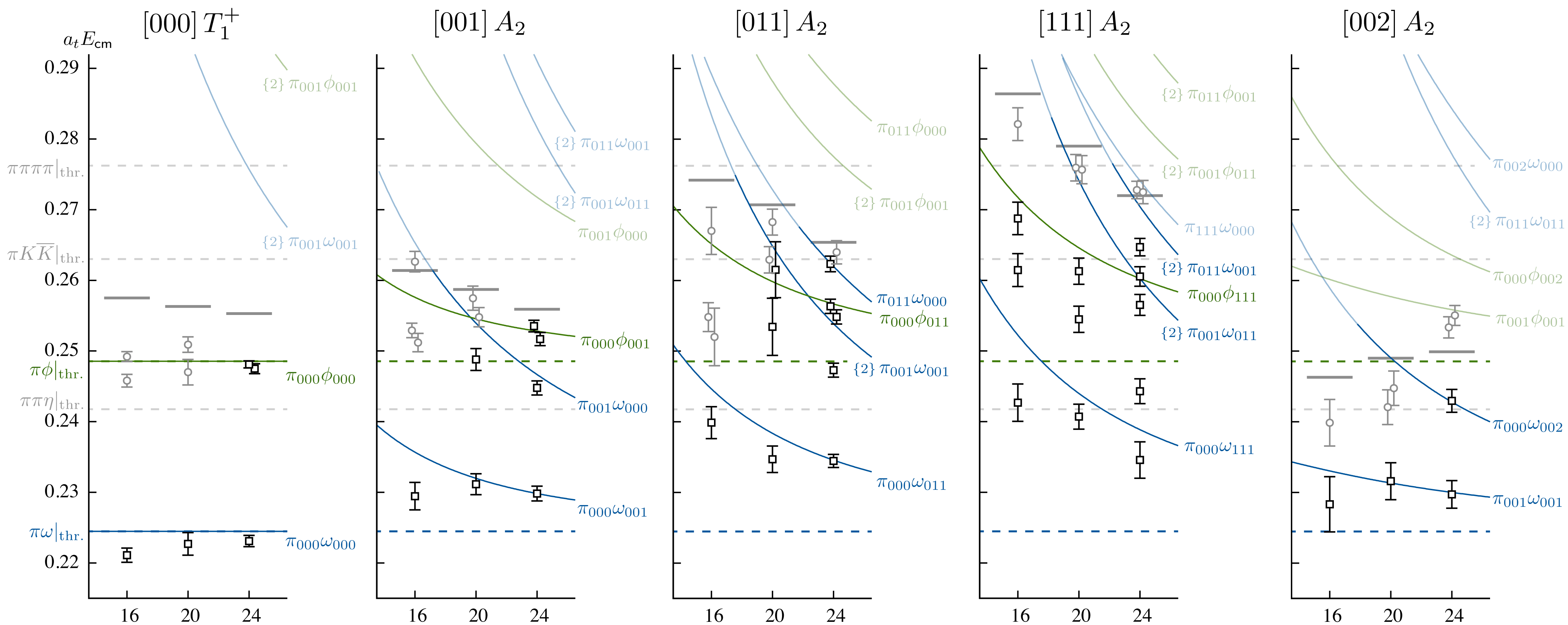
b_1(1235) DECAY MODES	Fraction (Γ_i/Γ)	Confidence level	p (MeV/c)
$\omega\pi$	dominant		348
[D/S amplitude ratio = 0.277 ± 0.027]			

in the quark model it's the $u\bar{d}(^1P_1)$ state

$\pi\omega$ scattering

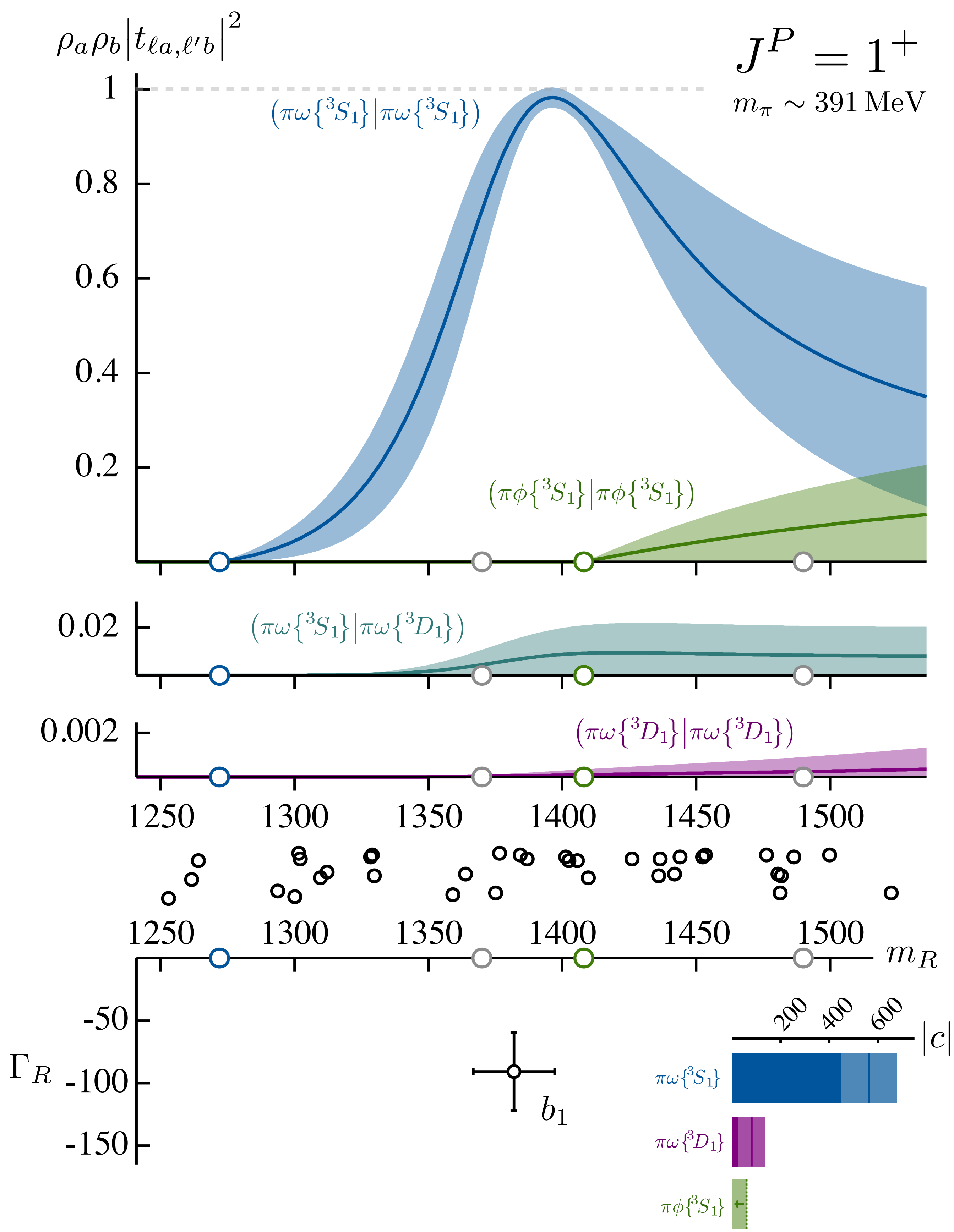
at m_π~391 MeV, the ω is a stable meson

m_π~391 MeV finite-volume spectrum



at low energies, a coupled $\pi\omega$, $\pi\varphi$ scattering system …

$\pi\omega/\pi\varphi$ scattering

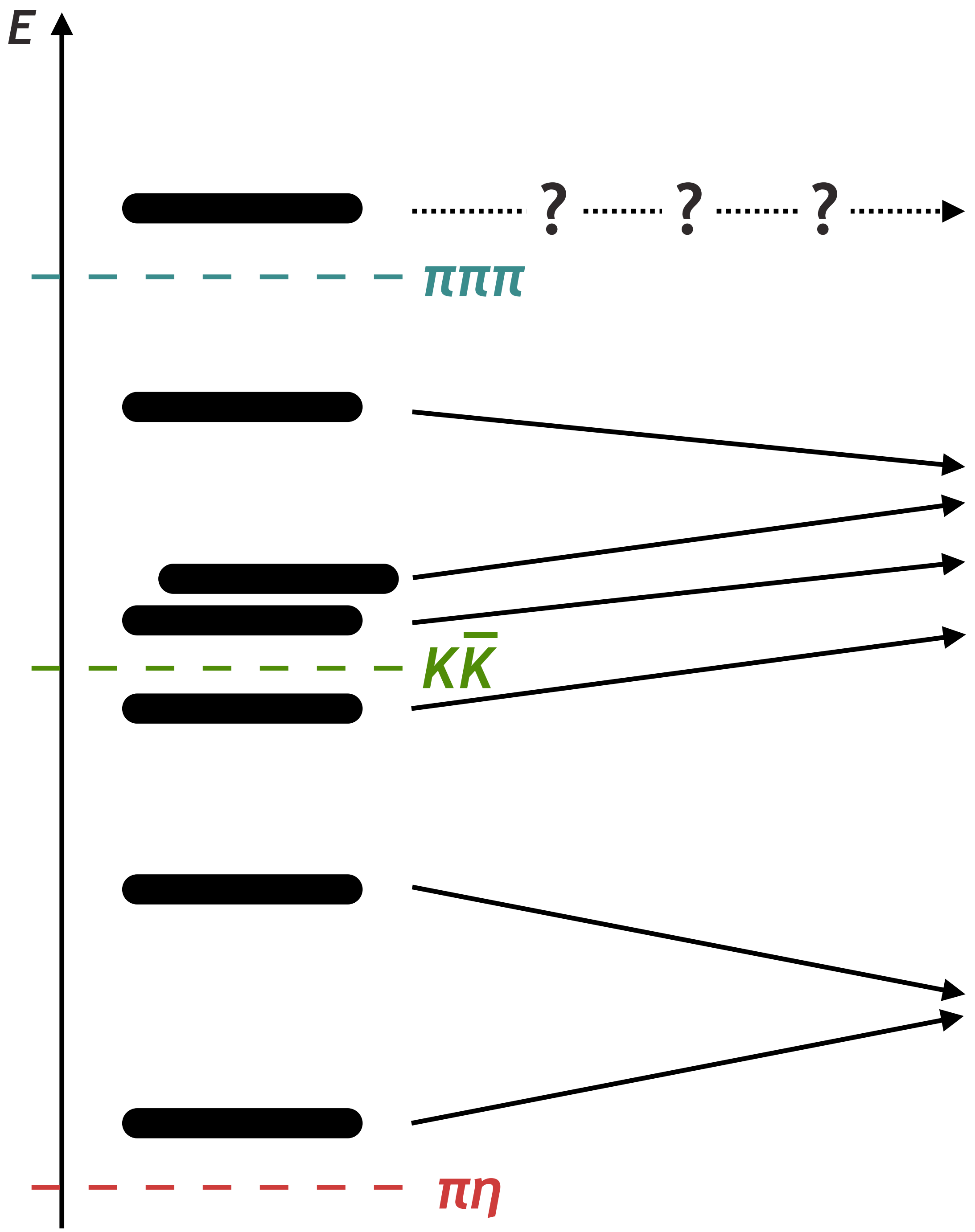


nothing in $\pi\varphi$
— no low-lying "Z_s"

clear b_1 resonance
- strong $\pi\omega$ S-wave
- weak $\pi\omega$ D-wave
- negligible $\pi\varphi$

where are the limitations ?

three-body and higher channels…



$$\begin{pmatrix} t_{\pi\eta,\pi\eta} & t_{\pi\eta,K\bar{K}} & t_{\pi\eta,\pi\pi\pi} \\ t_{\pi\eta,K\bar{K}} & t_{K\bar{K},K\bar{K}} & t_{K\bar{K},\pi\pi\pi} \\ t_{\pi\eta,\pi\pi\pi} & t_{K\bar{K},\pi\pi\pi} & t_{\pi\pi\pi,\pi\pi\pi} \end{pmatrix}$$

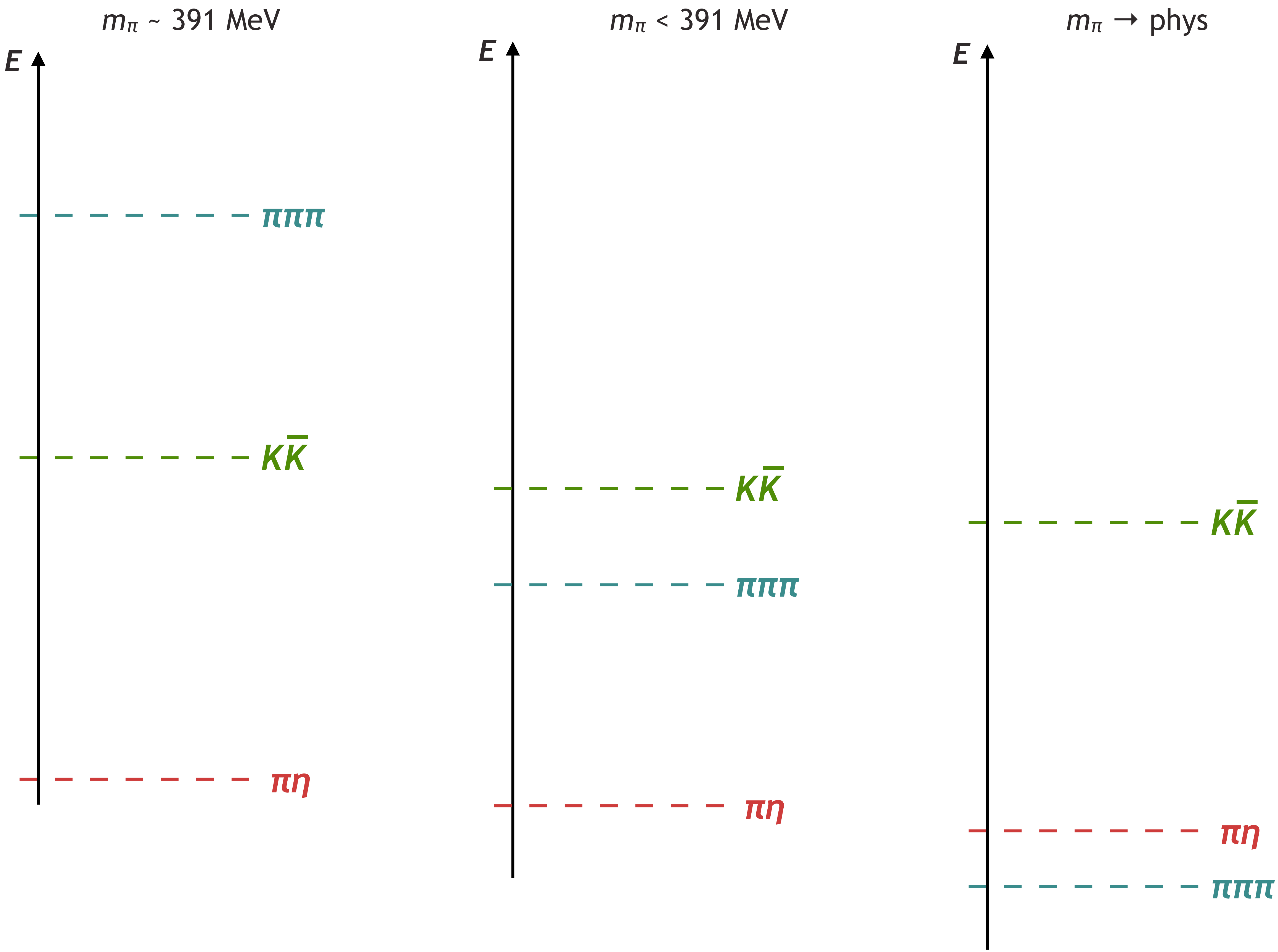
all the complications of infinite-volume three-body (+ more?)

$$\begin{pmatrix} t_{\pi\eta,\pi\eta} & t_{\pi\eta,K\bar{K}} \\ t_{\pi\eta,K\bar{K}} & t_{K\bar{K},K\bar{K}} \end{pmatrix}$$

$$t_{\pi\eta,\pi\eta}$$

finite-volume formalism(s) for three-body under development, first applications appearing

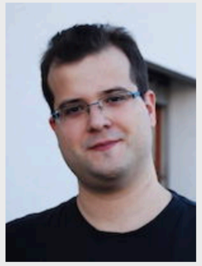
physical pion masses = low-lying multipion channels



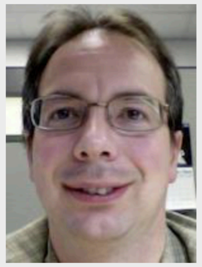
hadron spectrum collaboration

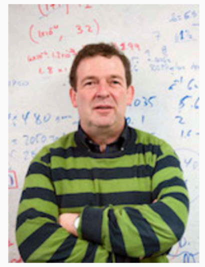
hadron spectrum collaboration
hadspec.org

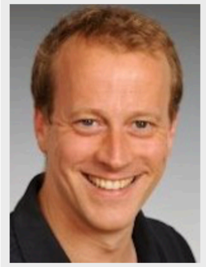
Jefferson Lab


Luka Leskovec
Postdoctoral Fellow

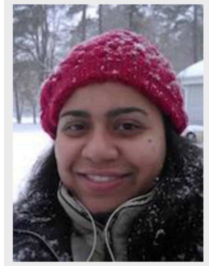

Robert Edwards
Senior Staff Scientist


Bálint Joó
Staff Scientist (HPC)


David Richards
Senior Staff Scientist


Frank Winter
Staff Scientist (HPC)

DAMTP, Cambridge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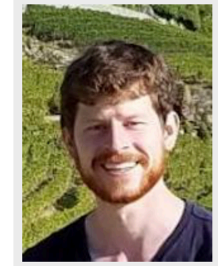
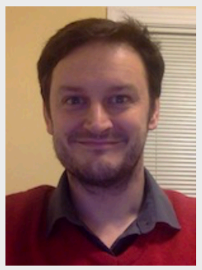
Christopher Thomas
University Lecturer

Graduate Students
James Delaney
Antoni Woss

CERN


Max Hansen
Scientific Staff

College of William and Mary

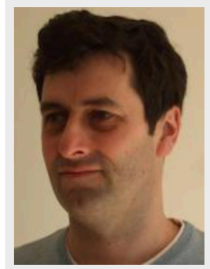

Jozef Dudek
Assistant Professor
& Staff Scientist (JLab)

Graduate Students
Christopher Johnson
Archana Radhakrishnan
Felipe Ortega

Old Dominion University


Raúl Briceño
Assistant Professor
& Staff Scientist (JLab)

Trinity College, Dublin


Michael Peardon
Associate Professor


Sinéad Ryan
Professor


David Wilson
Royal Society and
Science Foundation Ireland
University Research Fellow

Graduate Students
Nicolas Lang
David Tims

Tata Institute, Mumbai


Nilmani Mathur
Associate Professor

PRD93 094506 (2016)

An a_0 resonance in strongly coupled $\pi\eta$, $K\bar{K}$ scattering from lattice QCD

Jozef J. Dudek,[1,2,*] Robert G. Edwards,[1,†] and David J. Wilson[2,3,‡]
(Hadron Spectrum Collaboration)

PRD97 054513 (2018)

Isoscalar $\pi\pi, K\bar{K}, \eta\eta$ scattering and the σ, f_0, f_2 mesons from QCD

Raul A. Briceño,[1,2,*] Jozef J. Dudek,[1,3,†] Robert G. Edwards,[1,‡] and David J. Wilson[4,§]

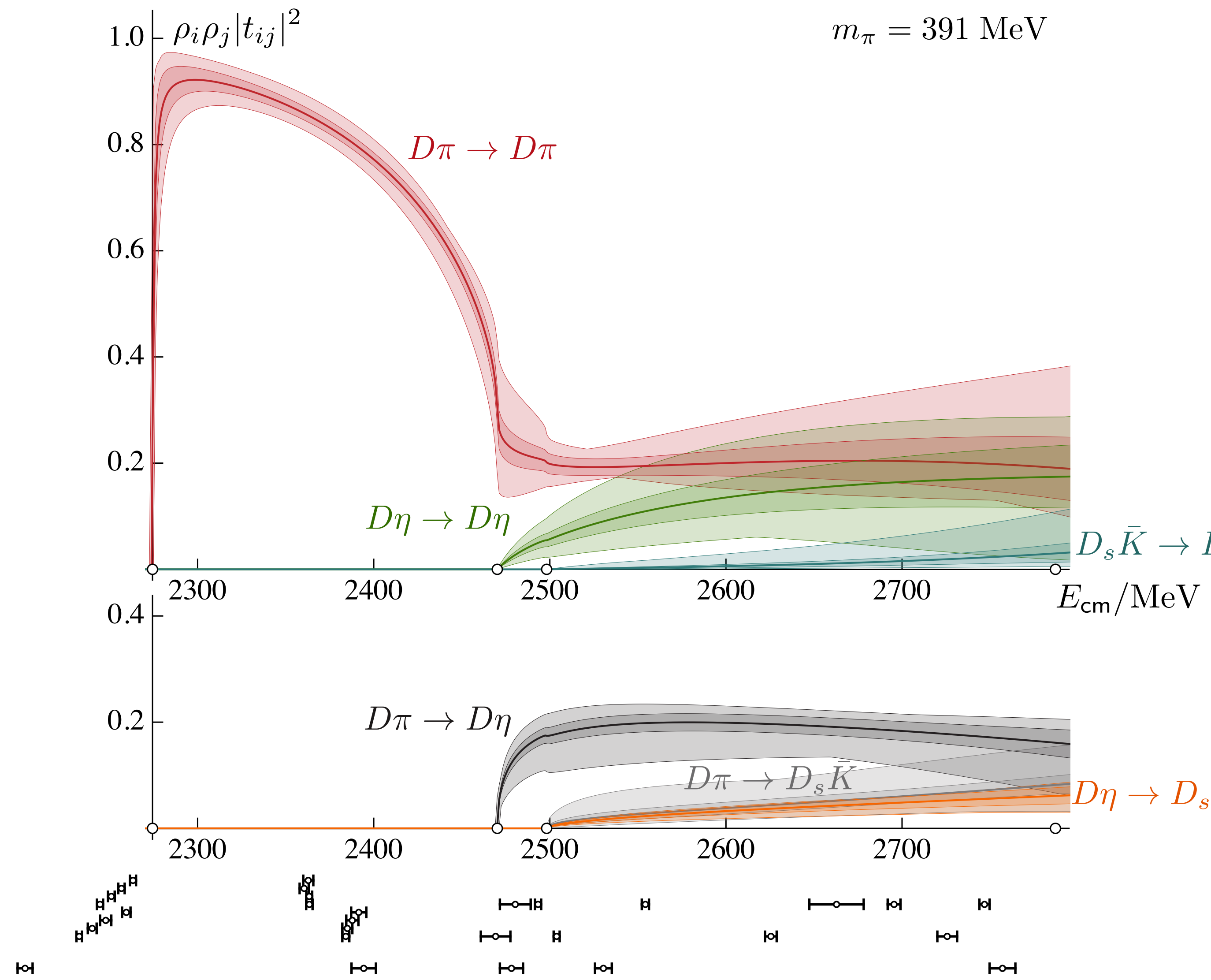
(for the Hadron Spectrum Collaboration)

PRD100 054506 (2019)

b_1 resonance in coupled $\pi\omega$, $\pi\phi$ scattering from lattice QCD

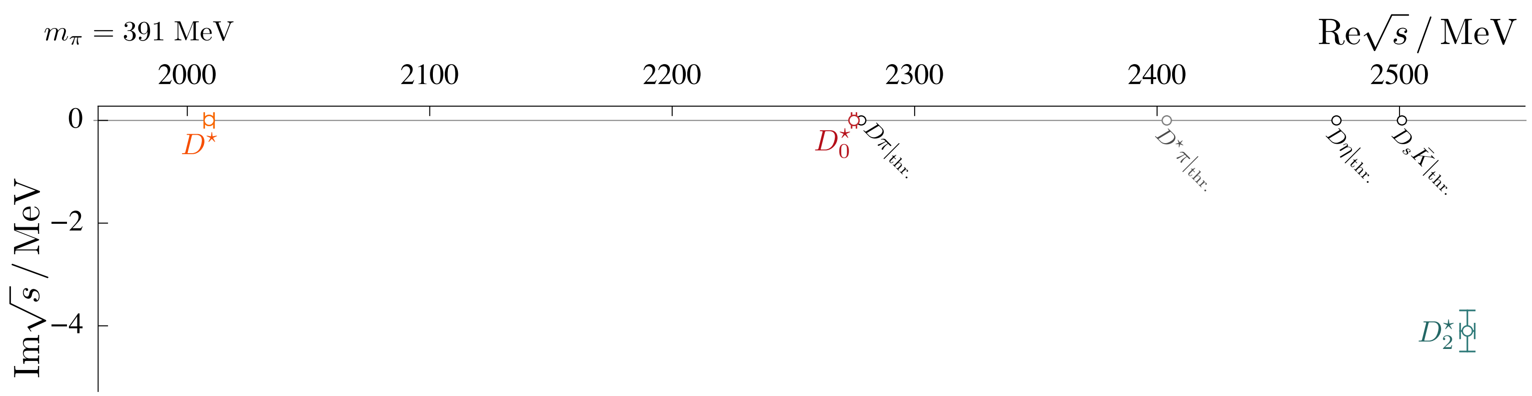
Antoni J. Woss,[1,*] Christopher E. Thomas,[1,†] Jozef J. Dudek,[2,3,‡] Robert G. Edwards,[2,‡] and David J. Wilson[4,‖]

$D\pi$, $D\eta$, $D_s\bar{K}$ scattering



D_0^* manifests as a bound-state at threshold

c.f. $D_s(2317)$



Luescher finite-volume functions

$$0 = \det\left[\mathbf{1} + i\boldsymbol{\rho}(E)\cdot\mathbf{t}(E)\cdot\big(\mathbf{1} + i\boldsymbol{\mathcal{M}}(E,L)\big)\right]$$

$$\overline{\mathcal{M}}_{\ell J m,\ell' J' m'} = \sum_{m_\ell, m'_\ell, m_S} \langle \ell m_\ell; 1 m_S | J m\rangle \langle \ell' m'_\ell; 1 m_S | J' m'\rangle$$

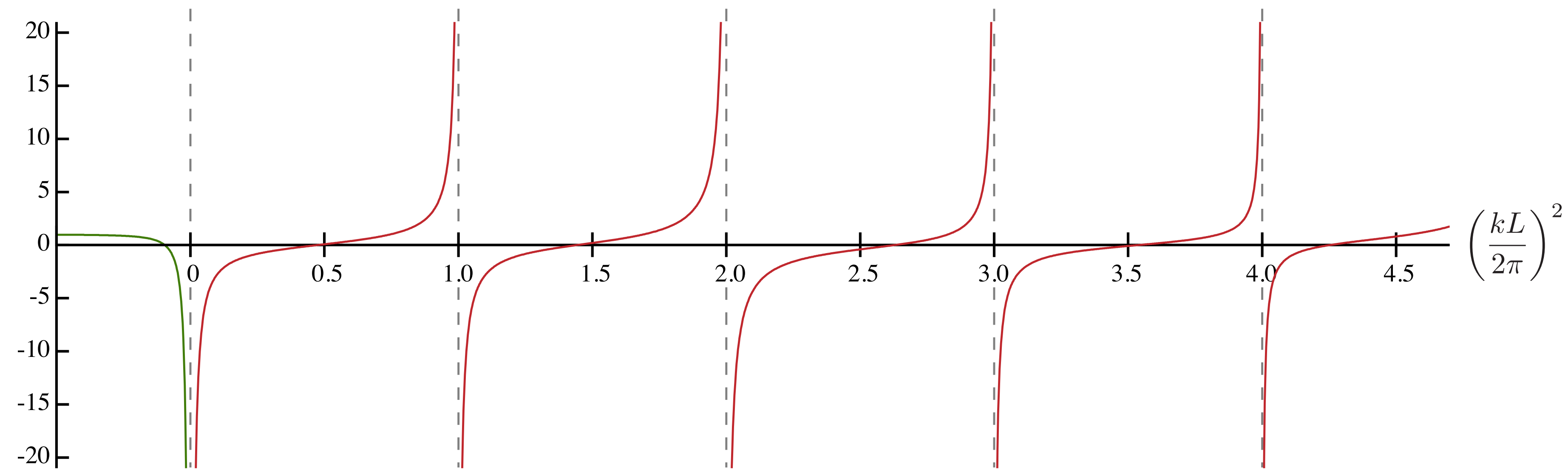
$$\times \sum_{\bar\ell,\bar m_\ell} \frac{(4\pi)^{3/2}}{k_{\mathsf{cm}}^{\bar\ell+1}}\, c^{\vec n}_{\bar\ell,\bar m_\ell}(k^2_{\mathsf{cm}};L) \int d\Omega\, Y^*_{\ell m_\ell} Y^*_{\bar\ell \bar m_\ell} Y_{\ell' m'_\ell}$$

"spinless" Luescher functions

to respect the lattice symmetries,
need to subduce into irreducible representations

$$\overline{\mathcal{M}}^{\vec n,\Lambda}_{\ell J n,\ell' J' n'}\,\delta_{\Lambda,\Lambda'}\delta_{\mu,\mu'} = \sum_{\substack{m,\lambda\\ m',\lambda'}} \mathcal{S}^{J\lambda *}_{\Lambda\mu n}\, D^{(J)*}_{m\lambda}(R)\ \overline{\mathcal{M}}^{\vec n}_{\ell J m,\ell' J' m'}\ \mathcal{S}^{J'\lambda'}_{\Lambda'\mu' n'}\, D^{(J')}_{m'\lambda'}(R)$$

e.g. $\mathcal{M}_0$



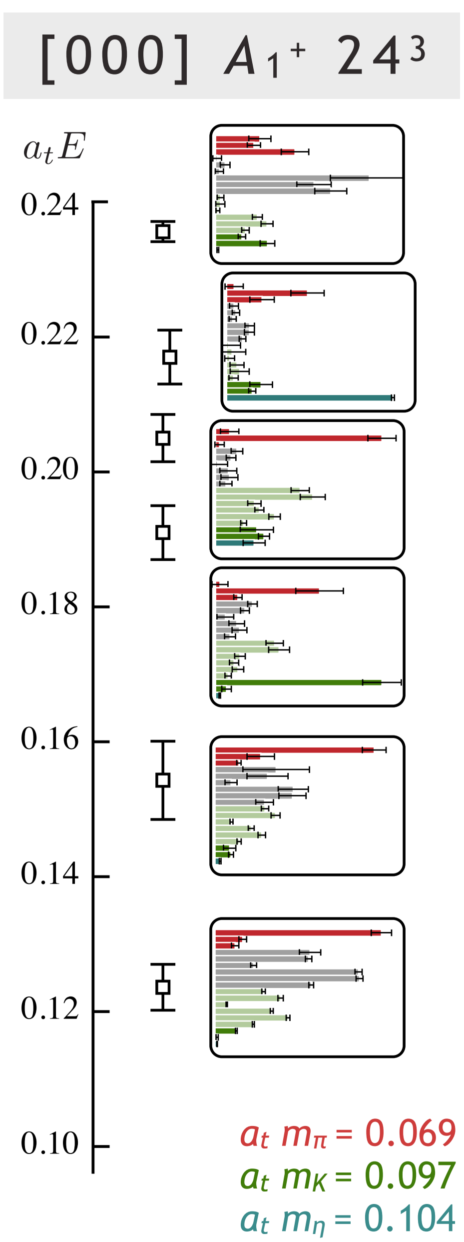
operator basis

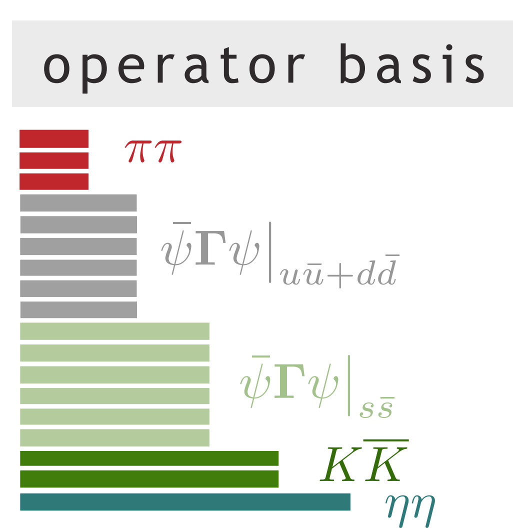
operator basis: ‘single-meson’ + ‘meson-meson’

$$\bar{\psi}\mathbf{\Gamma}\psi \qquad + \qquad \sum_{\hat{\mathbf{p}}_1,\hat{\mathbf{p}}_2} C(\mathbf{p}_1,\mathbf{p}_2;\mathbf{p})\, M_1(\mathbf{p}_1)\, M_2(\mathbf{p}_2)$$

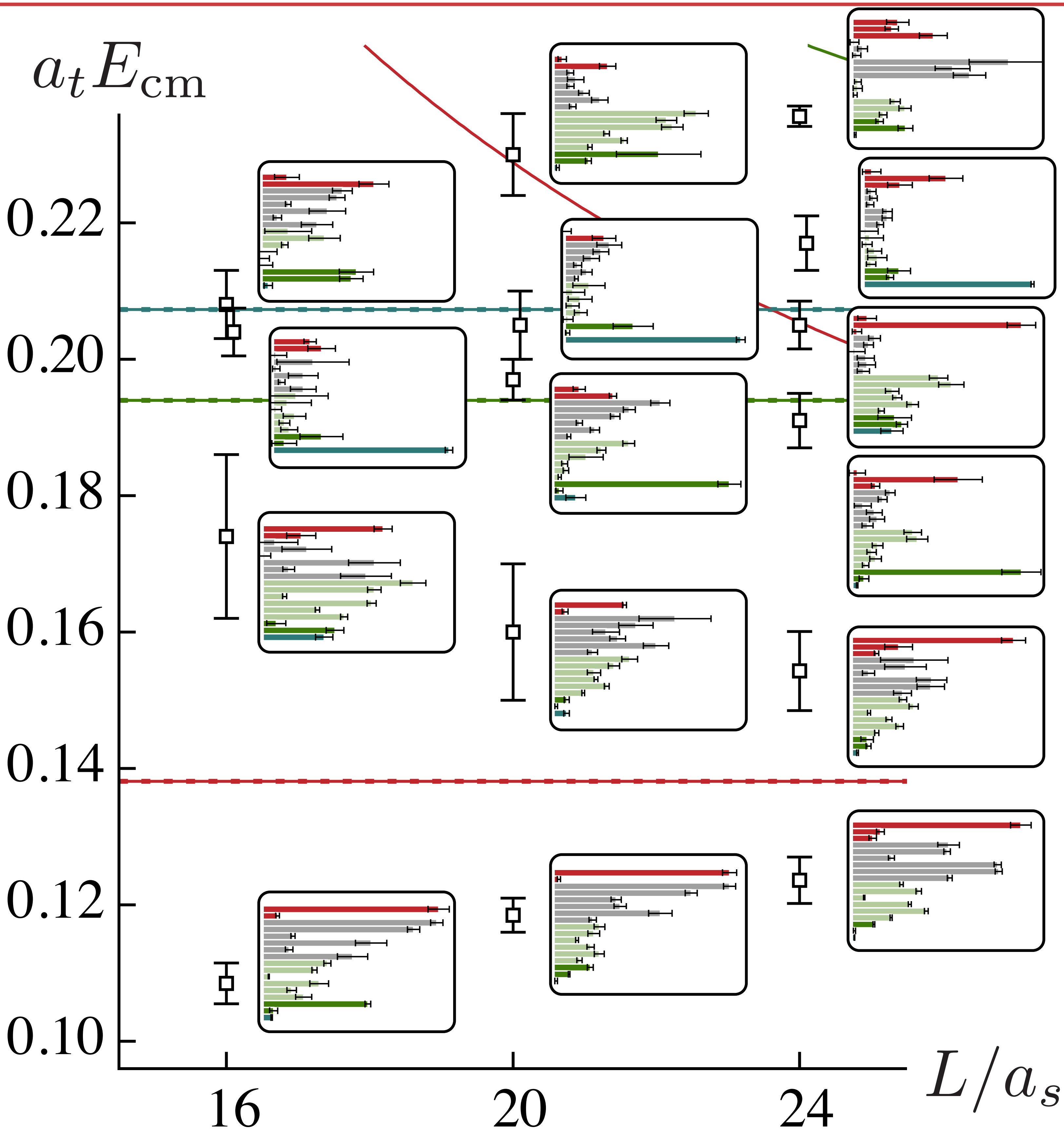
$$\sum_{\mathbf{x}} e^{i\mathbf{p}_1\cdot\mathbf{x}}\, \bar{\psi}_{\mathbf{x}}\mathbf{\Gamma}\psi_{\mathbf{x}} \sum_{\mathbf{y}} e^{i\mathbf{p}_2\cdot\mathbf{y}}\, \bar{\psi}_{\mathbf{y}}\mathbf{\Gamma}'\psi_{\mathbf{y}}$$

sampling the whole lattice volume



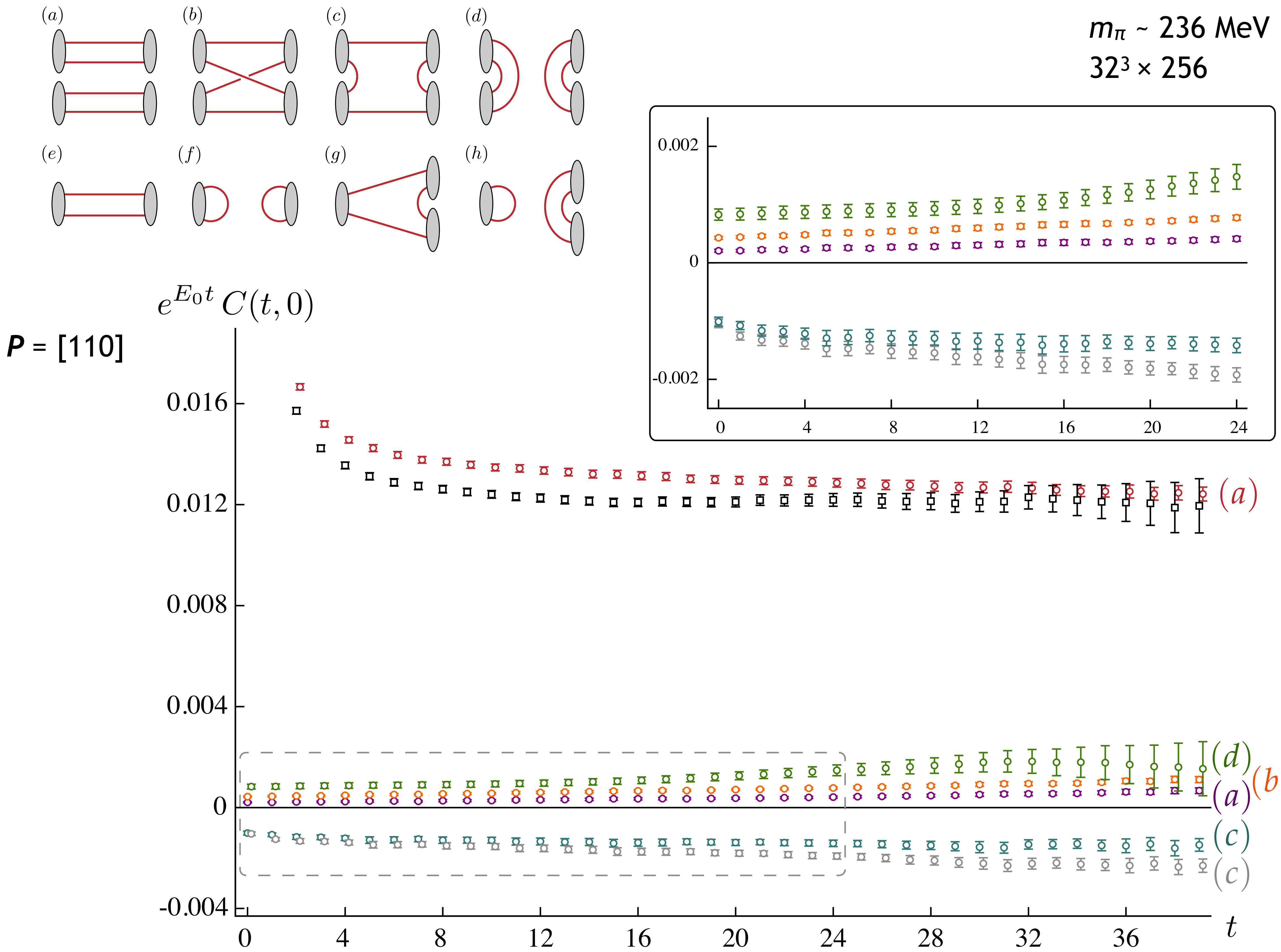


prefer to use
optimized single-meson operators …

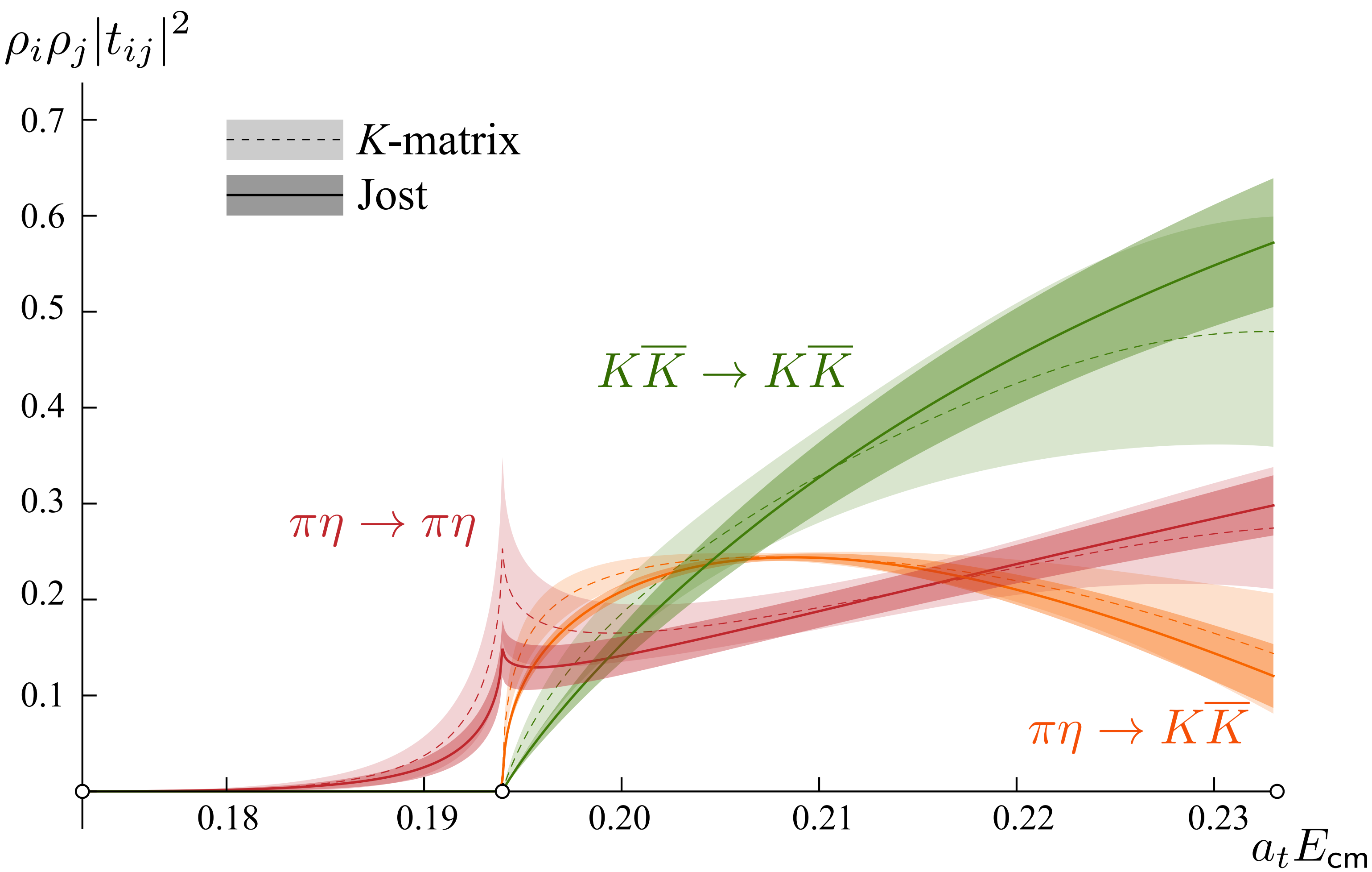

$a_t E_{cm}$
0.22
0.20
0.18
0.16
0.14
0.12
0.10
16
20
24
L/a_s

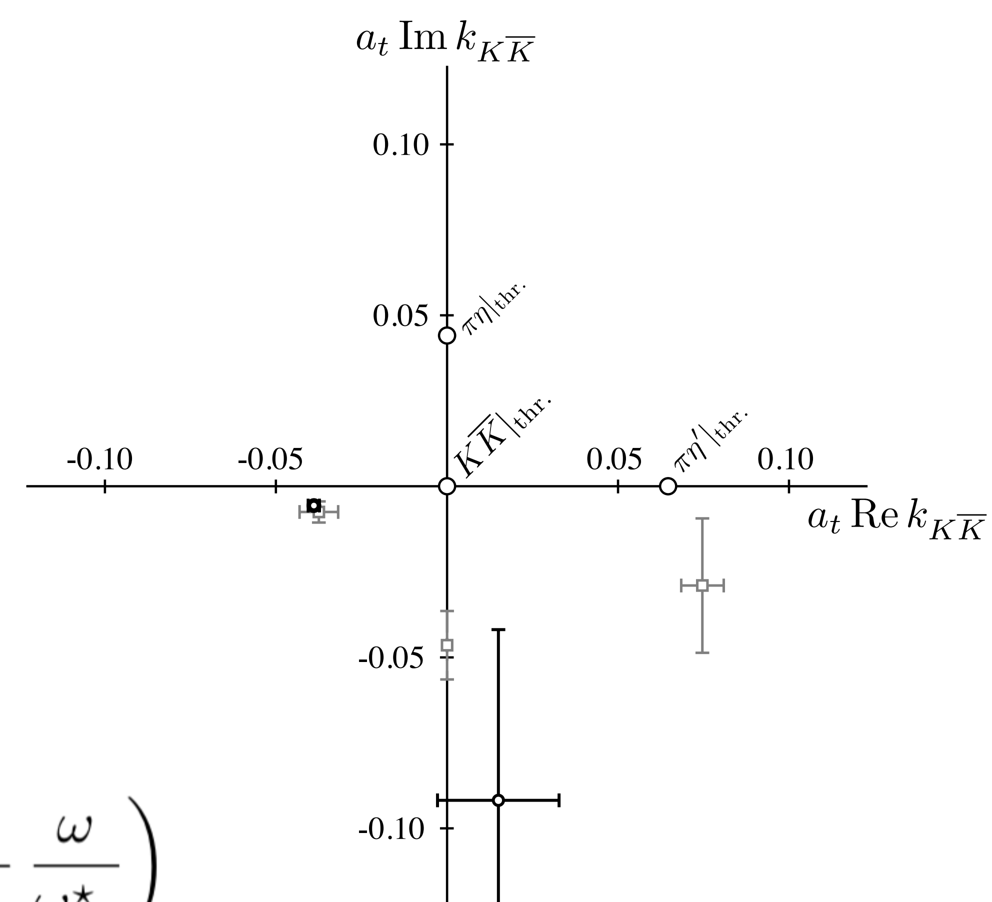
I=0 $\pi\pi$ Wick contraction contributions

m_π ~ 236 MeV
$32^3 \times 256$



$\pi\eta/K\bar{K}$ extras — Jost





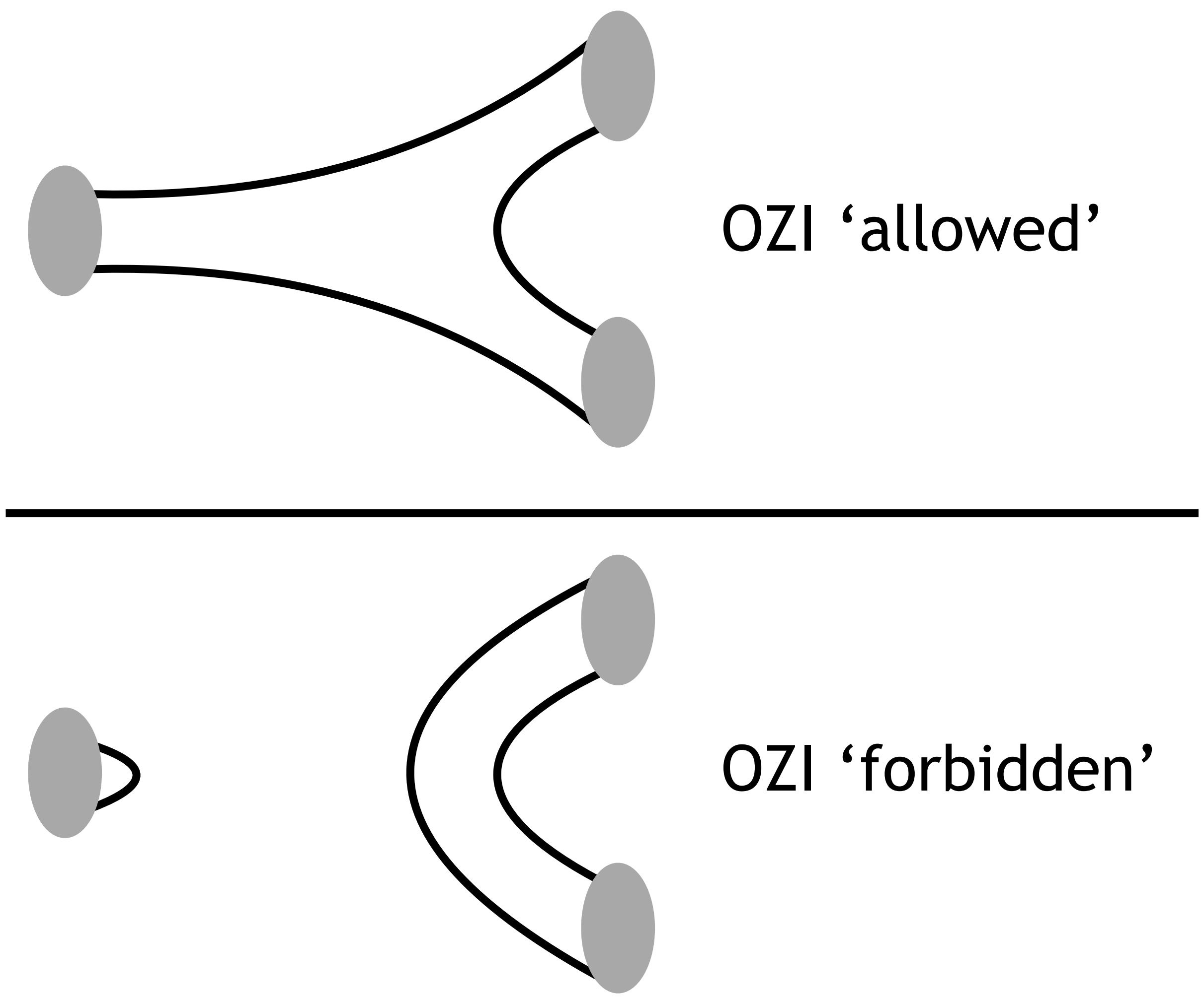
$$S_{11} = \frac{\mathfrak{D}(-\omega^{-1})}{\mathfrak{D}(\omega)}, \qquad \omega = \frac{k_1 + k_2}{\sqrt{k_1^2 - k_2^2}}$$

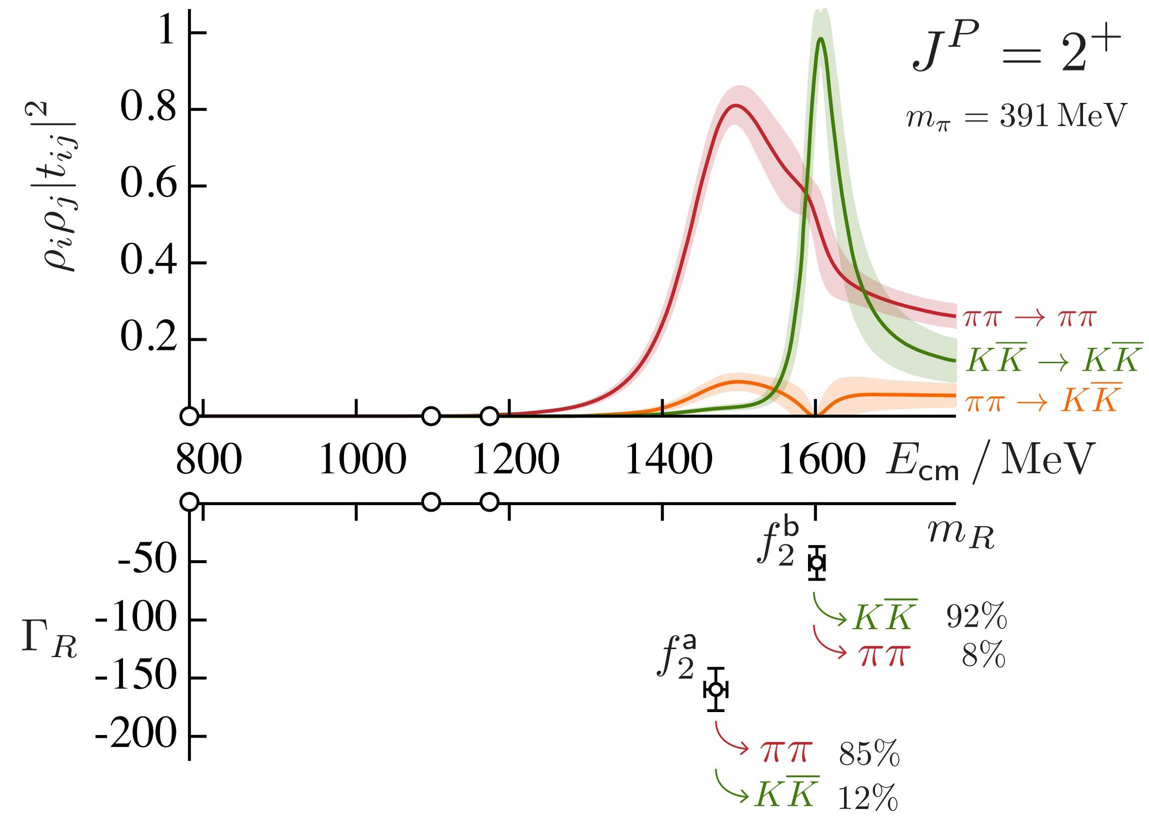
$$S_{22} = \frac{\mathfrak{D}(\omega^{-1})}{\mathfrak{D}(\omega)},$$

$$\det \mathbf{S} = \frac{\mathfrak{D}(-\omega)}{\mathfrak{D}(\omega)}.$$

$$\mathfrak{D}(\omega) = \frac{1}{\omega^2}\left(1 - \frac{\omega}{\omega_{p_1}}\right)\left(1 + \frac{\omega}{\omega^\star_{p_1}}\right)\left(1 - \frac{\omega}{\omega_{p_2}}\right)\left(1 + \frac{\omega}{\omega^\star_{p_2}}\right)$$

f_2 resonances — decay couplings & OZI





$f_2^{\mathsf{a}} \sim u\bar{u} + d\bar{d} \qquad f_2^{\mathsf{b}} \sim s\bar{s}$

couplings from pole residue

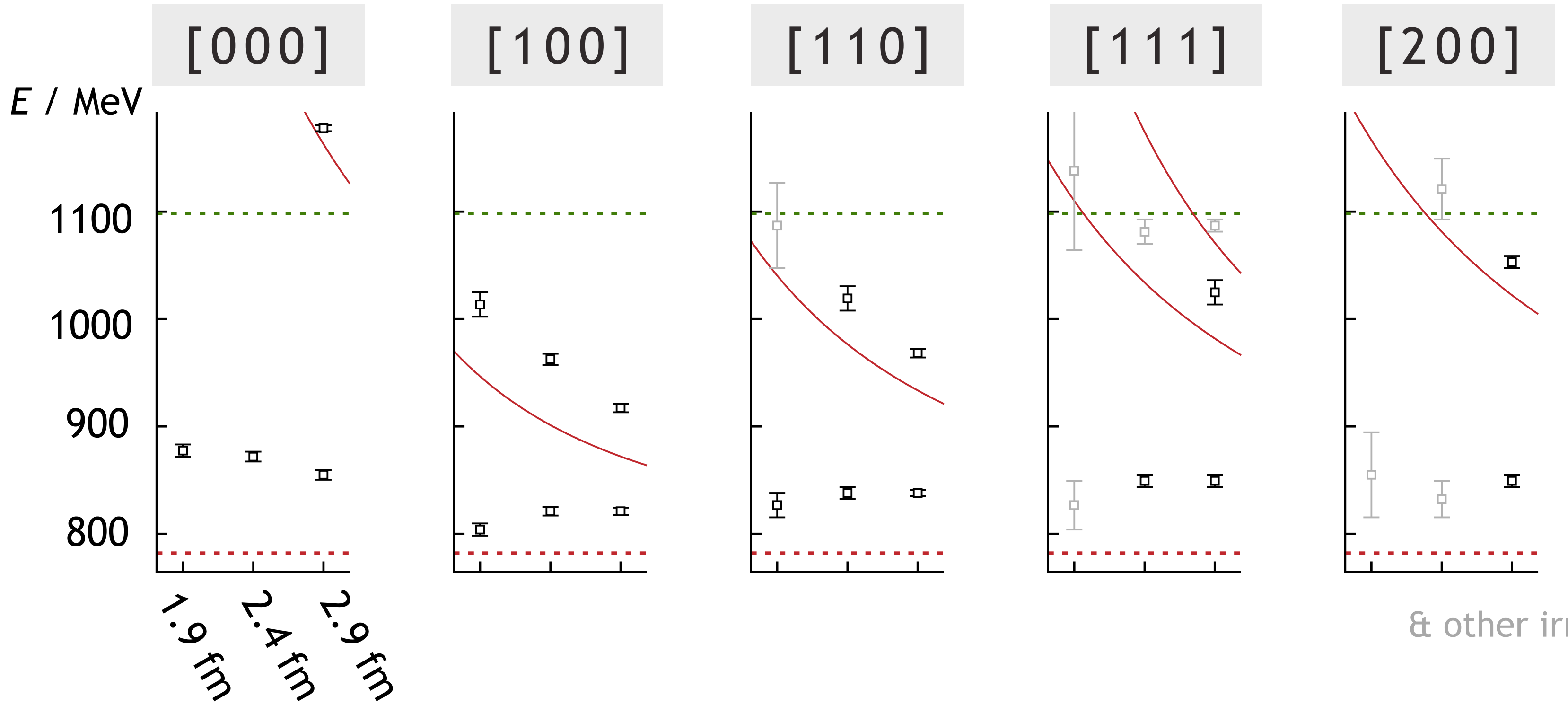
	$\frac{a_t \lvert c_{\pi\pi} \rvert}{(a_t k_{\pi\pi})^2}$	$\frac{a_t \lvert c_{K\bar{K}} \rvert}{(a_t k_{K\bar{K}})^2}$
f_2^{a}	7.1(4)	4.8(9)
f_2^{b}	1.0(3)	5.5(8)

zero in 'OZI' limit
— requires $s\bar{s}$ annihilation

an elastic resonance – the ρ in $\pi\pi$ – lattice QCD

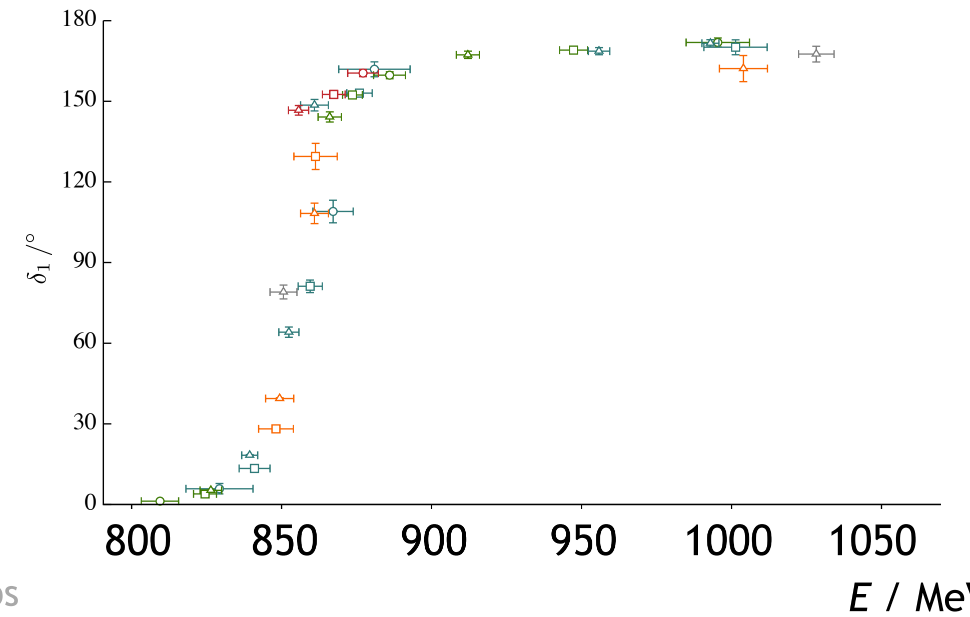
PRD87 034505 (2013)

$m_\pi \sim 391$ MeV



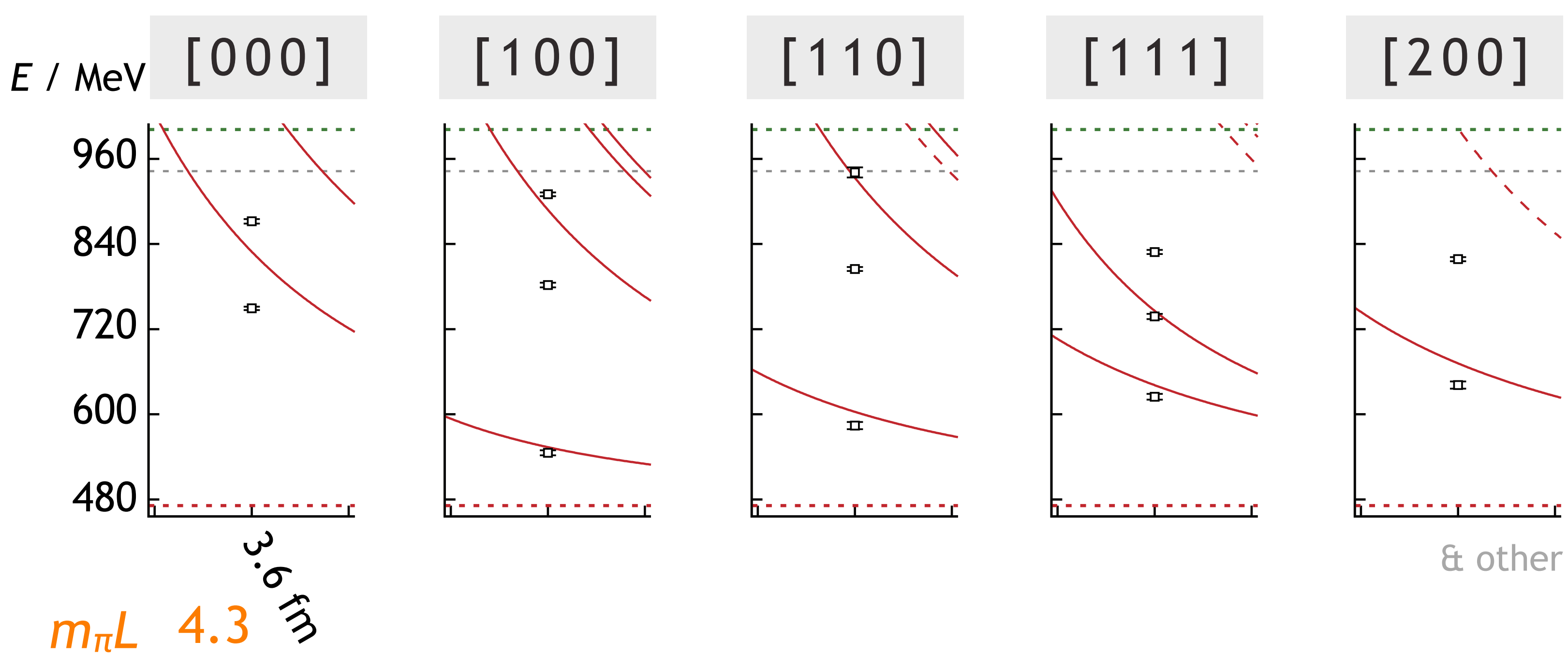
& other irreps

scattering phase-shift



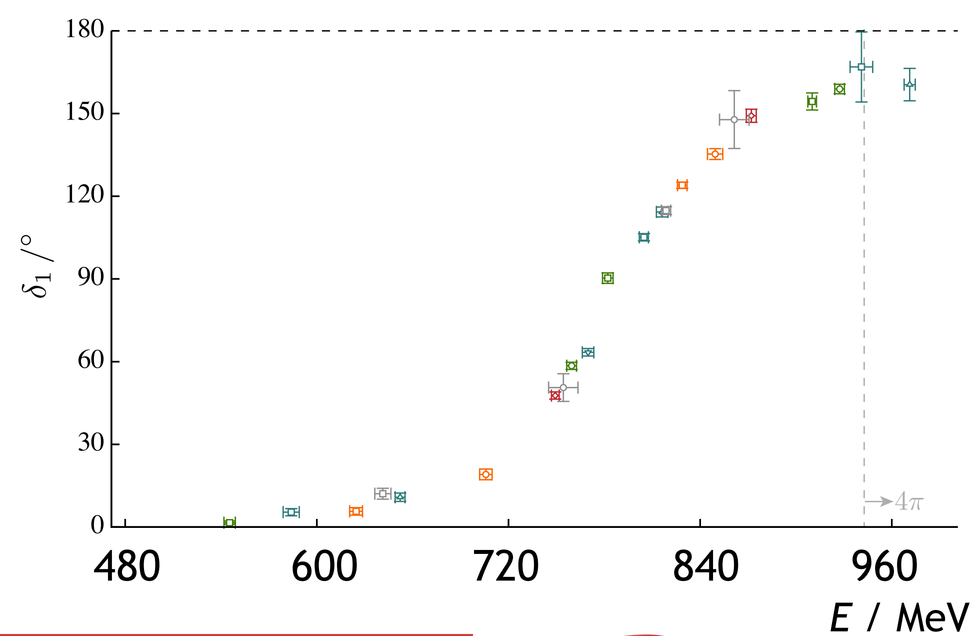
PRD92 094502 (2015)

$m_\pi \sim 236$ MeV



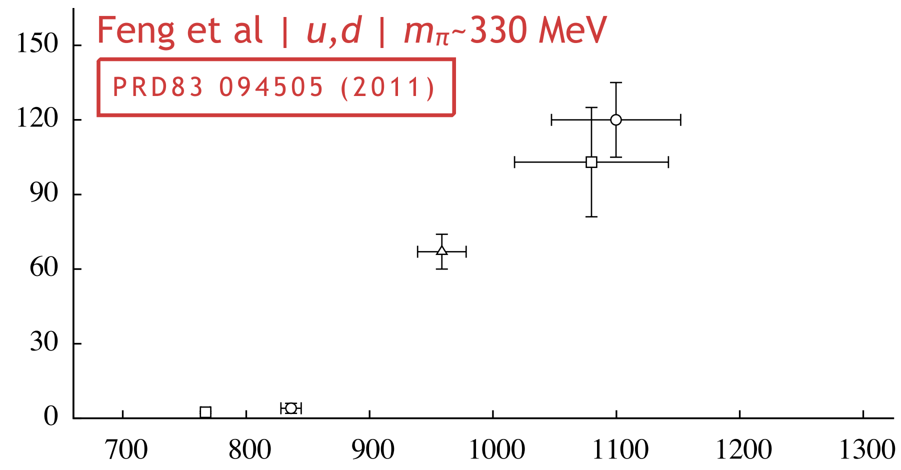
& other irreps

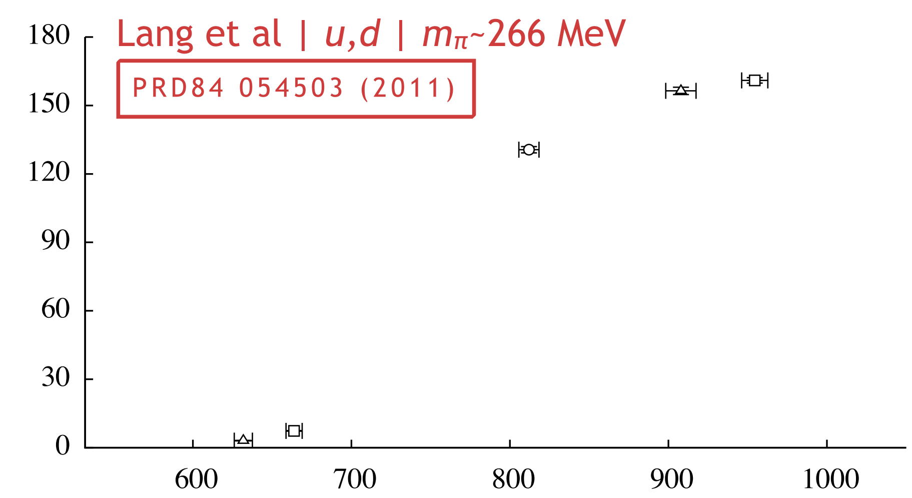
scattering phase-shift

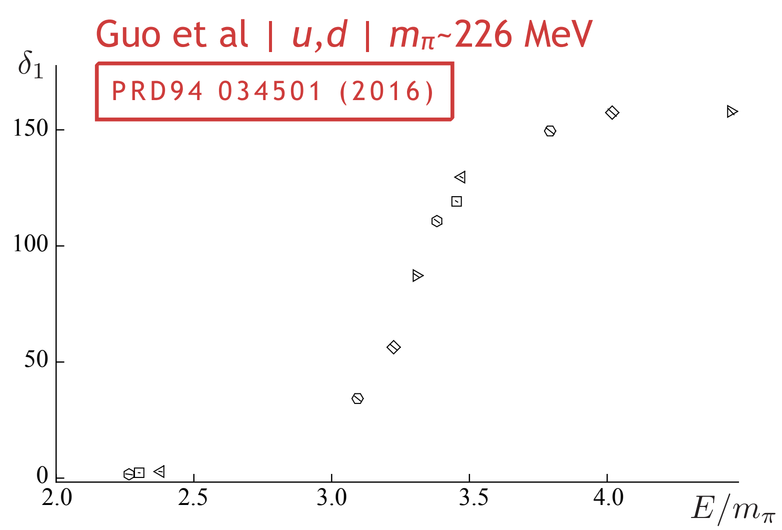


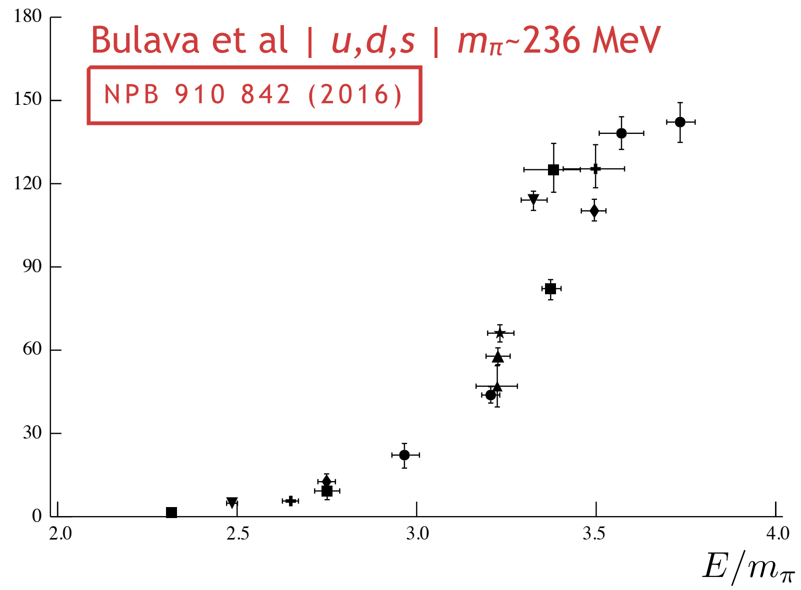
$m_\pi L$ 4.3

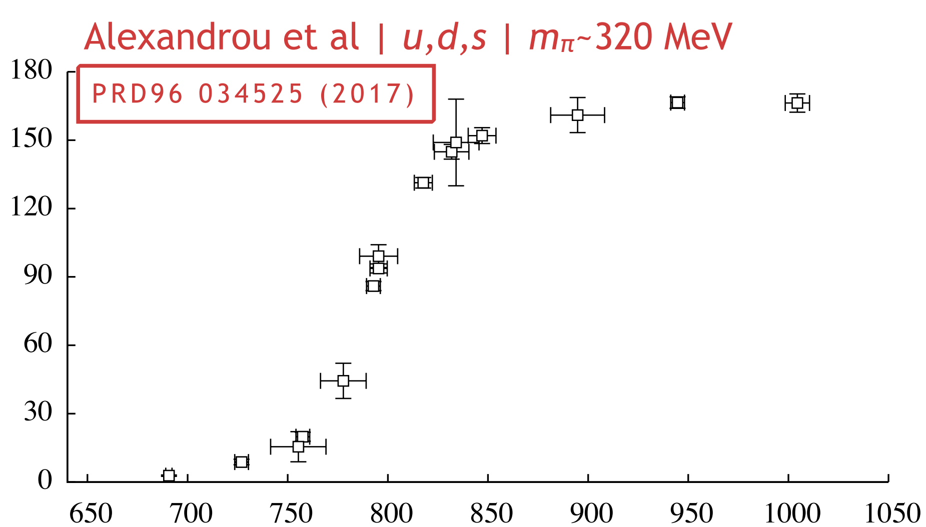
an elastic resonance — the ρ in $\pi\pi$ — lattice QCD

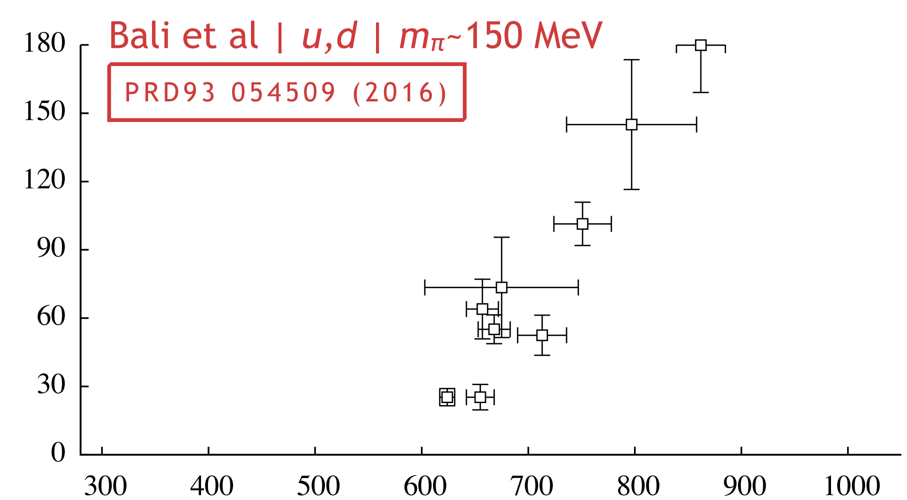












relates scattering length, effective range
to compositeness measure, Z

$$a = -2\frac{1-Z}{2-Z}\frac{1}{\sqrt{m_\pi \epsilon}} + \dots$$
$$r = -2\frac{Z}{1-Z}\frac{1}{\sqrt{m_\pi \epsilon}} + \dots$$

with corrections whose size is set by
the range of the interaction

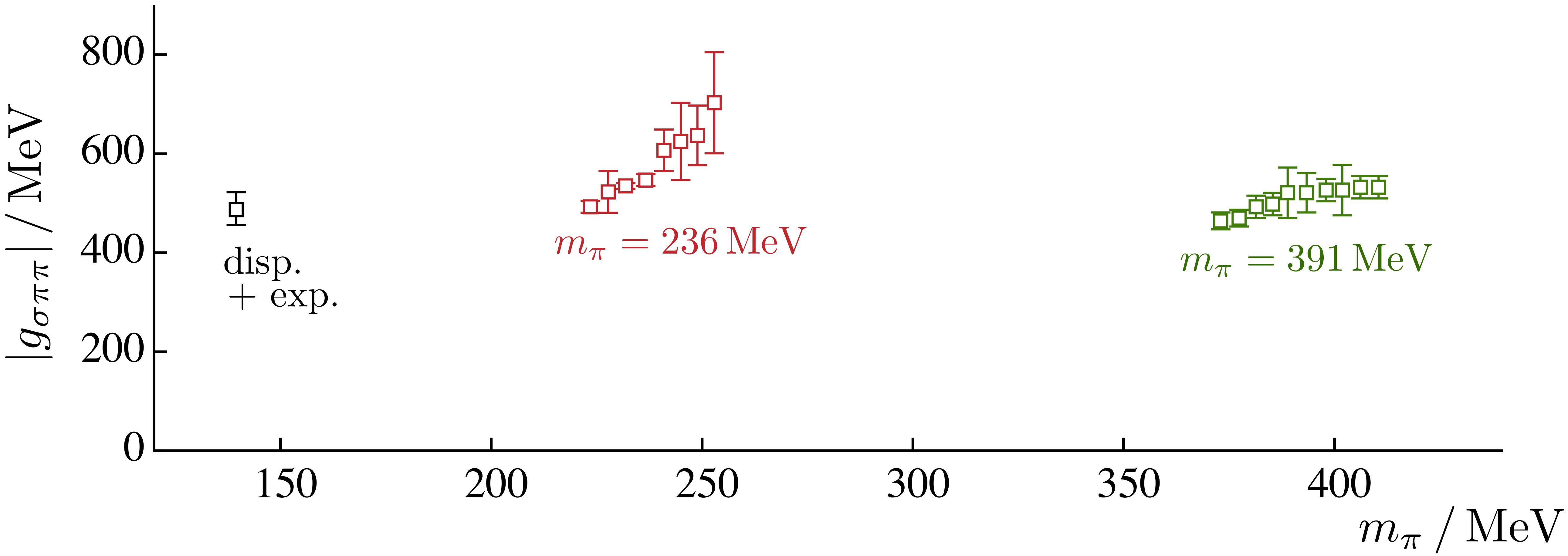
Z=1 compact state
Z=0 a $\pi\pi$ molecule

$$\epsilon = 37(5)\,\mathrm{MeV}$$

$$a = -0.0071(11)\,\mathrm{MeV}^{-1} = -1.4(2)\,\mathrm{fm}$$
$$r = -0.0041(14)\,\mathrm{MeV}^{-1} = -0.8(3)\,\mathrm{fm}$$

$$\boxed{Z \sim 0.3(1)}$$

$\sigma \to \pi\pi$ coupling from the pole residue



WILLIAM & MARY

Jefferson Lab
Thomas Jefferson National Accelerator Facility

σ pole in unitarized chiral perturbation theory

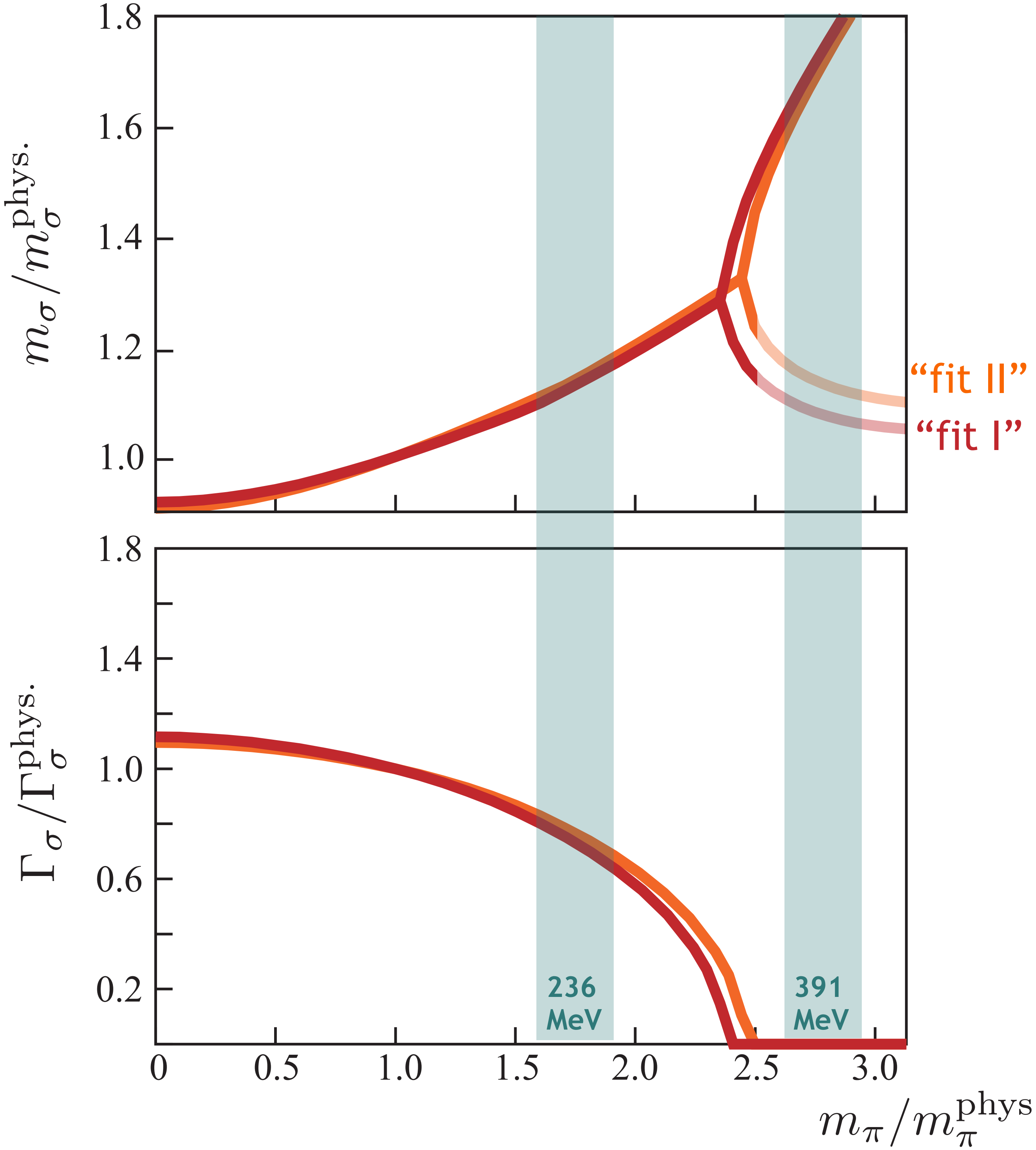
J.R. Pelaez ...

PRD81 054035 (2010)

resonance becomes a **virtual bound state** near m_π~350 MeV ...

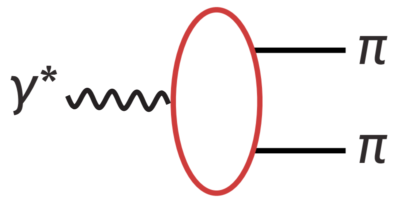
... then a **bound state** near m_π~420 MeV

"the exact m_π value when this happens is not very reliable"



coupling resonances to currents

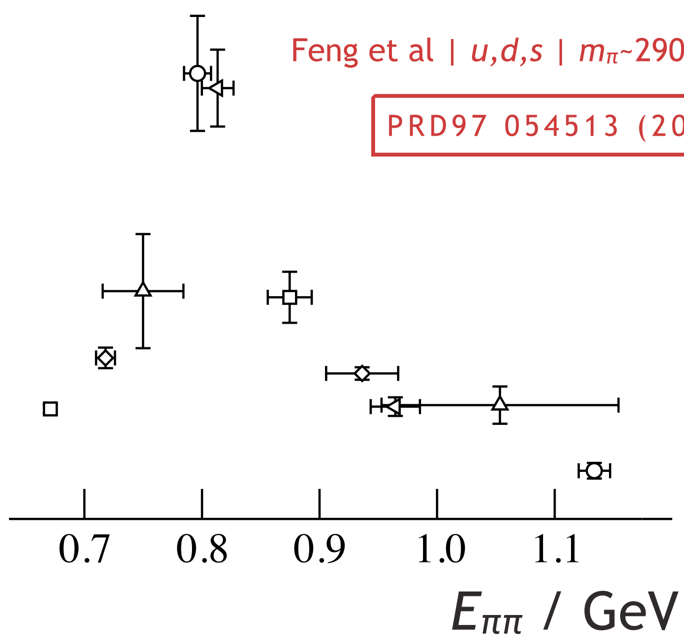
$\gamma^* \to \pi\pi$



$F_\pi(E_{\pi\pi})$

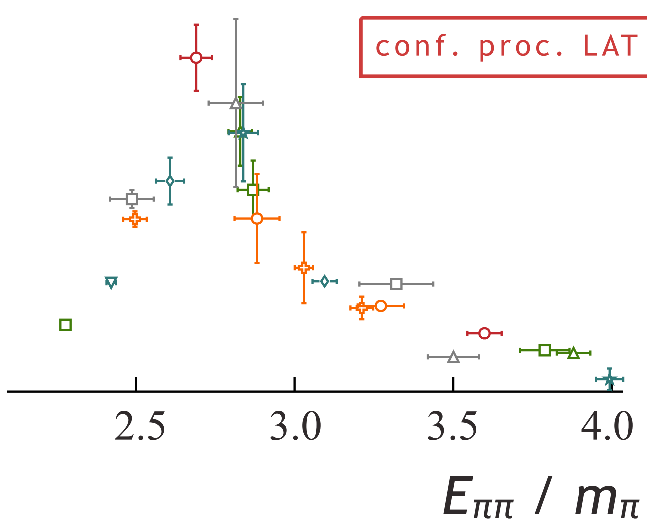
Feng et al | *u,d,s* | m_π~290 MeV

PRD97 054513 (2018)

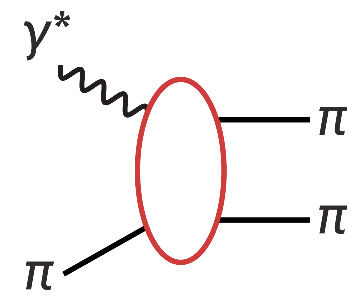


Bulava et al | *u,d,s* | m_π~280 MeV

conf. proc. LAT '15



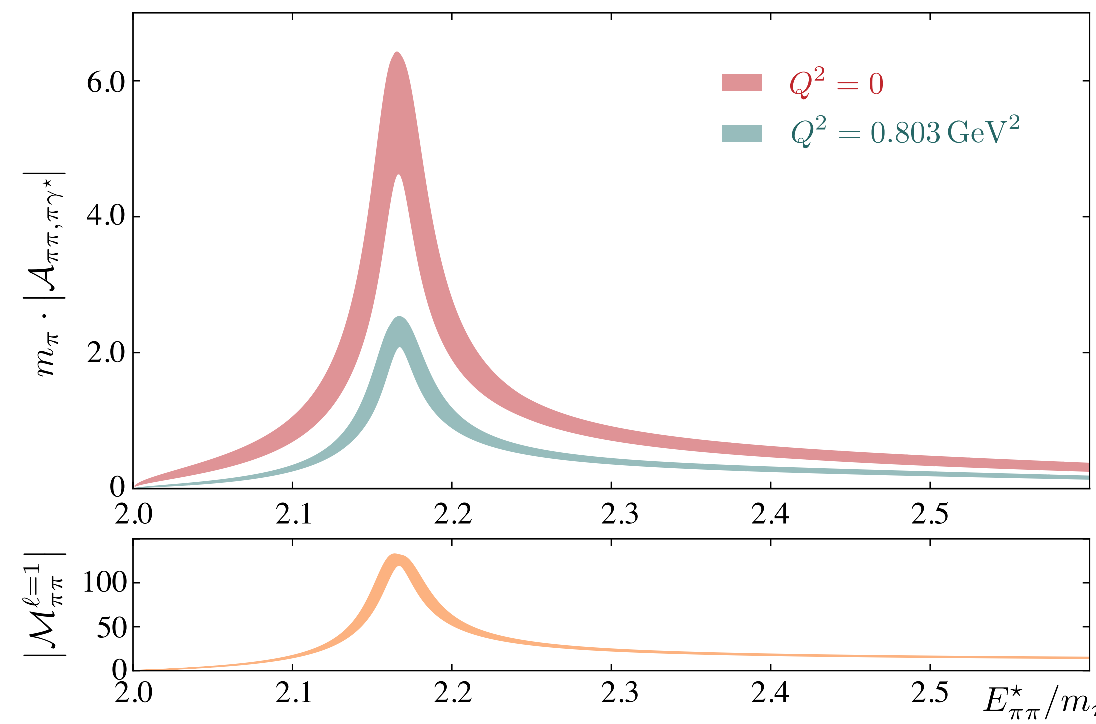
$\gamma^* \pi \to \pi\pi$

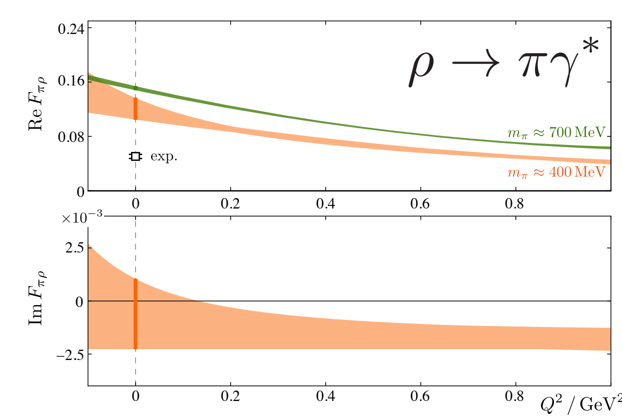


$\mathcal{A}(E_{\pi\pi}, Q^2)$

Briceno et al | *u,d,s* | m_π~391 MeV

PRL115 242001 (2015)

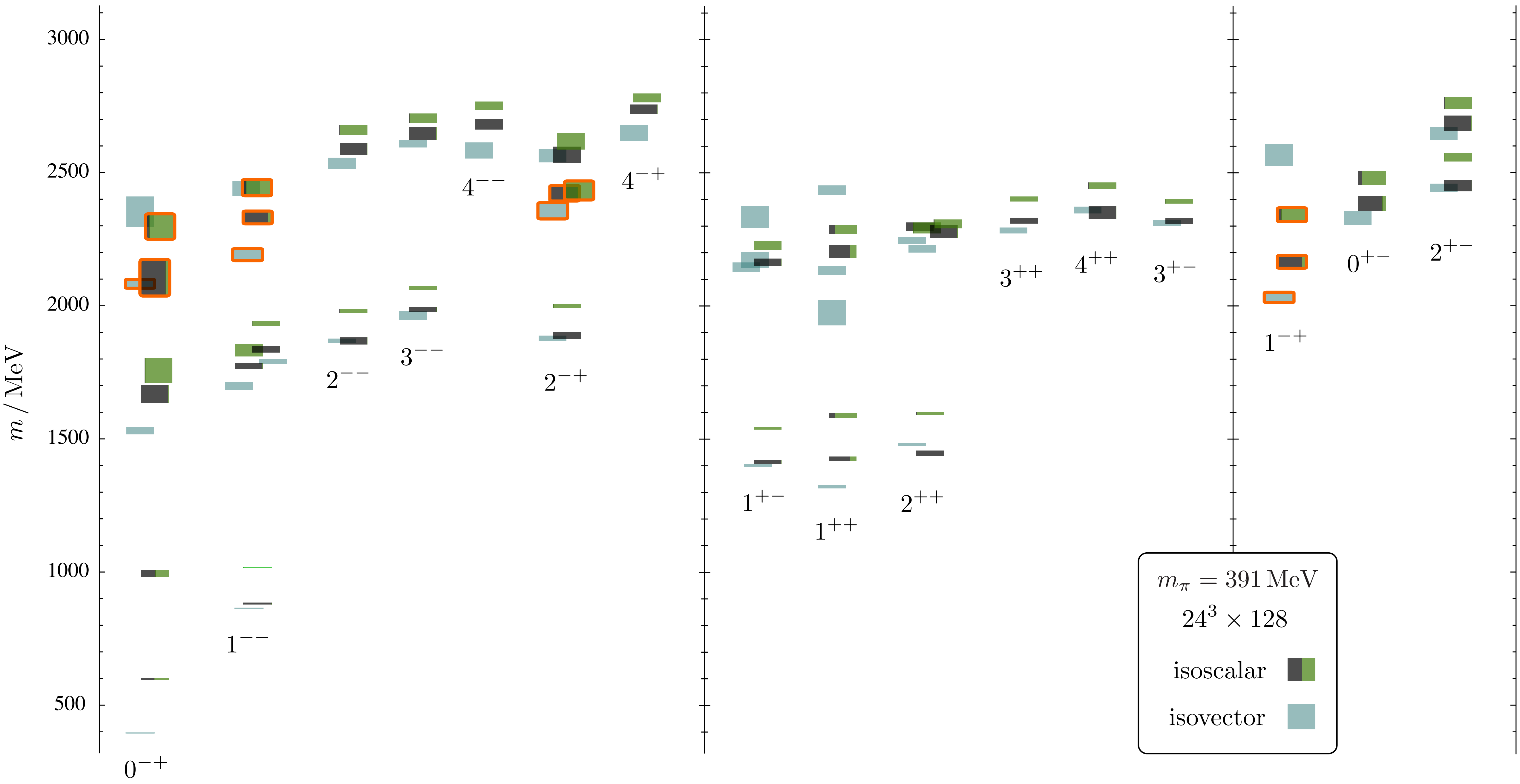




meson spectrum ‘ignoring’ decays

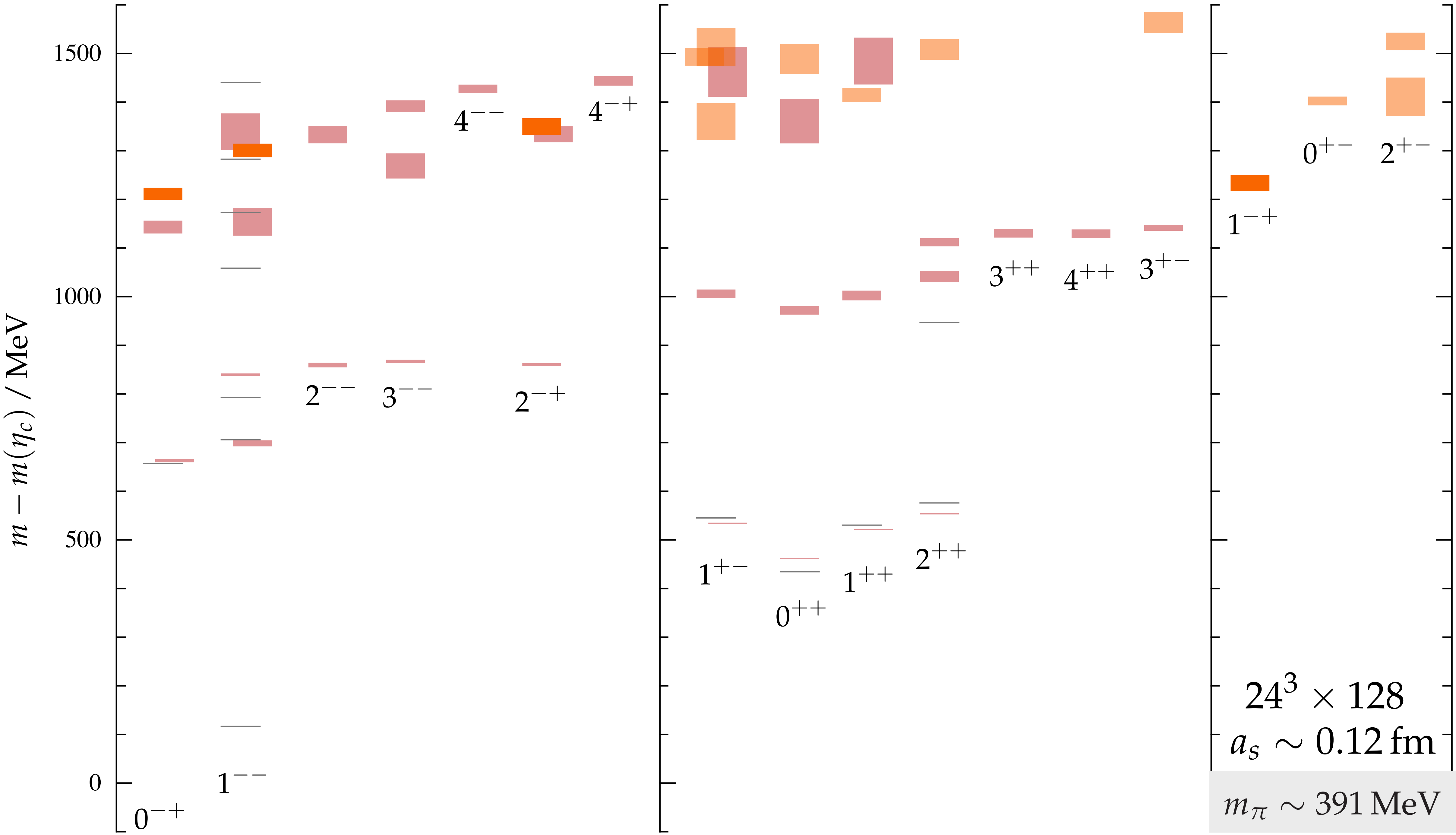
PRD 88 094505 (2013)

‘ignoring’ here is not a controlled approximation

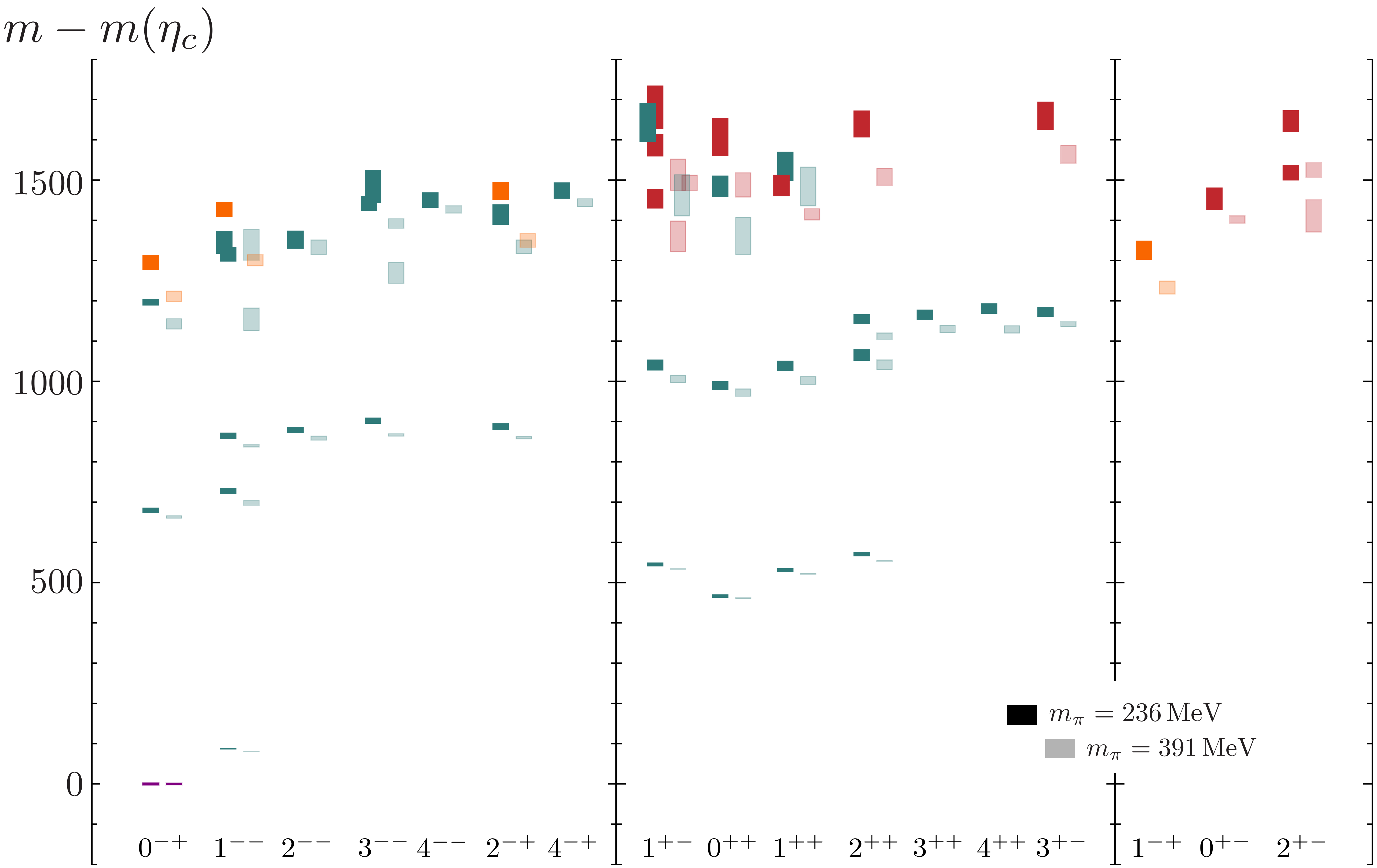


- two 'super'-multiplets of hybrid mesons

this work lead by our Dublin collaborators

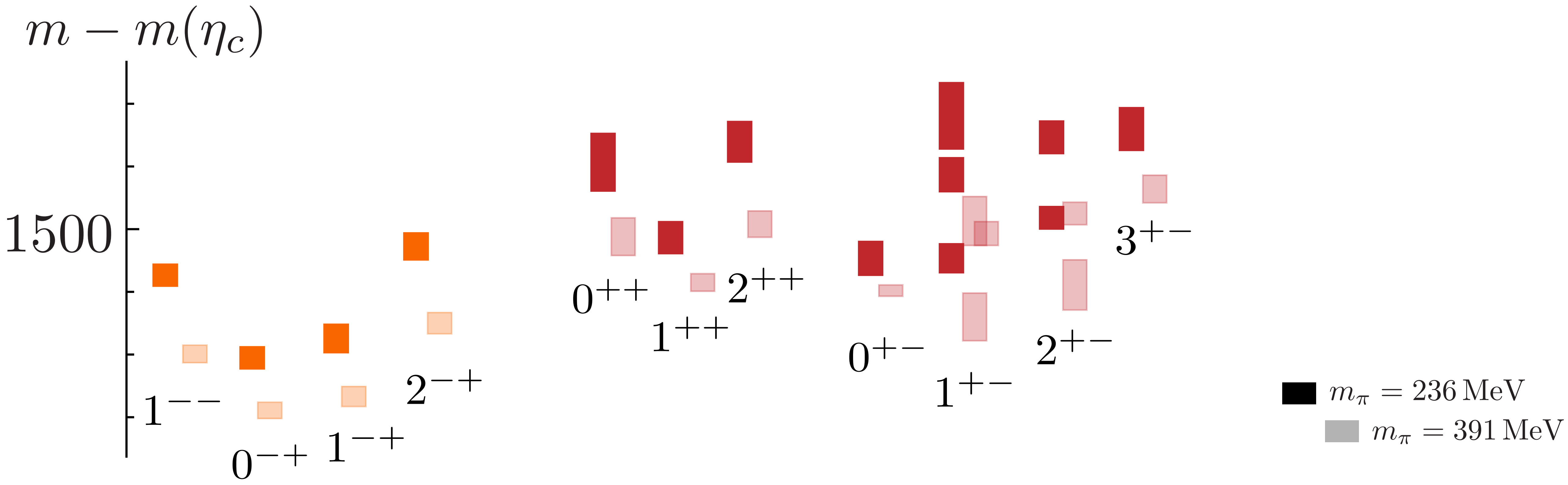


JHEP 1207 126 (2012)

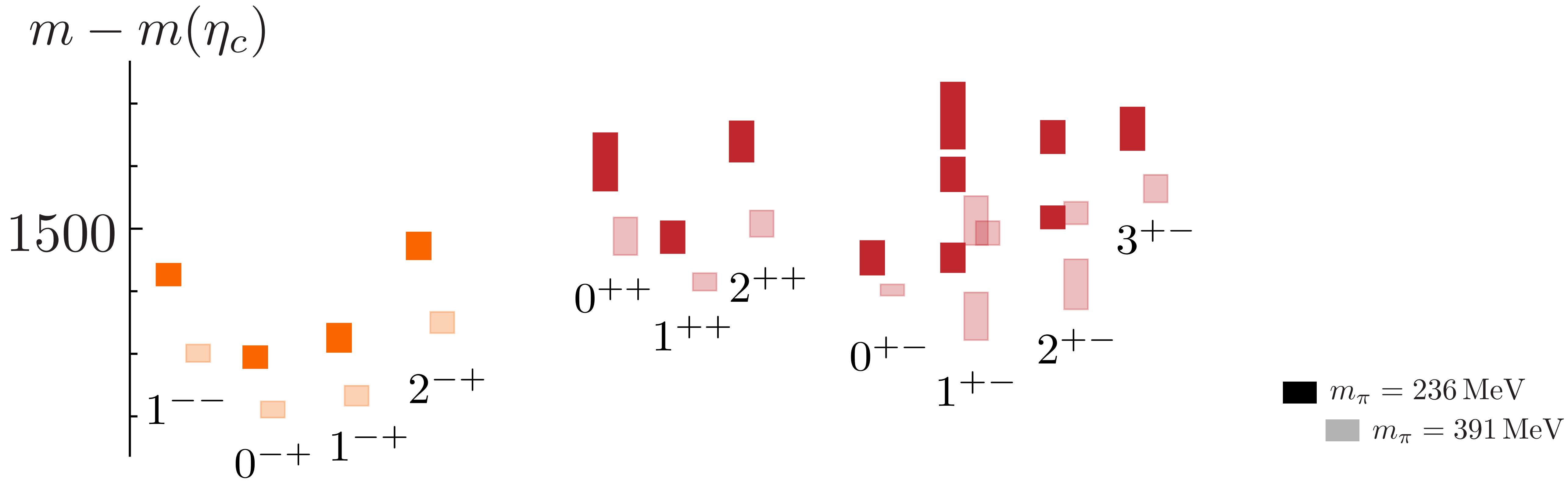


$m - m(\eta_c)$
1500
1000
500
0
0^{-+} 1^{--} 2^{--} 3^{--} 4^{--} 2^{-+} 4^{-+}
1^{+-} 0^{++} 1^{++} 2^{++} 3^{++} 4^{++} 3^{+-}
1^{-+} 0^{+-} 2^{+-}
$m_\pi = 236\,\mathrm{MeV}$
$m_\pi = 391\,\mathrm{MeV}$

a ‘minimal’ model of hybrid mesons



a ‘minimal’ model of hybrid mesons



hybrid meson model $q\bar{q}_{\mathbf{8}} \otimes G_{\mathbf{8}}(1^{+-})$

‘chromo-magnetic’ excitation

$q\bar{q}(^1S_0) \otimes G(1^{+-}) \to 1^{--}$

$q\bar{q}(^3S_1) \otimes G(1^{+-}) \to (0,1,2)^{-+}$

$q\bar{q}(^1P_1) \otimes G(1^{+-}) \to (0,1,2)^{++}$

$q\bar{q}(^3P_{0,1,2}) \otimes G(1^{+-}) \to (0,1^3,2^2,3)^{+-}$

different to the flux-tube model (now disfavored)

we now know the lowest hybrid meson content of QCD

- lightest set of hybrid mesons appear to contain a *1*$^{+-}$ gluonic excitation

quarks in an *S*-wave

$$\left[q\bar{q}_{\mathbf{8_c}}\left[{}^1S_0\right]\,G^\star_{\mathbf{8_c}}\,[B]\right]_{\mathbf{1_c}} \to 1^{--}_{\mathrm{hyb.}}$$

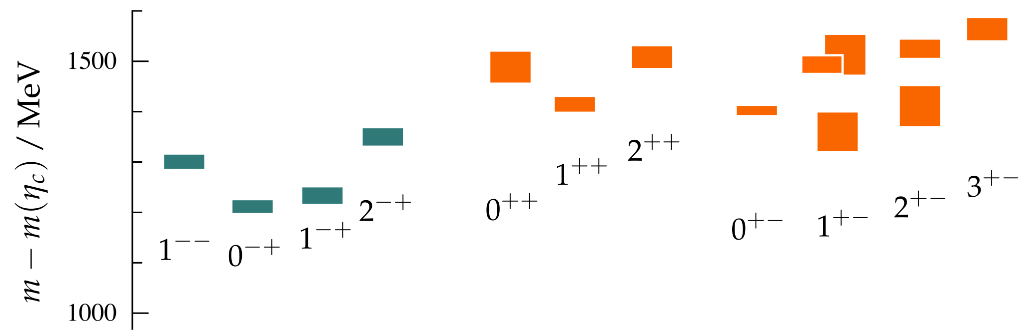
$$\left[q\bar{q}_{\mathbf{8_c}}\left[{}^3S_1\right]\,G^\star_{\mathbf{8_c}}\,[B]\right]_{\mathbf{1_c}} \to (0,1,2)^{-+}_{\mathrm{hyb.}}$$

quarks in a *P*-wave

$$\left[q\bar{q}_{\mathbf{8_c}}\left[{}^1P_1\right]\,G^\star_{\mathbf{8_c}}\,[B]\right]_{\mathbf{1_c}} \to (0,1,2)^{++}_{\mathrm{hyb.}}$$

$$\left[q\bar{q}_{\mathbf{8_c}}\left[{}^3P_{0,1,2}\right]\,G^\star_{\mathbf{8_c}}\,[B]\right]_{\mathbf{1_c}} \to (0,1^3,2^2,3)^{+-}_{\mathrm{hyb.}}$$

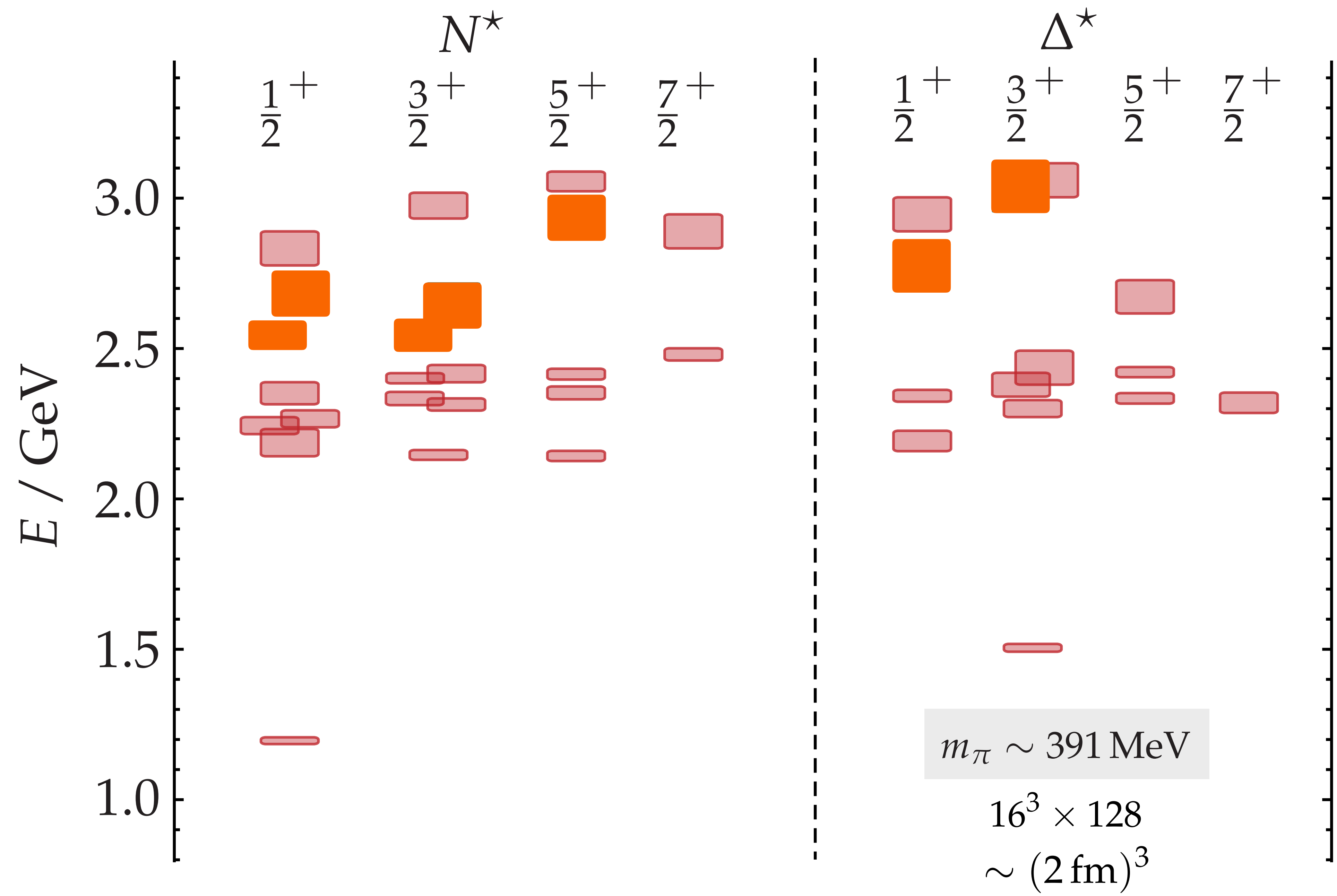
CHARMONIUM HYBRIDS



- some models have similar systematics
  - bag model also has *1*$^{+-}$ lowest in energy
  - *1*$^{+-}$ in a Coulomb-gauge approach

excited baryons

- a ‘super’-multiplet of **hybrid baryons**



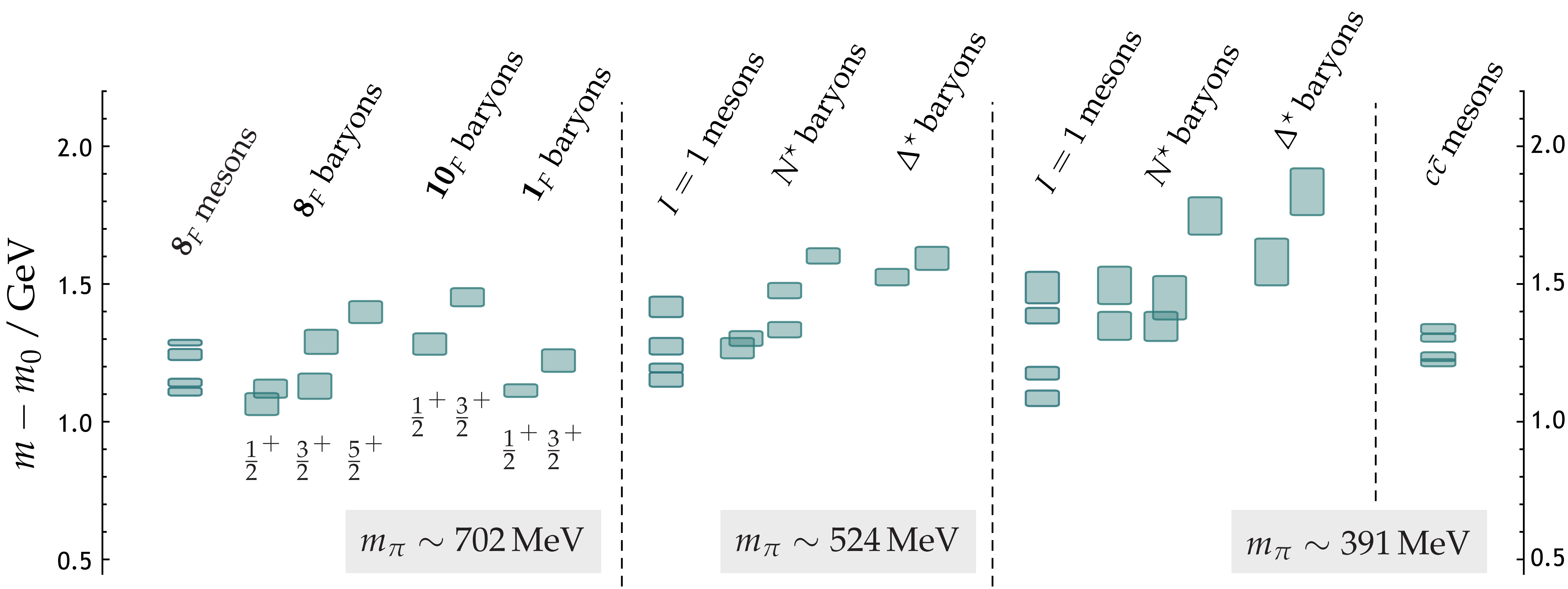
spectrum from large basis of baryon operators

$$\epsilon_{abc}\left(D^{n_1}\tfrac{1}{2}(1\pm\gamma_0)\psi\right)^a\left(D^{n_2}\tfrac{1}{2}(1\pm\gamma_0)\psi\right)^b\left(D^{n_3}\tfrac{1}{2}(1\pm\gamma_0)\psi\right)^c$$

PRD84 074508 (2011)
PRD85 054016 (2012)

- subtract the 'quark mass' contribution

$$m_0^{\text{mes}} = m_\rho$$
$$m_0^{\text{bar}} = m_N$$
$$m_0^{c\bar{c}} = m_{\eta_c}$$



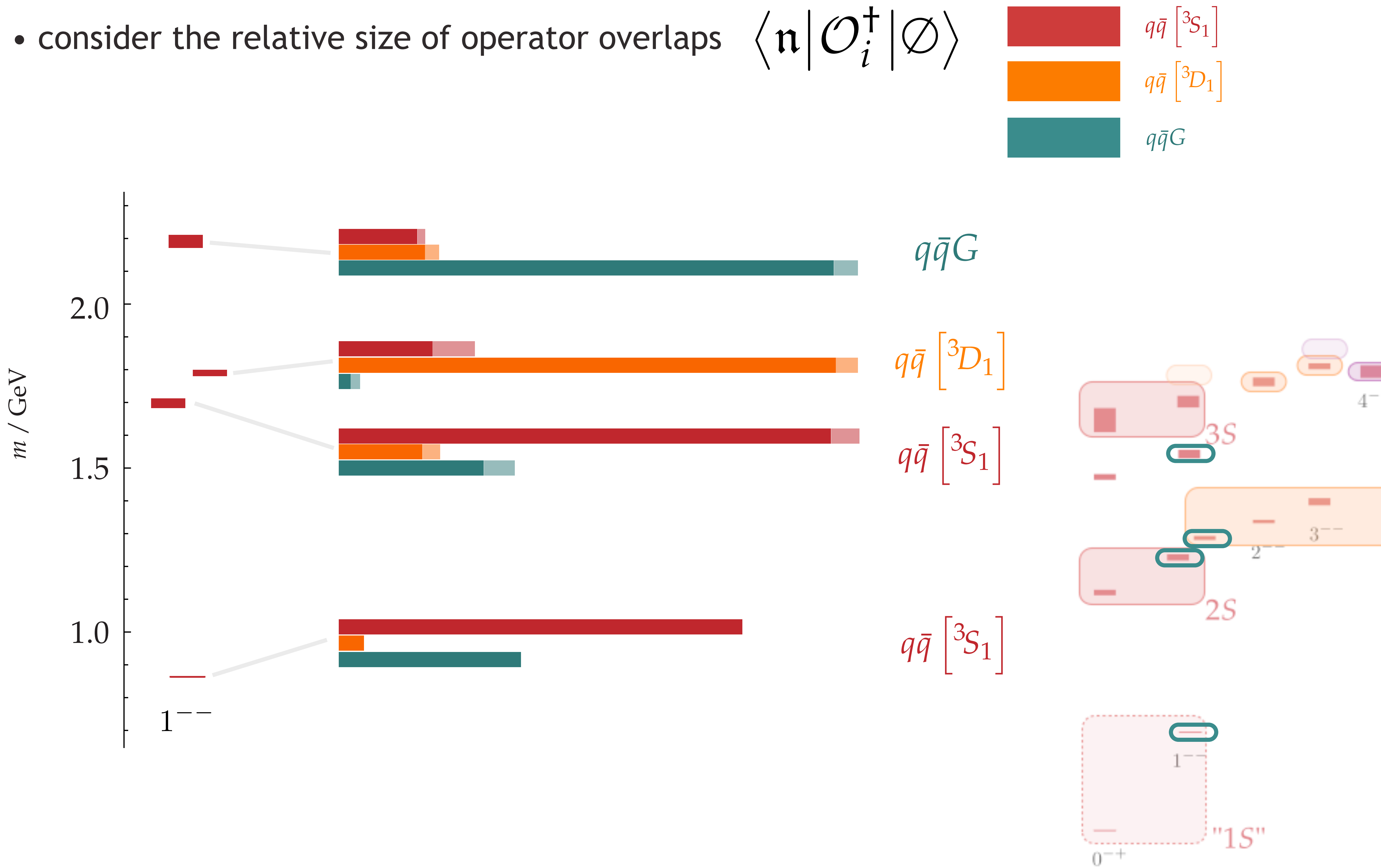
SU(3)F point

common energy scale of chromomagnetic gluonic excitation $\sim 1.3\,\text{GeV}$

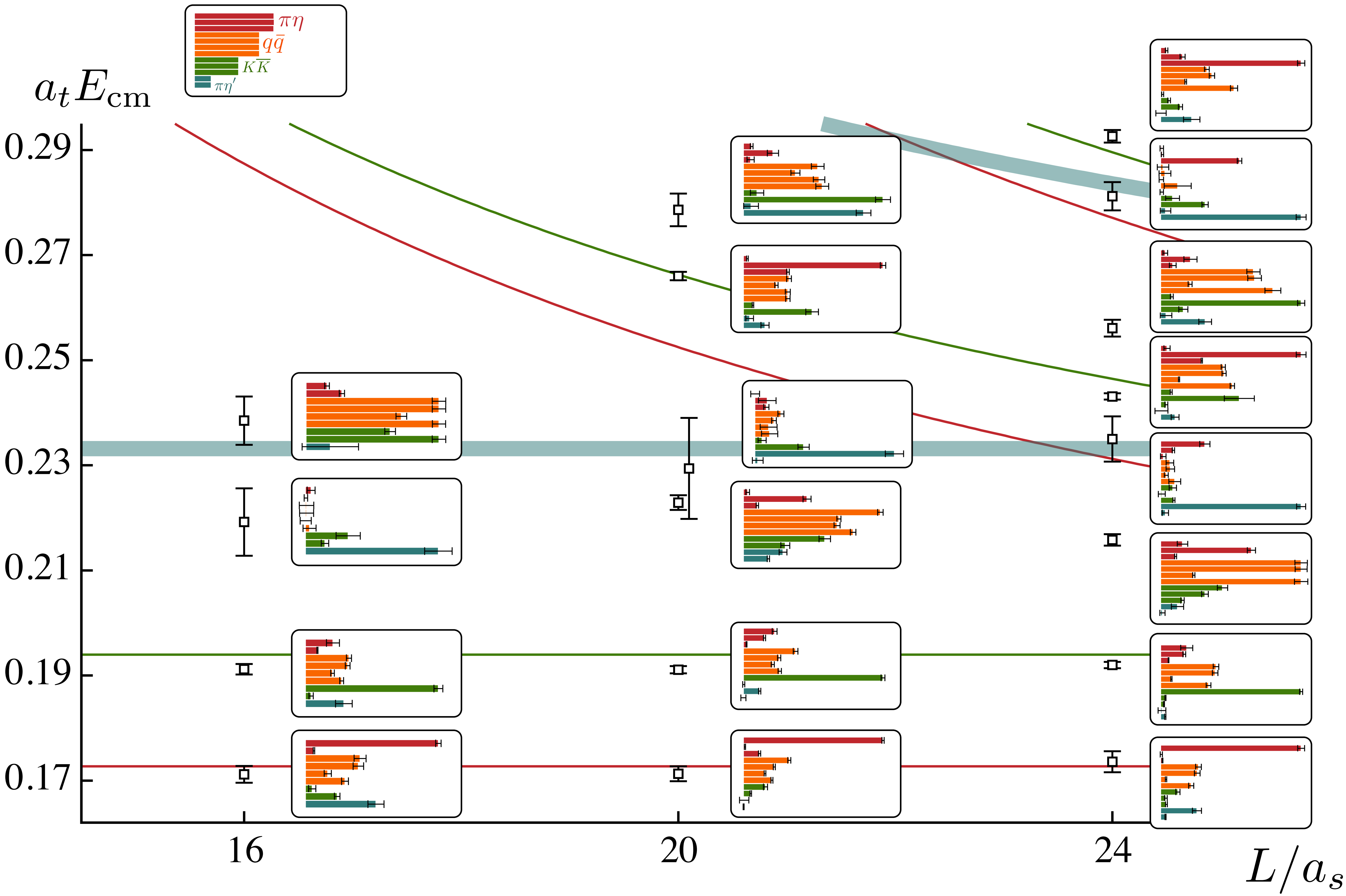
lowest gluonic excitation in QCD now determined ?

1^{--} operator overlaps

- consider the relative size of operator overlaps $\langle \mathfrak{n} | \mathcal{O}_i^\dagger | \varnothing \rangle$

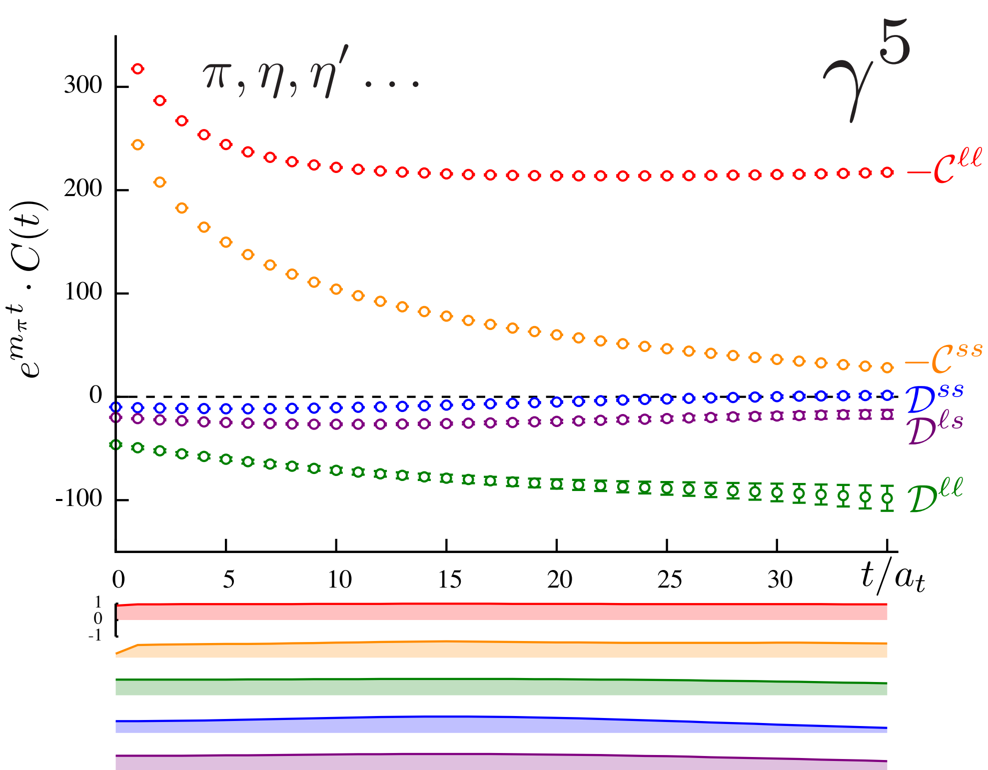


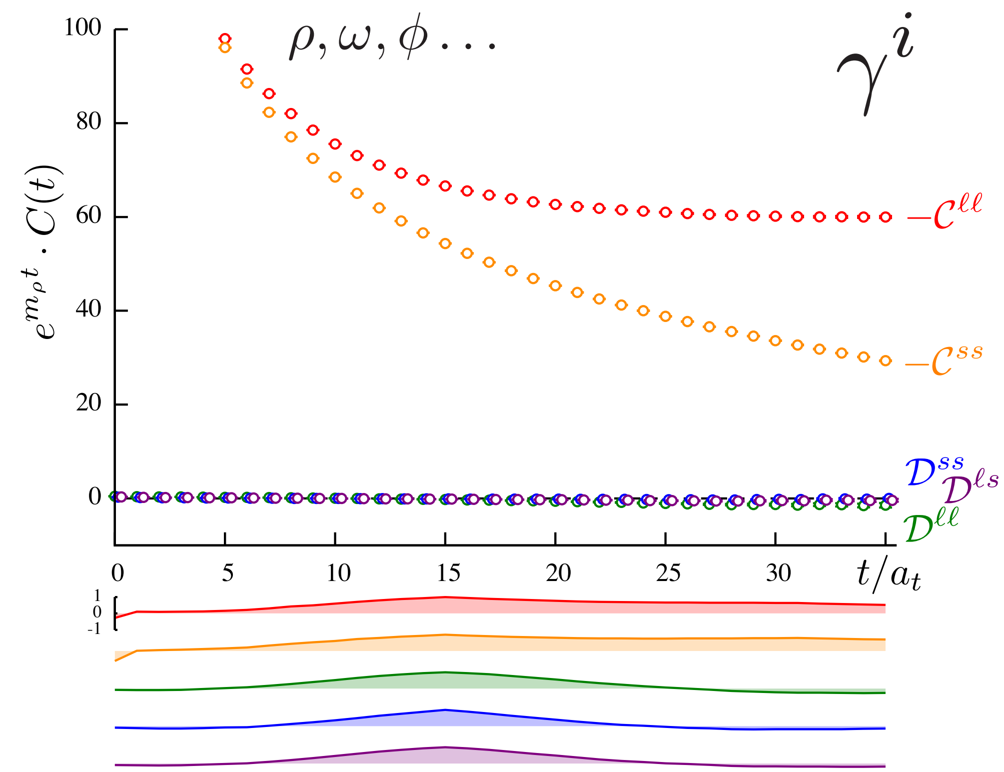
operator overlaps — $\pi\eta / K\bar{K}$

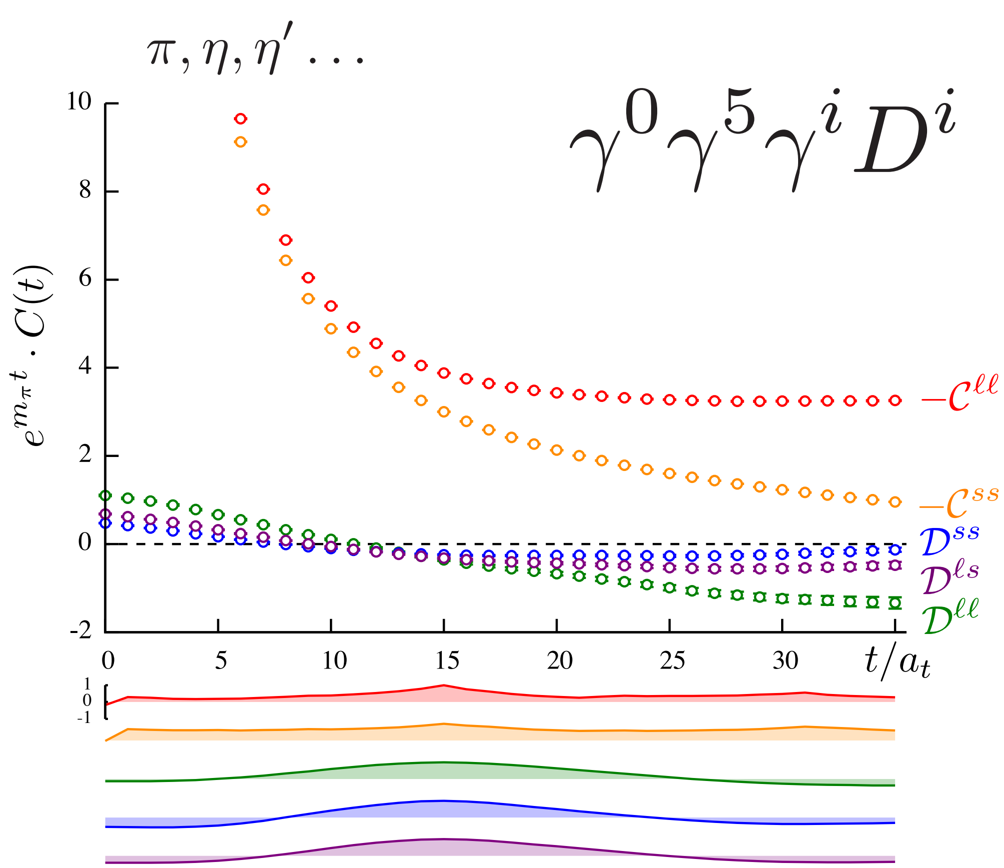


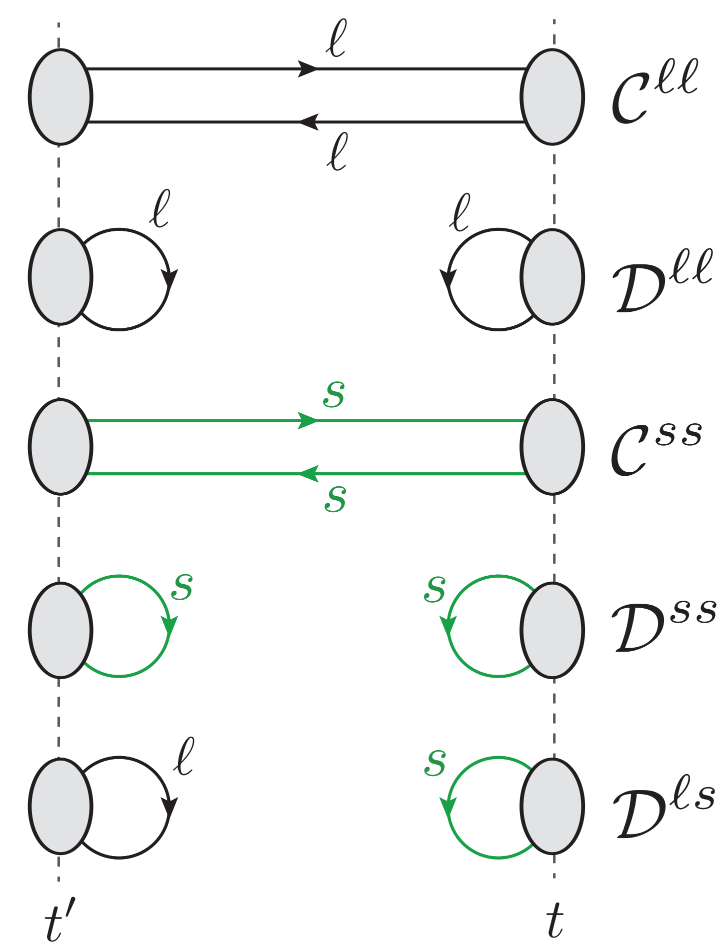
‘annihilation’ contributions

PRD 88 094505 (2013)

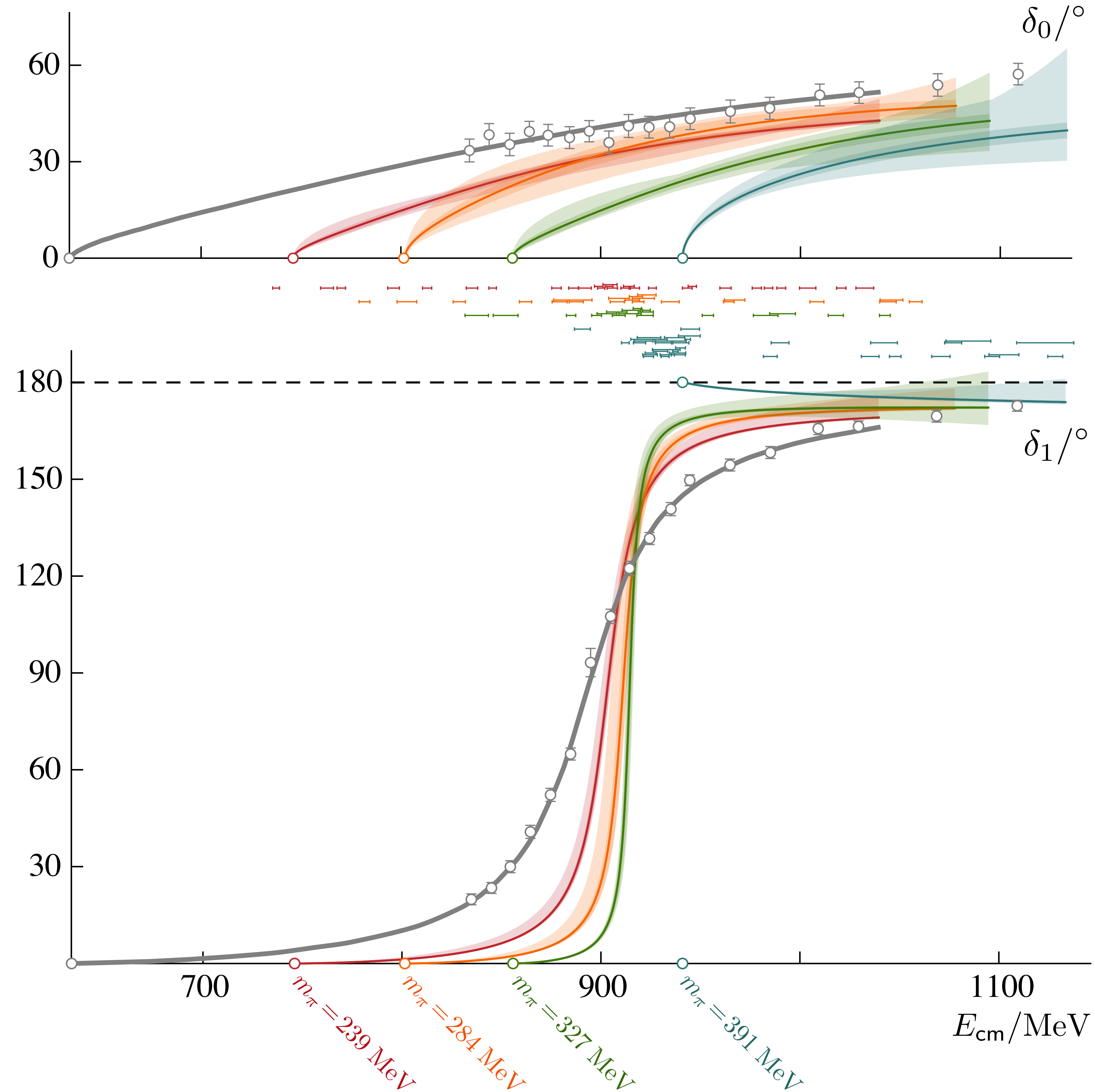








πK elastic scattering at four light quark masses



m_π ~

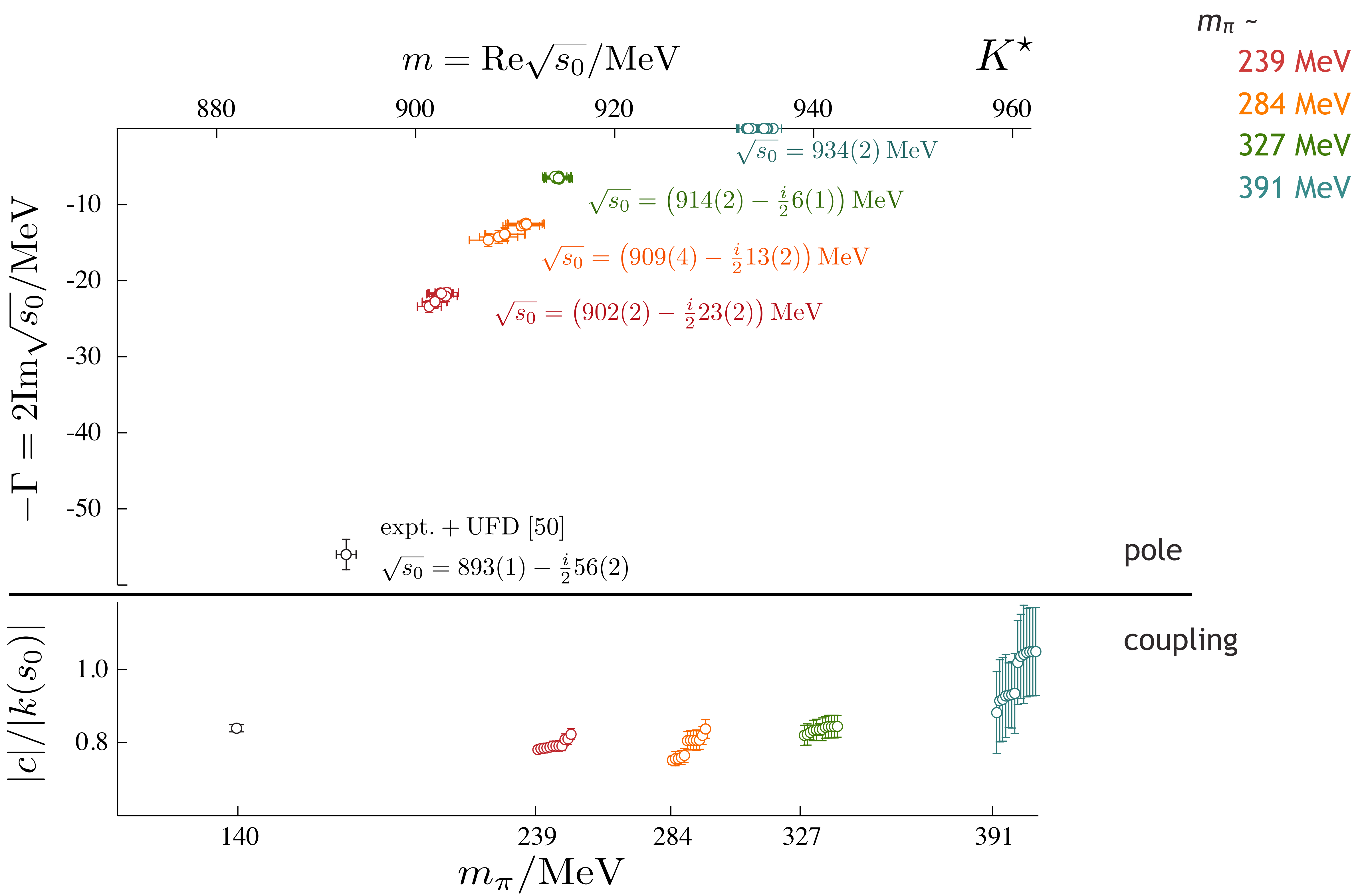
239 MeV

284 MeV

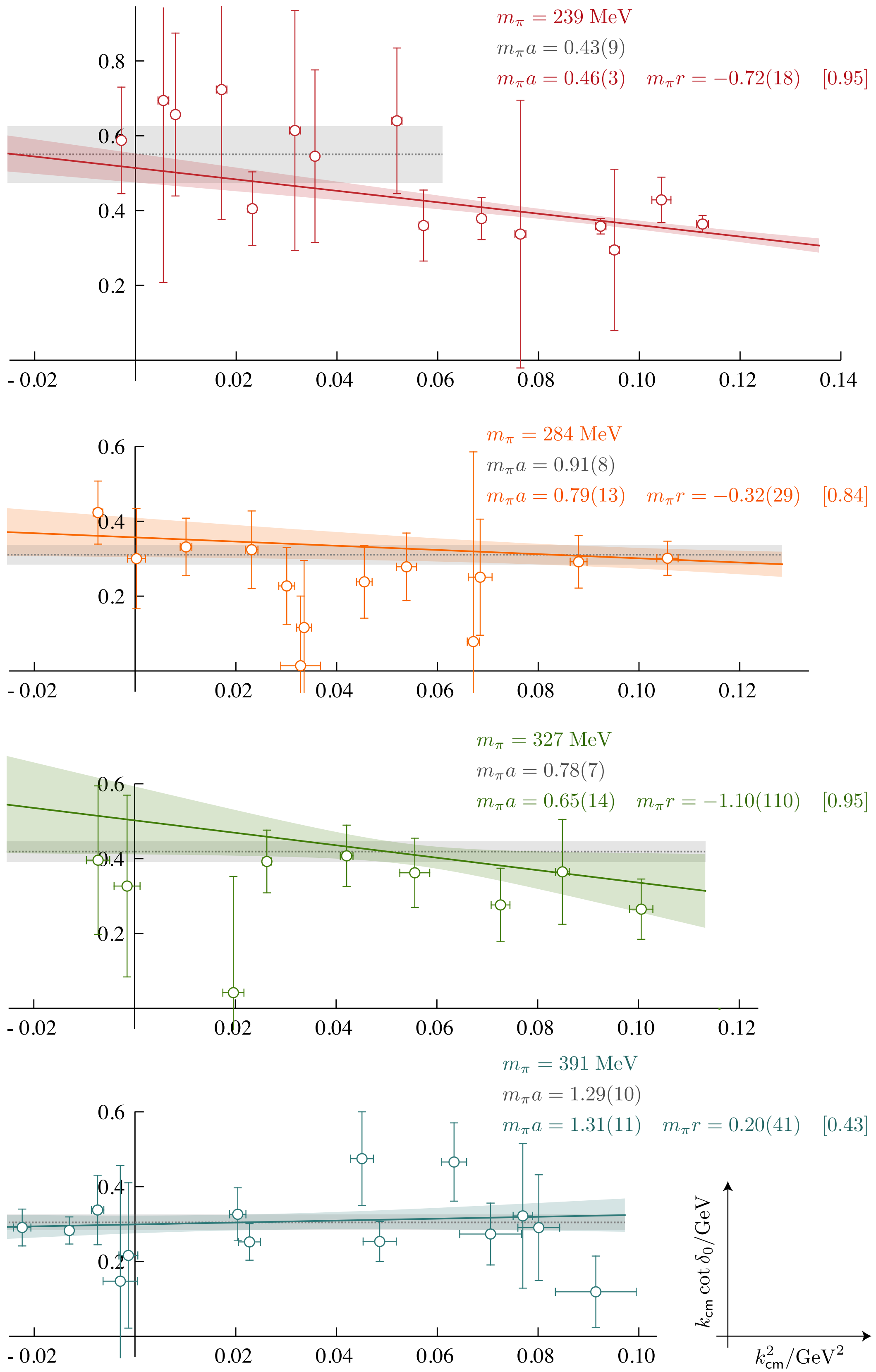
327 MeV

391 MeV

πK elastic scattering at four light quark masses

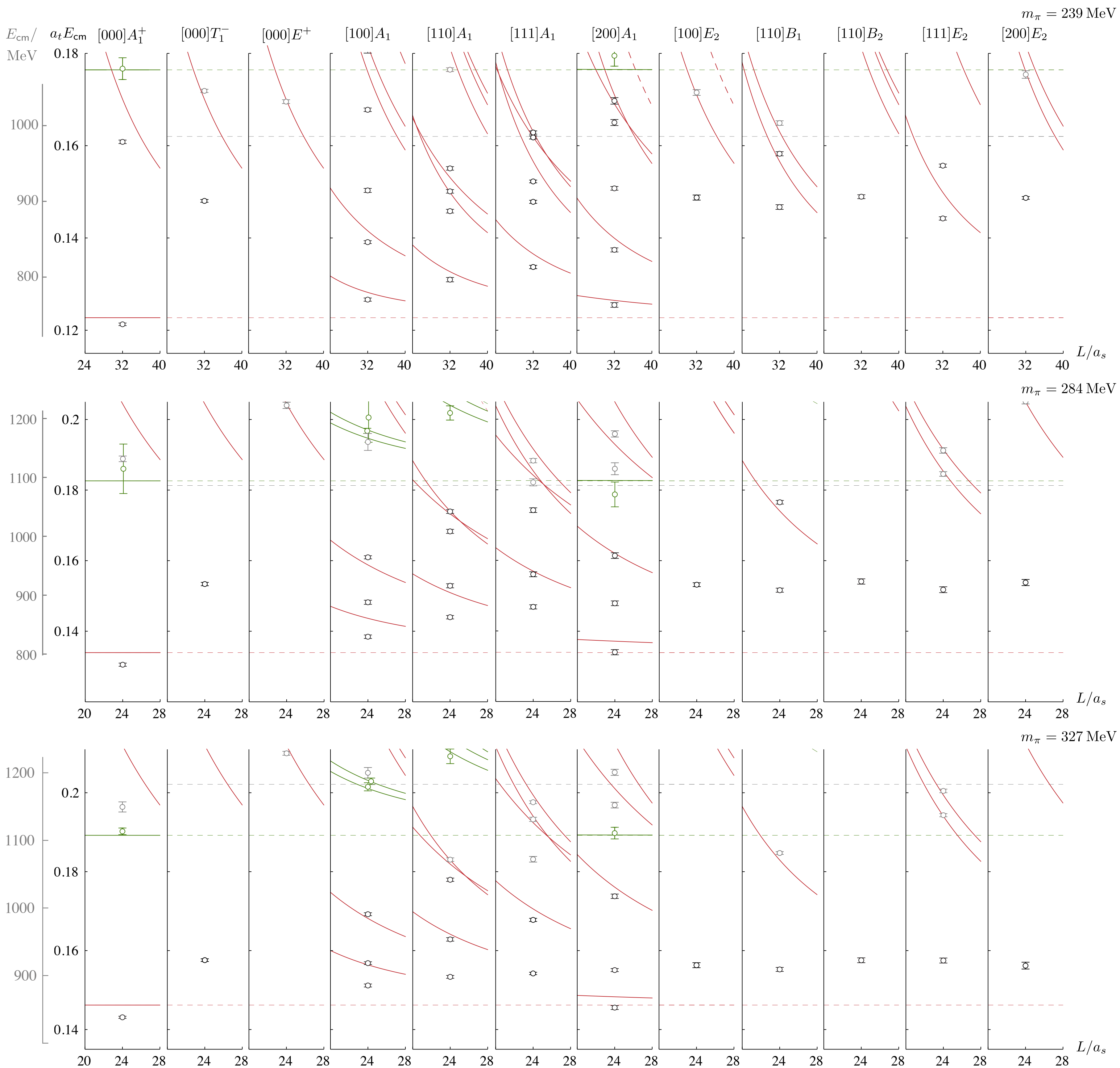


πK elastic scattering at four light quark masses

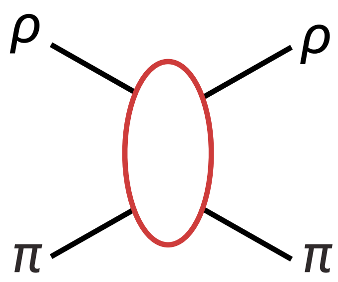


m_π ~
239 MeV
284 MeV
327 MeV
391 MeV

πK elastic scattering at four light quark masses



spin-scattering – $\pi\rho$ isospin=2



ρ has J=1

more partial wave possibilities from coupling ρ-spin to orbital angular momentum

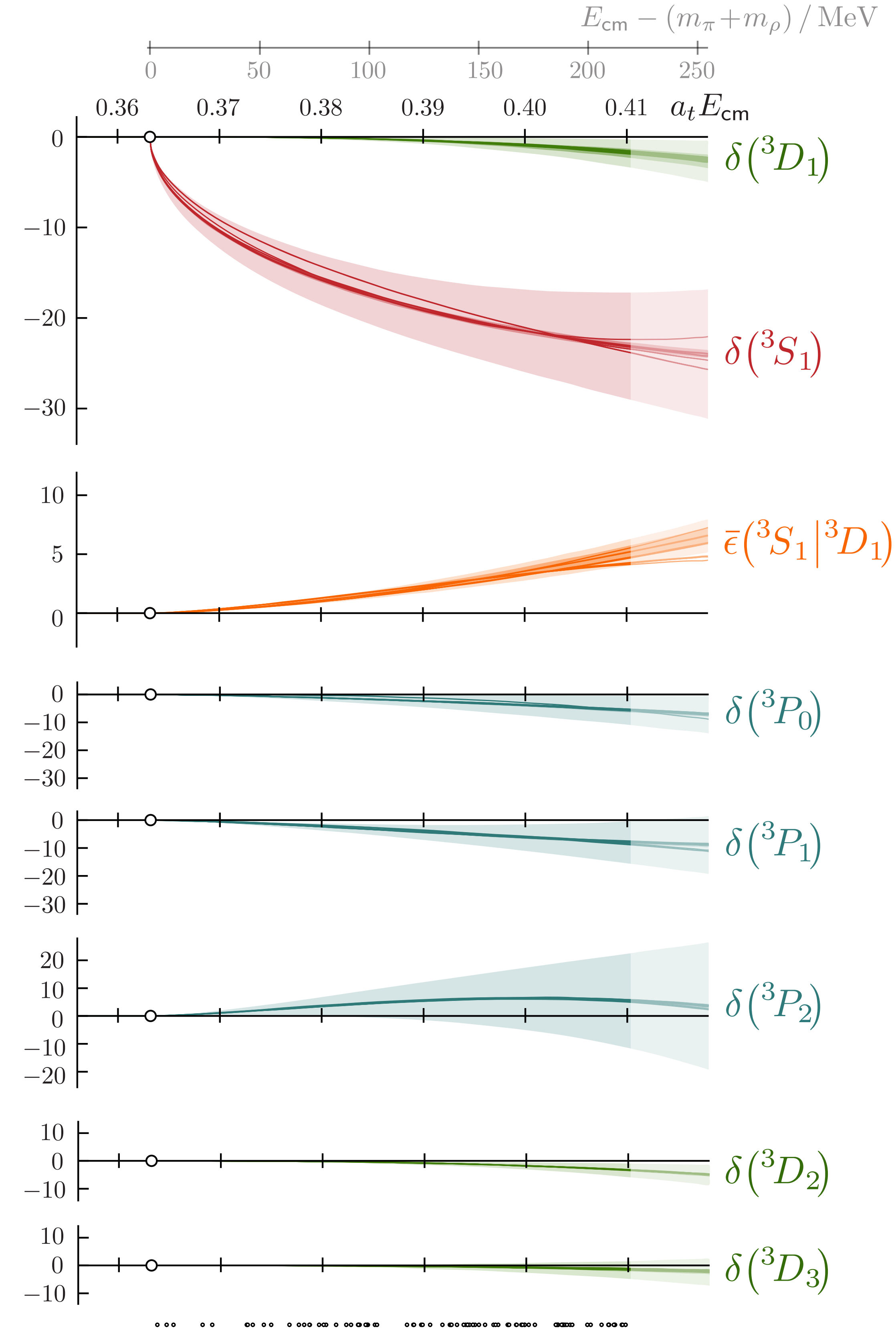
rather heavy quarks here (m_u=m_d=m_s)

m_π~700 MeV, m_ρ~1020 MeV

this ρ is stable against decay

coupled $^3S_1/^3D_1$ scattering system

$$\mathbf{S} = \begin{pmatrix} \cos 2\bar{\epsilon}\; e^{2i\delta_S} & i\sin 2\bar{\epsilon}\; e^{i(\delta_S+\delta_D)} \\ i\sin 2\bar{\epsilon}\; e^{i(\delta_S+\delta_D)} & \cos 2\bar{\epsilon}\; e^{2i\delta_D} \end{pmatrix}$$



tetraquark operators

generic local diquark operator

$$\delta^{J[\Gamma]}_{RF} = \langle \mathbf{3} r_a; \mathbf{3} r_b | R r\rangle \langle F_a f_a; F_b f_b | F f\rangle \, q^T_{r_a f_a} (C\Gamma)\, q_{r_b f_b}$$

color reps. $R = \bar{\mathbf{3}}, \mathbf{6}$

spins $J^P = 0^\pm, 1^\pm$

no assumptions made at this point about good/bad diquarks

generic local tetraquark operator

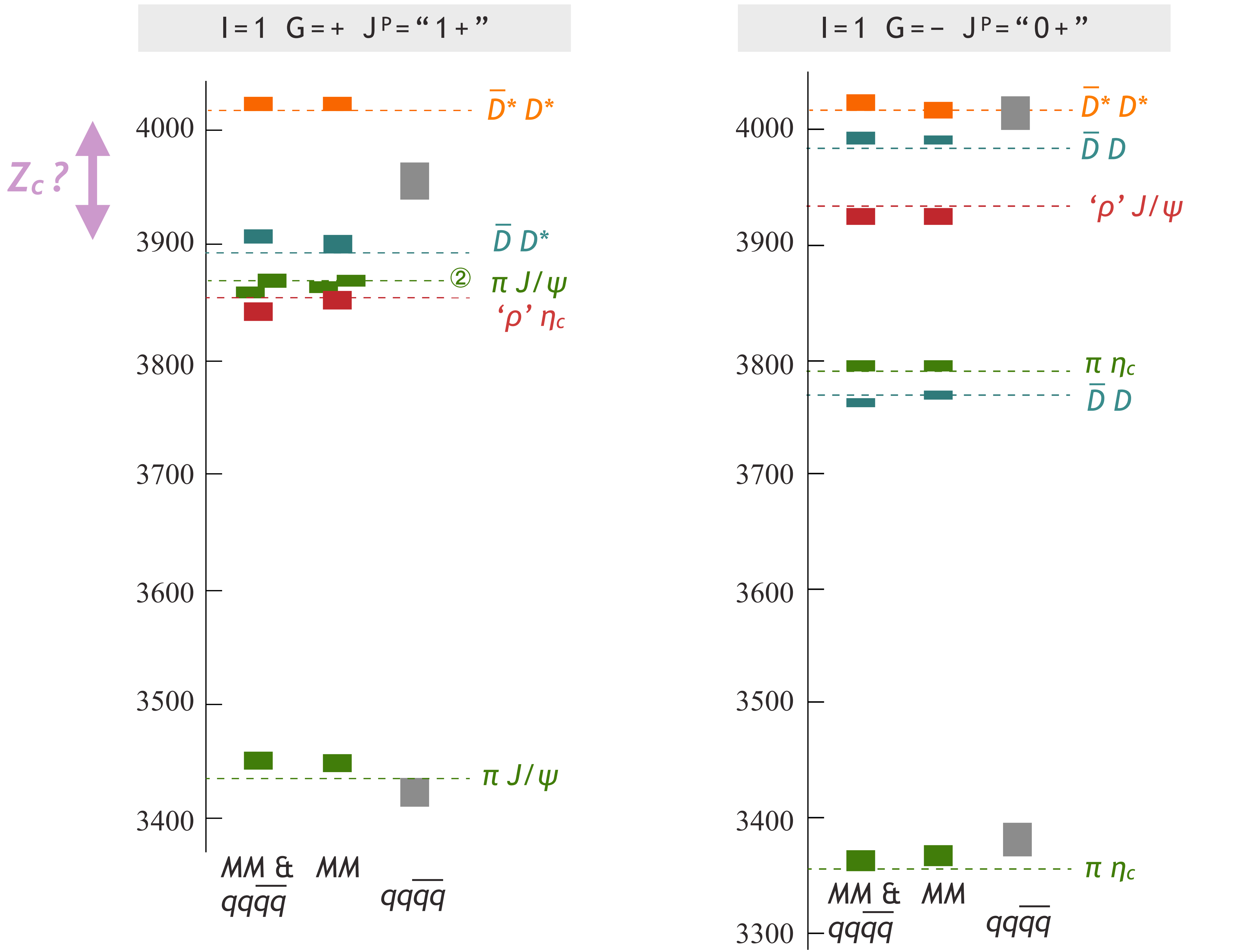
$$\mathcal{T}^{J[\Gamma_1\Gamma_2]}_{\mathbf{1}[R_1 R_2]\, F[F_1 F_2]} = \langle J_1 m_1; J_2 m_2 | J m\rangle \langle R_1 r_1; R_2 r_2 | \mathbf{1}\rangle \langle F_1 f_1; F_2 f_2 | F f\rangle \, \delta^{J_1[\Gamma_1]}_{R_1 F_1} \, \bar{\delta}^{J_2[\Gamma_2]}_{R_2 F_2}$$

(+ *C*/*G*-parity symmetrisation ...)

spins $J \leq 2$

smeared quark fields, but otherwise **local**,
certainly not sampling the whole lattice volume

(diquark construction just makes fermion antisymmetry manifest)

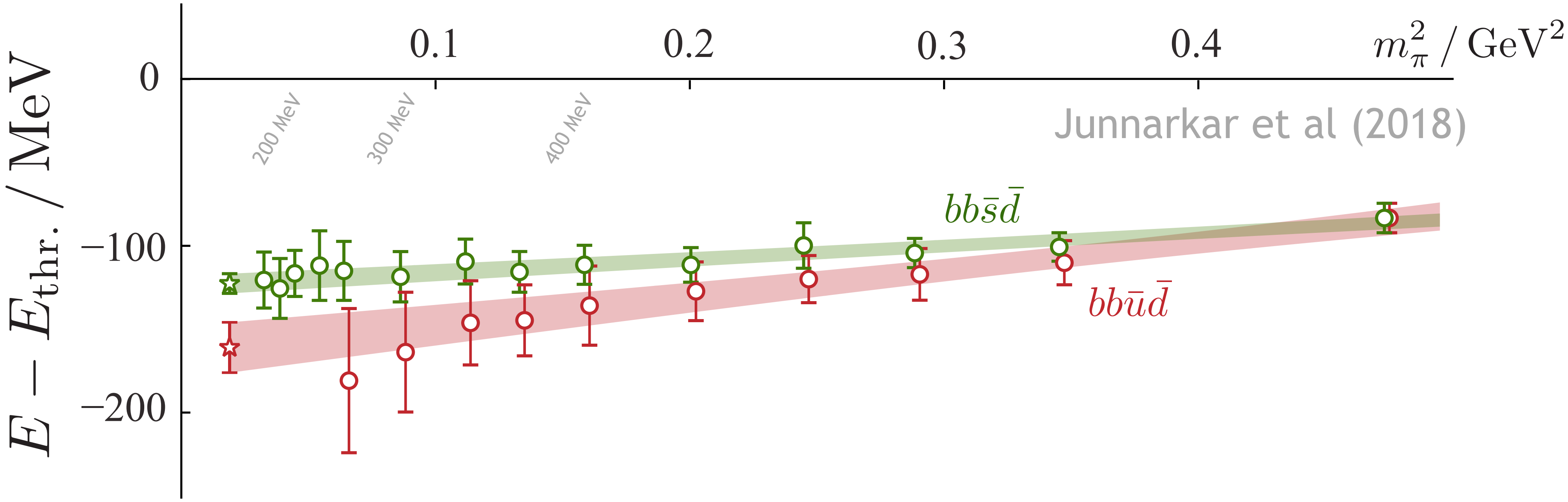

I = 1 G = + J^P = "1+"
I = 1 G = − J^P = "0+"
Zc ?
D̄* D*
D̄ D*
π J/ψ
'ρ' ηc
D̄ D
'ρ' J/ψ
π ηc
4000
3900
3800
3700
3600
3500
3400
3300
MM & qqq̄q̄
MM
qqq̄q̄

truly flavor-exotic hadrons

one recent observation in lattice QCD that has a good chance of being robust:

a double-bottom bound-state $bb\bar{u}\bar{d}$ (I=0, J^P=1$^+$, lying well below $B\,B^*$ threshold)

and probably a strange partner

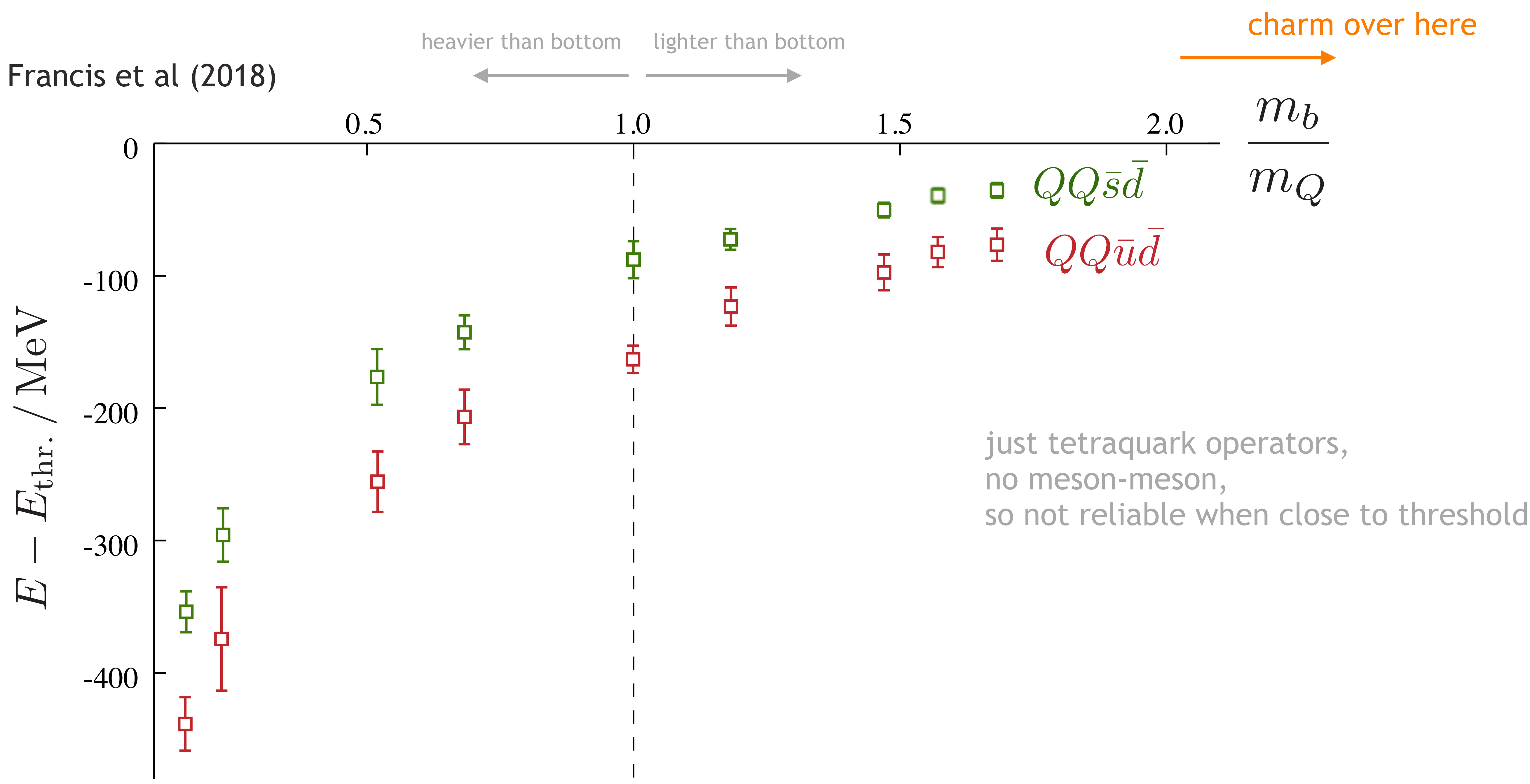


Francis et al (2017)
Junnarkar et al (2018)
Leskovec et al (2018)

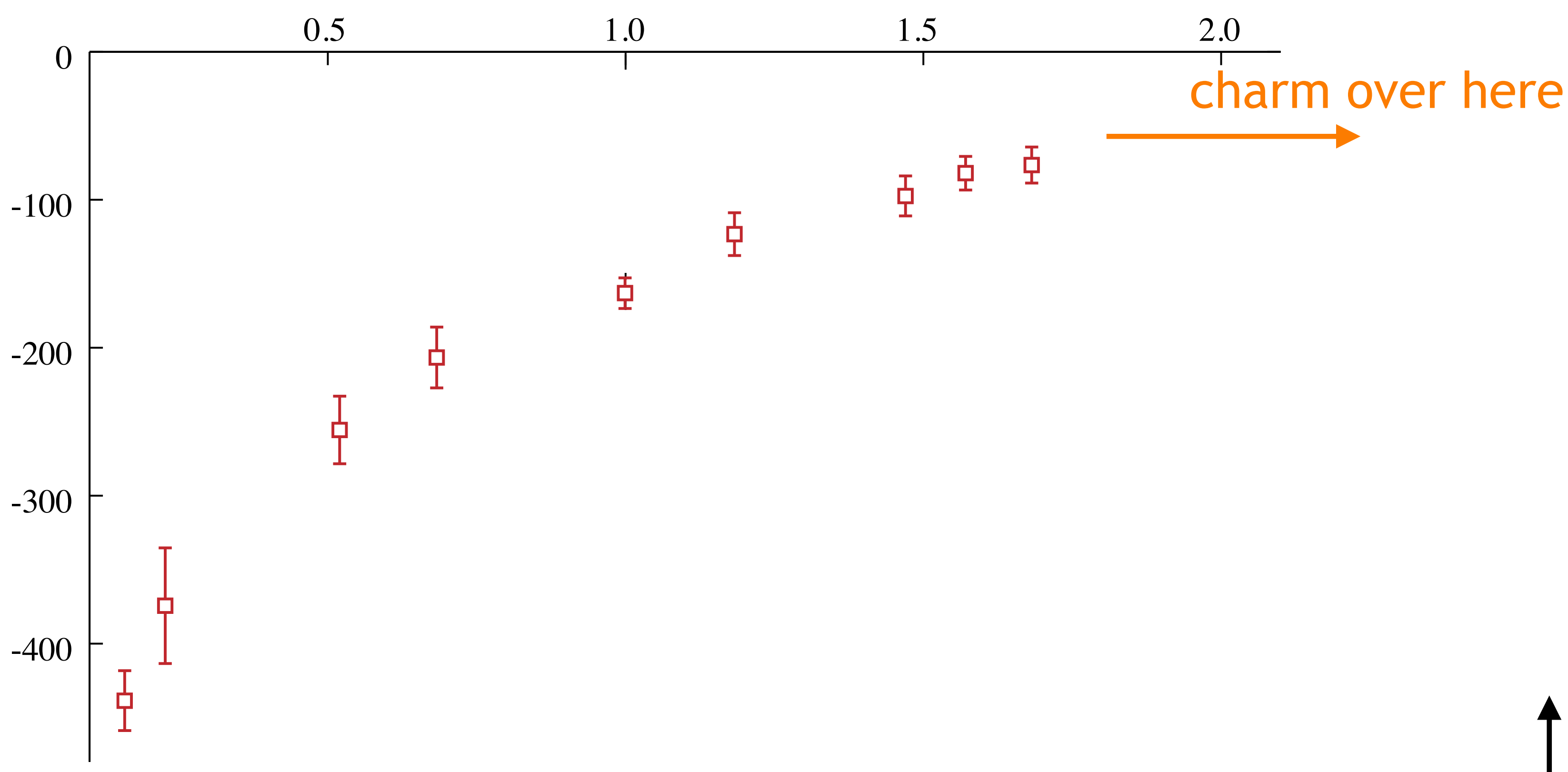
$$\left|\Delta E(bb\bar{u}\bar{d})\right| \sim 100 - 200\,\mathrm{MeV}$$
$$\left|\Delta E(bb\bar{s}\bar{d})\right| \sim 90 - 120\,\mathrm{MeV}$$

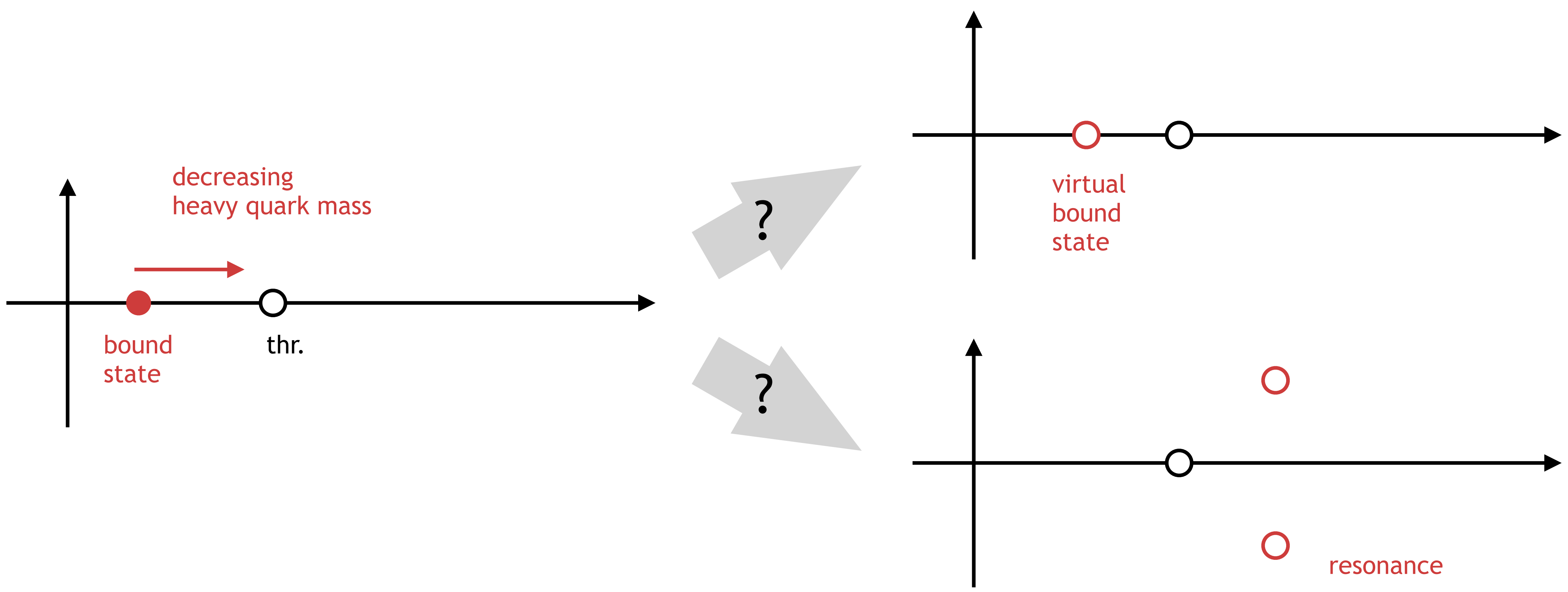
see Eichten & Quigg (2017) for heavy quark symmetry argument

binding energy with changing heavy quark mass



what happens for $cc\overline{ud}$?

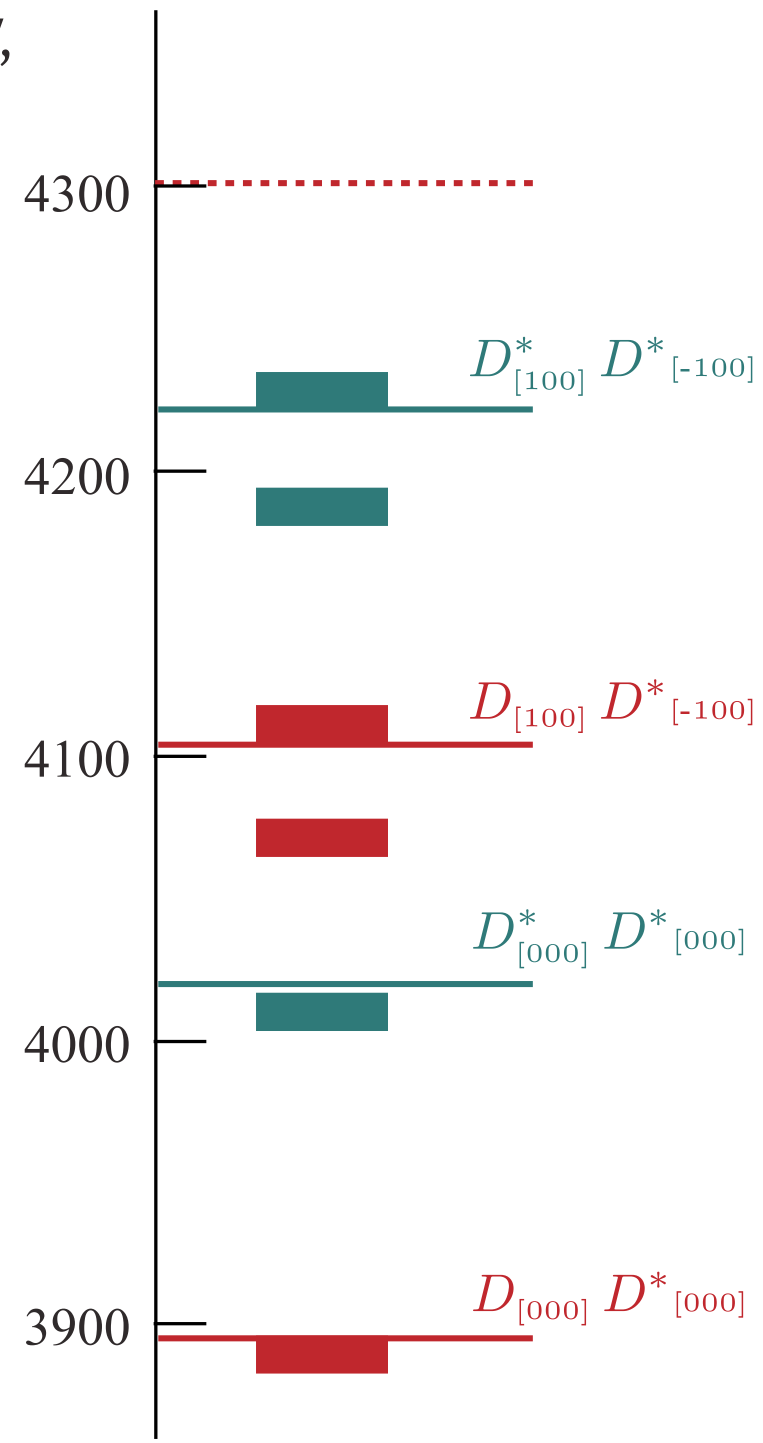




direct calculation of $cc\bar{u}\bar{d}$

tetraquark operators
& meson-meson operators

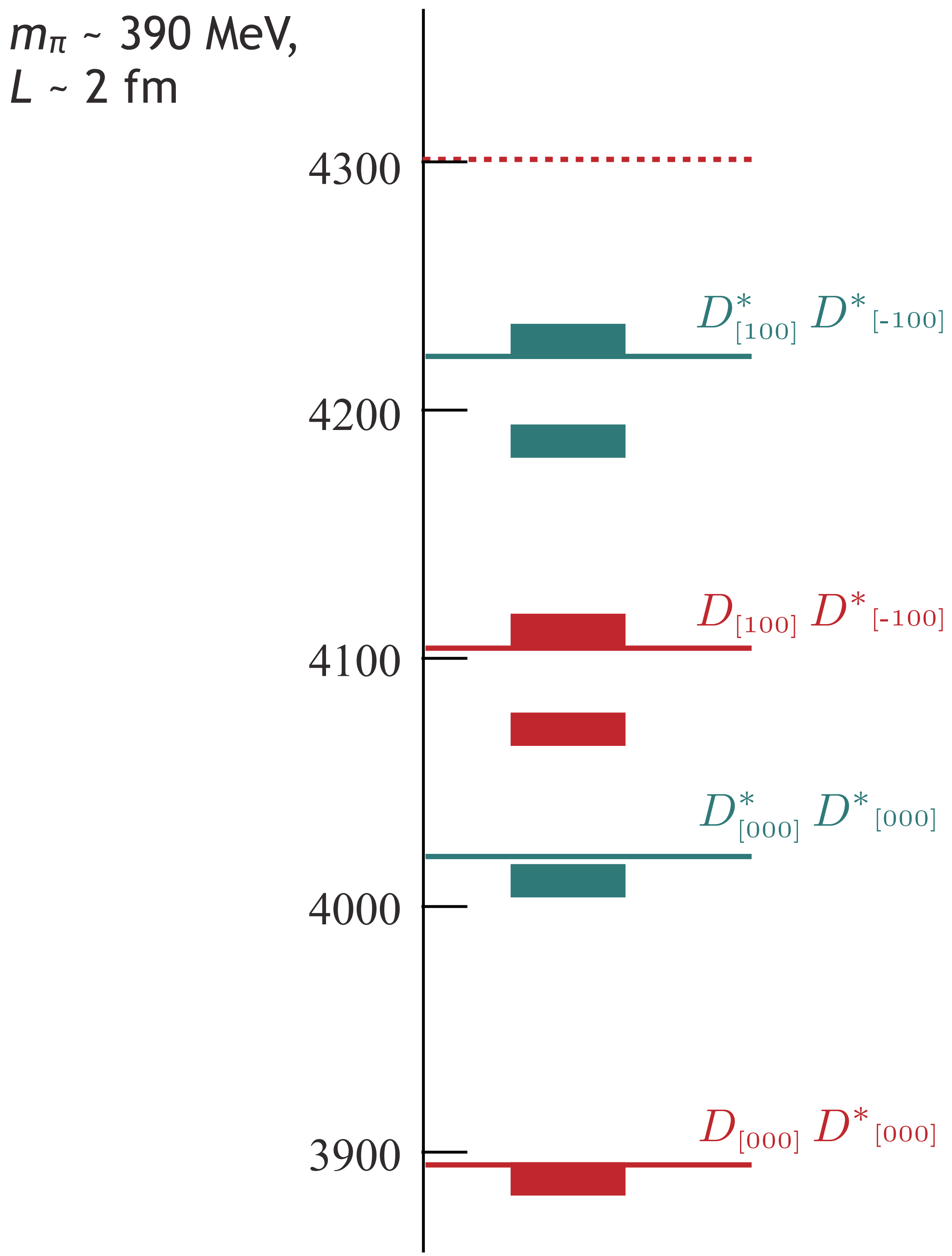
m_π ~ 390 MeV,
L ~ 2 fm



no obvious sign of a narrow resonance …

direct calculation of $cc\bar{u}\bar{d}$

tetraquark operators
& meson-meson operators



just tetraquark ops

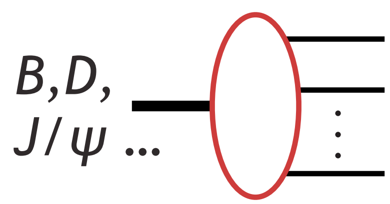
completely misleading spectrum

BE CAREFUL WITH SMALL OPERATOR BASES !

producing meson resonances

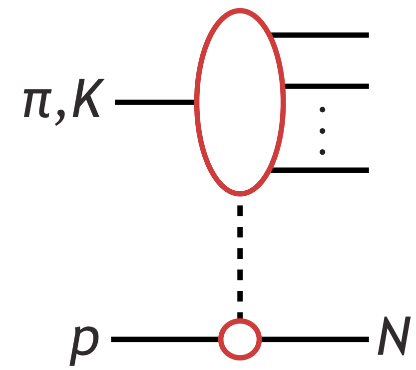
some example processes:

heavy flavour decays



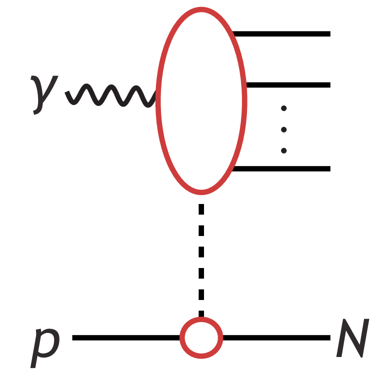
e.g. LHCb

peripheral meson hadroproduction



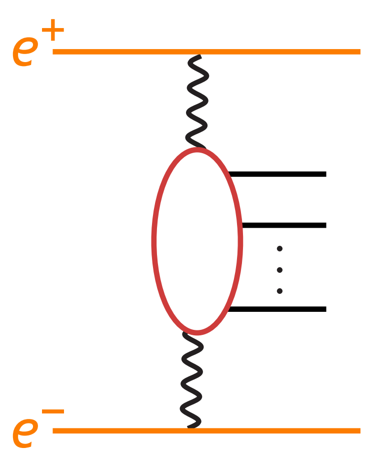
e.g. COMPASS

peripheral meson photoproduction



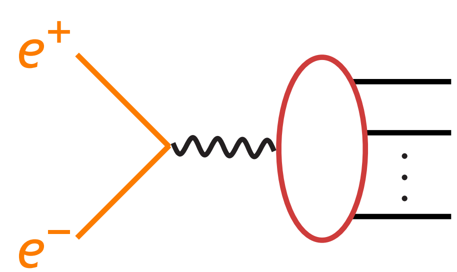
e.g. GlueX

two photon fusion



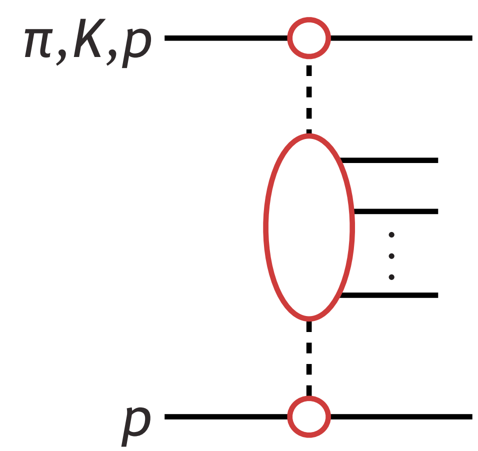
e.g. Belle

e^+e^- annihilation



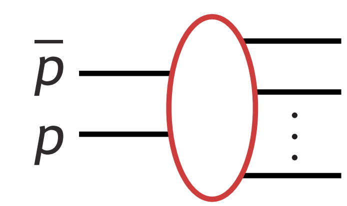
e.g. BES III

central production



e.g. WA102

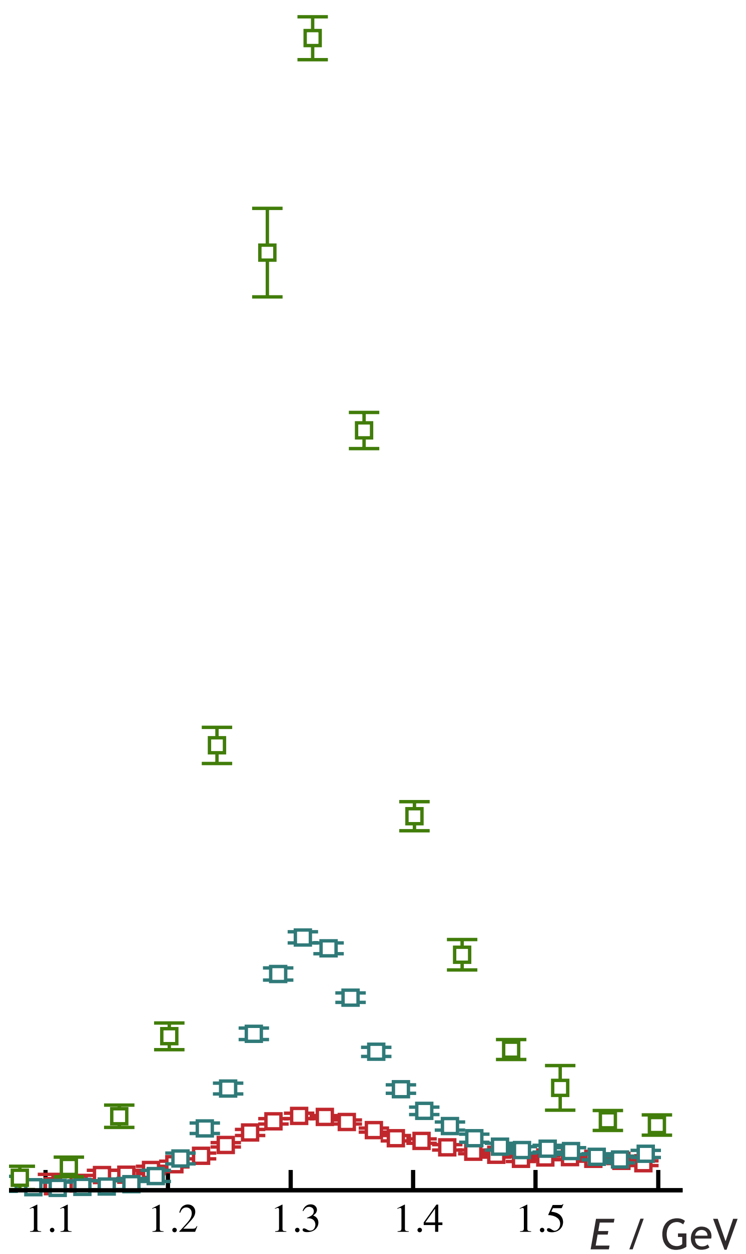
$p\bar{p}$ annihilation



e.g. Crystal Barrel

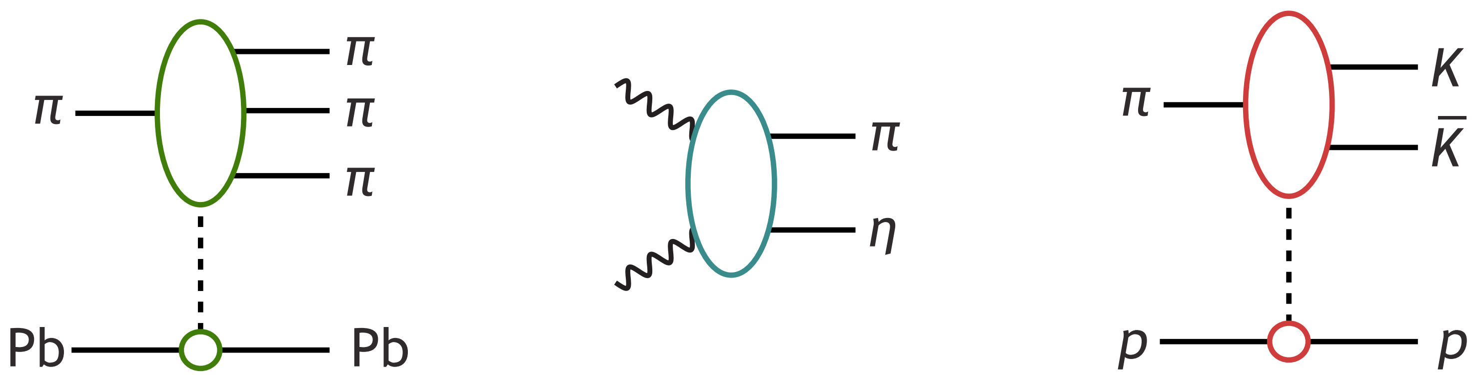
many decades of accumulated data …

'straightforward' coupled-channel resonances



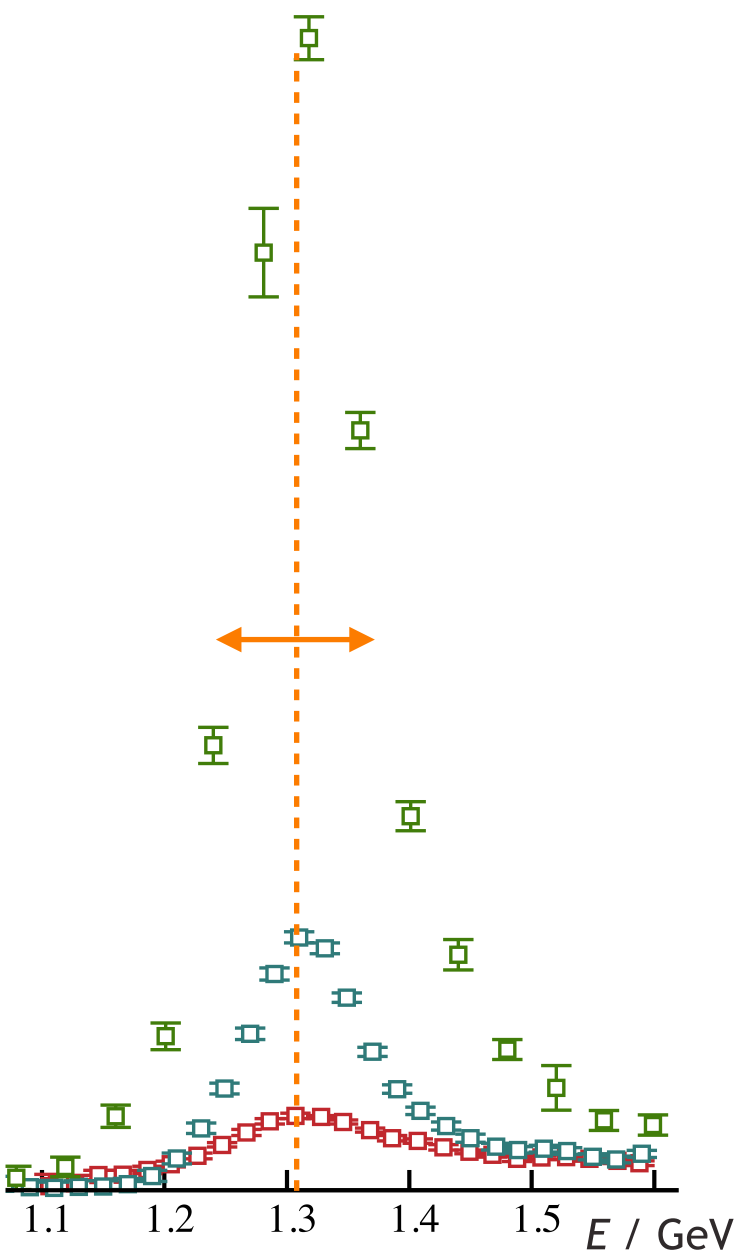
$\pi\,\mathrm{Pb} \to \pi\rho\,\mathrm{Pb}$	COMPASS
$\gamma\gamma \to \pi\eta$	Belle
$\pi p \to K\overline{K}\,p$	CERN SPS

same 'bump' appears in multiple different processes



‘straightforward’ coupled-channel resonances

$a_2(1320)$



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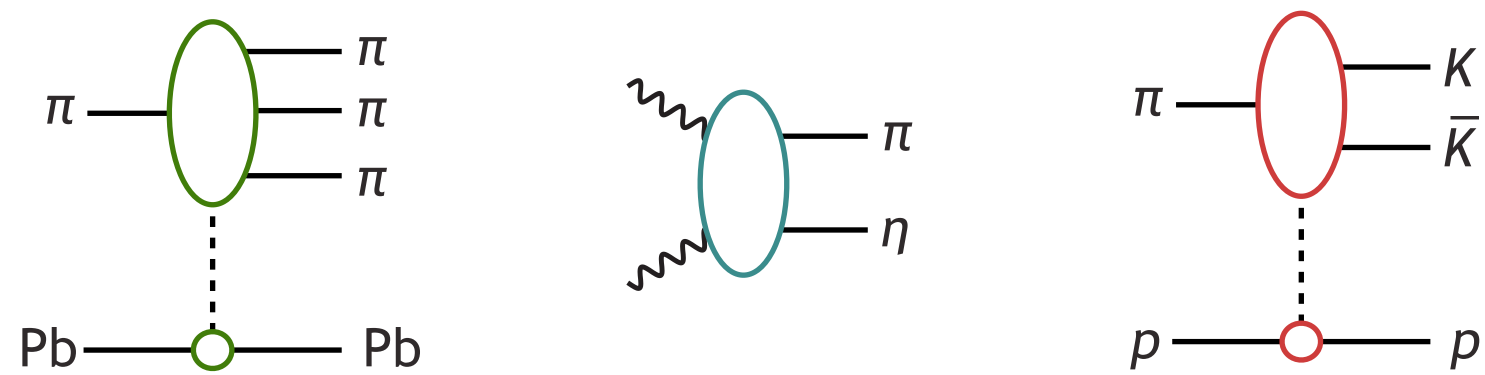
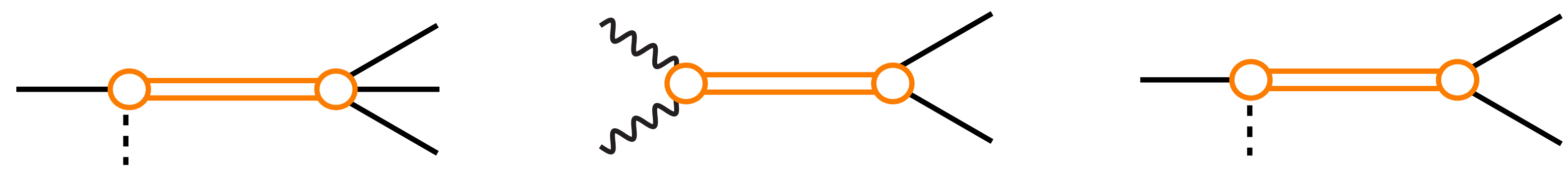


… due to same **a_2 resonance**



pdg summary entry

$a_2(1320)$ $\qquad I^G(J^{PC}) = 1^-(2^{++})$

Mass $m = 1318.3^{+0.5}_{-0.6}$ MeV
Full width $\Gamma = 107 \pm 5$ MeV

$a_2(1320)$ DECAY MODES	Fraction (Γ_i/Γ)
3π	(70.1 ± 2.7) %
$\eta\pi$	(14.5 ± 1.2) %
$\omega\pi\pi$	(10.6 ± 3.2) %
$K\overline{K}$	(4.9 ± 0.8) %
$\eta'(958)\pi$	$(5.5 \pm 0.9) \times 10^{-3}$
$\pi^\pm\gamma$	$(2.91 \pm 0.27) \times 10^{-3}$
$\gamma\gamma$	$(9.4 \pm 0.7) \times 10^{-6}$

the experimental excited meson spectrum

pdg meson listings

LIGHT UNFLAVORED ($S = C = B = 0$)

	$I^G(J^{PC})$		$I^G(J^{PC})$
• $\pi^\pm$	$1^-(0^-)$	• $\rho_3(1690)$	$1^+(3^{--})$
• π^0	$1^-(0^{-+})$	• $\rho(1700)$	$1^+(1^{--})$
• η	$0^+(0^{-+})$	$a_2(1700)$	$1^-(2^{++})$
• $f_0(500)$	$0^+(0^{++})$	• $f_0(1710)$	$0^+(0^{++})$
• $\rho(770)$	$1^+(1^{--})$	$\eta(1760)$	$0^+(0^{-+})$
• $\omega(782)$	$0^-(1^{--})$	• $\pi(1800)$	$1^-(0^{-+})$
• $\eta'(958)$	$0^+(0^{-+})$	$f_2(1810)$	$0^+(2^{++})$
• $f_0(980)$	$0^+(0^{++})$	$X(1835)$	$?^?(0^{-+})$
• $a_0(980)$	$1^-(0^{++})$	$X(1840)$	$?^?(?^{??})$
• $\phi(1020)$	$0^-(1^{--})$	$a_1(1420)$	$1^-(1^{++})$
• $h_1(1170)$	$0^-(1^{+-})$	• $\phi_3(1850)$	$0^-(3^{--})$
• $b_1(1235)$	$1^+(1^{+-})$	$\eta_2(1870)$	$0^+(2^{-+})$
• $a_1(1260)$	$1^-(1^{++})$	• $\pi_2(1880)$	$1^-(2^{-+})$
• $f_2(1270)$	$0^+(2^{++})$	$\rho(1900)$	$1^+(1^{--})$
• $f_1(1285)$	$0^+(1^{++})$	$f_2(1910)$	$0^+(2^{++})$
• $\eta(1295)$	$0^+(0^{-+})$	$a_0(1950)$	$1^-(0^{++})$
• $\pi(1300)$	$1^-(0^{-+})$	• $f_2(1950)$	$0^+(2^{++})$
• $a_2(1320)$	$1^-(2^{++})$	$\rho_3(1990)$	$1^+(3^{--})$
• $f_0(1370)$	$0^+(0^{++})$	• $f_2(2010)$	$0^+(2^{++})$
$h_1(1380)$	$?^-(1^{+-})$	$f_0(2020)$	$0^+(0^{++})$
• $\pi_1(1400)$	$1^-(1^{-+})$	• $a_4(2040)$	$1^-(4^{++})$
• $\eta(1405)$	$0^+(0^{-+})$	• $f_4(2050)$	$0^+(4^{++})$
• $f_1(1420)$	$0^+(1^{++})$	$\pi_2(2100)$	$1^-(2^{-+})$
• $\omega(1420)$	$0^-(1^{--})$	$f_0(2100)$	$0^+(0^{++})$
$f_2(1430)$	$0^+(2^{++})$	$f_2(2150)$	$0^+(2^{++})$
• $a_0(1450)$	$1^-(0^{++})$	$\rho(2150)$	$1^+(1^{--})$

STRANGE ($S = \pm1$, $C = B = 0$)

	$I(J^P)$
• $K^\pm$	$1/2(0^-)$
• K^0	$1/2(0^-)$
• K^0_S	$1/2(0^-)$
• K^0_L	$1/2(0^-)$
$K^*_0(800)$	$1/2(0^+)$
• $K^*(892)$	$1/2(1^-)$
• $K_1(1270)$	$1/2(1^+)$
• $K_1(1400)$	$1/2(1^+)$
• $K^*(1410)$	$1/2(1^-)$
• $K^*_0(1430)$	$1/2(0^+)$
• $K^*_2(1430)$	$1/2(2^+)$
$K(1460)$	$1/2(0^-)$
$K_2(1580)$	$1/2(2^-)$
$K(1630)$	$1/2(?^?)$
$K_1(1650)$	$1/2(1^+)$
• $K^*(1680)$	$1/2(1^-)$
• $K_2(1770)$	$1/2(2^-)$
• $K^*_3(1780)$	$1/2(3^-)$
• $K_2(1820)$	$1/2(2^-)$
$K(1830)$	$1/2(0^-)$
$K^*_0(1950)$	$1/2(0^+)$
$K^*_2(1980)$	$1/2(2^+)$
• $K^*_4(2045)$	$1/2(4^+)$
$K_2(2250)$	$1/2(2^-)$
$K_3(2320)$	$1/2(3^+)$

CHARMED, STRANGE ($C = S = \pm1$)

	$I(J^P)$
• $D^\pm_s$	$0(0^-)$
• $D^{*\pm}_s$	$0(?^?)$
• $D^*_{s0}(2317)^\pm$	$0(0^+)$
• $D_{s1}(2460)^\pm$	$0(1^+)$
• $D_{s1}(2536)^\pm$	$0(1^+)$
• $D_{s2}(2573)$	$0(2^+)$
• $D^*_{s1}(2700)^\pm$	$0(1^-)$
$D^*_{s1}(2860)^\pm$	$0(1^-)$
$D^*_{s3}(2860)^\pm$	$0(3^-)$
$D_{sJ}(3040)^\pm$	$0(?^?)$

BOTTOM ($B = \pm1$)

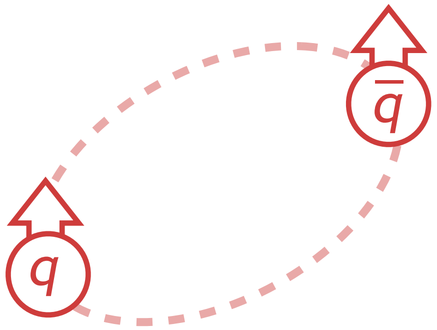
• $B^\pm$	$1/2(0^-)$
• B^0	$1/2(0^-)$
• $B^\pm/B^0$ ADMIXTURE	
• $B^\pm/B^0/B^0_s/b$-baryon ADMIXTURE	
V_{cb} and V_{ub} CKM Matrix Elements	
• B^*	$1/2(1^-)$
• $B_1(5721)^+$	$1/2(1^+)$
• $B_1(5721)^0$	$1/2(1^+)$
$B^*_J(5732)$	$?(?^?)$
• $B^*_2(5747)^+$	$1/2(2^+)$
• $B^*_2(5747)^0$	$1/2(2^+)$

$c\bar{c}$

	$I^G(J^{PC})$
• $\eta_c(1S)$	$0^+(0^{-+})$
• $J/\psi(1S)$	$0^-(1^{--})$
• $\chi_{c0}(1P)$	$0^+(0^{++})$
• $\chi_{c1}(1P)$	$0^+(1^{++})$
• $h_c(1P)$	$?^?(1^{+-})$
• $\chi_{c2}(1P)$	$0^+(2^{++})$
• $\eta_c(2S)$	$0^+(0^{-+})$
• $\psi(2S)$	$0^-(1^{--})$
• $\psi(3770)$	$0^-(1^{--})$
• $\psi(3823)$	$?^?(2^{--})$
• $X(3872)$	$0^+(1^{++})$
• $X(3900)$	$1^+(1^{+-})$
• $X(3915)$	$0^+(0/2^{++})$
• $\chi_{c2}(2P)$	$0^+(2^{++})$
$X(3940)$	$?^?(?^{??})$
• $X(4020)$	$1(?^?)$
• $\psi(4040)$	$0^-(1^{--})$
$X(4050)^\pm$	$?(?^?)$
$X(4055)^\pm$	$?(?^?)$
• $X(4140)$	$0^+(1^{++})$
• $\psi(4160)$	$0^-(1^{--})$
$X(4160)$	$?^?(?^{??})$
$X(4200)^\pm$	$?(1^+)$
$X(4230)$	$?^?(1^{--})$
$X(4240)^\pm$	$?^?(0^-)$
$X(4250)^\pm$	$?(?^?)$
• $X(4260)$	$?^?(1^{--})$

pdg.lbl.gov

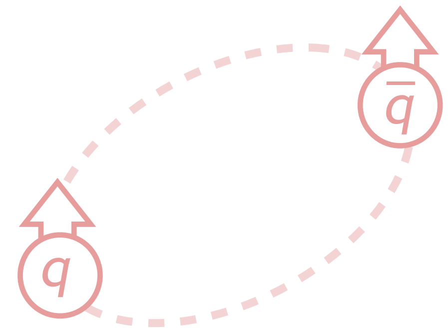
the canonical view of the meson spectrum provided by the $q\bar{q}$ constituent quark model



$^{2S+1}\ell_J$	J^{PC}	light mesons	charmonium
$^1S_0$	0^{-+}	π, η, η'	η_c
$^3S_1$	1^{--}	ρ, ω, ϕ	J/ψ
$^1P_1$	1^{+-}	b_1, h_1	h_c
$^3P_{0,1,2}$	$(0,1,2)^{++}$	a_J, f_J	χ_{cJ}
$^1D_2$	2^{-+}	π_2, η_2	$?$
$^3D_{1,2,3}$	$(1,2,3)^{--}$	$\rho, \rho_3, \omega, \omega_3 \ldots$	ψ'

gets the gross features of the spectrum right ...

but treats excited hadrons as essentially stable the canonical view of the meson spectrum provided by the $q\bar{q}$ constituent quark model



$^{2S+1}\ell_J$	J^{PC}	light mesons	charmonium
$^1S_0$	0^{-+}	π, η, η'	η_c
$^3S_1$	1^{--}	ρ, ω, ϕ	J/ψ
$^1P_1$	1^{+-}	b_1, h_1	h_c
$^3P_{0,1,2}$	$(0,1,2)^{++}$	a_J, f_J	χ_{cJ}
$^1D_2$	2^{-+}	π_2, η_2	?
$^3D_{1,2,3}$	$(1,2,3)^{--}$	$\rho, \rho_3, \omega, \omega_3 \ldots$	ψ'

gets the gross features of the spectrum right ...

but treats excited hadrons as essentially stable

is there more than this ?

why doesn't QCD have meson states where the gluonic field is 'active' ?

glueballs = states where quarks are not required

hybrids = states where quark colour neutralized by gluonic field

lattice QCD can be used to study such speculations rigorously ...

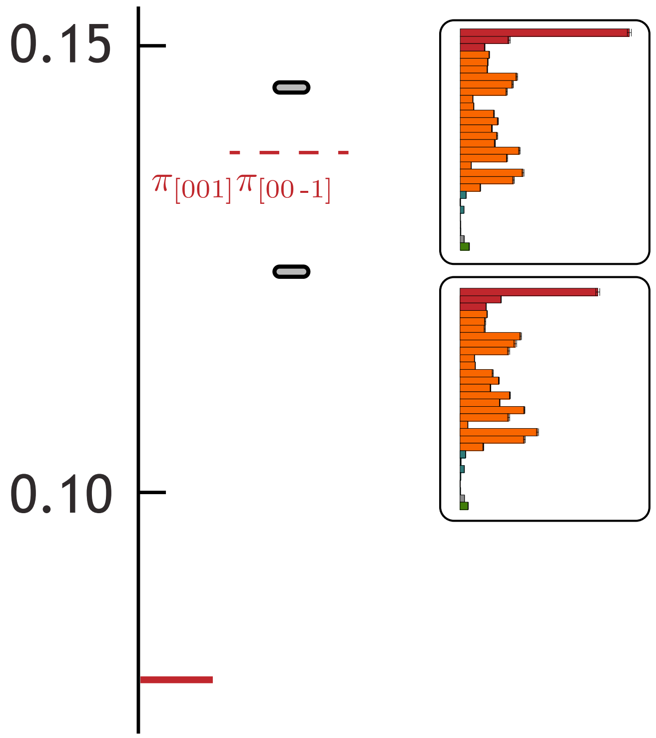
what happens if we vary the operator basis ?

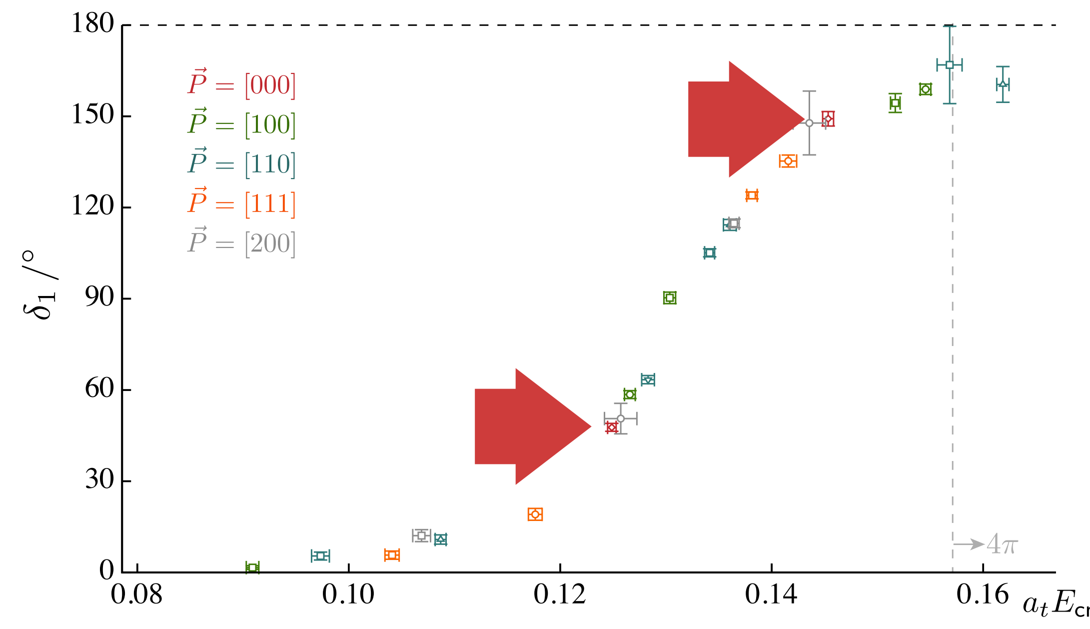
$m_\pi = 0.039$
$m_K = 0.083$

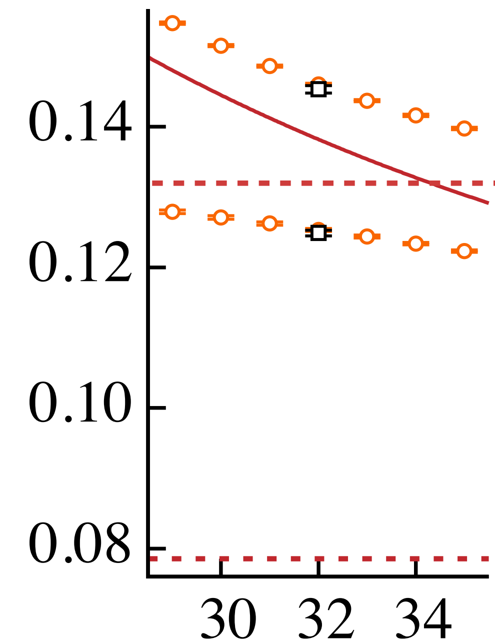
L ~ 3.8 fm

what's happening here ?

focus on the lowest two states

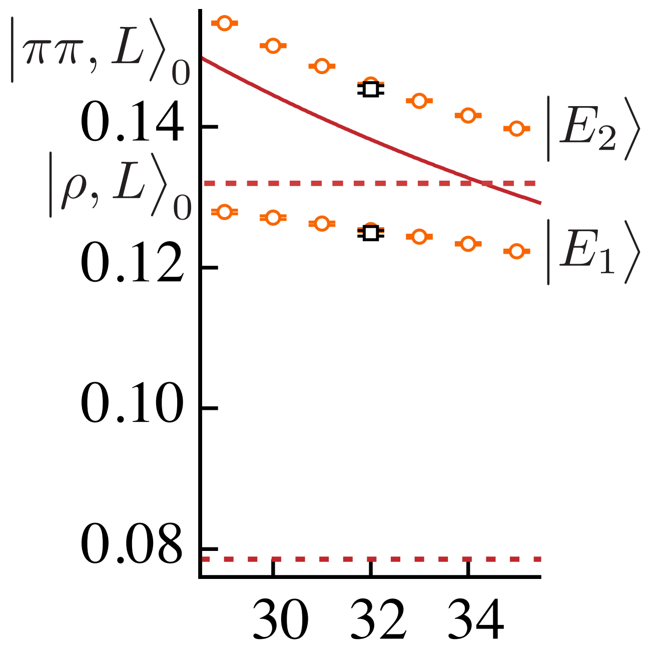






an avoided level crossing

what's happening here ?



think about this as a **two-state problem**

imagine we could turn off the coupling so
a 'bound-state' and a 'meson-meson' state were eigenstates

$$|\rho, L\rangle_0 \qquad\qquad |\pi\pi, L\rangle_0$$

with the coupling turned on, the eigenstates are admixtures

$$|E_1\rangle = \cos\theta\,|\rho, L\rangle_0 + \sin\theta\,|\pi\pi, L\rangle_0$$
$$|E_2\rangle = -\sin\theta\,|\rho, L\rangle_0 + \cos\theta\,|\pi\pi, L\rangle_0$$

with operators that 'look-like' $|\rho, L\rangle_0$ and $|\pi\pi, L\rangle_0$ in the basis, the variational method separates $|E_1\rangle$,$|E_2\rangle$

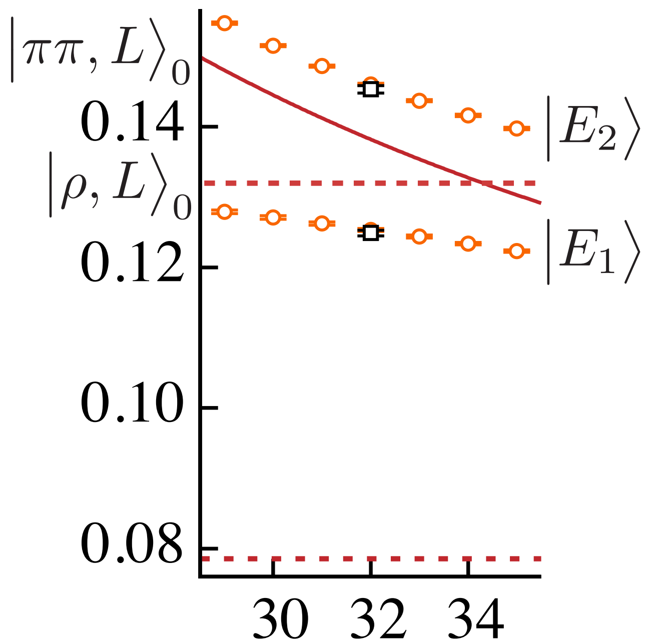
$$\begin{pmatrix} C_{\rho,\rho}(t) & C_{\rho,\pi\pi}(t) \\ C_{\pi\pi,\rho}(t) & C_{\pi\pi,\pi\pi}(t) \end{pmatrix} = \begin{pmatrix} Z_\rho & 0 \\ 0 & Z_{\pi\pi} \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} e^{-E_1 t} & 0 \\ 0 & e^{-E_2 t} \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} Z_\rho & 0 \\ 0 & Z_{\pi\pi} \end{pmatrix}$$

$$\mathcal{O}_\rho|0\rangle = Z_\rho|\rho, L\rangle_0 + \epsilon|\pi\pi, L\rangle_0$$
$$\mathcal{O}_{\pi\pi}|0\rangle = Z_{\pi\pi}|\pi\pi, L\rangle_0 + \epsilon|\rho, L\rangle_0$$

GEVP eigenvectors will find the rotation

and the principal correlators
$$\lambda_1(t) \sim e^{-E_1 t}$$
$$\lambda_2(t) \sim e^{-E_2 t}$$

what's happening here ?



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$$|E_2\rangle = -\sin\theta\,|\rho, L\rangle_0 + \cos\theta\,|\pi\pi, L\rangle_0$$

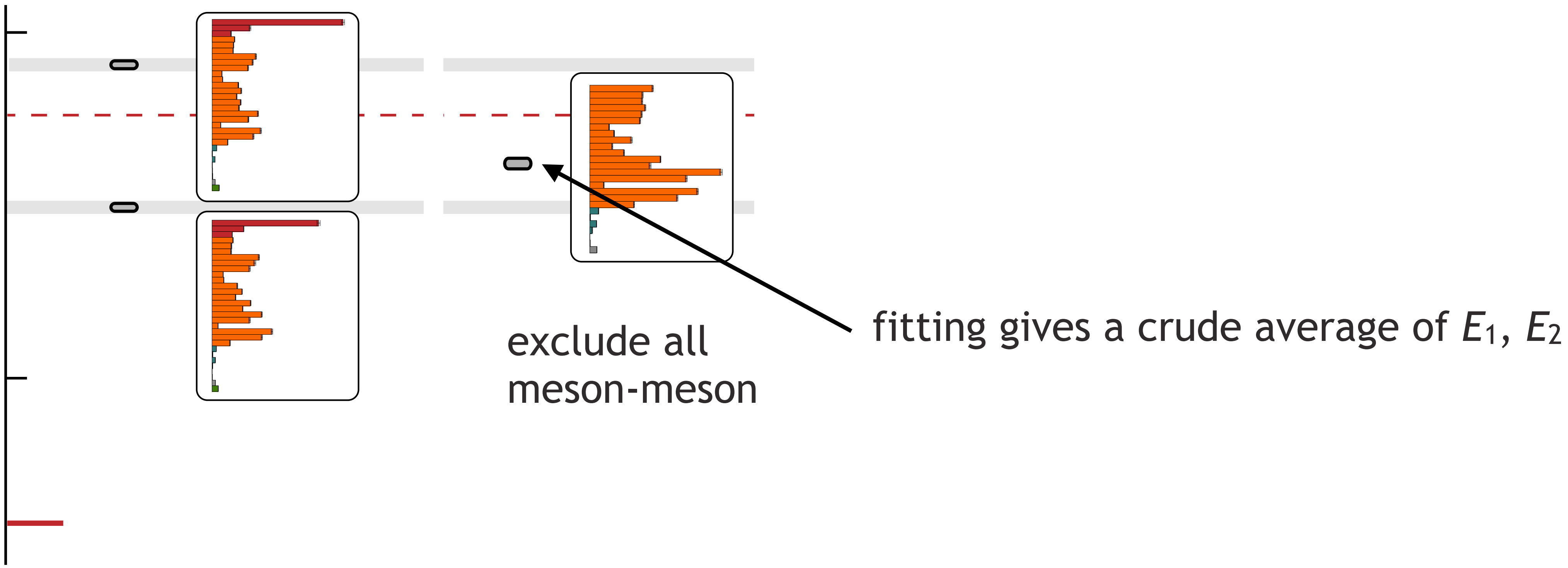
now suppose we used only the $\mathcal{O}_\rho$ operators

then $C(t) \propto \cos^2\theta\, e^{-E_1 t} + \sin^2\theta\, e^{-E_2 t}$ and there'll be two energies present …

… and they're very hard to separate

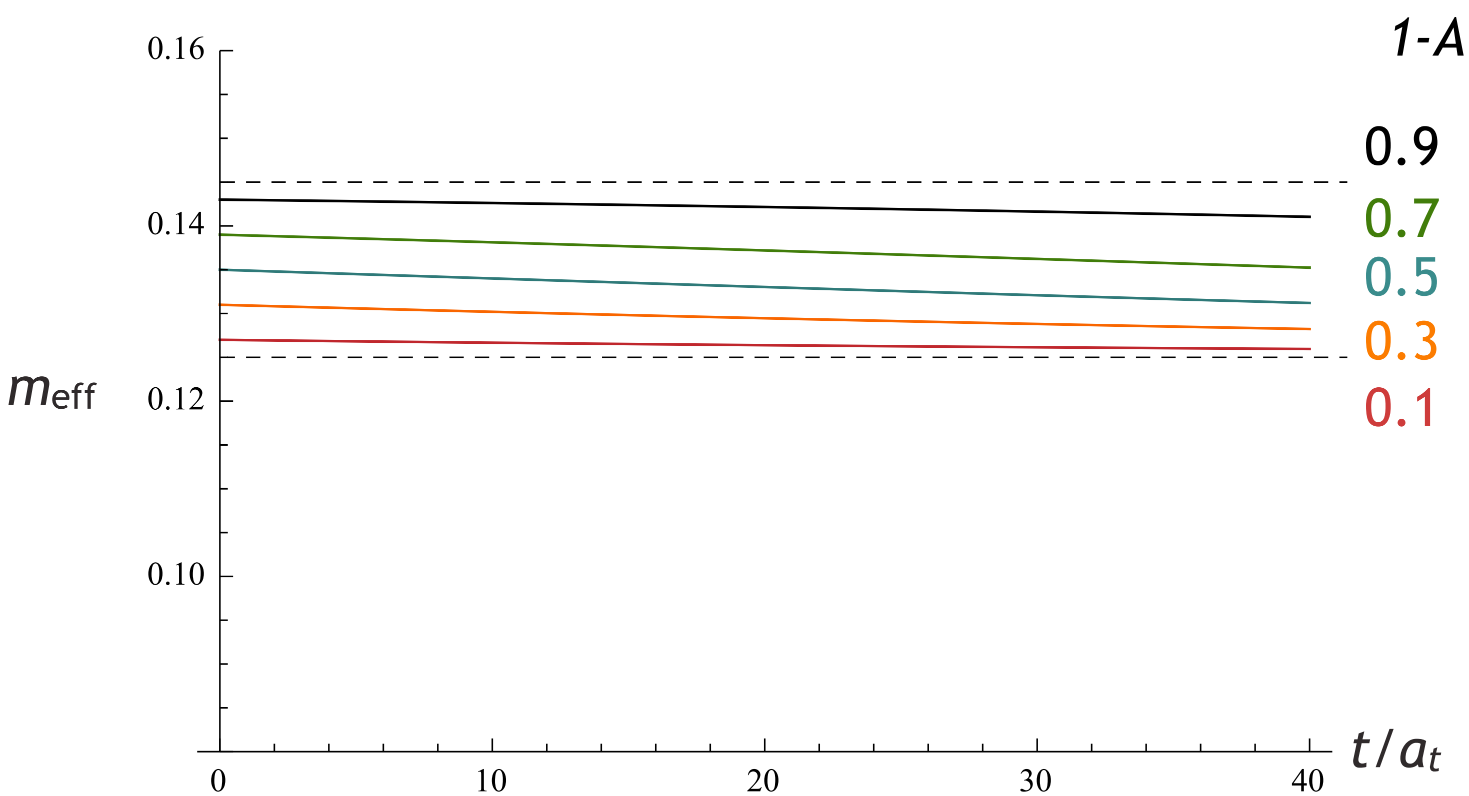
what's happening here ?

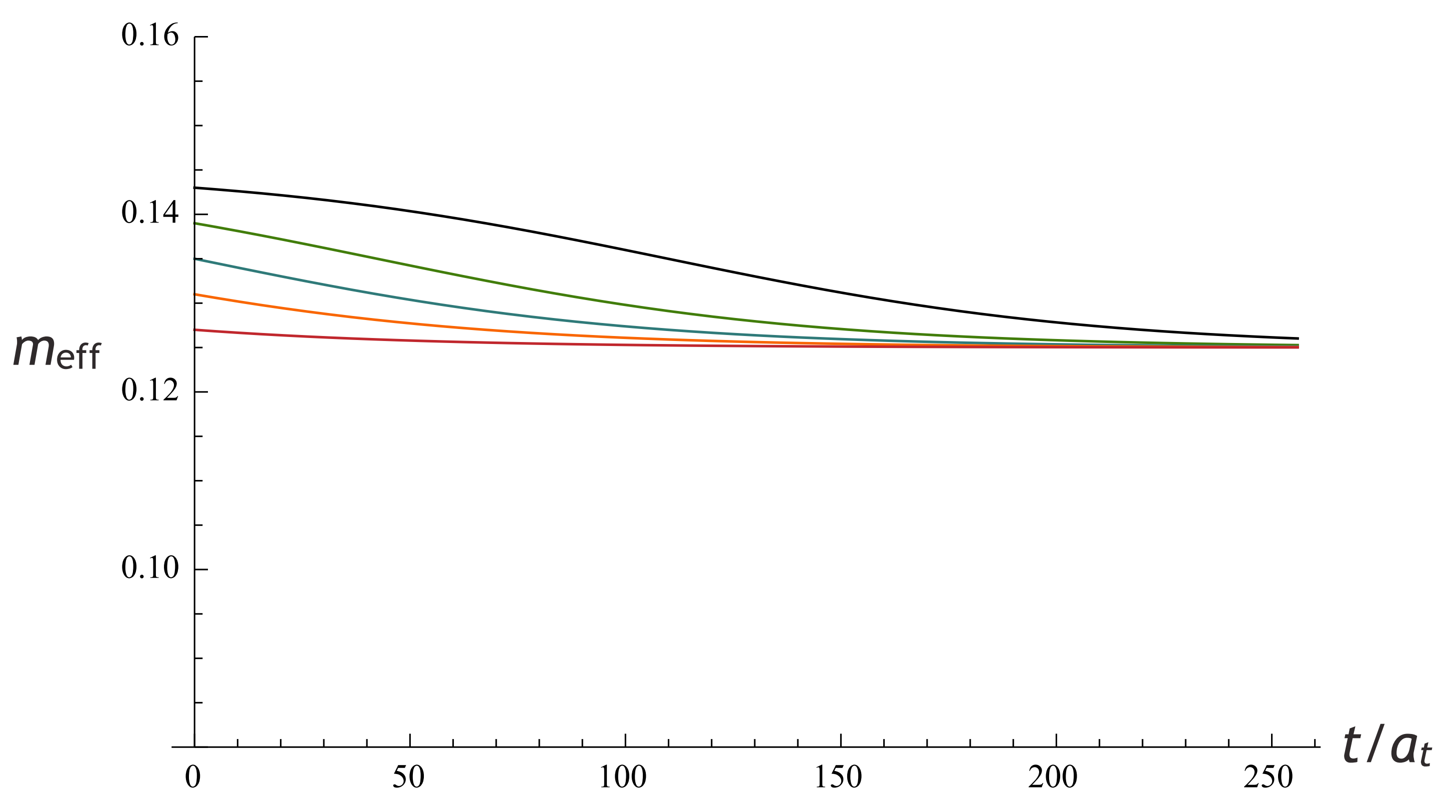
it looks like this is what's happening

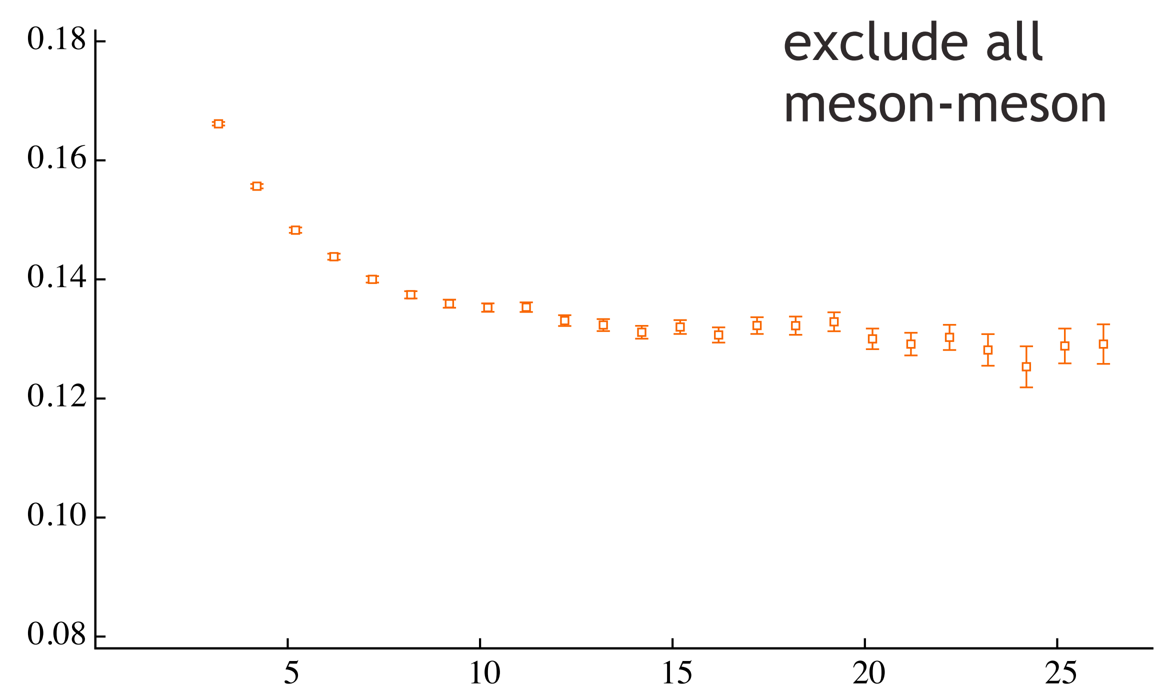


two-state admixture

$$C(t) = A\, e^{-0.125\, t/a_t} + (1-A)\, e^{-0.145\, t/a_t}$$

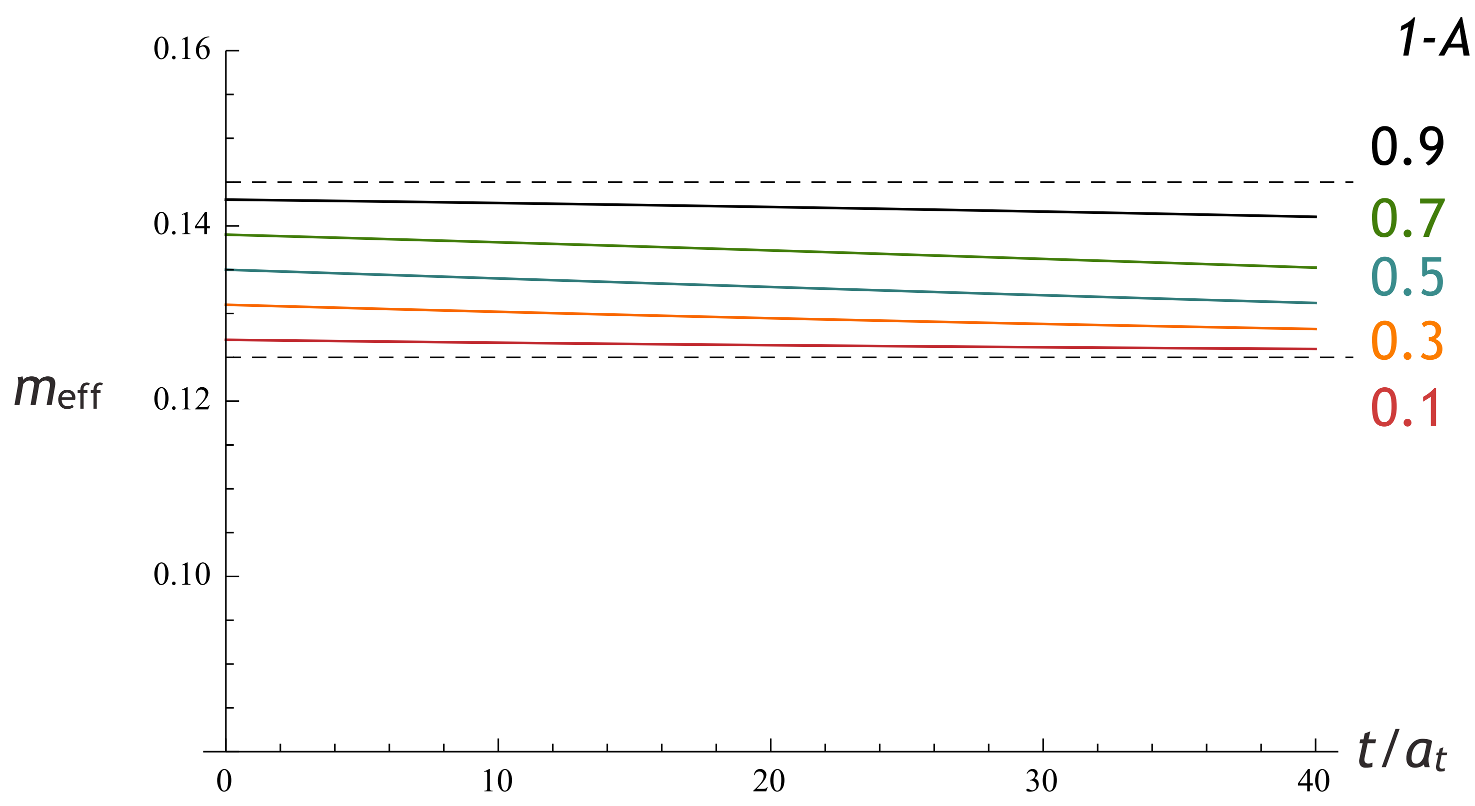


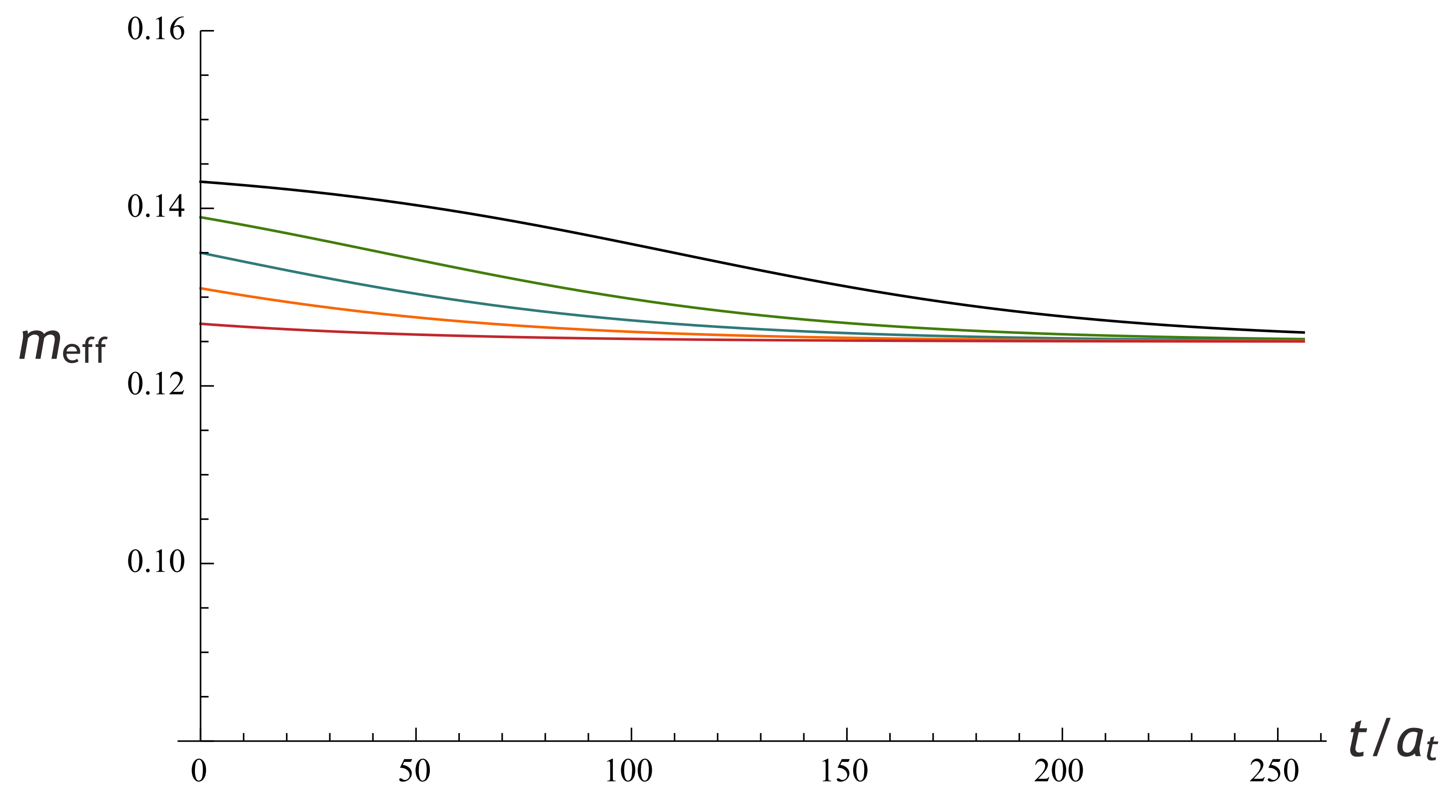


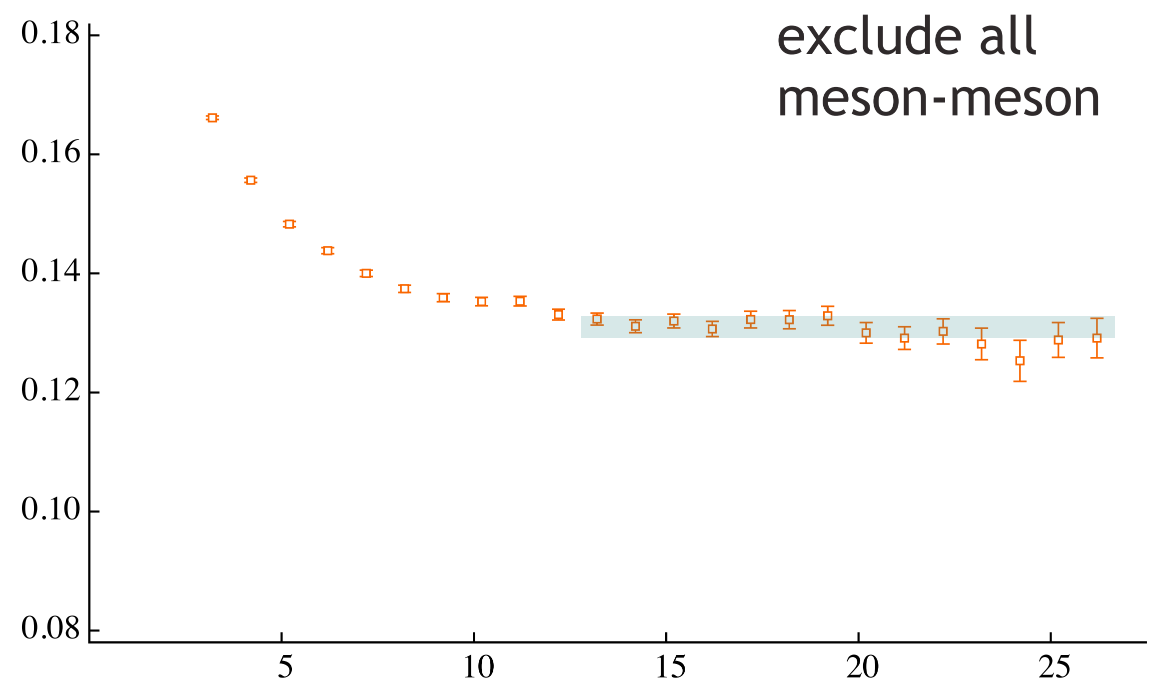


two-state admixture

$$C(t) = A\, e^{-0.125\, t/a_t} + (1-A)\, e^{-0.145\, t/a_t}$$







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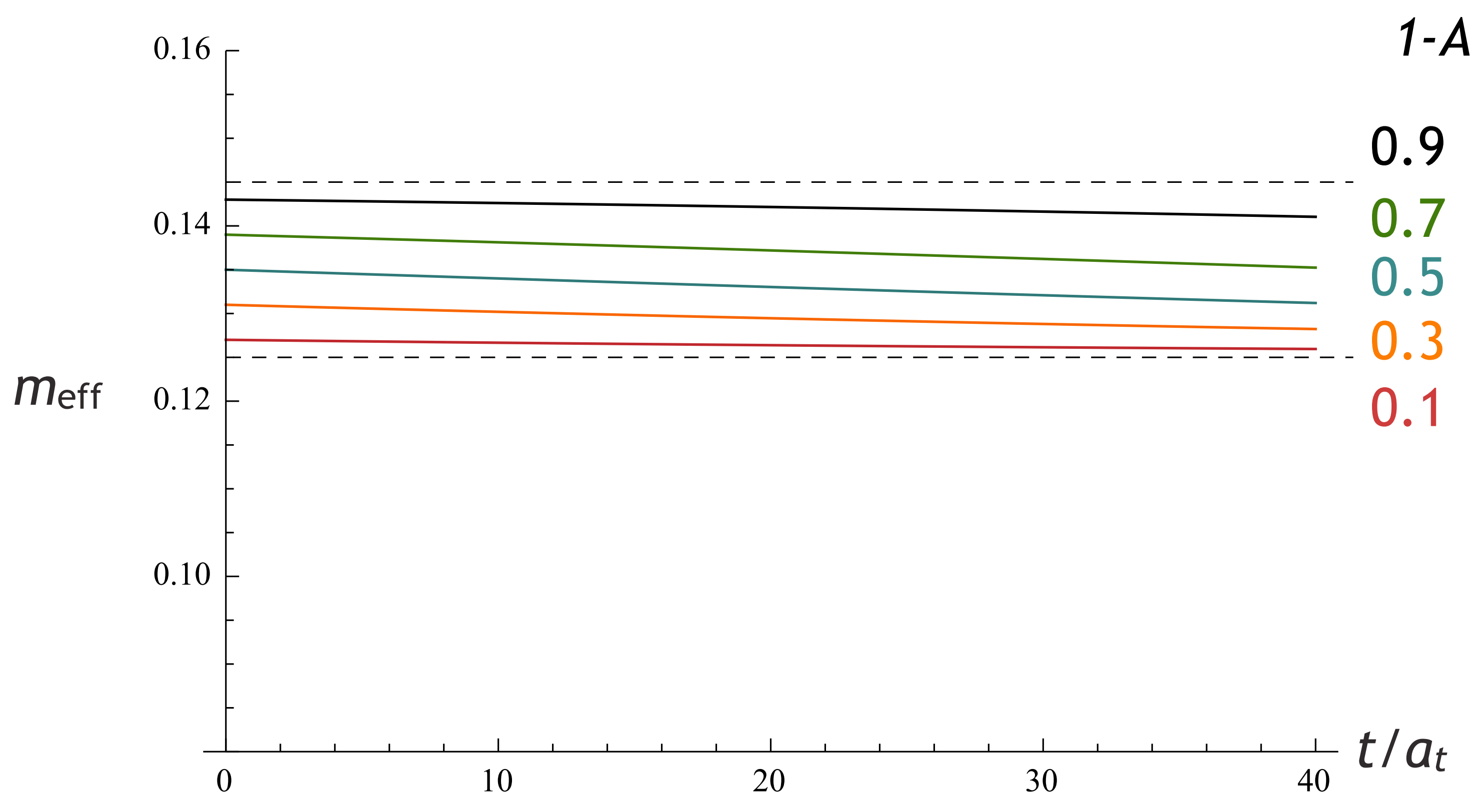
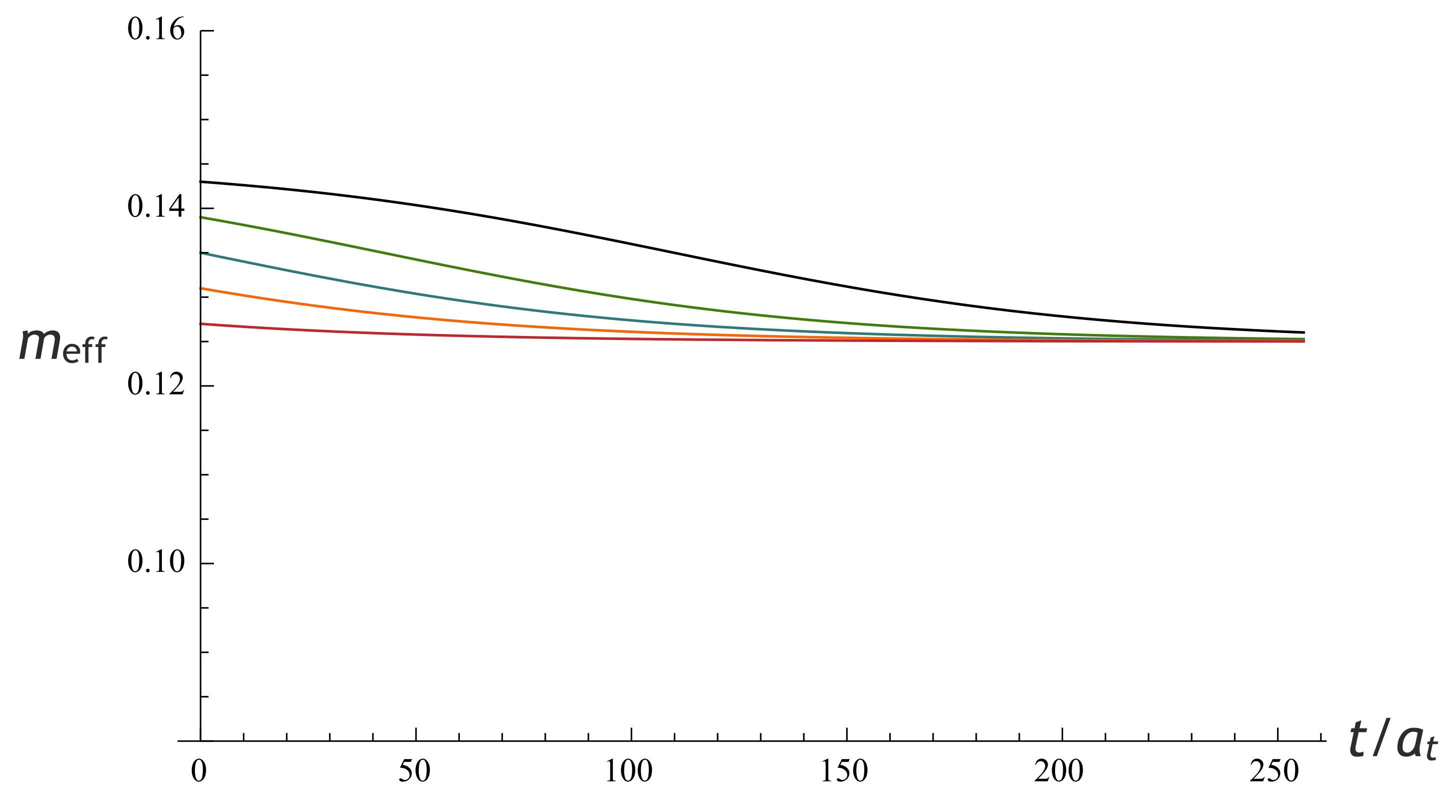
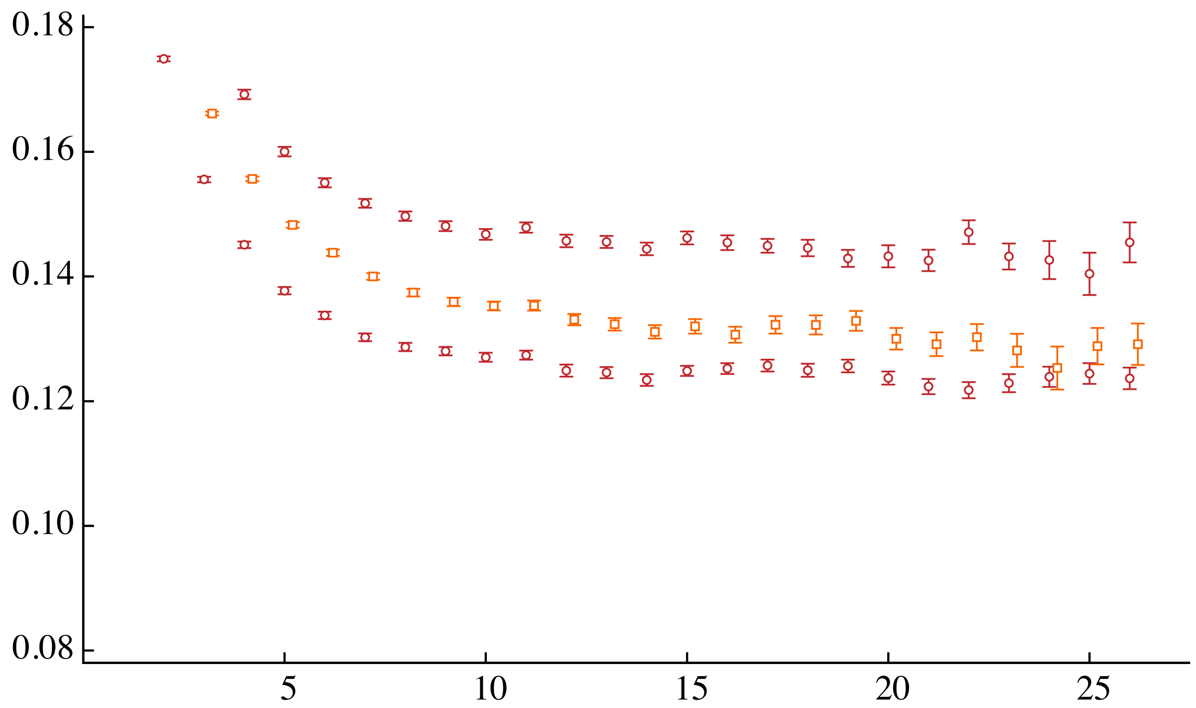






this explanation requires 'single-meson'-like operators
to have negligible overlap onto 'meson-meson' basis states …

… why would that be ?

volume dependence !

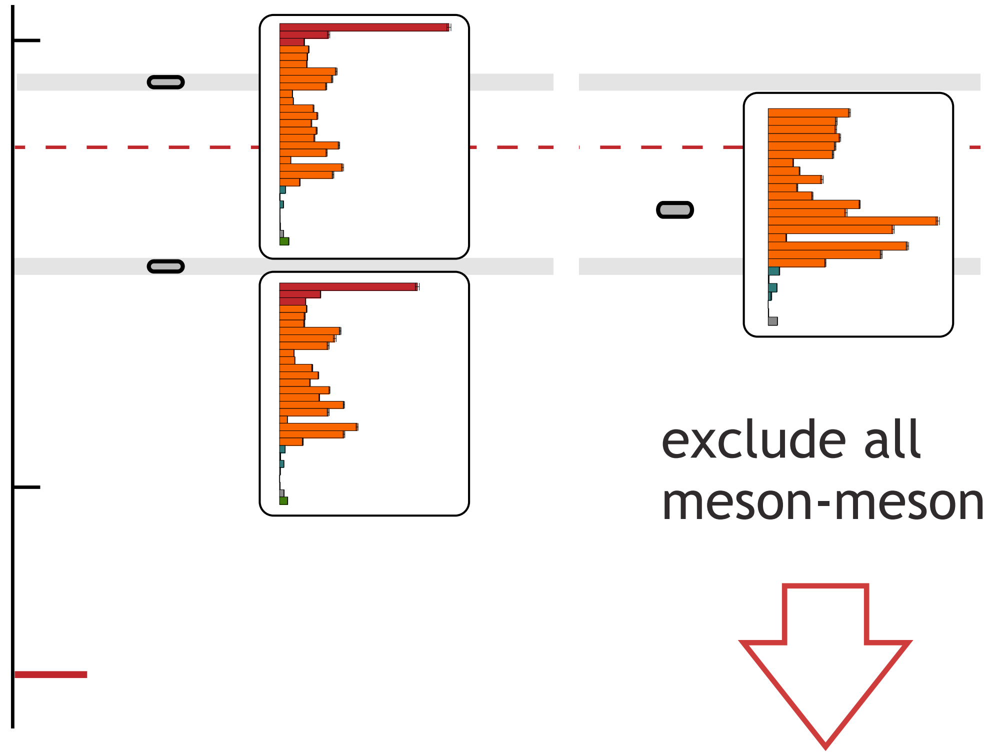
'meson-meson'-like $\sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}}\, \bar{\psi}_{\mathbf{x}} \mathbf{\Gamma} \psi_{\mathbf{x}} \sum_{\mathbf{y}} e^{i\mathbf{q}\cdot\mathbf{y}}\, \bar{\psi}_{\mathbf{y}} \mathbf{\Gamma}' \psi_{\mathbf{y}}$ samples the whole volume of the lattice

'single-meson'-like $\sum_{\mathbf{x}} e^{i\mathbf{P}\cdot\mathbf{x}}\, \bar{\psi}_{\mathbf{x}} \mathbf{\Gamma} \psi_{\mathbf{x}}$ samples a single point (translated)

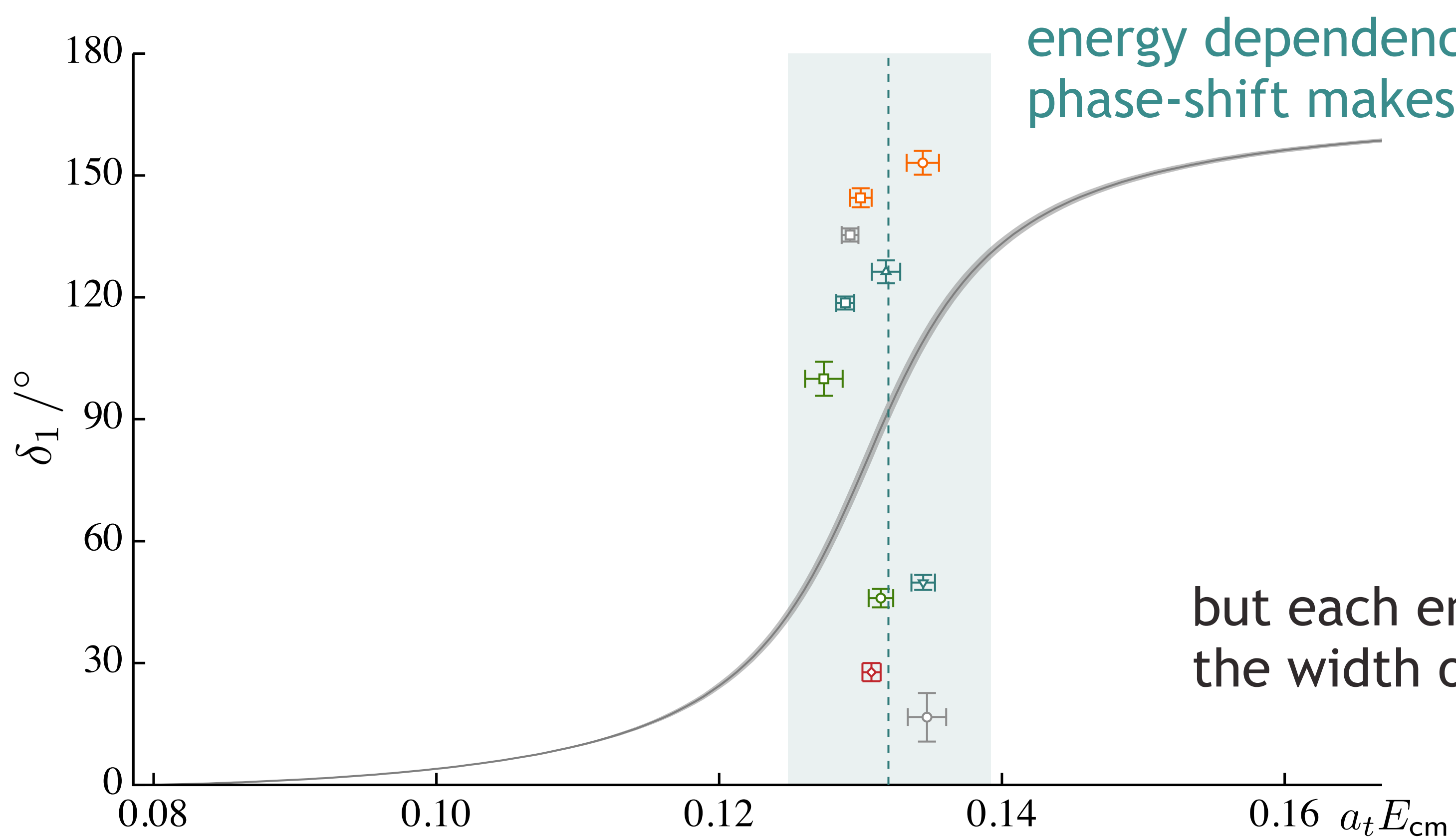
so: 'looks-like' = 'has the same volume sampling as'

interesting side note:
tetraquark operators won't work well for interpolating meson-meson components — wrong volume sampling

how bad is it really to get the wrong energies ?



exclude all
meson-meson



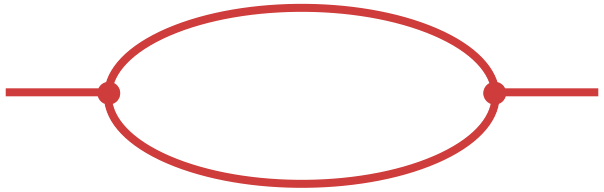
energy dependence of the
phase-shift makes no sense

but each energy within
the width of the resonance ... ?

- equal mass case

$$I(s) = -C(s)$$

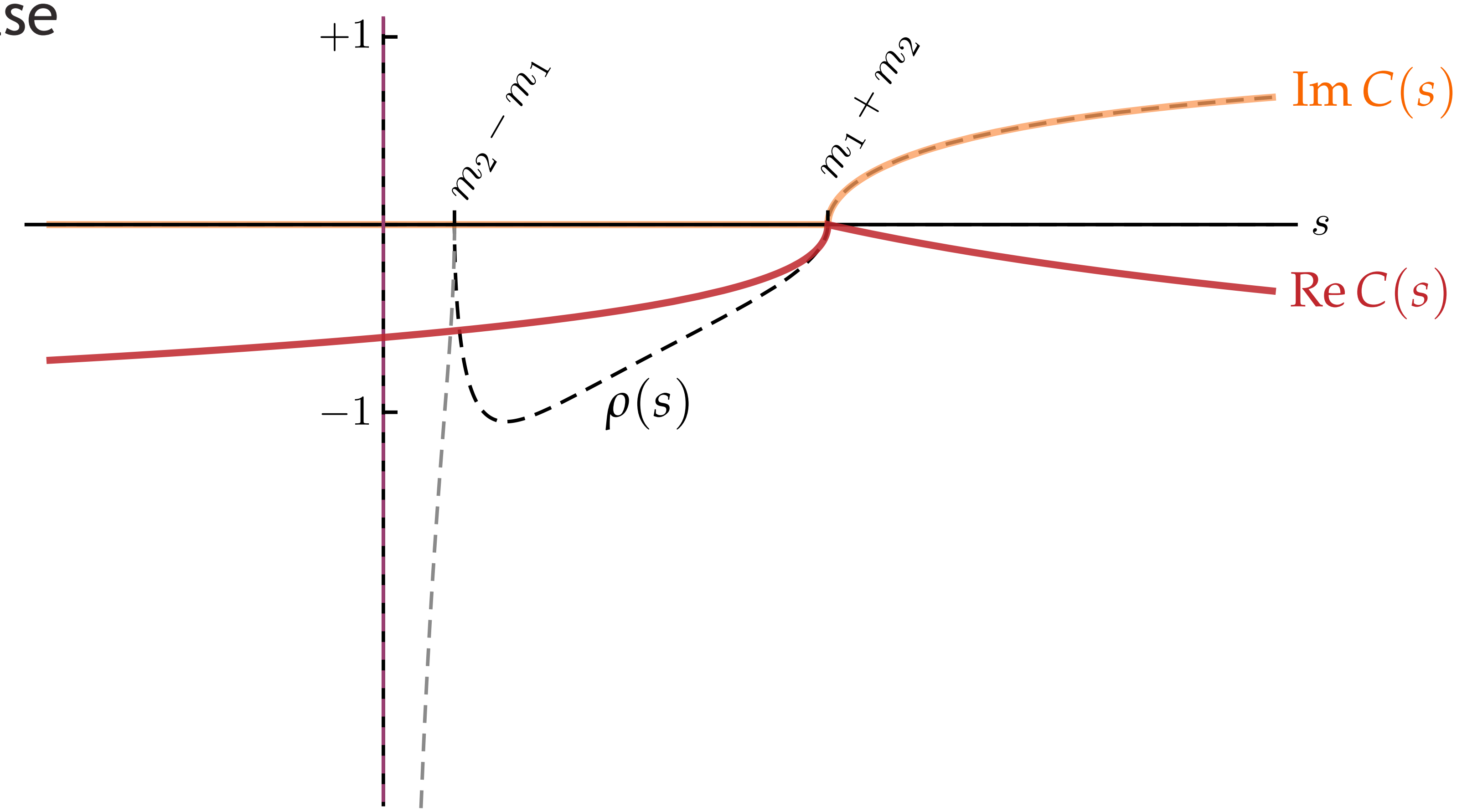
$$C(s) = C(0) + \frac{s}{\pi}\int_{s_{\rm th}}^{\infty} ds' \, \sqrt{1 - \frac{s_{\rm th}}{s'}} \, \frac{1}{s'(s'-s)}$$



$$C(s) = \frac{\rho(s)}{\pi} \log\left[\frac{\rho(s)-1}{\rho(s)+1}\right]$$

subtracting at threshold $C(s_{\rm th}) = 0$

- unequal mass case



- multiple possible scattering channels (still just spin-0—spin-0)
  - the quantisation condition is

$$0 = \det\left[\delta_{\ell n;\ell' n'}\delta_{\alpha\beta} + i\rho_\alpha t^{[\ell]}_{\alpha\beta}\left(\delta_{\ell n;\ell' n'} + i\mathcal{M}^{\vec{d};\Lambda}_{\ell n,\ell' n'}(k_\alpha)\right)\right]$$

  - must we include all possible channels ?

no, only channels which are kinematically open, or close to opening

e.g. $E < E_{\mathsf{thr}}$, $k = i\kappa$ $\qquad \mathcal{M}^{\vec{0};A_1}_{01,01}(i\kappa) = i - \frac{i}{\kappa}\sum_{\vec{n}\neq 0}\frac{e^{-\kappa|\vec{n}|L}}{|\vec{n}|L}$

e.g. two-channels, *S*-wave

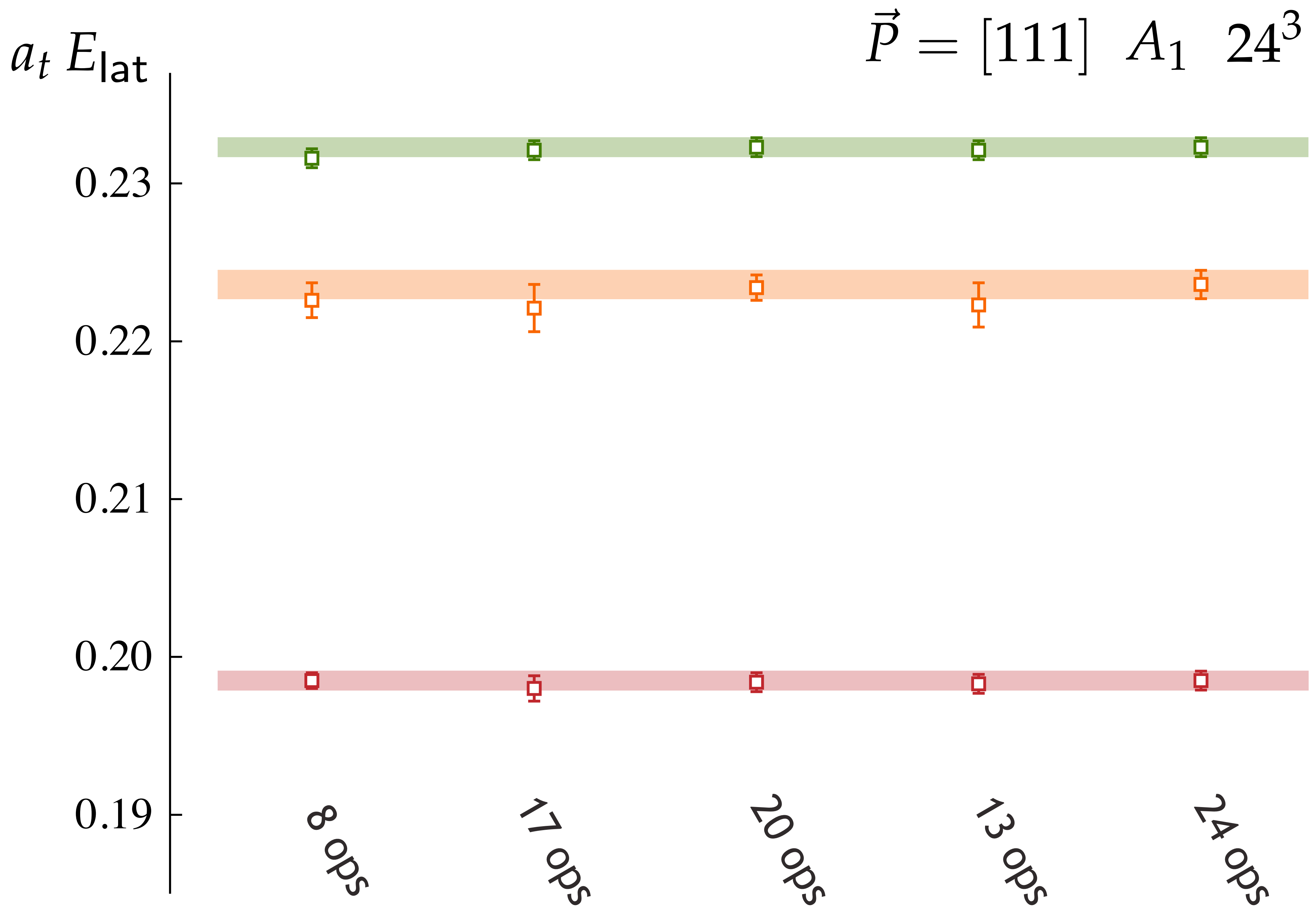
$$\begin{vmatrix} 1 + i\rho_1 t_{11}(1+i\mathcal{M}_1) & i\rho_1 t_{12}(1+i\mathcal{M}_1) \\ i\rho_2 t_{12}(1+i\mathcal{M}_2) & 1 + i\rho_1 t_{11}(1+i\mathcal{M}_2) \end{vmatrix} \xrightarrow{\text{well below threshold 2}} 1 + i\rho_1 t_{11}(1+i\mathcal{M}_1)$$

well below threshold 2

elastic condition

varying the bilinear operator basis

- varying the $\bar{\psi}\Gamma\psi$ content of the operator basis

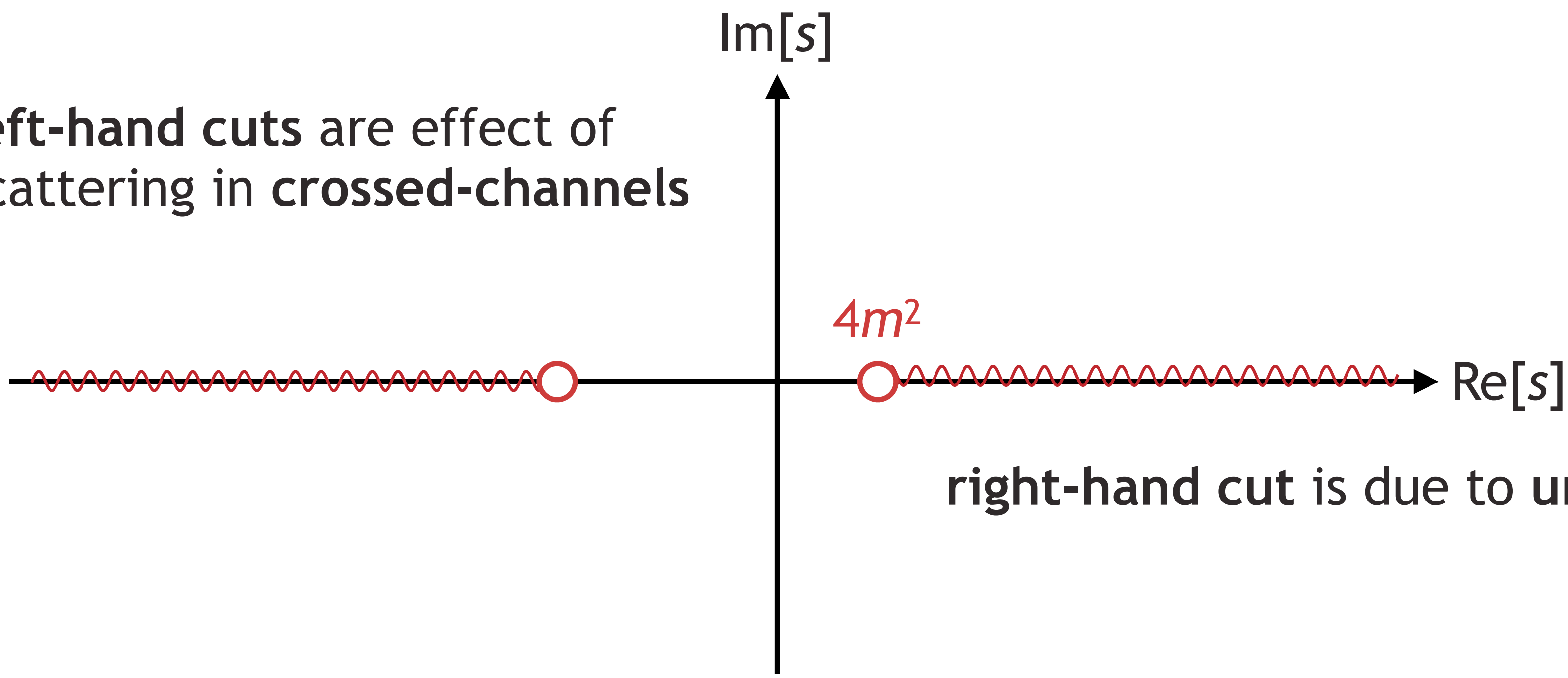


3× ππ ops

$m_\pi \sim 391\,\mathrm{MeV}$

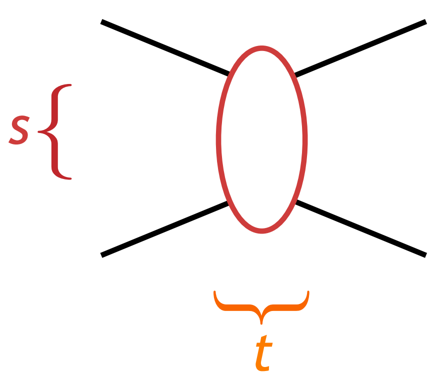
left-hand cuts

left-hand cuts are effect of scattering in **crossed-channels**

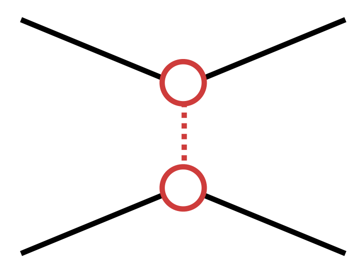


right-hand cut is due to **unitarity**

a very simple-minded toy model illustrating a left-hand cut:



suppose there is a *t*-channel stable meson exchange



$$f(s,t) \sim \frac{1}{t - M^2}$$

S-wave partial-wave projection

$$t_\ell(s) \sim \int d\cos\theta \; f\big(s, t(\cos\theta)\big)$$

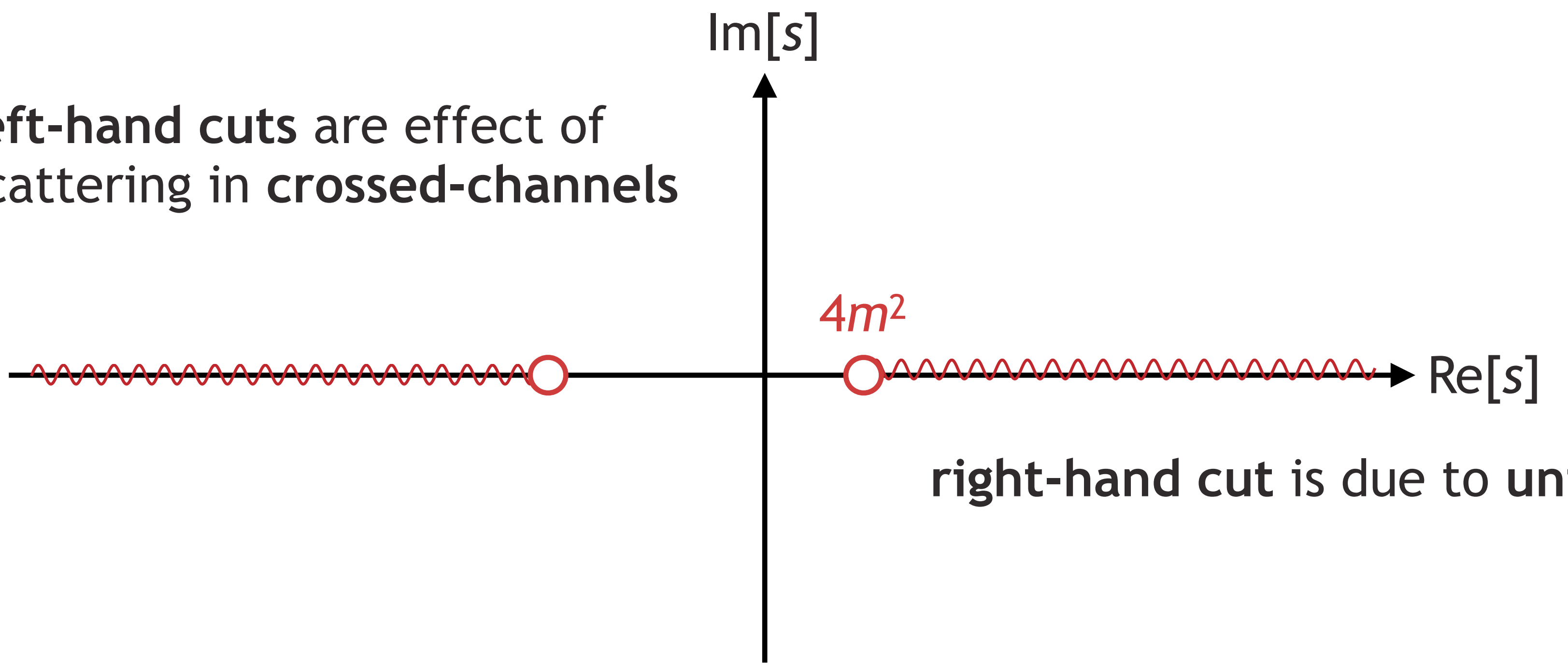
$$= -\tfrac{1}{2k^2} \log\left[s - (4m^2 - M^2)\right]$$

$$t = -2k^2(1-\cos\theta)$$
$$s = 4(m^2 + k^2)$$

log branch cut from $4m^2 - M^2$

left-hand cuts

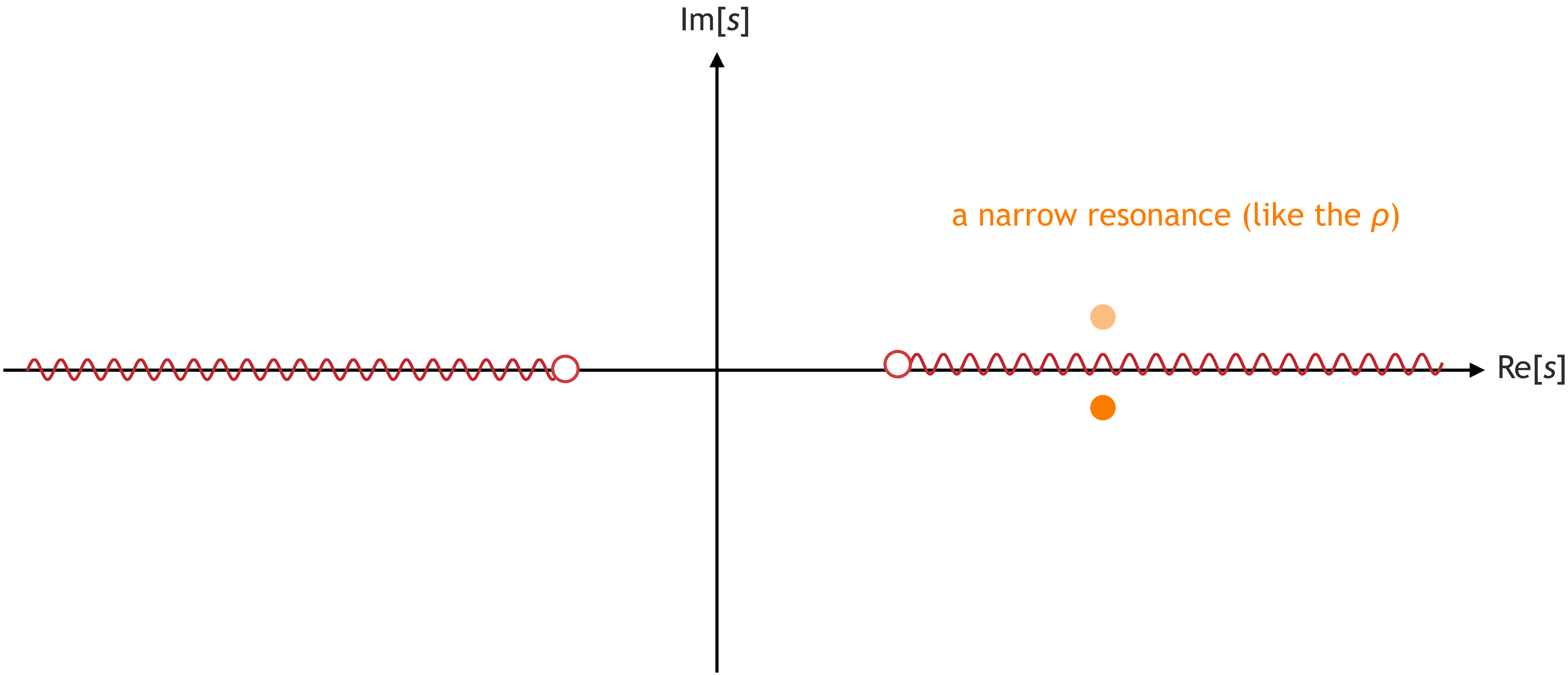
left-hand cuts are effect of scattering in **crossed-channels**



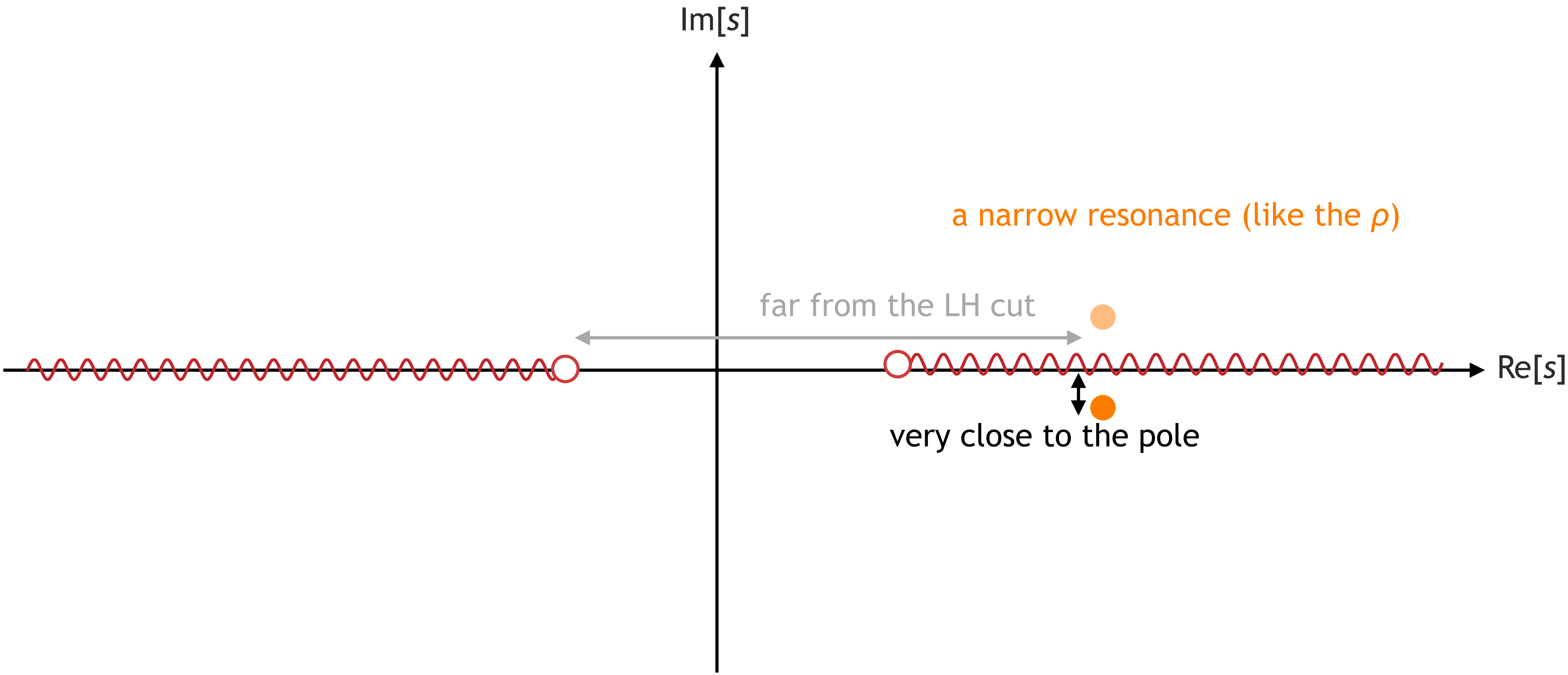
right-hand cut is due to **unitarity**

more generally **unitarity** in the *t*-channel
ensures there will be a left-hand cut in partial-wave amplitudes

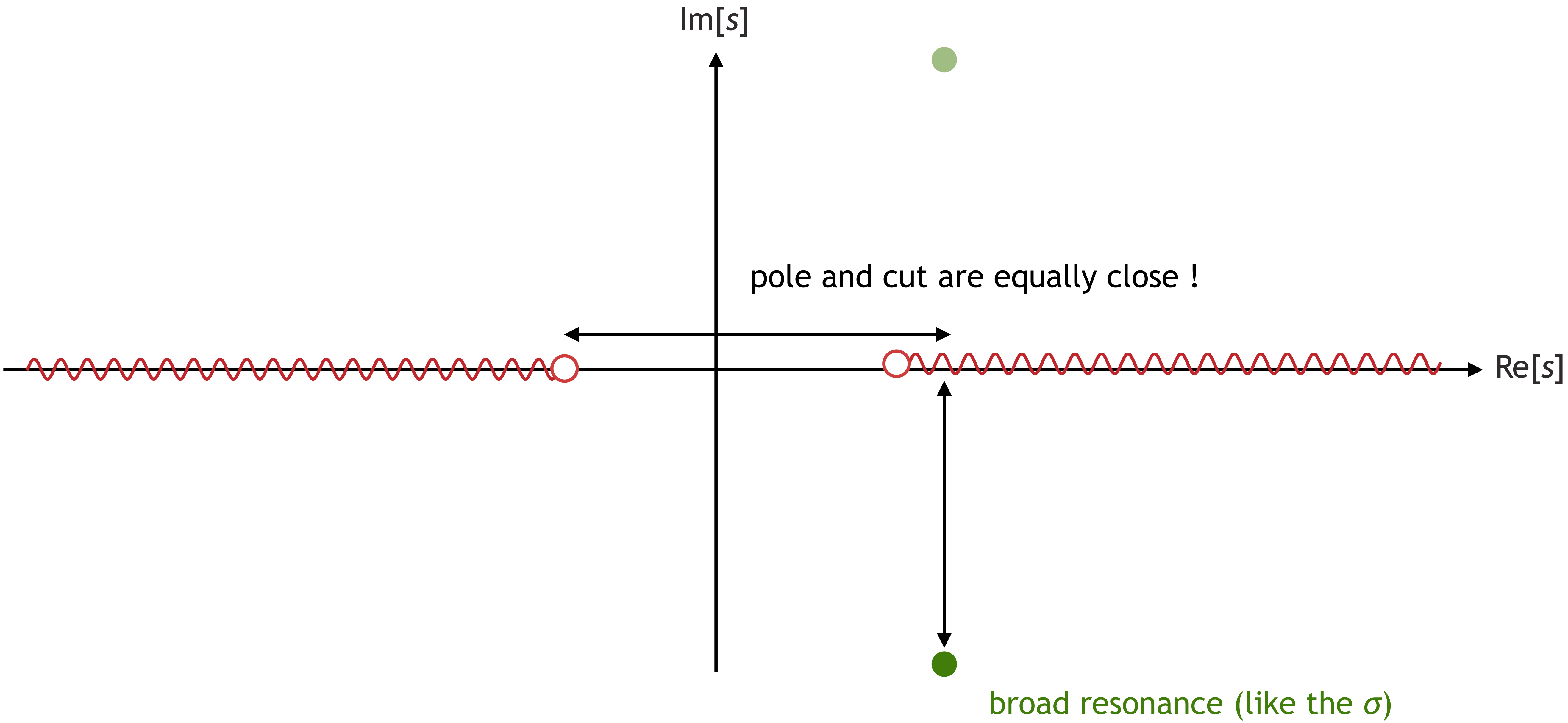
where left-hand cuts don't matter



where left-hand cuts don't matter

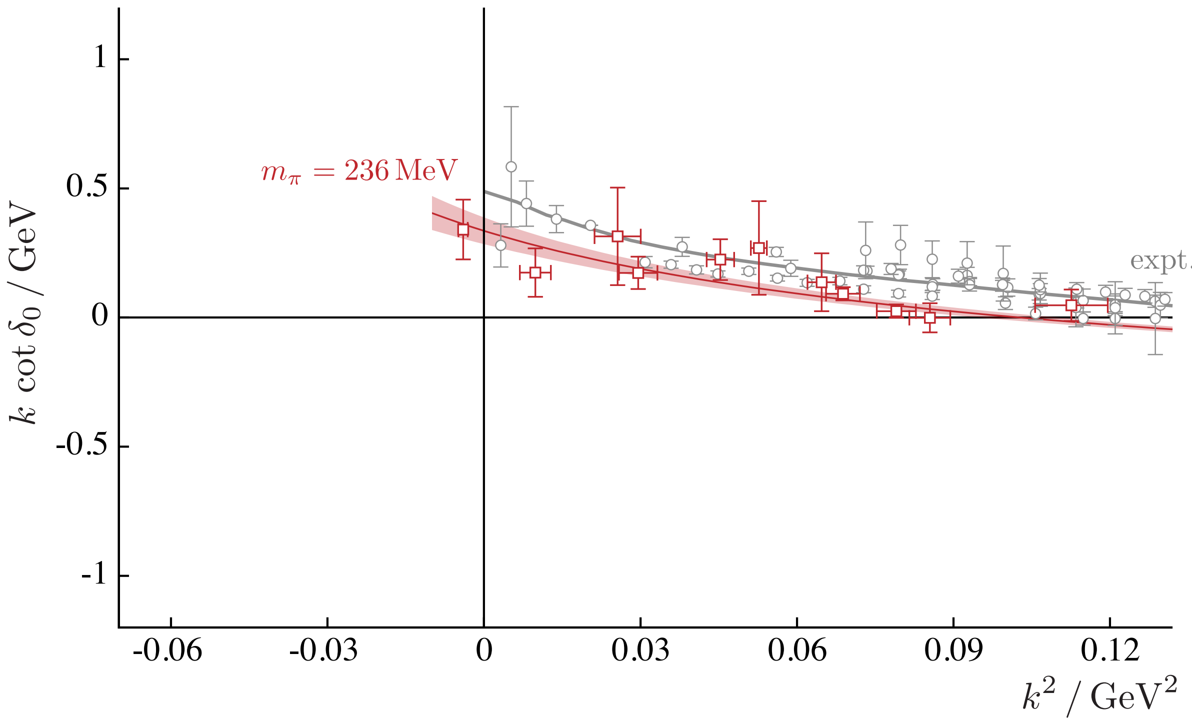


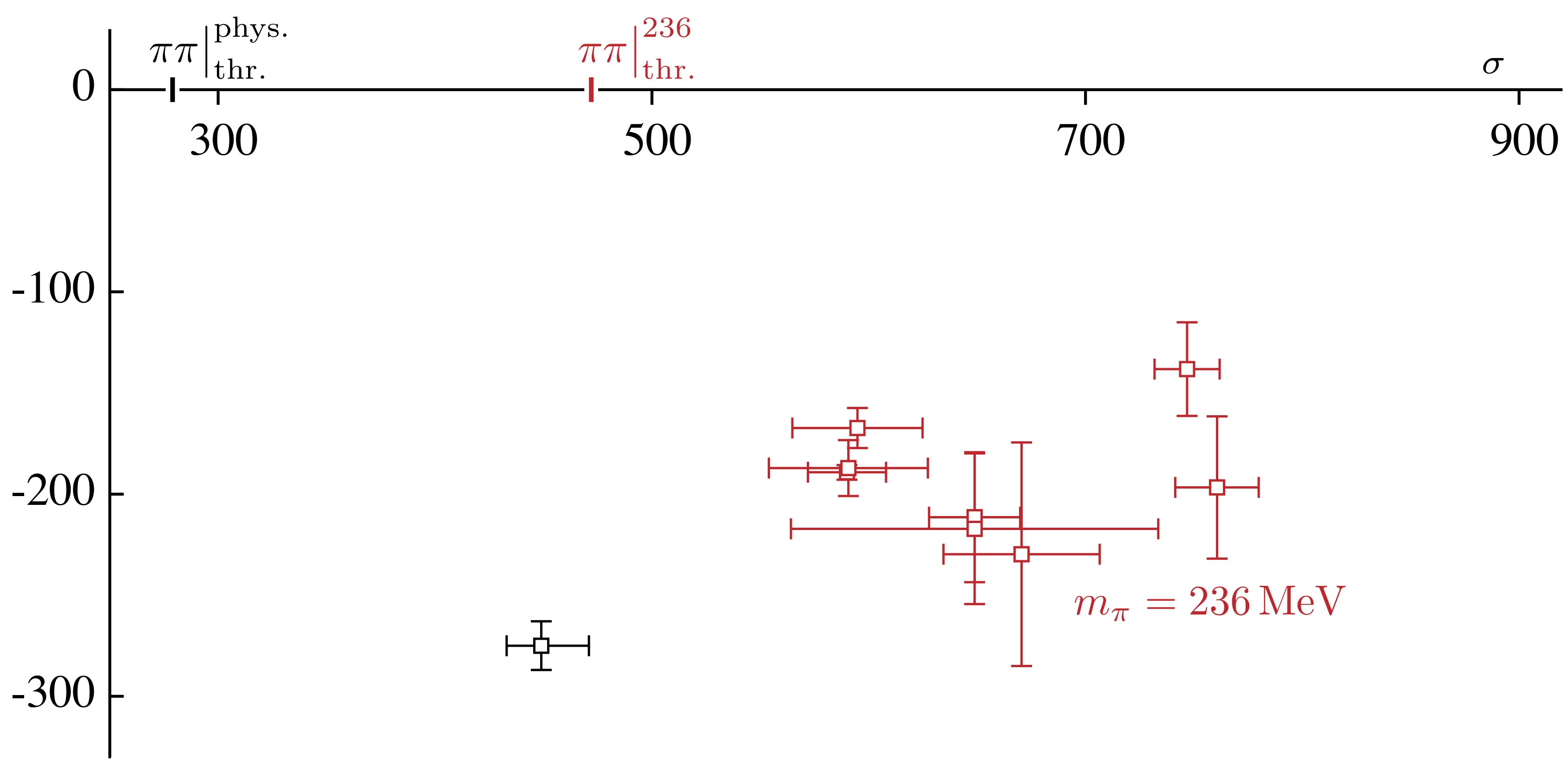
where left-hand cuts matter (maybe ?)



big scatter — no left-hand cut constraint

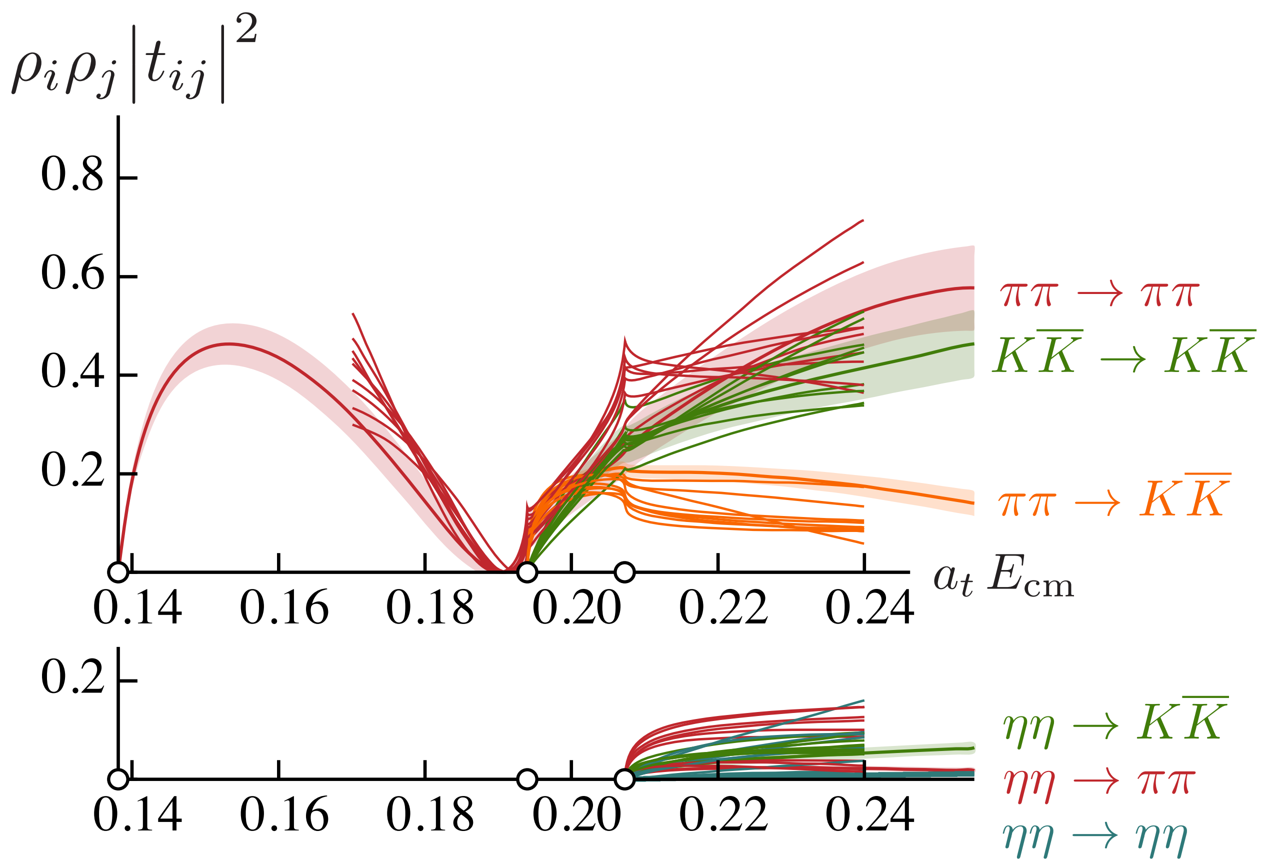
ππ isospin=0

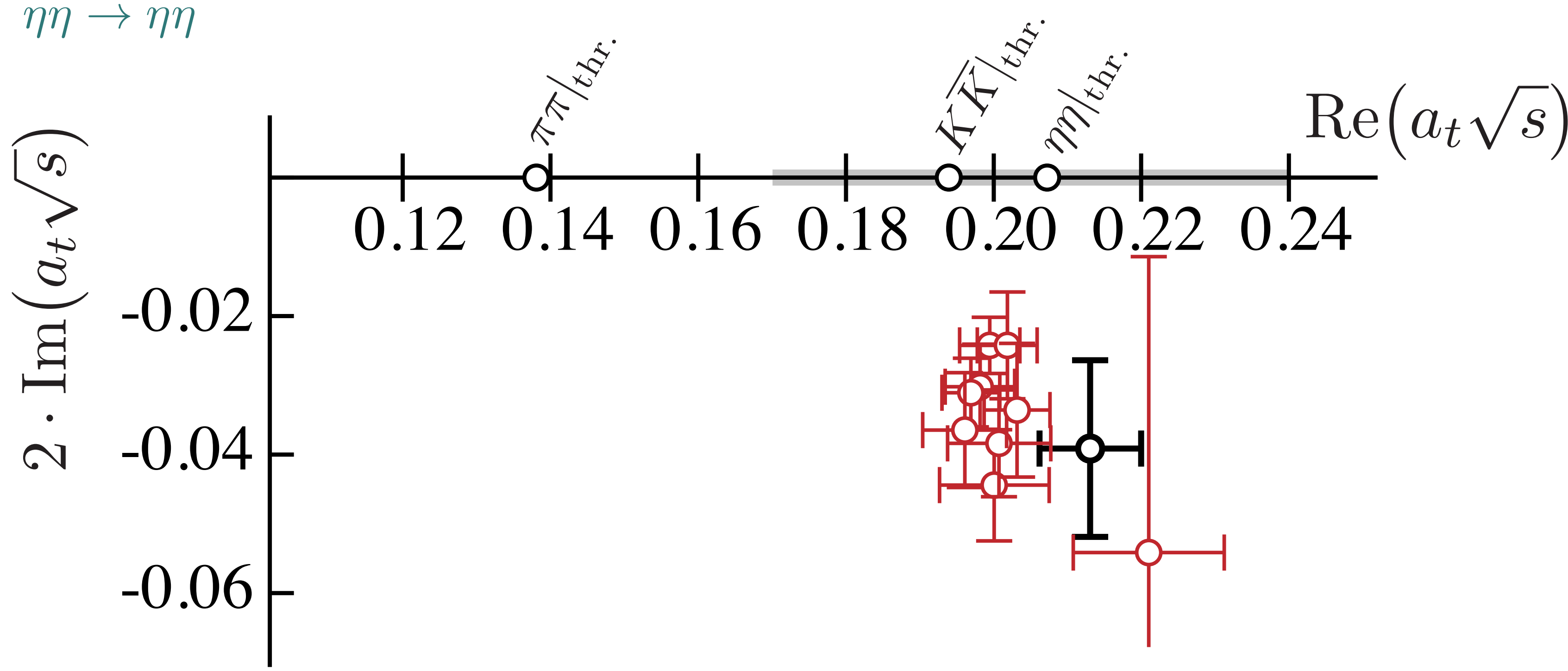




limited energy region about $K\bar{K}$ threshold

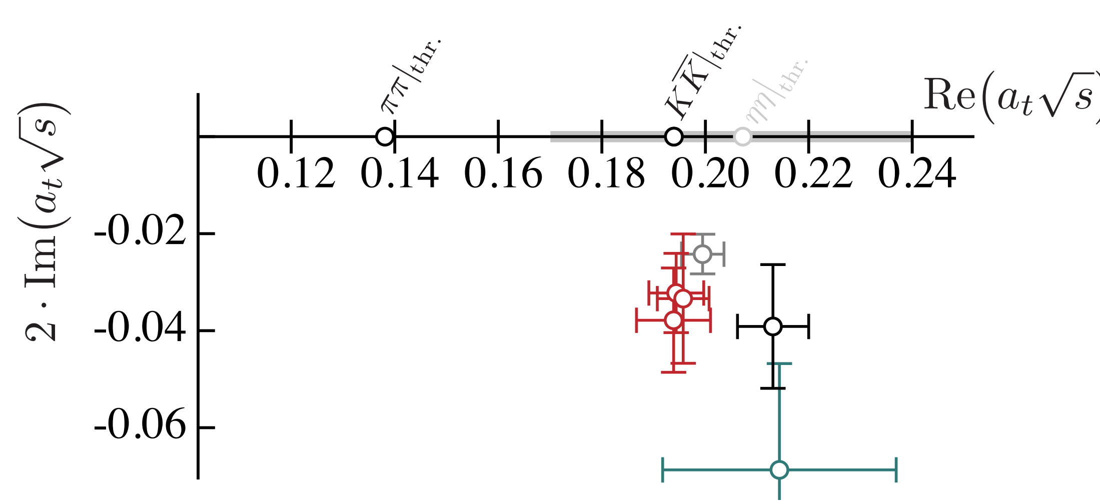
σ contributes only a slowly varying ‘background’ in this energy region



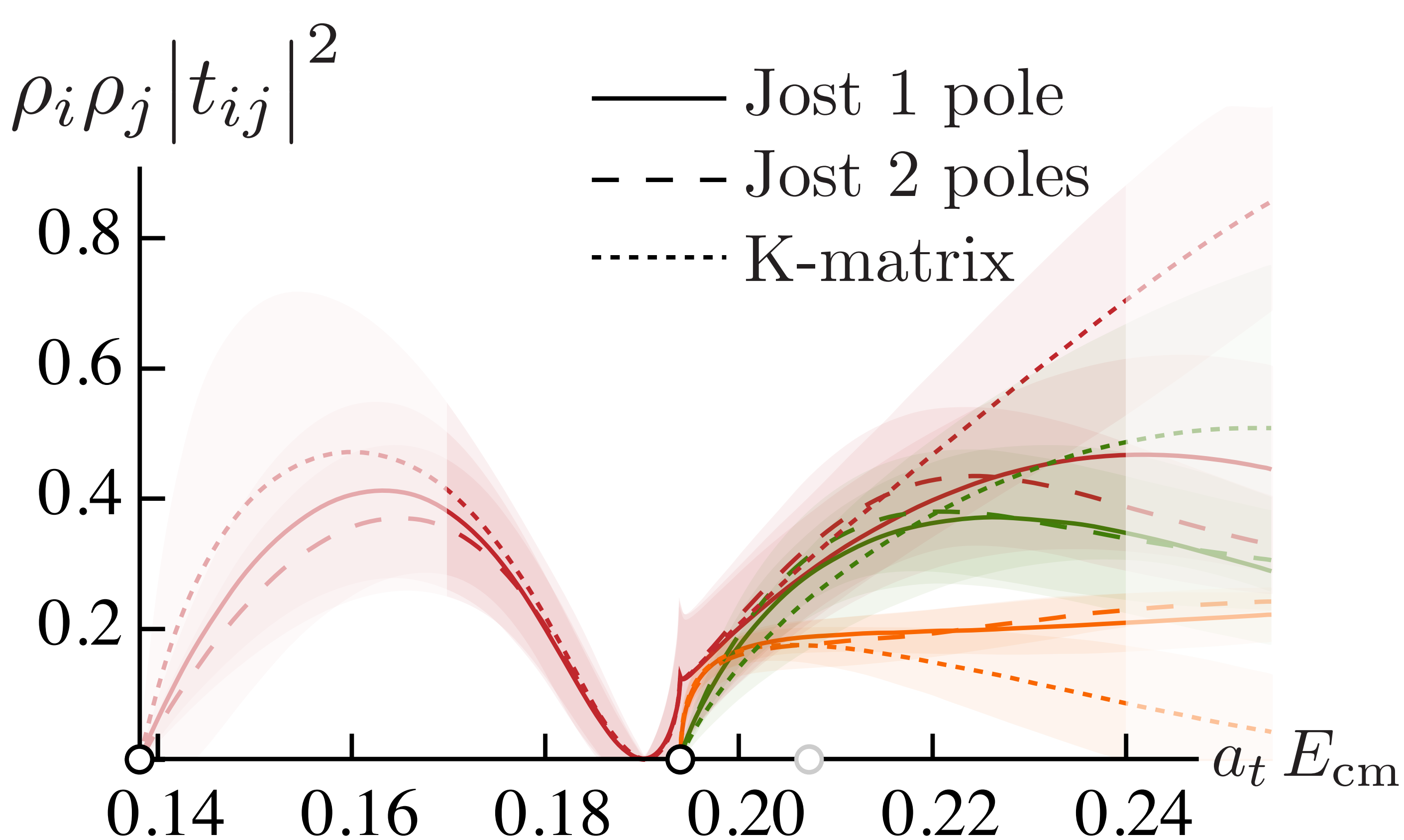


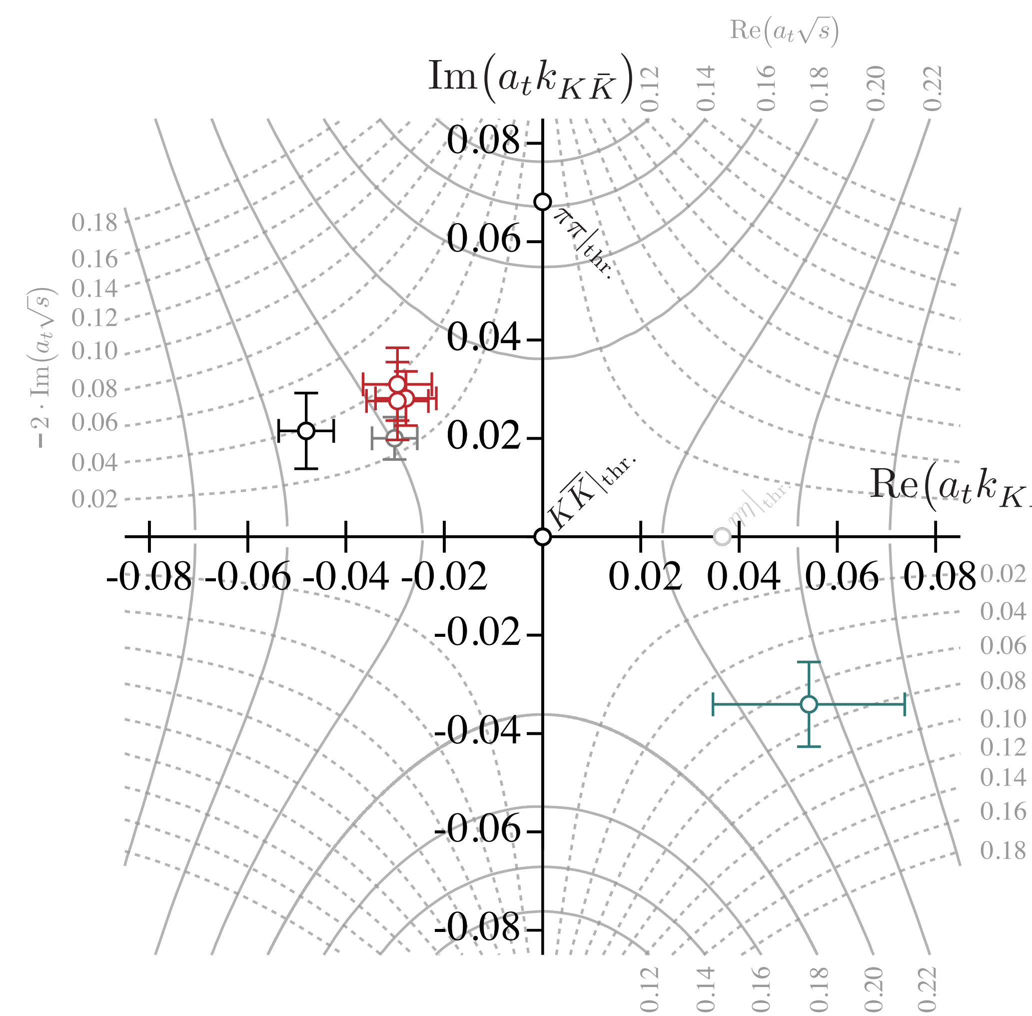
$$\omega = \frac{k_1 + k_2}{\sqrt{k_1^2 - k_2^2}}, \qquad \omega^{-1} = \frac{k_1 - k_2}{\sqrt{k_1^2 - k_2^2}},$$

$$S_{11} = \frac{\mathfrak{D}(-\omega^{-1})}{\mathfrak{D}(\omega)}, \quad S_{22} = \frac{\mathfrak{D}(\omega^{-1})}{\mathfrak{D}(\omega)}, \quad \det \mathbf{S} = \frac{\mathfrak{D}(-\omega)}{\mathfrak{D}(\omega)}.$$



$$\mathfrak{D}(\omega) = \exp\left(\sum_{b=1}^{n_b} \gamma_b\, \omega^b\right) \frac{1}{\omega^2} \prod_{p=1}^{n_p} \left(1 - \frac{\omega}{\omega_p}\right)\left(1 + \frac{\omega}{\omega_p^*}\right).$$





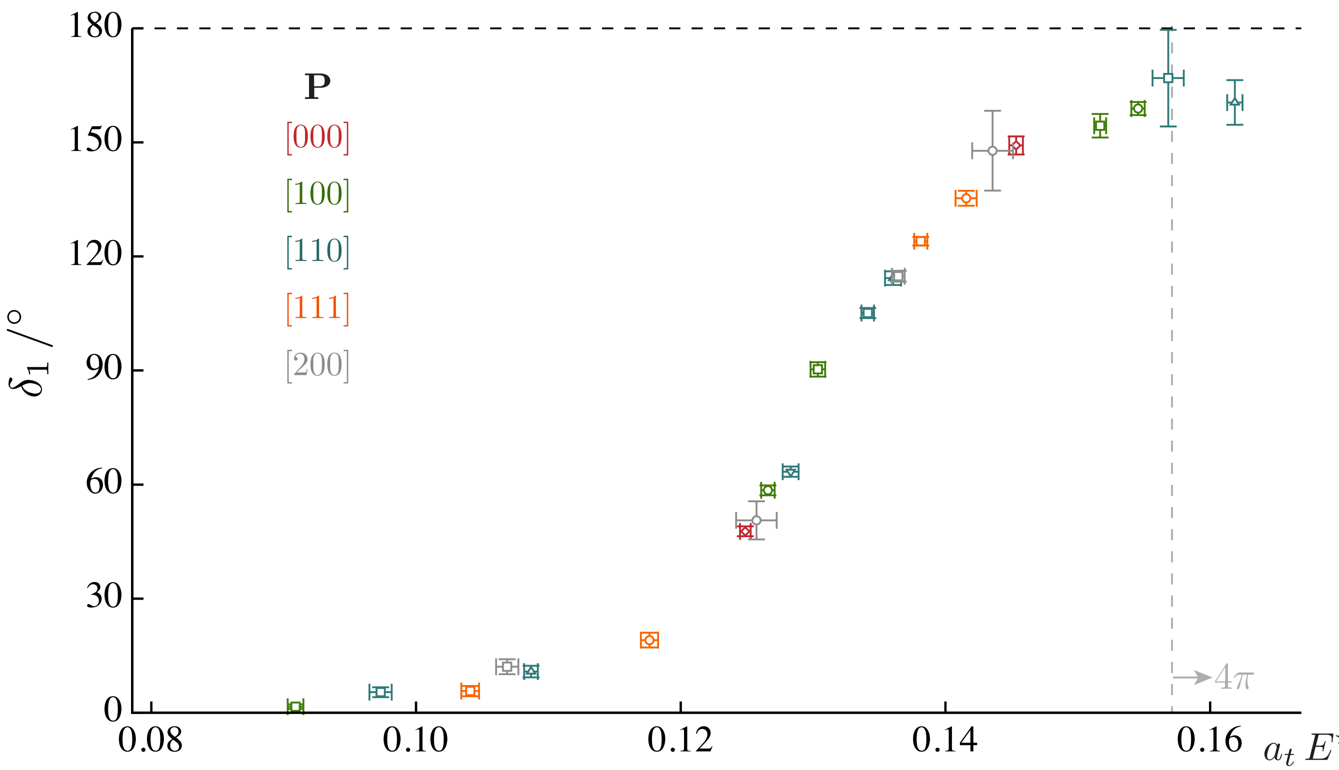
standard text size 20 pt

paper references

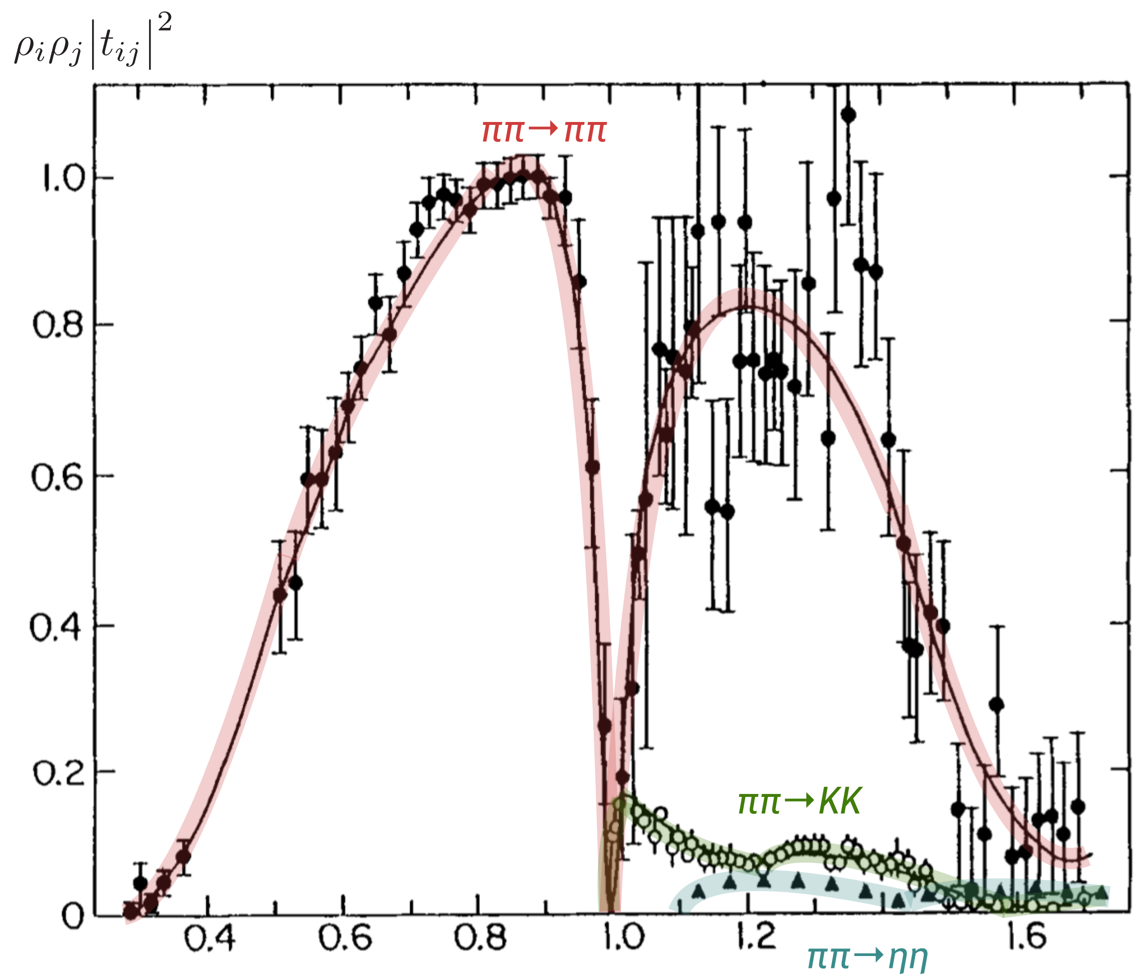
PRD92 0124567 (2007)

arXiv:1234.56789

loosened figure header

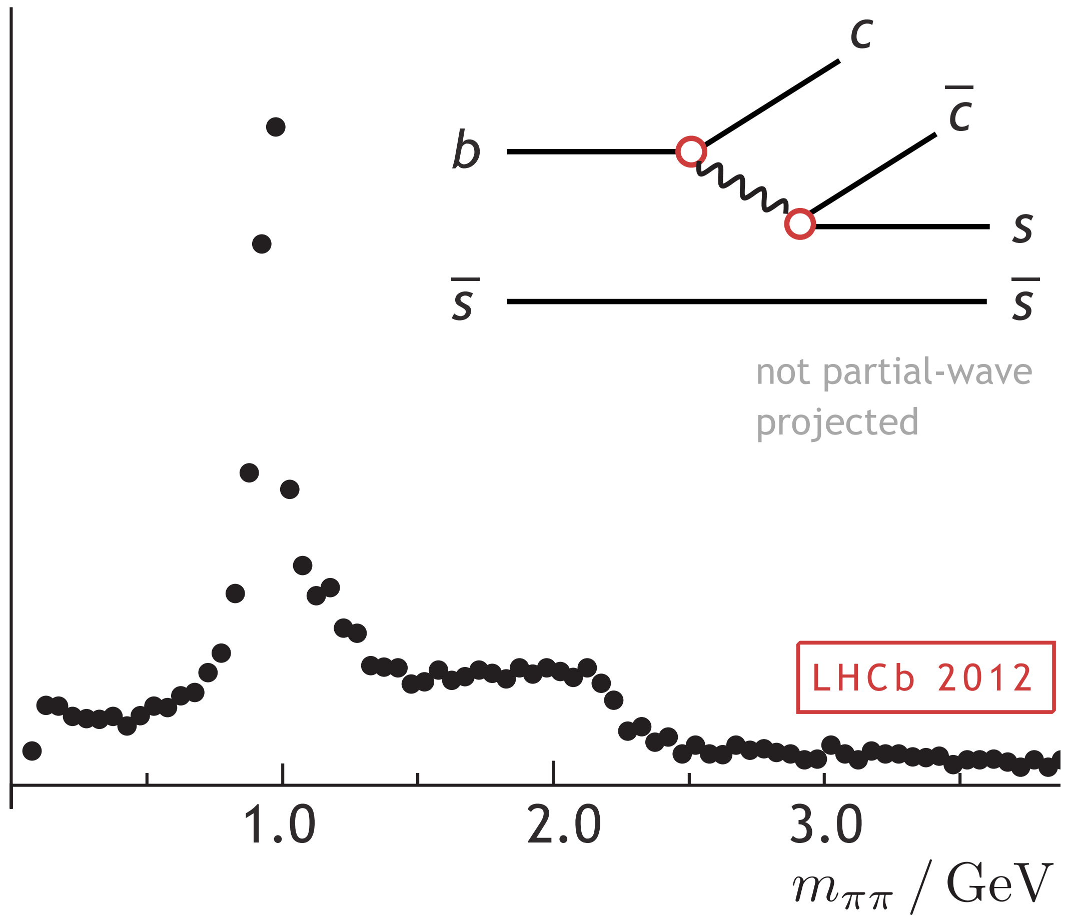


$f_0(980)$ as a dip in elastic scattering

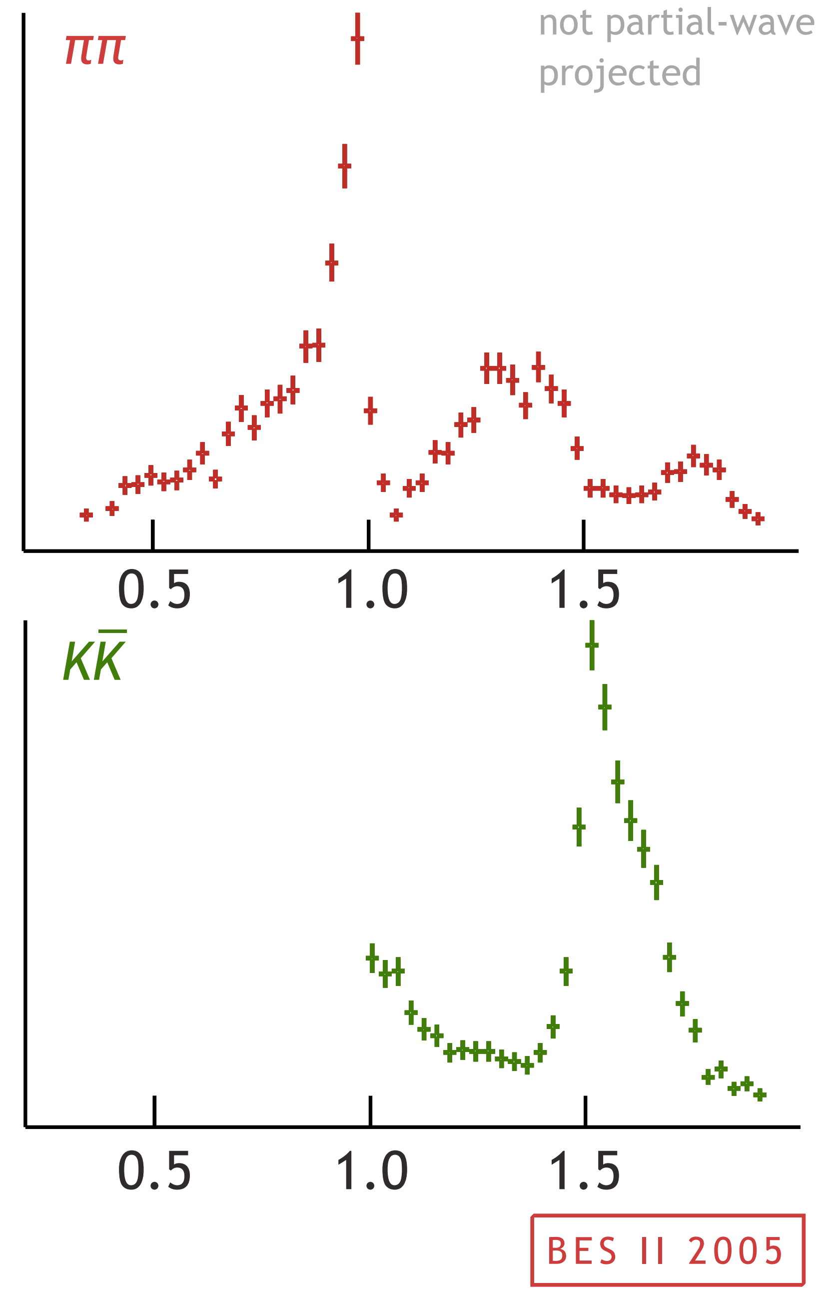


$f_0(980)$ as a peak in "$s\bar{s}$" production

$\bar{B}^0_s \to J/\psi\ \pi^+\pi^-$



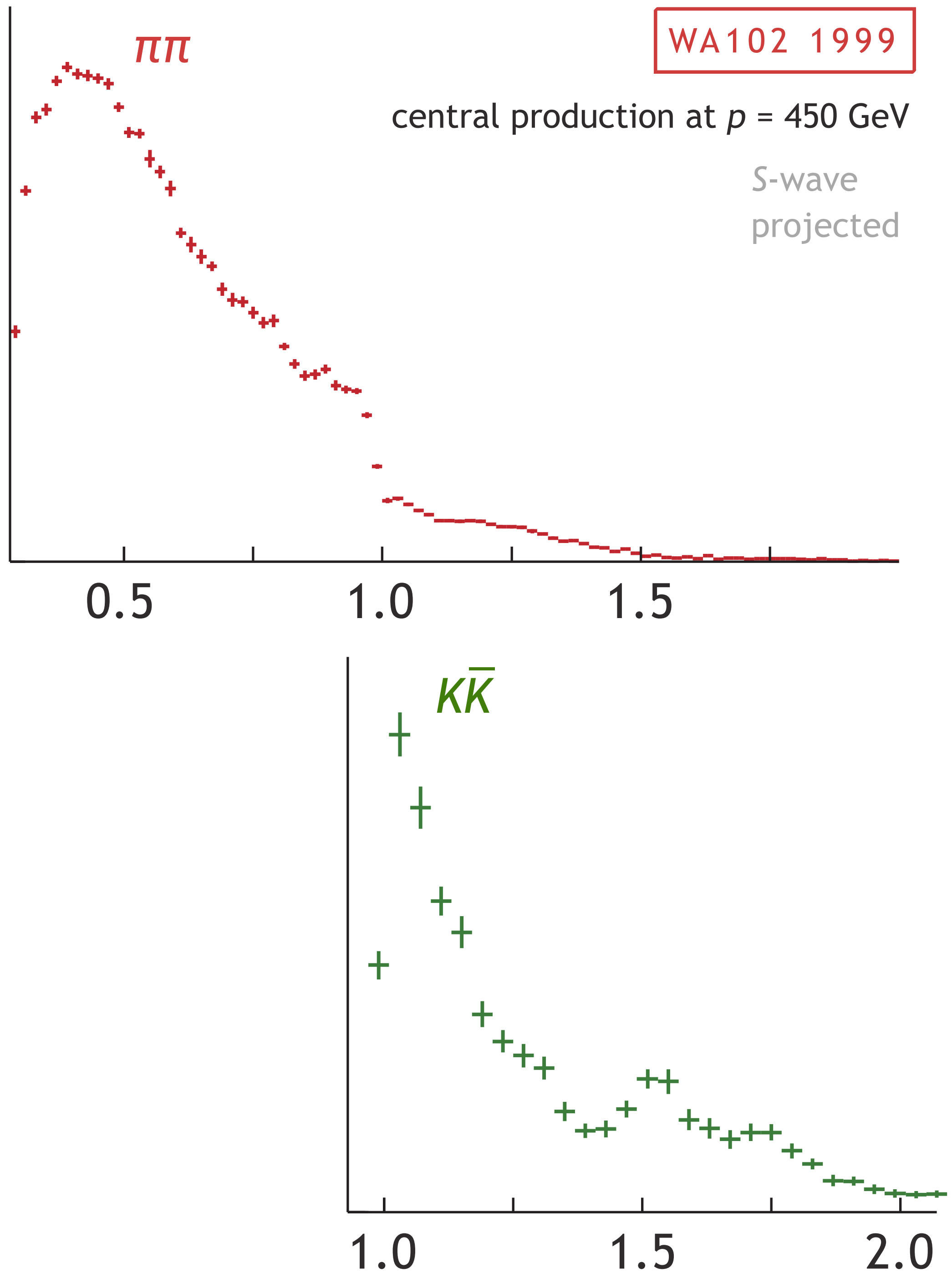
$J/\psi \to \varphi(\pi\pi, K\bar{K})$



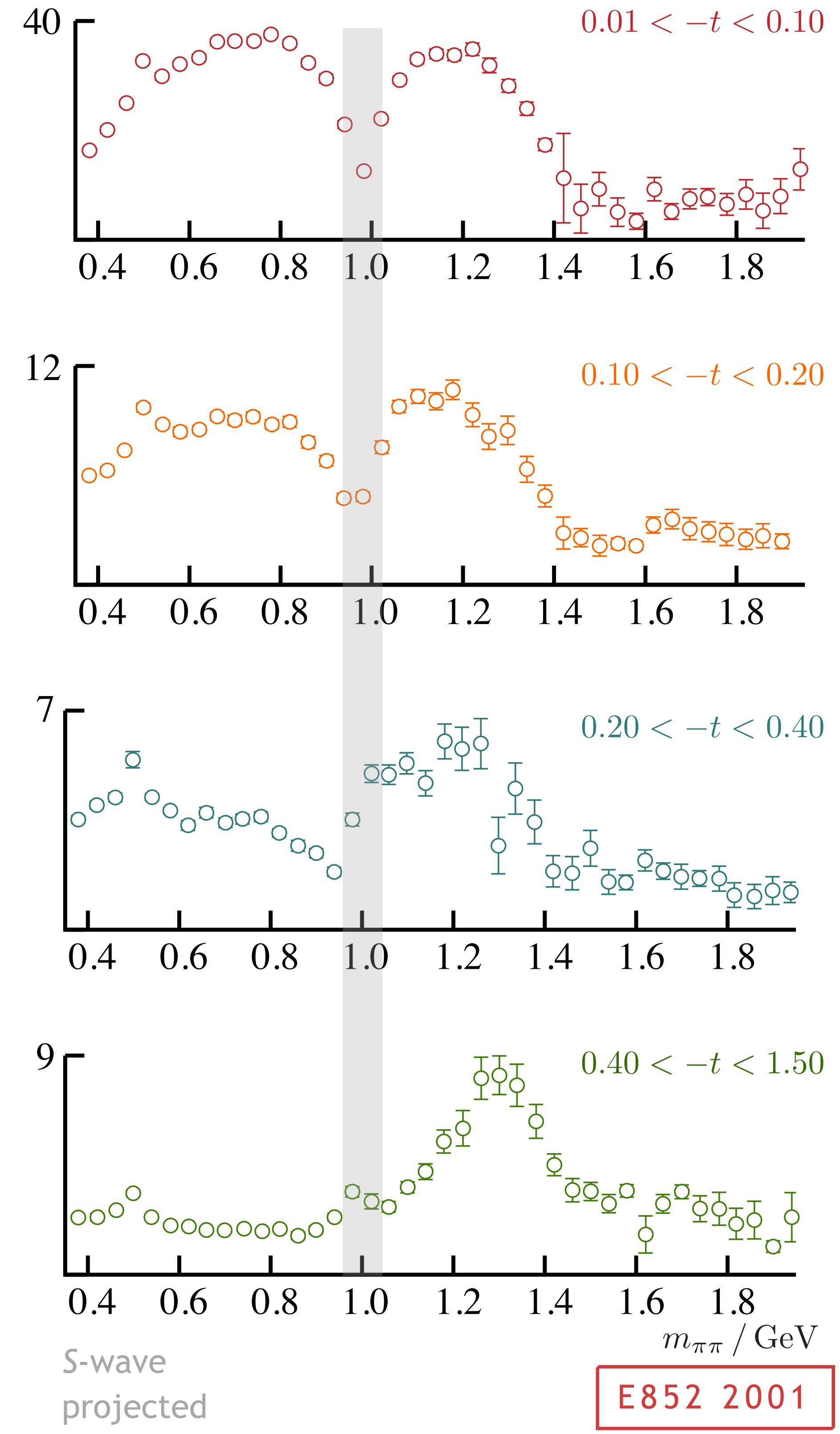
note the rapid turn-on of $K\bar{K}$ at threshold

$f_0(980)$ as ?

$pp \to p(\pi\pi, K\bar{K})p$

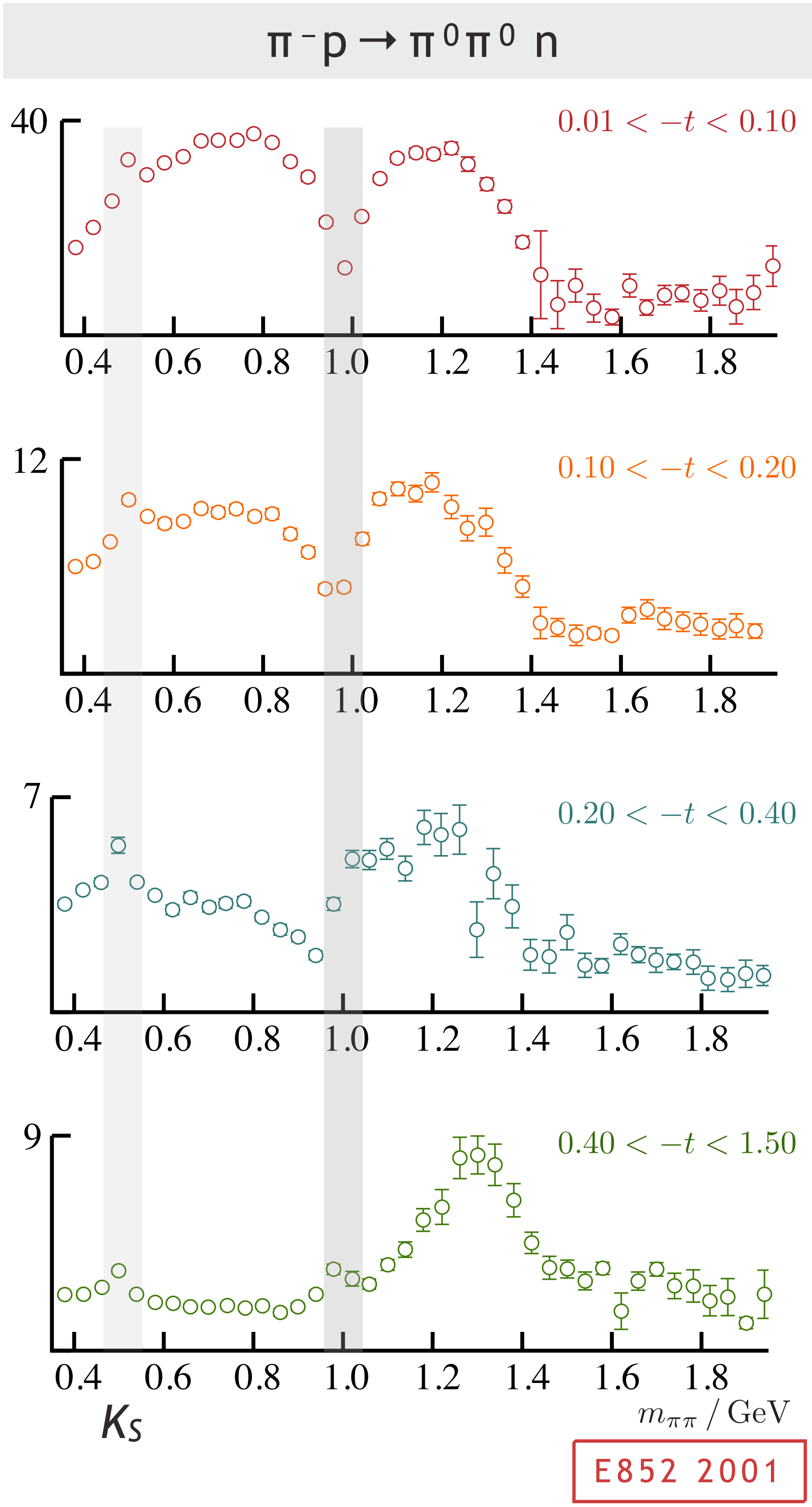


$\pi^- p \to \pi^0 \pi^0 n$



$f_0(980)$ as a shoulder on a large σ 'background'

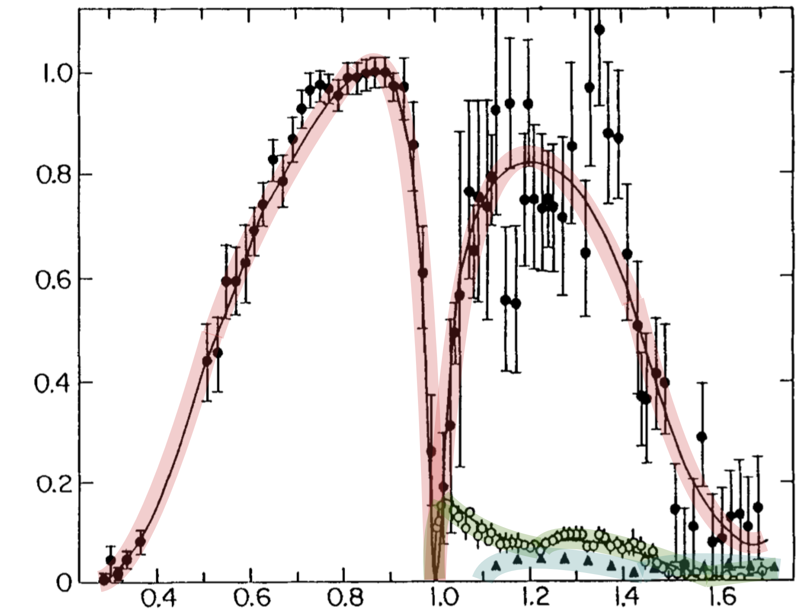
S—wave $\pi\pi$ production



dominated by π exchange
— looks like the the 1970s elastic phase-shift data

other (non-π) exchanges becoming significant, $f_0(980)$ dip less pronounced

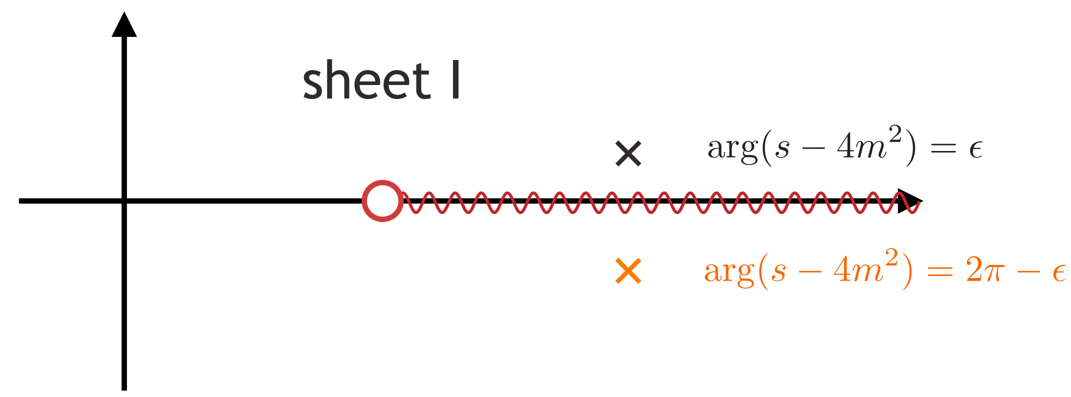
σ no longer large,
$f_0(980)$ starting to be a peak ?



Breit-Wigner pole is on the unphysical sheet

$$\frac{1}{t(s)} \propto M^2 - s - ig^2\rho(s) \qquad \rho(s) = \frac{\sqrt{s-4m^2}}{\sqrt{s}}$$

$$\propto -s + \left(M^2 - ig^2\rho(s)\right) \quad \text{[D]}$$



$\rho = \frac{\sqrt{|s+4m^2|}}{\sqrt{s}}\, e^{i\epsilon/2}$

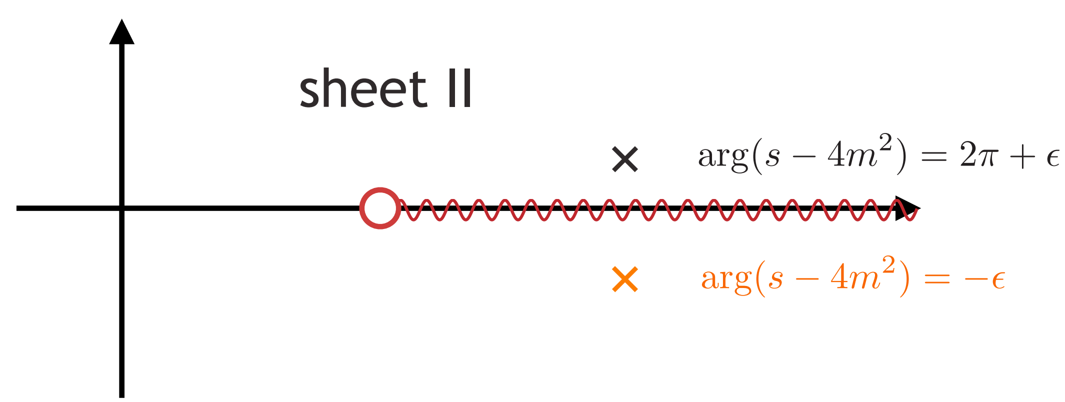
$\mathrm{Re}\,\rho > 0$

so no way for [D] to be zero

$\rho = \frac{\sqrt{|s+4m^2|}}{\sqrt{s}}\, e^{i\pi}\, e^{i\epsilon/2}$

$\mathrm{Re}\,\rho < 0$

so no way for [D] to be zero



$\rho = \frac{\sqrt{|s+4m^2|}}{\sqrt{s}}\, e^{i\pi}\, e^{i\epsilon/2}$

$\mathrm{Re}\,\rho < 0$

so [D] can be zero

$\rho = \frac{\sqrt{|s+4m^2|}}{\sqrt{s}}\, e^{-i\epsilon/2}$

$\mathrm{Re}\,\rho > 0$

so [D] can be zero

(c.f. Burkhardt pg 41 –)

CAUSALITY & ANALYTICITY – simple minded justification

the responce of a physical system $G(t)$ to an input effect $g(t)$ can be described by a convolution

$$G(t) = \int_{-\infty}^{\infty} dt' \, f(t-t') \, g(t')$$

where $f(t-t')$ describes the effect time t' has on time t.

In order for the responce to be CAUSAL we require that $f(t-t')=0$ for $t<t'$

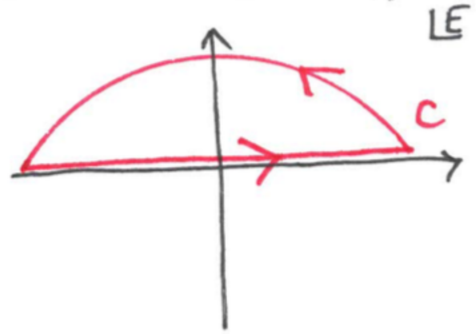
i.e. that $f(\tau)=0$ for $\tau<0$.

Now consider the fourier transform of $f(\tau)$: $\tilde{f}(E) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau \, e^{iE\tau} f(\tau)$

and the inverse fourier transform $f(\tau) = \int_{-\infty}^{\infty} dE \, e^{-iE\tau} \tilde{f}(E)$

and suppose that $\tilde{f}(E)$ is ANALYTIC in the upper half plane of COMPLEX E

– if we close a contour



then $\oint_C dE\, e^{-iE\tau}\tilde{f}(E) = 0$

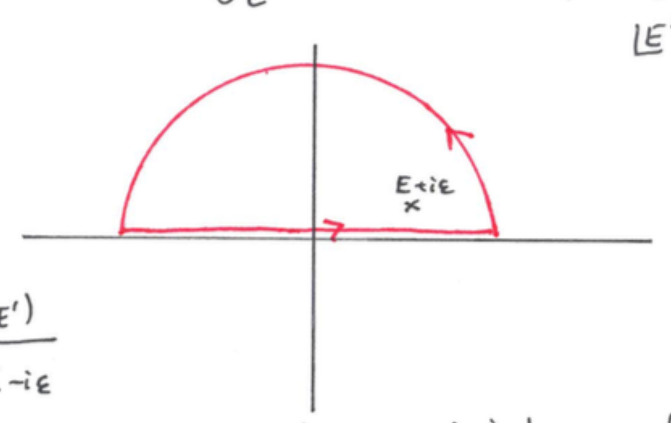
$$= \underbrace{\int_{-\infty}^{\infty} dE\, e^{-iE\tau}\tilde{f}(E)}_{f(\tau)} + \int_{\substack{\text{semi}\\\text{circle}\\\text{at }\infty}} dE\, e^{-iE\tau}\tilde{f}(E)$$

Since we closed in the upper half plane, on the semicircle $\mathrm{Im}(E) > 0$ so $e^{-iE\tau} \sim e^{\mathrm{Im}(E)\tau}$

which tends to zero for $\tau < 0$ $\Rightarrow$ $\underline{f(\tau<0) = 0}$

inverting the logic we conclude that CAUSAL RESPONSE FUNCTIONS have fourier transforms which are ANALYTIC IN THE UPPER HALF-PLANE the cauchy integral formula states that $f(z) = \frac{1}{2\pi i}\oint_C dw \, \frac{f(w)}{w-z}$ if $f(w)$ is analytic inside and on C

∴ consider $\tilde{f}(E+i\epsilon)$ with E real



$$\tilde{f}(E+i\epsilon) = \frac{1}{2\pi i}\oint_C dE' \, \frac{\tilde{f}(E')}{E'-E-i\epsilon}$$

where the contour is in the upper half plane so $\tilde{f}(E')$ is analytic

as we take $\epsilon \to 0$, the pole comes to lie on the contour, so we need the principal value decomposition $\frac{1}{E'-E-i\epsilon} = P\frac{1}{E'-E} + i\pi\,\delta(E'-E)$

and

$$\tilde{f}(E) = \frac{1}{2\pi i} P\int_{-\infty}^{\infty} dE' \, \frac{\tilde{f}(E')}{E'-E} + \frac{i\pi}{2\pi i}\tilde{f}(E)$$

[if the contribution from the semicircle at infinity vanishes]

$\lim_{\epsilon\to 0^+} \tilde{f}(E+i\epsilon)$

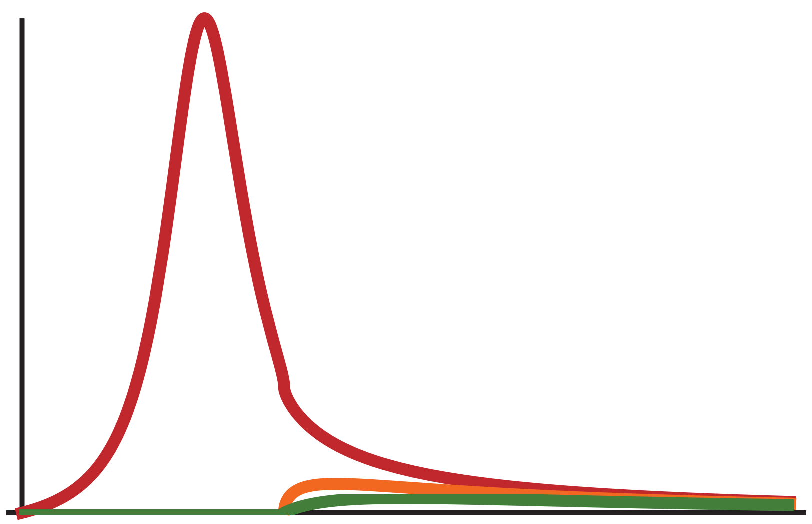
$$= \frac{1}{2\pi i} P\int_{-\infty}^{\infty} dE' \, \frac{\tilde{f}(E')}{E'-E} + \frac{1}{2}\tilde{f}(E)$$

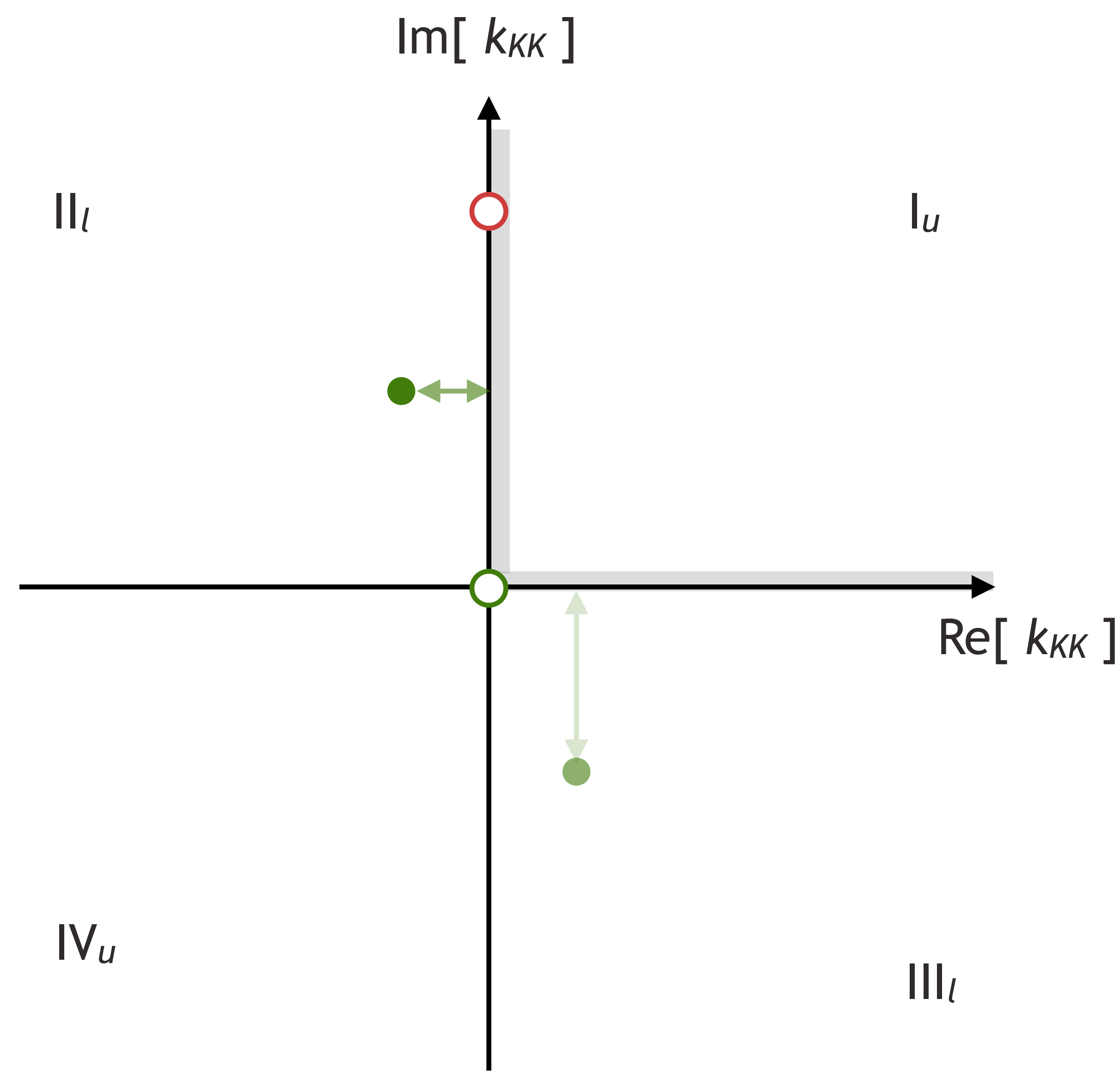
so

$$\tilde{f}(E) = \frac{-i}{\pi} P\int_{-\infty}^{\infty} dE' \, \frac{\tilde{f}(E')}{E'-E} \longrightarrow \begin{cases} \mathrm{Re}\,\tilde{f}(E) = \frac{1}{\pi} P\int_{-\infty}^{\infty} dE' \, \frac{\mathrm{Im}\,\tilde{f}(E')}{E'-E} \\ \mathrm{Im}\,\tilde{f}(E) = -\frac{1}{\pi} P\int_{-\infty}^{\infty} dE' \, \frac{\mathrm{Re}\,\tilde{f}(E')}{E'-E} \end{cases}$$

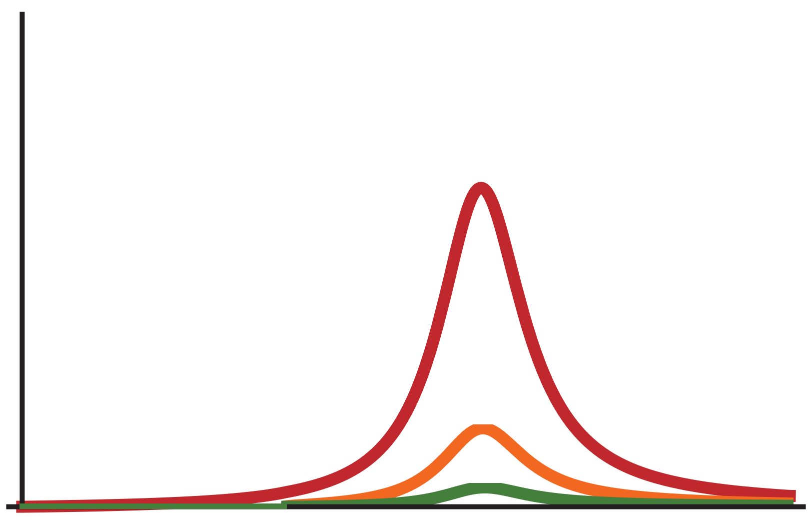
"dispersion relations"

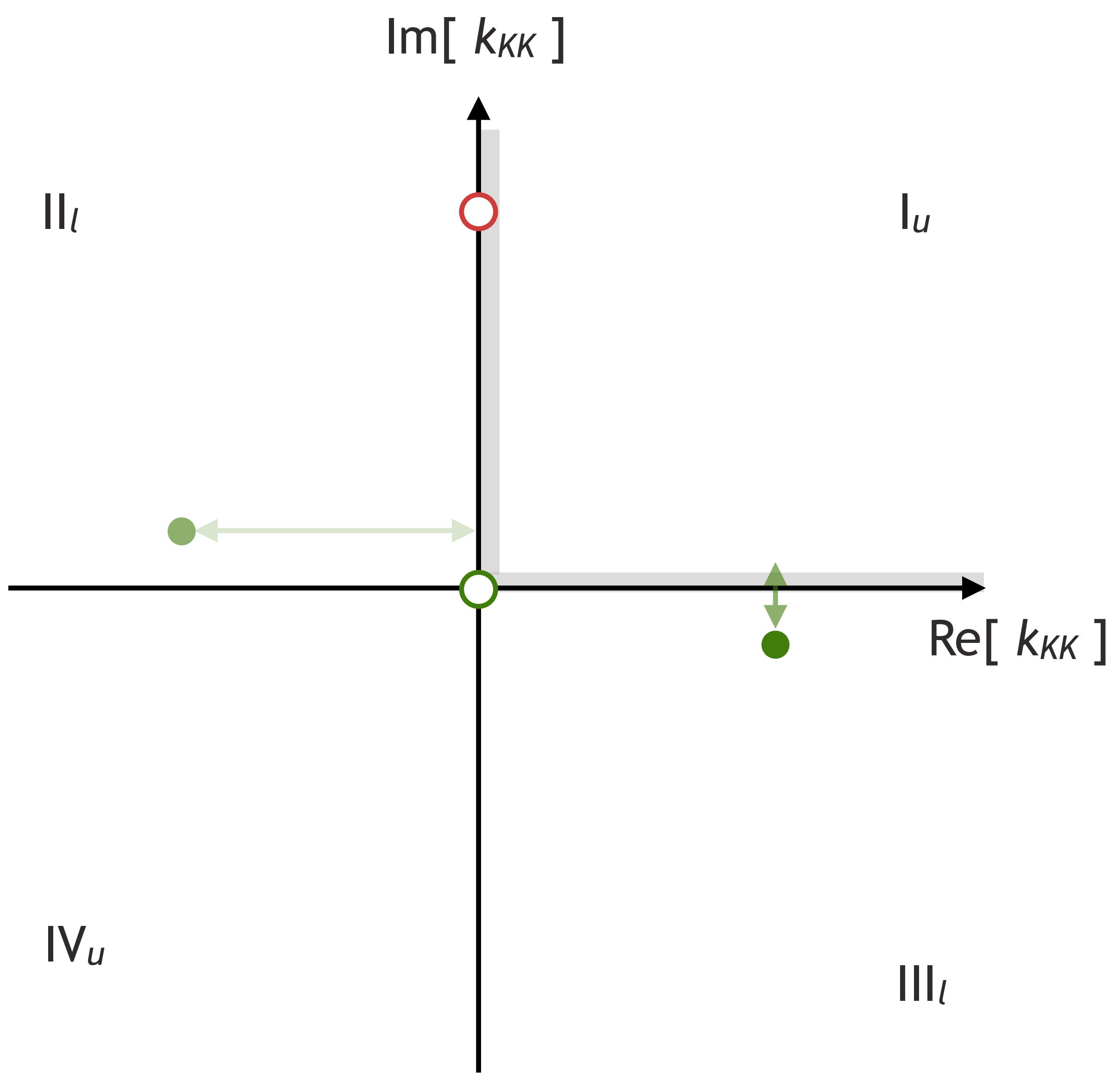
poles in two-channel scattering



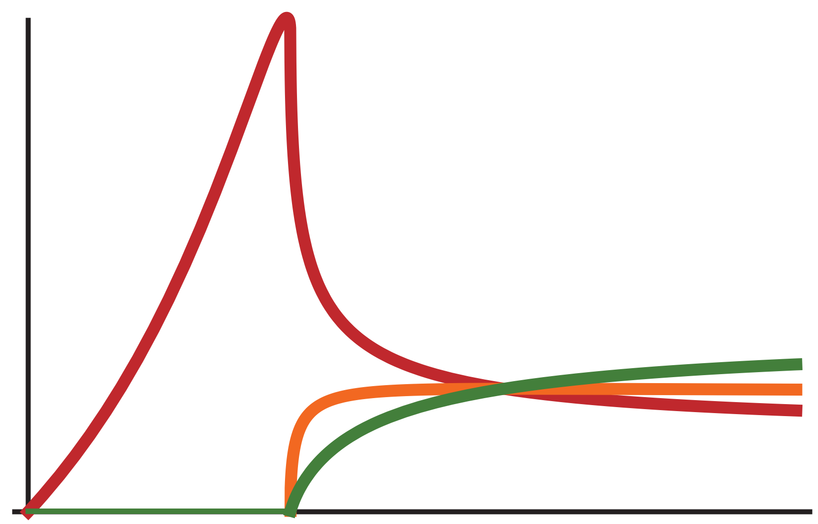


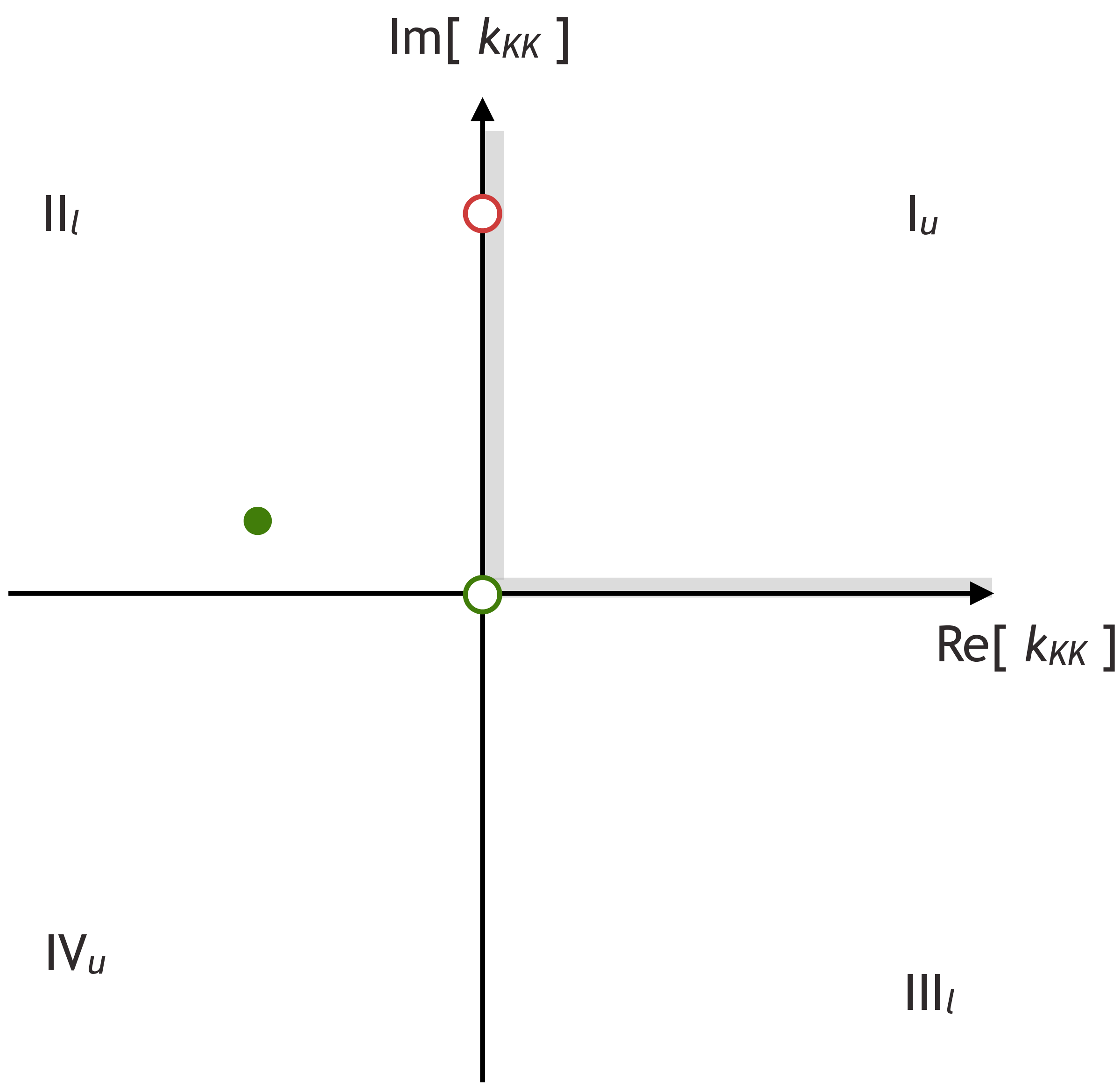
poles in two-channel scattering



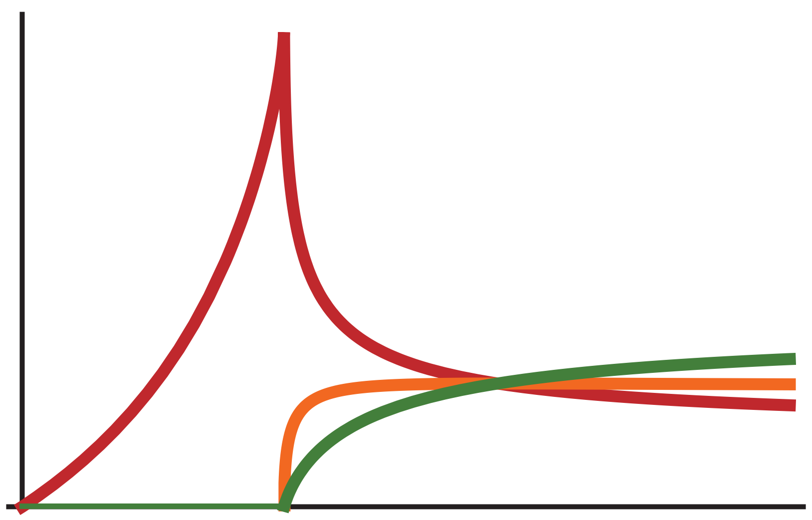


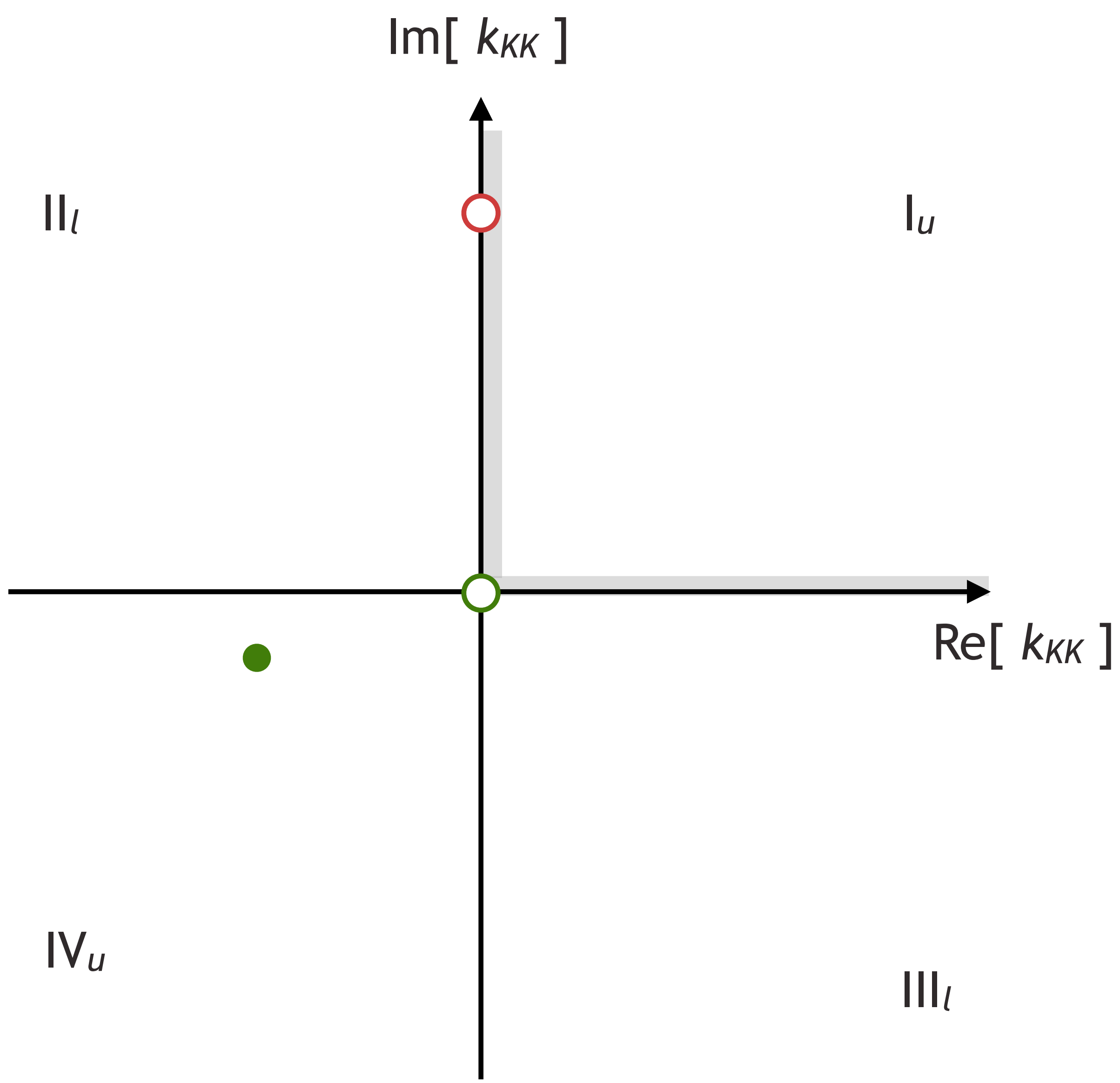
poles in two-channel scattering





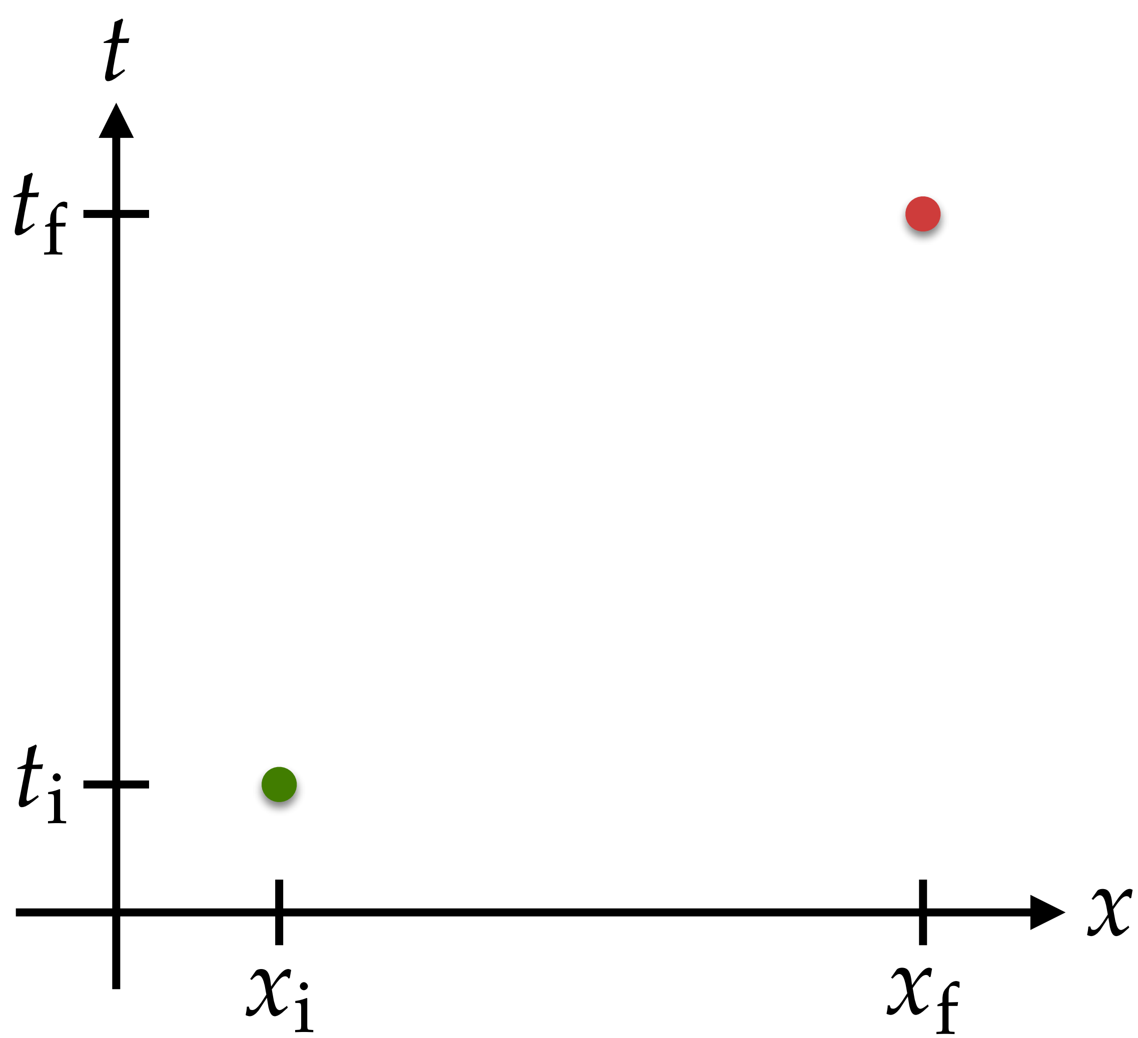
poles in two-channel scattering





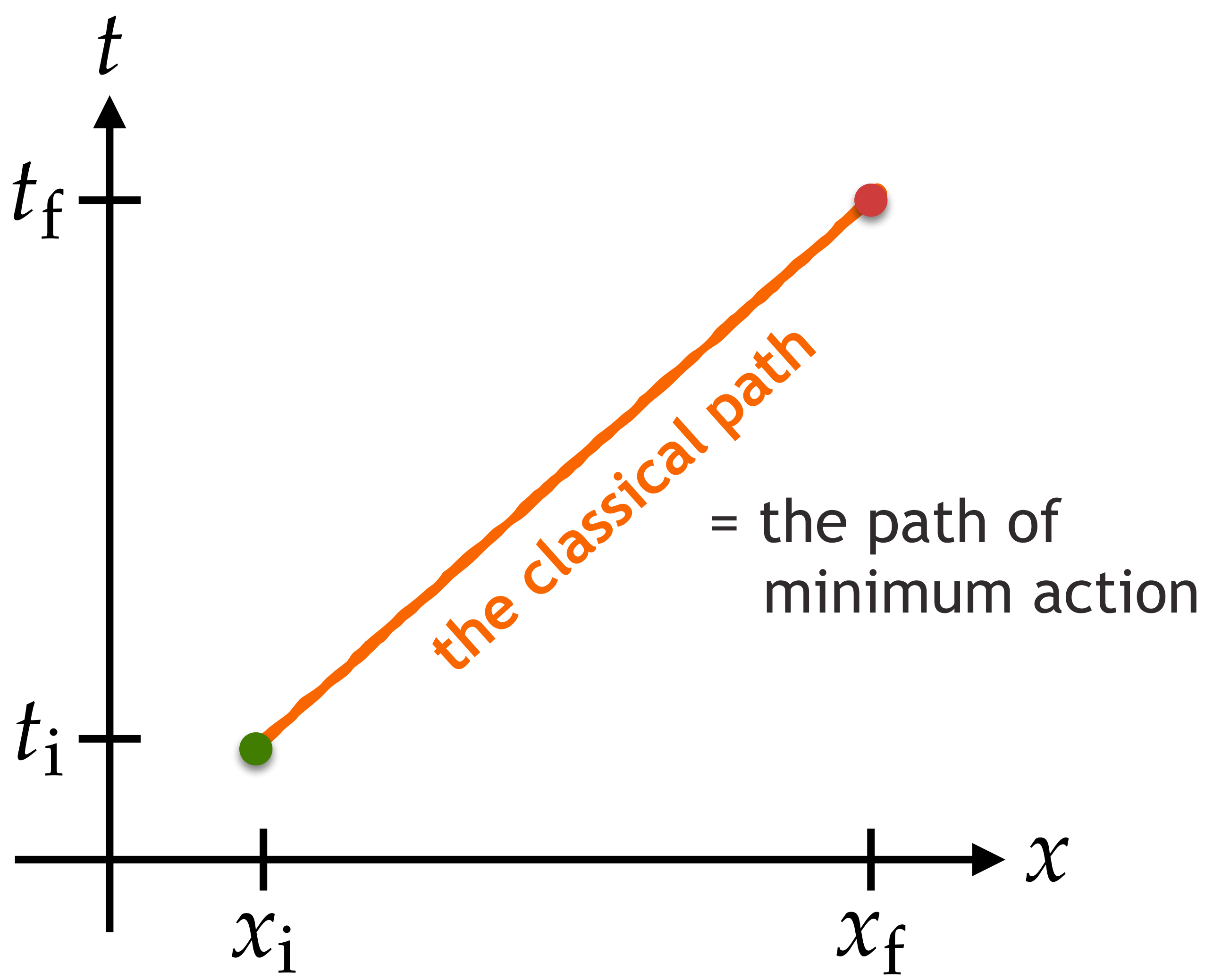
e.g. a free particle moving between a
fixed initial position (x_i,t_i)
and a
fixed final position (x_f,t_f)

SPACE-TIME DIAGRAM



e.g. a free particle moving between a
fixed initial position (x_i,t_i)
and a
fixed final position (x_f,t_f)

SPACE-TIME DIAGRAM



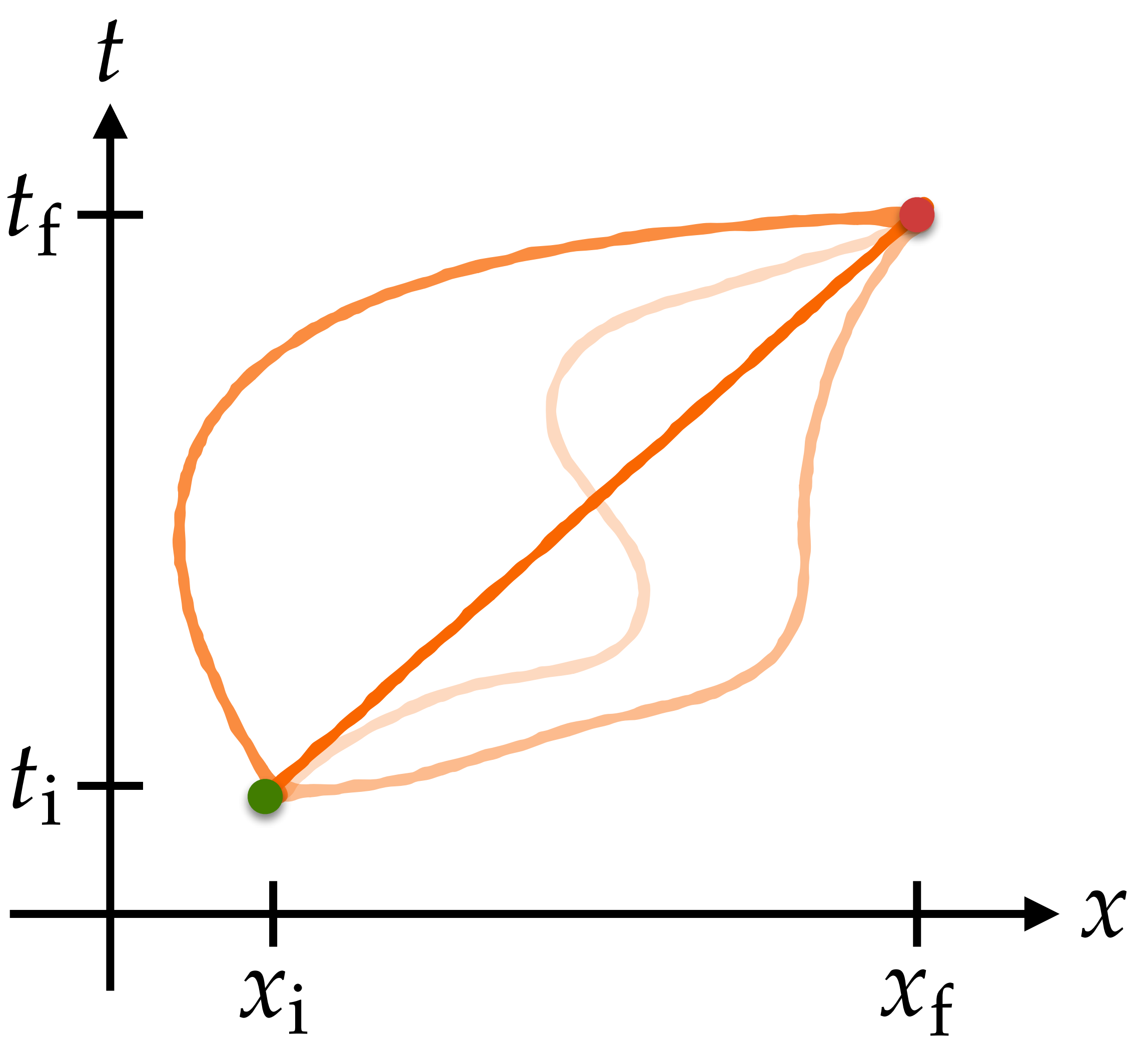
the action

$$S[x(t)] = \int_{t_i}^{t_f} dt\, L(x, \dot{x})$$

$$L = \frac{1}{2} m \dot{x}^2$$

e.g. a free particle moving between a
fixed initial position (x_i, t_i)
and a
fixed final position (x_f, t_f)

SPACE-TIME DIAGRAM



quantum mechanical amplitude

$$\left\langle x_f \middle| e^{-i\hat{H}(t_f - t_i)} \middle| x_i \right\rangle$$

$$= \int \mathcal{D}x \, e^{-iS[x(t)]}$$

'sum' over all paths

... the usual rules of quantum mechanics follow ...

consider a scalar field theory

REAL SCALAR FIELD LAGRANGIAN

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \tfrac{1}{2} m^2 \varphi^2 + V[\varphi]$$

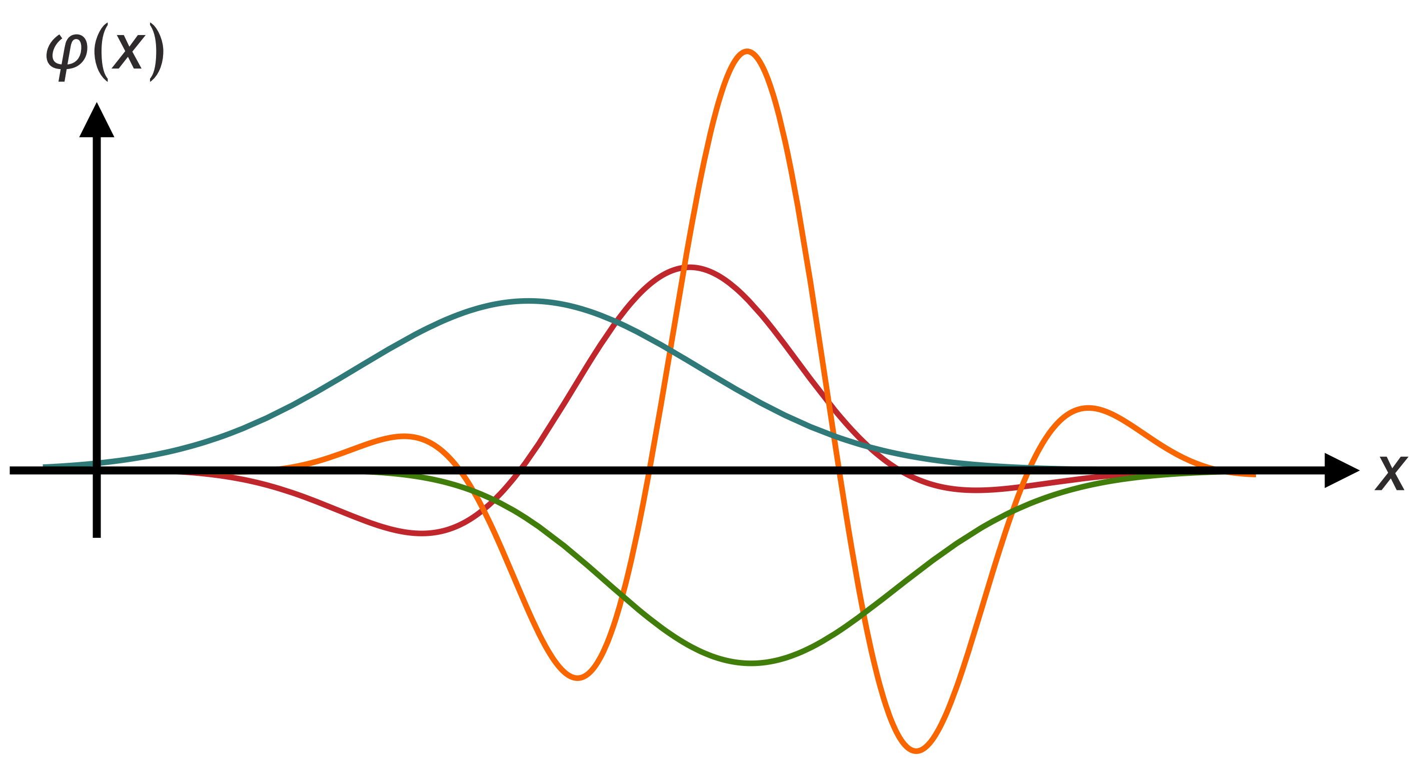
can define a path integral

$$Z = \int \mathcal{D}\varphi(x)\, e^{-iS[\varphi(x)]}$$

'sum' over all field configurations

with action $S[\varphi] = \int d^4x\, \mathcal{L}[\varphi(x)]$

e.g. in one-dimension



path-integrals in quantum field theory

consider a scalar field theory

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'sum' over all
field configurations

with action $S[\varphi] = \int d^4x\, \mathcal{L}[\varphi(x)]$

and correspondingly **correlation functions**

$$\langle 0 | \hat{\varphi}(x'') \hat{\varphi}(x') | 0 \rangle = \frac{1}{Z} \int \mathcal{D}\varphi(x)\; \varphi(x'')\, \varphi(x')\, e^{-iS[\varphi(x)]}$$

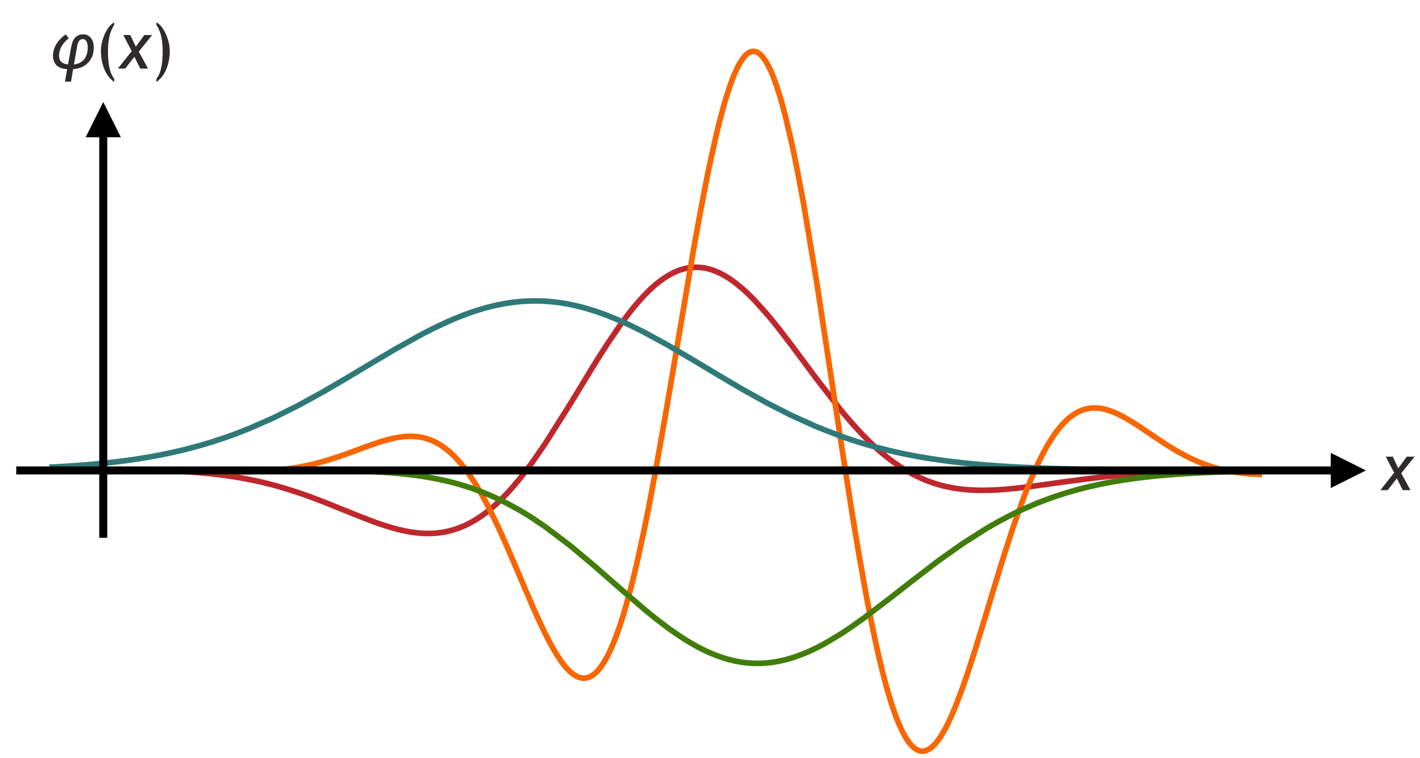
a concrete meaning for $\int \mathcal{D}\varphi(x)$

comes from considering the fields on a space-time grid

$$\int \mathcal{D}\varphi(x) = \prod_x \int d\varphi_x$$

do an integral over all values the field can take at each point on the grid

e.g. in one-dimension



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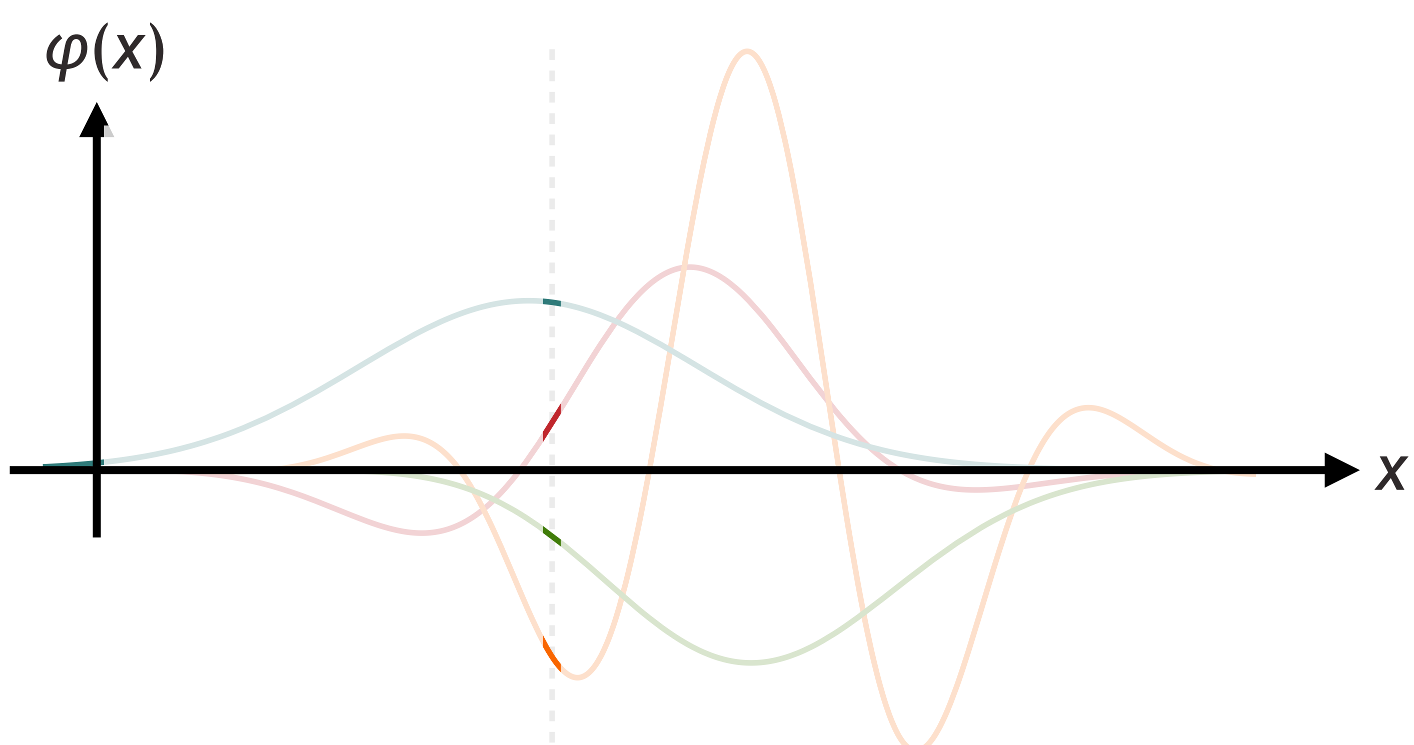

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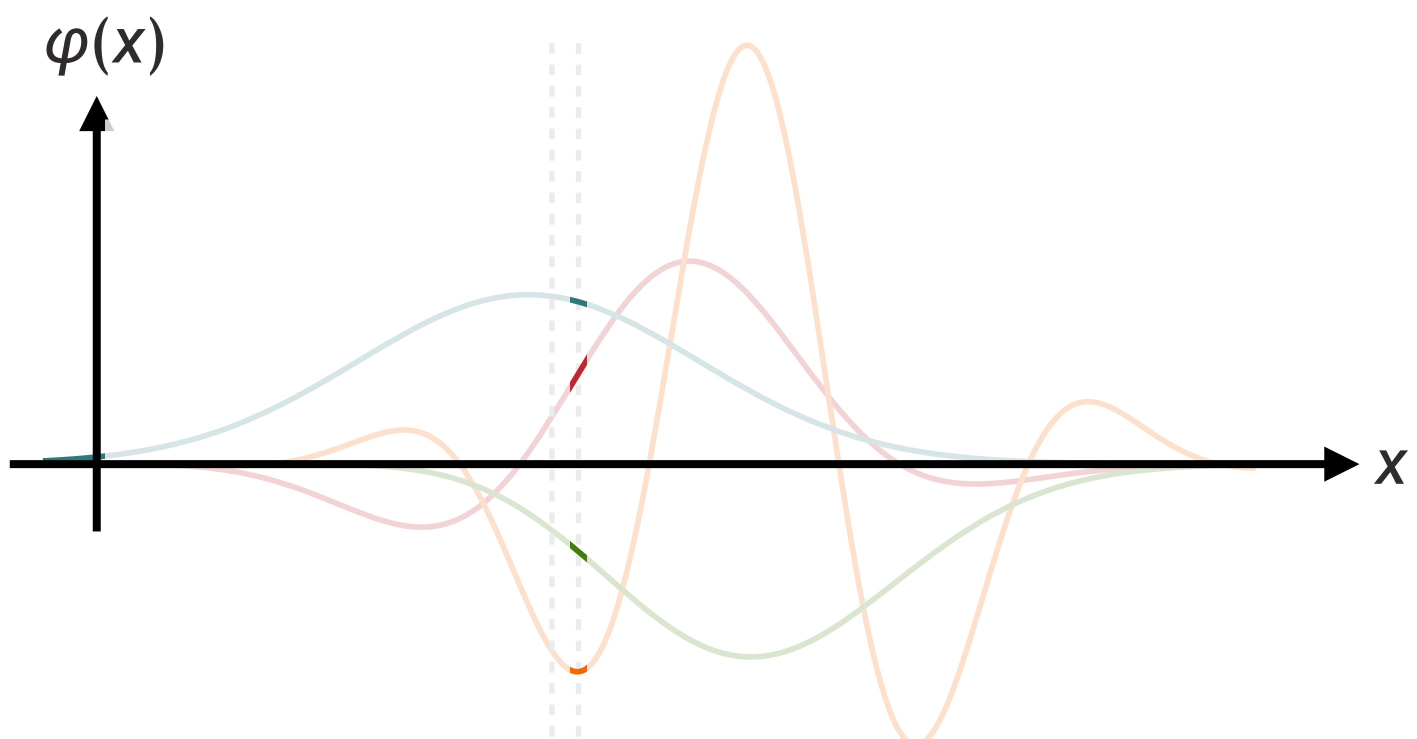

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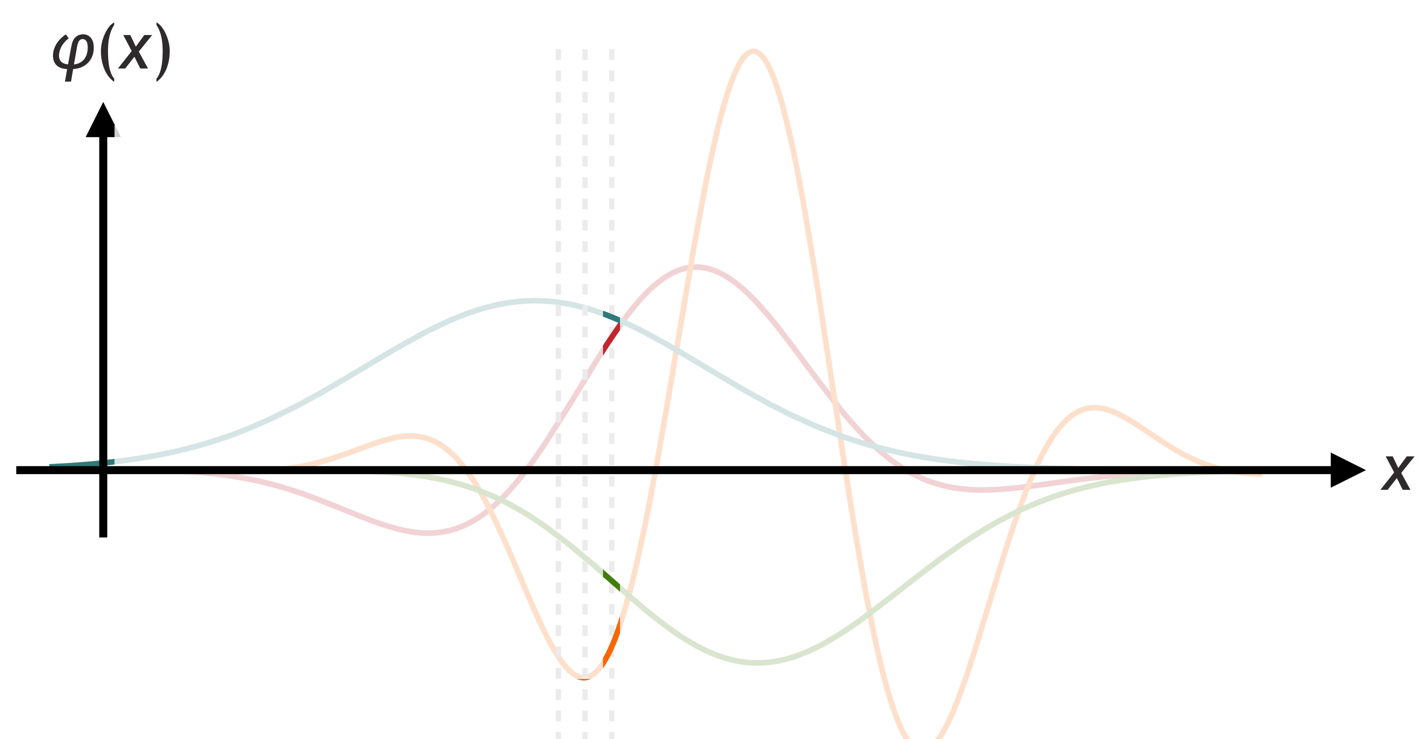

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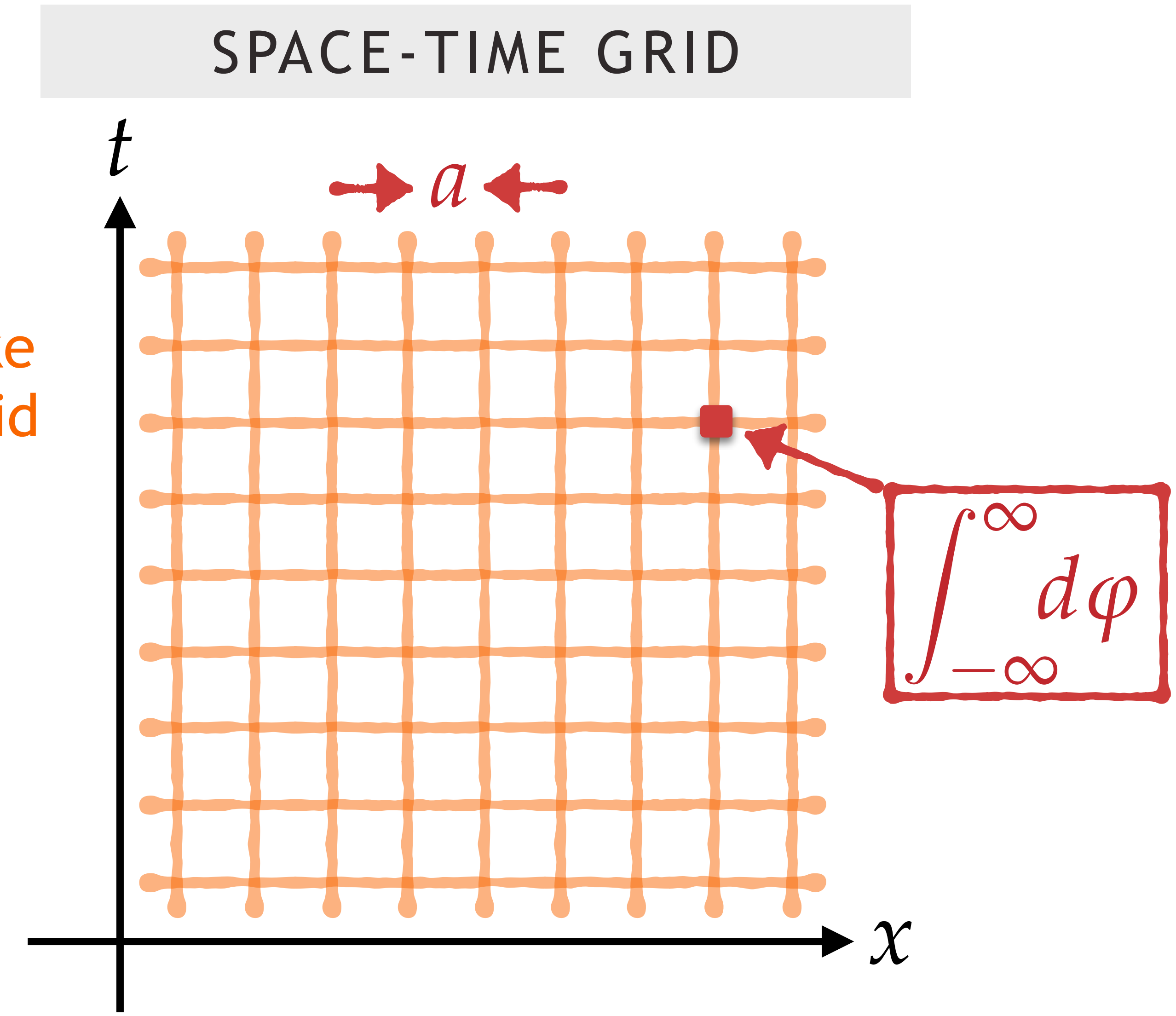

$$Z = \int \mathcal{D}\varphi(x)\; e^{-iS[\varphi(x)]}$$

now make a transform to an *imaginary time variable* $t \to -i\tau$

then the argument of the exponential becomes

$$-iS = -i\int d^3x\, dt\, \mathcal{L} = -\int d^3x\, d\tau\, \mathcal{L}_{\mathrm{E}} = -S_{\mathrm{E}}$$

and the integrand transforms

$$e^{-iS} \to e^{-S_{\mathrm{E}}}$$

EUCLIDEAN PATH INTEGRAL

$$Z_{\mathrm{E}} = \int \mathcal{D}\varphi(x)\, e^{-S_{\mathrm{E}}[\varphi]}$$

EUCLIDEAN PATH INTEGRAL

$$Z_{\mathrm{E}} = \int \mathcal{D}\varphi(x)\, \boxed{e^{-S_{\mathrm{E}}[\varphi]}}$$

probability for a field configuration $\varphi(x)$

importance sampled Monte Carlo

generate field configurations (on the space-time grid) according to the probability above

obtain an ensemble of configurations $\{\varphi_x\}^{i=1\ldots N}$

an observable function of the field (e.g. a correlation function)

$$\langle 0 | O[\hat{\varphi}] | 0 \rangle = \int \mathcal{D}\varphi \, O[\varphi] \, e^{-S_{\mathrm{E}}[\varphi]}$$

can now be estimated as an average over the ensemble

$$\langle 0 | O[\hat{\varphi}] | 0 \rangle \approx \overline{O} = \frac{1}{N} \sum_{i=1}^{N} O[\varphi^{(i)}]$$

and the uncertainty due to the finite ensemble can be estimated via the variance on the mean

$$\epsilon(O) = \sqrt{\frac{1}{N(N-1)} \sum_{i=1}^{N} \left(O[\varphi^{(i)}] - \overline{O} \right)^2}$$

ENSEMBLE MEAN & ERROR

$$\langle 0 | O[\hat{\varphi}] | 0 \rangle \approx \overline{O} \pm \epsilon(O)$$

consider $\langle 0 | \mathcal{O}_f(t)\, \mathcal{O}_i^\dagger(0) | 0 \rangle$

and since time evolution in Euclidean time is $\mathcal{O}(t) = e^{\hat{H}t}\mathcal{O}(0)e^{-\hat{H}t}$

we have $$\langle 0 | \mathcal{O}_f(t)\, \mathcal{O}_i^\dagger(0) | 0 \rangle = \langle 0 | \mathcal{O}_f(0)\, e^{-\hat{H}t}\, \mathcal{O}_i^\dagger(0) | 0 \rangle$$

now let's assume the Hamiltonian has a complete set of discrete eigenstates

$$\hat{H}|\mathfrak{n}\rangle = E_\mathfrak{n}|\mathfrak{n}\rangle$$
$$1 = \sum_\mathfrak{n} |\mathfrak{n}\rangle\langle\mathfrak{n}|$$

and thus $$\langle 0 | \mathcal{O}_f(t)\, \mathcal{O}_i^\dagger(0) | 0 \rangle = \sum_\mathfrak{n} e^{-E_\mathfrak{n} t} \langle 0 | \mathcal{O}_f(0) | \mathfrak{n} \rangle \langle \mathfrak{n} | \mathcal{O}_i^\dagger(0) | 0 \rangle$$

what about QCD ?

gauge theory with SU(3) ‘color’ symmetry

QCD LAGRANGIAN

$$\mathcal{L} = \overline{\psi}\left(i\gamma^{\mu} D_{\mu} - m\right)\psi - \tfrac{1}{2}\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right)$$

gauge covariant derivative $D_{\mu} = \partial_{\mu} + igA_{\mu}$

field strength tensor

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + ig\left[A_{\mu}, A_{\nu}\right]$$

QCD FIELDS

quark field $\psi^{i}_{\alpha}(x)$

color index *i = 1...3*

Dirac spin index *α = 1...4*

gluon field $A^{ij}_{\mu}(x)$

traceless matrix in color space

Lorentz vector

$$= \sum_{a=1}^{8} A^{a}_{\mu}(x)\, t^{a}$$

gauge theory with SU(3) 'color' symmetry

QCD LAGRANGIAN

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gauge covariant derivative $$D_\mu = \partial_\mu + igA_\mu$$

field strength tensor $$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\left[A_\mu, A_\nu\right]$$

relativistic fermions

$$\overline{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi$$

color vector current

$$g\,\left(\overline{\psi}\gamma^\mu t^a \psi\right)\,A_\mu^a$$

massless gluons

$$\left(\partial_\mu A_\nu - \partial_\mu A_\nu\right)^2$$

gluon self interactions

$$g[A,A]\,\partial A\,,\; g^2\left([A,A]\right)^2$$

QCD on a lattice

the QCD action in Euclidean space-time reads

$$\mathcal{S}_\mathrm{E} = \int d^4x_\mathrm{E}\, \overline{\psi}(\gamma_\mu D_\mu + m)\psi + \tfrac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}$$

and we'd like to discretize this on a hypercubic grid

quark fields take (spinor) values on the sites of the grid $\psi^i_\alpha(x_\mu = a\, n_\mu)$

derivatives can be constructed as finite differences

e.g. $\partial f(x) \to \frac{1}{2a}\big(f(x+a) - f(x-a)\big)$

but what shall we do with the gluon fields … ?

‘parallel transporters’ & gauge invariance

in the continuum theory - consider a quark-antiquark field pair separated by some distance

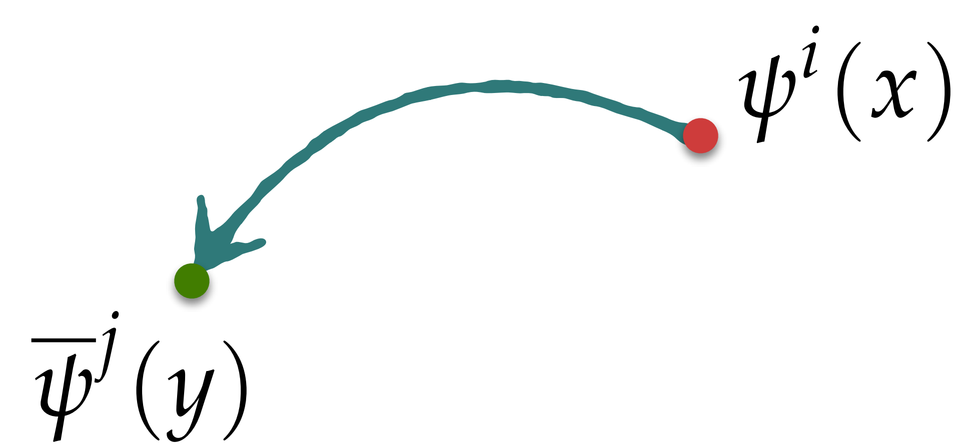
the combination $\overline{\psi}^j(y)\,\delta_{ji}\,\psi^i(x)$ is not gauge-invariant

can perform **different** local gauge transformations at *x* and *y*

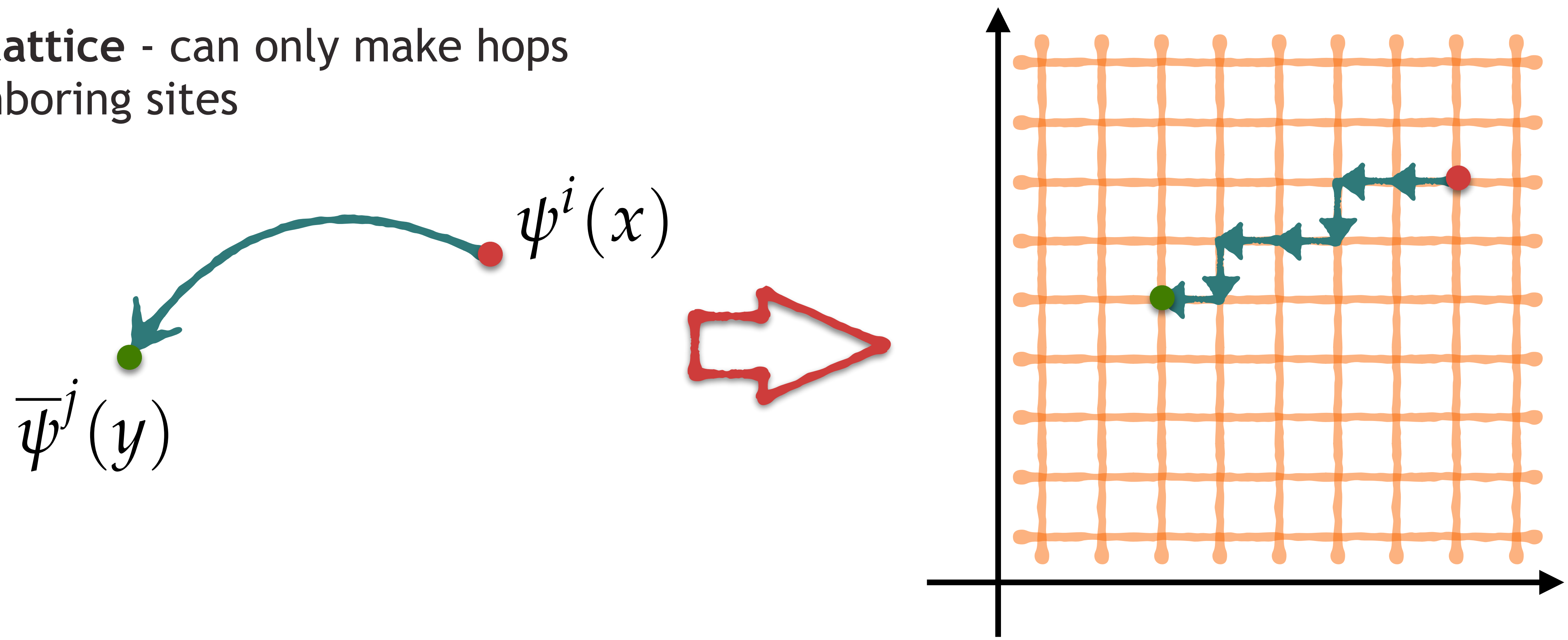
a gauge-invariant combination is

$$\overline{\psi}^j(y)\left[e^{ig\int_x^y dz_\mu A^\mu(z)}\right]_{ji}\psi^i(x)$$

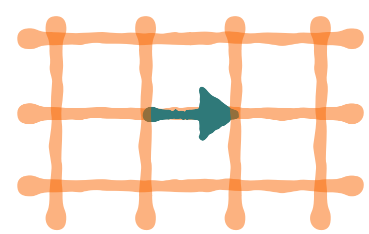
‘Wilson line’
transports the color



on the lattice - can only make hops to neighboring sites



shortest path between neighboring sites = a 'link'



$$\left[e^{igaA_\mu(x)}\right]_{ji}$$

$$U_\mu(x) = e^{igaA_\mu(x)}$$

SU(3) matrix on each **link** of the lattice gauge invariant version of a finite difference:

$$\overline{\psi}(x)\,\gamma_\mu\,U_\mu(x)\,\psi(x+\hat{\mu}a) - \overline{\psi}(x)\,\gamma_\mu\,U^\dagger_\mu(x-\hat{\mu}a)\,\psi(x-\hat{\mu}a)$$

$$\xrightarrow[a\,\to\,0]{} \;2a\,\overline{\psi}\gamma_\mu\big(\partial_\mu + igA_\mu\big)\psi$$

... using constructions like these can build discretized actions ...

$$S_{\mathrm{E}}^{\mathrm{ferm}} = \overline{\psi}^{i\alpha}_x\, M^{i\alpha,j\beta}_{x,y}[U]\,\psi^{j\beta}_y$$

it's possible to perform **exactly** the fermion integration in the path integral

$$S_{\mathrm{E}} = S_{\mathrm{E}}^{\mathrm{ferm}} + S_{\mathrm{E}}^{\mathrm{gauge}} = \overline{\psi} M[U] \psi + S_{\mathrm{E}}^{\mathrm{gauge}}[U]$$

$$\int \mathcal{D}\psi \mathcal{D}\overline{\psi} \mathcal{D}U \, e^{-S_{\mathrm{E}}} = \int \mathcal{D}U \, e^{-S_{\mathrm{E}}^{\mathrm{gauge}}[U]} \boxed{\int \mathcal{D}\psi \mathcal{D}\overline{\psi} \, e^{-\overline{\psi}\, M[U]\, \psi}}$$

$$= \det M[U]$$

$$\int \mathcal{D}\psi \mathcal{D}\overline{\psi} \mathcal{D}U \, e^{-S_{\mathrm{E}}} = \int \mathcal{D}U \boxed{\det M[U] \, e^{-S_{\mathrm{E}}^{\mathrm{gauge}}[U]}}$$

can treat this as the probability for configuration $U_\mu(x)$

what happens to correlation functions ?

$$\langle 0 | \hat{\psi}^{i\alpha}(x)\, \hat{\overline{\psi}}^{j\beta}(y) | 0 \rangle = \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \mathcal{D}U \; \psi^{i\alpha}(x)\, \overline{\psi}^{j\beta}(y)\, e^{-S_\mathrm{E}}$$

correlation between
a quark field at x of color i and spin α
and
a quark field at y of color j and spin β

$$\int \mathcal{D}\psi \mathcal{D}\overline{\psi} \mathcal{D}U \; \psi^{i\alpha}(x)\, \overline{\psi}^{j\beta}(y)\, e^{-S_\mathrm{E}} = \int \mathcal{D}U\, e^{-S_\mathrm{E}^\mathrm{gauge}[U]} \int \mathcal{D}\psi \mathcal{D}\overline{\psi}\, \psi^{i\alpha}(x)\, \overline{\psi}^{j\beta}(y)\, e^{-\overline{\psi}\, M[U]\, \psi}$$

$$= \int \mathcal{D}U \left[M^{-1}[U] \right]^{i\alpha,j\beta}_{x,y} \det M[U]\, e^{-S_\mathrm{E}^\mathrm{gauge}[U]}$$

the probability distribution

$$= \sum_{\{U\}} \left[M^{-1}[U] \right]^{i\alpha,j\beta}_{x,y}$$

will need to compute
this on every configuration

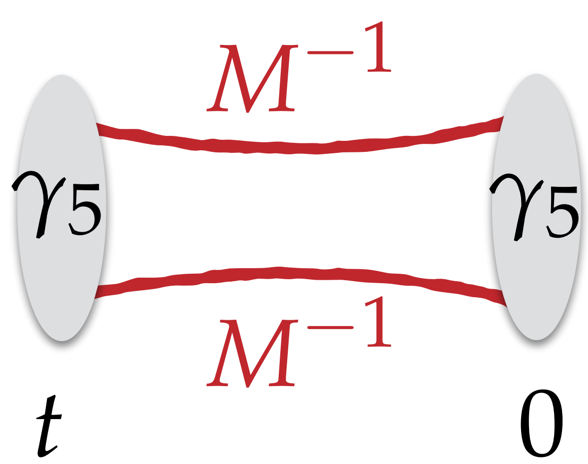
consider an actually useful correlation function

$$\langle 0 | \sum_{\vec{x}} \overline{\psi}\gamma_5\psi(\vec{x},t)\, \overline{\psi}\gamma_5\psi(\vec{0},0) | 0 \rangle$$

projected into zero momentum

pseudoscalar quantum numbers

$$= -\sum_{\{U\}} \mathrm{tr}\left[M^{-1}[U]\right]_{\vec{0}0,\vec{x}t}\gamma_5\left[M^{-1}[U]\right]_{\vec{x}t,\vec{0}0}\gamma_5$$



$$[M[U]]_{\vec{y}t',\vec{x}t}\,\psi_{\vec{x}t} = \delta_{\vec{y},\vec{0}}\,\delta_{t',0}$$

$$\psi_{\vec{x}t} = \left[M[U]^{-1}\right]_{\vec{x}t,\vec{0}0}$$

linear system
of the form **A**x=*b*

a simple meson correlation function

consider an actually useful correlation function

$$\langle 0 | \sum_{\vec{x}} \overline{\psi}\gamma_5\psi(\vec{x},t) \; \overline{\psi}\gamma_5\psi(\vec{0},0) | 0 \rangle$$

projected into zero momentum

pseudoscalar quantum numbers

$$= -\sum_{\{U\}} \mathrm{tr}\left[M^{-1}[U]\right]_{\vec{0}0,\vec{x}t} \gamma_5 \left[M^{-1}[U]\right]_{\vec{x}t,\vec{0}0} \gamma_5$$

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in fact there are much better ways to compute hadron correlation functions
… smearing the quark fields …
… *distillation* … *PRD80 054506 (2009)*

of course we are making approximations in order to make this practical

$a > 0$ the lattice spacing plays multiple roles:

it's a momentum/energy cutoff $\Lambda \sim \frac{1}{a}$

it appears as a scale when computing dimensionfull quantities $\hat{m} = a\, m$

its size controls discretization errors

$$X(a) = X(0) + a\,\Delta X_1 + a^2\,\Delta X_2 + \ldots$$

$L < \infty$ we calculate in a finite volume

provided $L \gg \frac{1}{m_\pi}$ the effects are manageable

in fact we'll use the finite volume as a tool of course we are making approximations in order to make this practical

$m_q > m_q^{\rm phys}$

calculating $\det M[U]$

or $M^{-1}[U]$ takes a lot of computer power

and the amount increases dramatically as the quark mass reduces

most current calculations use heavier than physical quarks

in principal all these are controlled approximations that can be overcome

e.g. compute at multiple *a* values and extrapolate