Effective Field Theories and Resonances

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OUTLINE

BIBLIOGRAPHY

- **Dynamics of the Standard Model**
  J.F. Donoghue, E. Golowich, B.R. Holstein (1992) (2nd ed. 2014)
- **Effective Lagrangians for the Standard Model**
  A. Dobado, A.Gómez Nicola, A. López Maroto, J.R. Peláez, (1997)
- **A Primer for Chiral Perturbation Theory**
  S. Scherer, M.R. Schindler, (2012)
- U.G. Meissner, Rep. Progr. Phys. 56 (1993) 903. hep-ph/9302247.
- A. Pich, Rep. Progr. Phys. 58 (1995) 563. hep-ph/9502366.
- J. R. Peláez, Phys. Rept. **658**, 1 (2016) [arXiv:1510.00653 [hep-ph]].

EFFECTIVE FIELD THEORIES (EFT)

Effective Field Theory

- Only relevant degrees of freedom below a scale Λ.
- Separation (mass/energy gap) from other states leading to well-defined $1/\Lambda^n$ power counting.
- At each order most general Lagrangian compatible with the symmetries of the underlying theory (if known) or system.
- Finite set of effective parameters fixed at every order.
- Loops increase order. Infinities absorbed in higher order parameters. (if renormalization scheme consistent with symmetries).
- Finite calculations order by order.
- Systematic and model independent approach.

Unfortunately, name not always used with this rigour...

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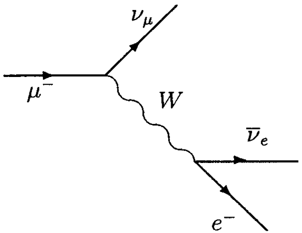
FERMI ELECTROWEAK THEORY (1934) (& FEYNMAN GELL-MANN 1958)

Electroweak processes with $E, m_i << M_W \equiv \Lambda$.

a)



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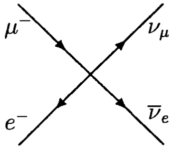

$$\frac{g^2}{2}\frac{g^{\rho\sigma}-k^\rho k^\sigma/M_W^2}{M_W^2-k^2} \rightarrow \frac{g^2}{2M_W^2} \equiv 2\sqrt{2}G_F$$

The W field propagator and vertices are reduced to an effective “contact term” and constant. It has been “integrated out”.

EFT BY INTEGRATING OUT A HEAVY STATE

This can be done rigorously and the heavy field is actually "integrated out" of the action. Schematically, if

$$S_{tot}[\phi,\Phi] = \int dx \mathcal{L}(\phi,\Phi) = S[\phi] + S[\phi,\Phi],$$

with $m_\phi << M_\Phi \equiv \Lambda$, then we define an "Effective action" through

$$e^{iS_{eff}[\phi]} = \int [d\Phi] e^{iS[\phi,\Phi]} = e^{iS[\phi]} \underbrace{\int [d\Phi] e^{iS[\phi,\Phi]}}_{\text{only depends on } \phi},$$

rewritten again formally as

$$S_{eff}[\phi] = \int dx \mathcal{L}_{eff}(\phi) = S[\phi] + S_{dec}[\phi] + S_{non-dec}[\phi],$$

where $S_{dec}[\phi] \equiv$ "decoupling terms" suppressed by $1/M$ powers.

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Integrating out a heavy state

Decoupling Theorem Appelquist-Carrazone (1975)

If $S[\phi, \Phi]$ is renormalizable, has no spontaneous symmetry breaking, no chiral fermions and heavy fermions form a complete multiplet, then the non-decoupling terms can be absorbed through renormalization in the Lagrangian of the light fields, up to decoupling terms suppressed by $1/M$.

Particularly interesting for vector gauge theories, where complete multiplets of non-chiral heavy fermions can be decoupled.

- In QED: Low energy theory of photons decoupling → Euler-Heisenberg Lagrangian
- In QCD we can decouple each heavy quark, one by one. We can safely consider QCD only with u, d, s or just u, d.

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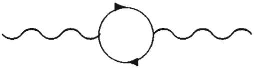
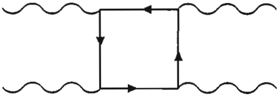
EXAMPLE: EULER-HEISENBERG LAGRANGIAN (1936)

Start from the usual QED action

$$S_{\rm QED}[A_\mu,\psi,\overline{\psi}] = -\frac{1}{4}\int dx F_{\mu\nu}F^{\mu\nu} + \int dx\overline{\psi}(i\,\not{D} - M_e)\psi$$





Integrate out the electron for photons with $E << M_e \equiv \Lambda$.

$$S_{\rm eff}[A] = \frac{-1}{4}\int dx F_{\mu\nu}F^{\mu\nu} - \frac{e^2}{3(4\pi)^2}\Delta\int dx F_{\mu\nu}F^{\mu\nu} \longleftarrow \text{non-decoupling and divergent}$$

$$-\frac{e^2}{15(4\pi)^2 M_e^2}\int dx F_{\mu\nu}\partial^\rho\partial_\rho F^{\mu\nu} + O\left(\frac{p^2}{M_e^2}\right)^2 \longleftarrow \text{new decoupling terms .}$$

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Other effective theories

- Heavy quark Effective Theory (HQET). For one heavy quark. $\Lambda = M_Q$.
- Non relativistic QCD (NRQCD), $\Lambda =$relative velocity of two heavy quarks.
- Soft collinear effective Theory (SCET). Only the hard parts of a field integrated out.
- For Electroweak Symmetry breaking sectors. $\Lambda =$scale of new particles. Lagrangian consistent with SM Lagrangian, widely considered an EFT.
- For Gravity. Other operators consistent with general covariance, expansion on $1/M_{Plank}$. Also non-relativistic effective theory.
- Effective Theories for Solid State Physics

But this lecture is focused on hadron resonances, we will concentrate on the low energy effective theory of QCD.

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Quantum Chromodynamics: non-Abelian $SU(3)_c$ gauge theory minimally coupled to quarks

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\begin{aligned}
\mathcal{L}_{QCD} &= \sum_{j=1}^{N_f} \bar{q}_j(x)\,(i\not{D} - m_j)\,q_j(x) - \frac{1}{4}\sum_{a=1}^{N_c^2-1} G^a_{\mu\nu}(x) G_a^{\mu\nu}(x) \\
G^a_{\mu\nu}(x) &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^a_{bc} A^b_\mu A^c_\nu, \qquad \not{D}_\mu = (\partial_\mu - i g T_a A^a_\mu/2)\gamma^\mu,
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with

$$m_u = 2.3^{+0.7}_{-0.5}\,\text{MeV}\,,\ m_d = 4.8^{+0.7}_{-0.3}\,\text{MeV}\,,\ m_s = 95 \pm 5\,\text{MeV}$$

$$m_c = 1.275 \pm 0.025\,\text{GeV}\,,\ m_b = 4.18 \pm 0.03\,\text{GeV}\,,\ m_t = 173.5 \pm 1.4\,\text{GeV}$$

Decoupling theorem:
Below 1-1.5 GeV we can keep the lightest quarks: $N_f = 2$ or 3.

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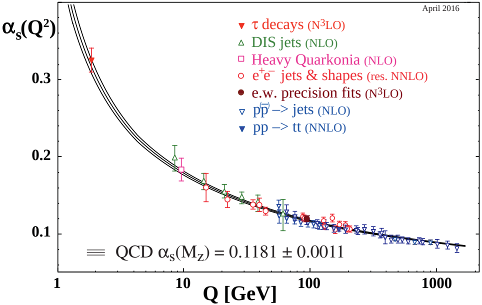
After renormaliztion $\alpha_s \equiv g^2/4\pi$ "runs":

$$\alpha_s(Q^2) = \frac{12\pi}{(33 - 2N_f)\log(Q^2/\Lambda^2_{QCD})}, \quad \Lambda_{QCD} \simeq 300 - 400\text{MeV} \qquad (1)$$



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- for $Q \leq \Lambda_{QCD}$ non-perturbative, confinement, hadrons!!

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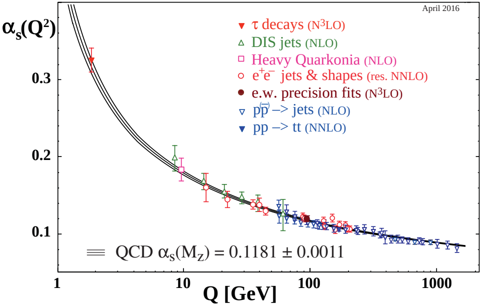

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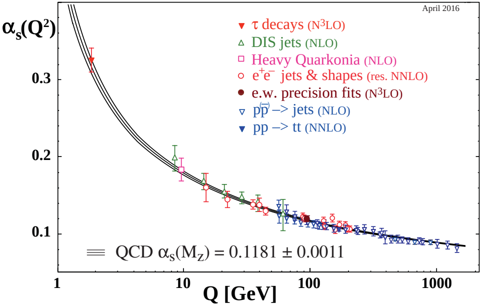

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- for $Q \leq \Lambda_{QCD}$ non-perturbative, confinement, hadrons!!

QCD AND CHIRAL SYMMETRY

Chiral limit $m_q \to 0$ interesting since $m_u, m_d, m_s << M_{hadrons}$.
Then $\mathcal{L}_{QCD}$ invariant under $SU(N_f)_L \times SU(N_f)_R$ Chiral Symmetry:

$$q_{L,R} \longrightarrow \underbrace{\exp\left(-i\theta_a^{L,R}\frac{T_a}{2}\right)}_{L,R \text{ transformations: } L^\dagger=L^{-1},R^\dagger=R^{-1}} q_{L,R}, \quad \text{with} \quad q_{L,R} = \left(\frac{1 \mp \gamma_5}{2}\right) q.$$

with $T_a = \lambda_a$ for $N_f = 3$ and $T_a = \tau_a$ (Pauli matrices) for $N_f = 2$.
Noether's Theorem $\Rightarrow$ Conserved currents:

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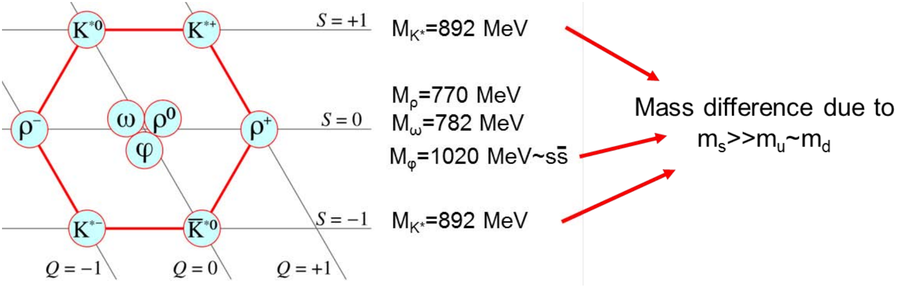

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...500 MeV too heavy!!

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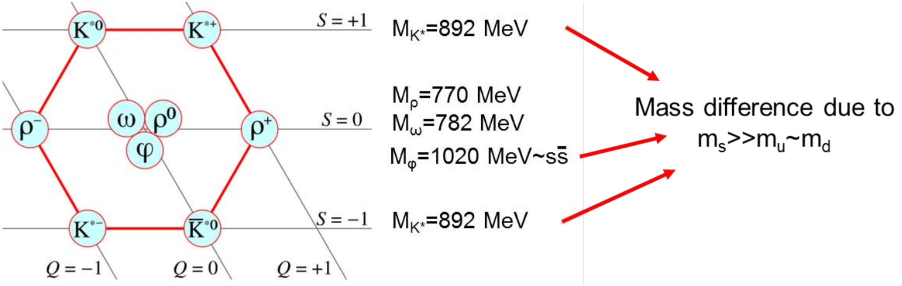

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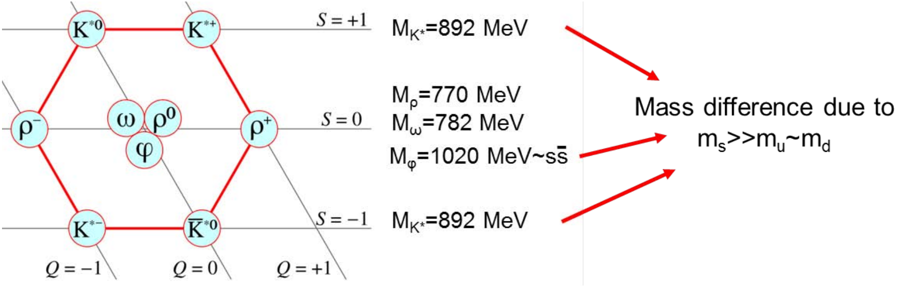

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Spontaneous Symmetry Breaking (SSB)

Noether's Theorem:

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Symmetry charges $Q_a = \int dx\, J_a^0(x)$ are group generators $U = e^{i\theta_a Q_a}$
If H is the Hamiltonian: $UHU^{-1} = H \Rightarrow [Q_a, H] = 0$.

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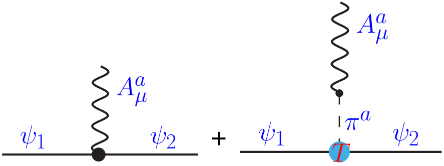
$$\langle 0 | A_a^\mu(0) | \pi_b(p_\mu) \rangle = i\, f_\pi p^\mu \delta_{ab} \neq 0, \quad f_\pi = \text{pion decay constant}$$

$\langle \Psi_1 |\, A_\mu^a(0)\, | \Psi_2 \rangle =$



$= R_\mu^a + f_\pi p_\mu \frac{1}{p^2} T_a$

Current conservation: $0 = p^\mu A_\mu^a = p^\mu R_\mu^a + f_\pi T_a = 0 \Rightarrow \lim_{p \to 0} T_a = 0$

NGB interactions vanish at low energies. Derivative couplings!

But since there is an explicit violation $m_q \neq 0$:

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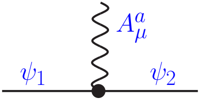

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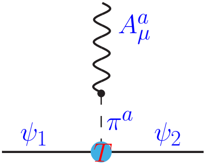

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It is a TOY MODEL, not QCD !!

Let $\Phi^A = (\sigma, \phi^a)$, $a = 1, 2, 3$ and $\Phi = |\vec{\Phi}|$

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$\mu^2 > 0$ case:

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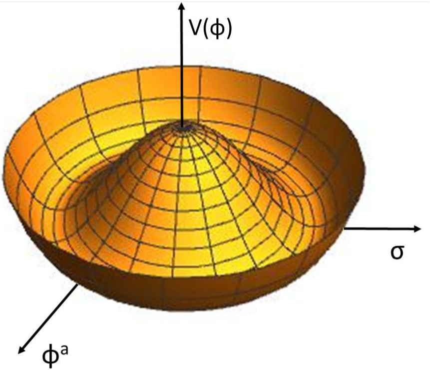
Choose perturbative vacuum at $\sigma = f \equiv \sqrt{\mu^2/\lambda}$



$O(4) \to O(3)$ Spontaneous Symmetry Breaking

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Recast (σ, ϕ^a) into $\Sigma = \sigma + i\tau^a\phi^a$. Then

$$\mathcal{L}_{L\sigma M} = \frac{1}{4} Tr(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma) + \frac{\mu^2}{4} Tr(\Sigma^\dagger \Sigma) - \frac{\lambda}{16} [Tr(\Sigma^\dagger \Sigma)]^2, \qquad (2)$$

invariant under linear $\Sigma \to L\Sigma R^\dagger$, with $L \in SU(2)_L$ and $R \in SU(2)_R$.

Degenerate vacua $Tr(\Sigma\Sigma^\dagger) = 2v^2$. Redefining fields...

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The LσM is just a toy model where the auxiliary σ is used to facilitate a linear representation of chiral symmetry and to build an invariant $\mathcal{L}$.

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FROM THE LσM TO THE NON-LINEAR-σ MODEL(NLσM)

- Integrating out the σ and expanding in powers of $1/M_\sigma$:

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Meson masses vs. quark masses

Due to explicit chiral symmetry breaking NGB→"pseudo-NGB"
Note that meson masses are $M^2_{NGB} \sim m_q$.
This ensures the Gell Mann-Okubo relation (GMOR):

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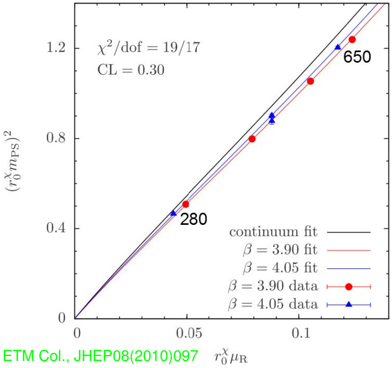

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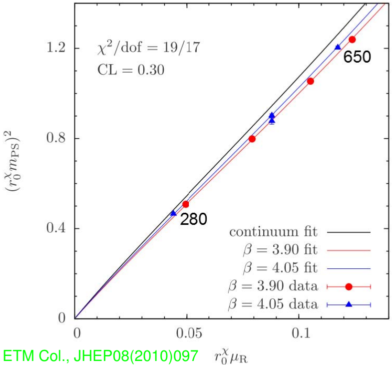

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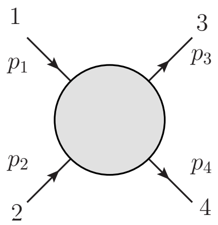
Scattering: Definitions and Notation



Mandelstamm variables:

$s = (p_1 + p_2)^2$

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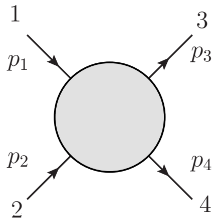

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- Partial waves: $T(s,t) = 16K\pi \sum_\ell (2\ell + 1) P_\ell(\cos\theta) t_\ell(s)$

  ($K = 1$ or $K = 2$ if particles identical)

  Then $t_\ell(s) = \frac{\sqrt{s}}{2k} \frac{\eta_\ell(s) e^{2i\delta_\ell(s)} - 1}{2i}$, $\delta_\ell(s) \equiv$ phase shift. $\eta_\ell(s) \equiv$ elasticity

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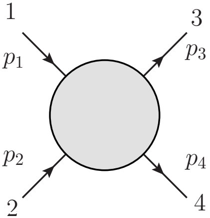

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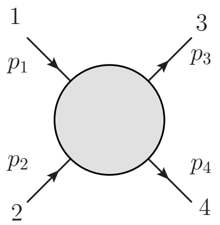

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For instance, the NLσM provides a universal leading order prediction for the $\pi^+\pi^- \to \pi^0\pi^0$ amplitude $T(s,t,u) = (s - M_\pi^2)/f_\pi^2$

Weinberg Low Energy Theorems (LET):

For $t_\ell^{(I)}(s)$ of definite isospin I

$$t_0^{(0)} = \frac{2s - M_\pi^2}{32\pi f_\pi^2}, \quad t_1^{(1)} = \frac{s - 4M_\pi^2}{96\pi f_\pi^2}, \quad t_0^{(2)} = \frac{2M_\pi^2 - s}{32\pi f_\pi^2}.$$

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For instance, the NLσM provides a universal leading order prediction for the $\pi^+\pi^- \to \pi^0\pi^0$ amplitude $T(s,t,u) = (s - M_\pi^2)/f_\pi^2$

Weinberg Low Energy Theorems (LET):

For $t_\ell^{(I)}(s)$ of definite isospin I

$$t_0^{(0)} = \frac{2s - M_\pi^2}{32\pi f_\pi^2}, \quad t_1^{(1)} = \frac{s - 4M_\pi^2}{96\pi f_\pi^2}, \quad t_0^{(2)} = \frac{2M_\pi^2 - s}{32\pi f_\pi^2}.$$

- If $M_\pi \to 0$, NO interaction at threshold.
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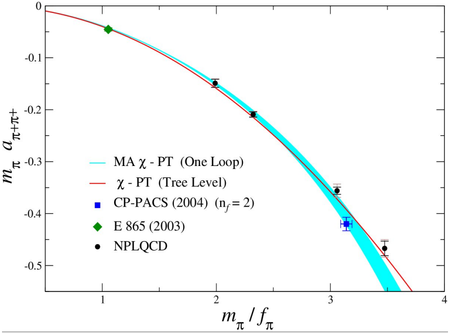
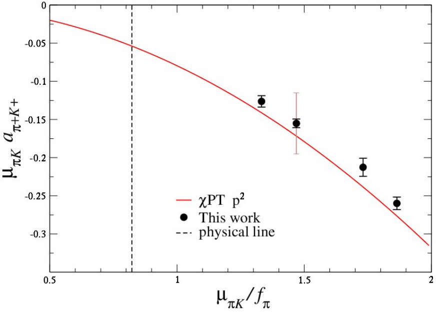
Actually, the NLσM at $O(p^2)$ describes rather well the quark mass dependence of some observables calculated on the lattice:





NPLQCD Phys.Rev.D77:014505,2008, and Phys.Rev.D77:094507,2008

THE QCD LOW-ENERGY EFFECTIVE THEORY

So far we only have an effective Lagrangian with the relevant d.o.f.

Weinberg's power counting (1979):

A Feynman diagram is $O\left(\frac{p}{4\pi f_0}\right)^D$, with $D = 2 + \sum_n N_n(n-2) + 2N_L$
$N_n \equiv$number of vertices with n derivatives (or masses).
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- $\mathcal{L}_{NL\sigma M} \equiv \mathcal{L}_2 \equiv$ leading order. Two derivatives or masses. No loops so far.
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CHIRAL PERTURBATION THEORY (CHPT) GASSER & LEUTWYLER

At next-to-leading order (NLO), within $SU(3)$:

$$\mathcal{L}_4 = L_1 Tr\left(\partial^\mu U^\dagger \partial_\mu U\right)^2 + L_2 Tr\left(\partial^\mu U^\dagger \partial^\nu U\right) Tr\left(\partial_\mu U^\dagger \partial_\nu U\right) + L_3 Tr\left(\partial^\mu U^\dagger \partial_\mu U \partial^\nu U^\dagger \partial_\nu U\right)$$
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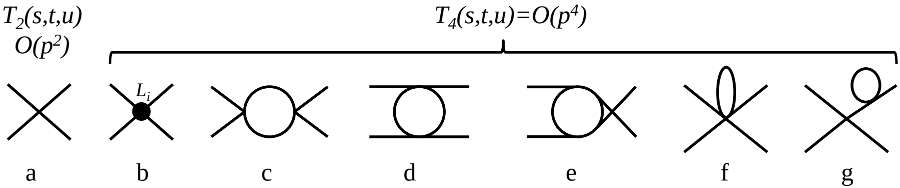
Meson-meson scattering at NLO ChPT



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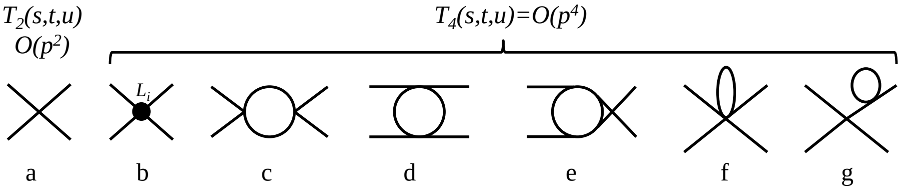

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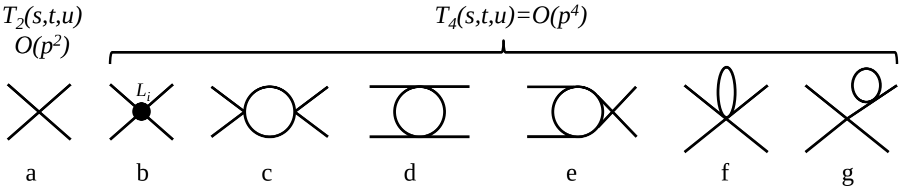

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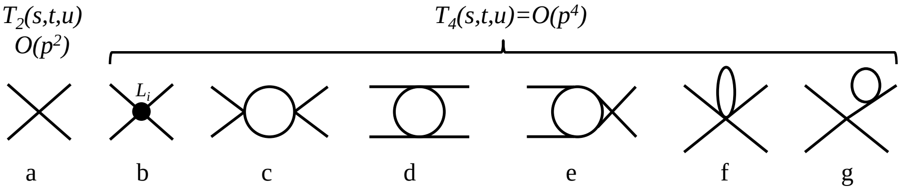

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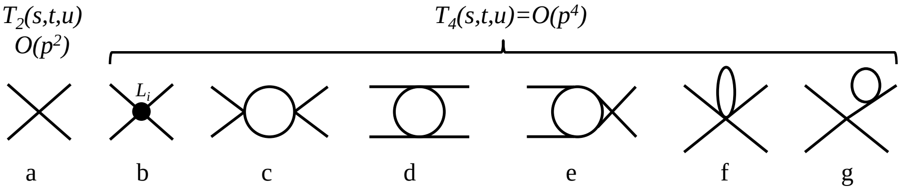

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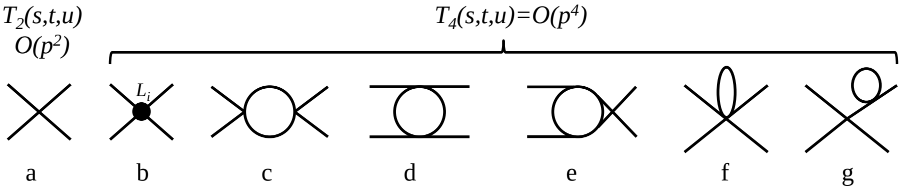

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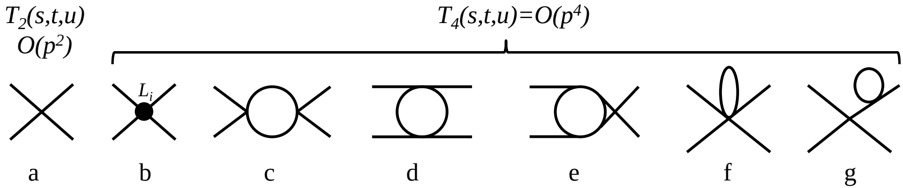

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LOW ENERGY CONSTANTS OBSERVED VALUES

Low Energy Constants (LECs) have been determined phenomenologically

(a few also from lattice)

10^3	GL [55]	NNLO [255]	NLO [255]
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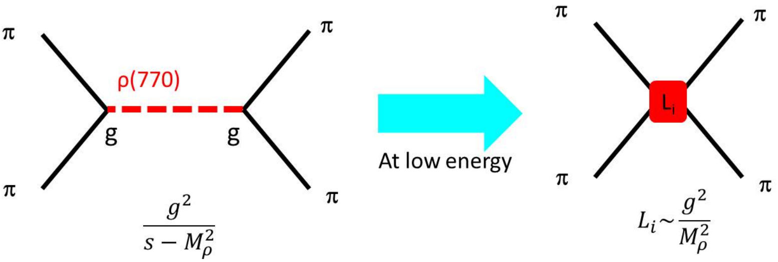
Typically $O(10^{-3})$
Uncertainties 10-20%

Resonances and Low Energy Constants

The LECs receive contributions from the integration of heavier resonances.



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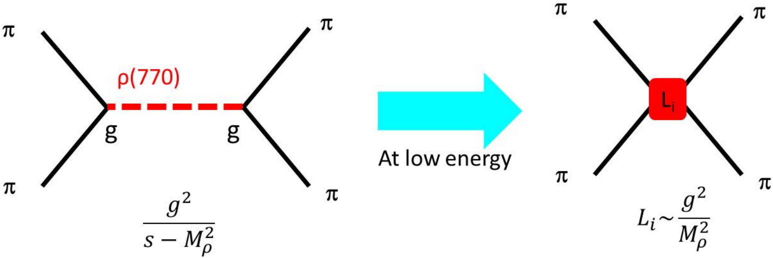

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Resonance Saturation Donoghue, Ecker, Gaser, Leutwyler, Pich, Valencia...

Integrating out vector(V), scalar (S), and singlet-scalar (S) multiplets from a general chirally invariant Lagrangian: $L_i = L_i^V + L_i^S + L_i^{S_1}$

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L_1^r	0.7(3)	0.53(06)	1.0(1)	0.6	0.6	−0.2	0.2	$O(N_c)$
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Single Resonance Approximation (SRA)

LEC values are saturated by the lowest multiplet of each kind.

Vector-Meson Dominance by the vector multiplet of the $\rho(770)$

Scalar contributions with $M_S \geq 1$ GeV. No LσM. No $f_0(500)$ contribution

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2. Only π, K, η in the Lagrangian, as NGB.
3. Explicit symmetry breaking $M_0^2 \sim m_q$ as perturbation
4. LO: massive NLσM
5. Systematic power counting. Loops $\sim 1/(4\pi f_\pi)^2$ supression
6. LECs absorb loop divergences. Finite results at each order.
7. LECs encode underlying QCD dynamics
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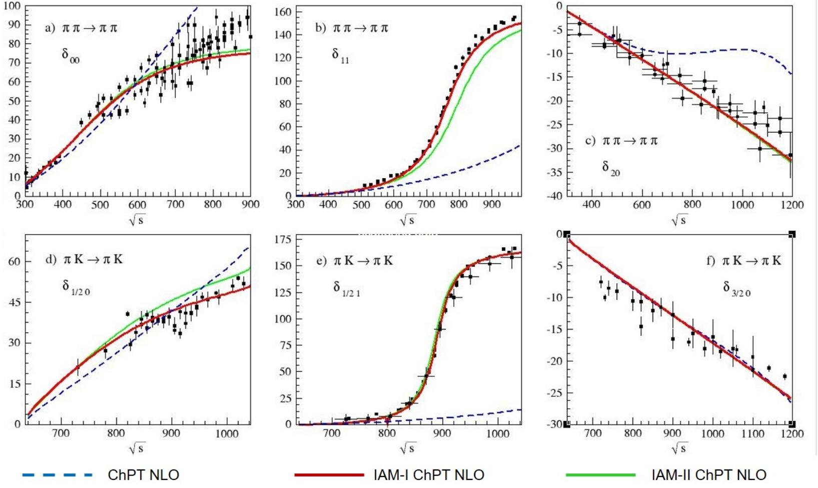
CHPT IN THE RESONANCE REGION

ChPT: good results up to $k = 100 - 200\,\text{MeV}$, beyond if no resonances.

But fails to describe resonances



Outline

1. Effective Field Theories
2. Resonances, unitarity and dispersion relations
3. Unitarity and unitarization of EFTs

BIBLIOGRAPHY

- **Elementary particle theory**. A. D. Martin and T. D. Spearman, North-Holland Pub. Co., 1970.
- **Strong Interactions of Hadrons at High Energies**. V. N. Gribov, Y. L. Dokshitzer and J. Nyri Cambridge University Press, 2009
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- J. R. Peláez, Phys. Rept. **658**, 1 (2016) [arXiv:1510.00653 [hep-ph]].

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ANALYTICITY, CUTS AND POLES

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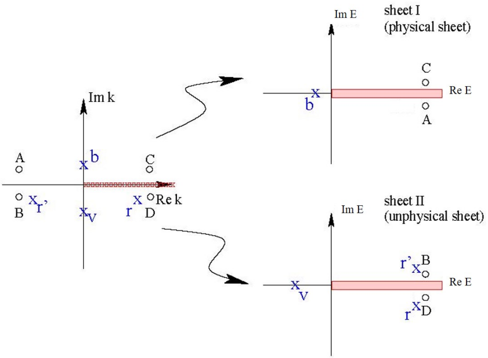
$u(k^2, f)$ is a function of k^2, but we used the double valued $k = \sqrt{2E}$. Two Riemann sheets to map k on $E-$plane.



Define $\kappa > 0$

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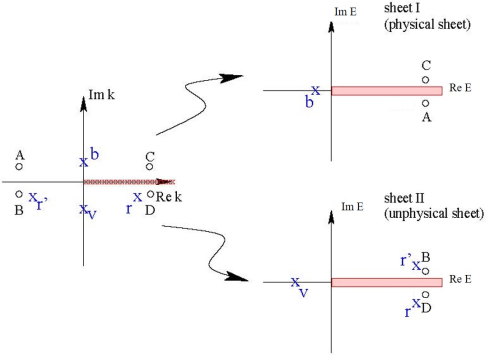

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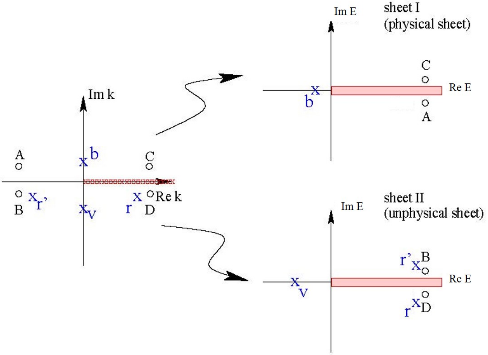

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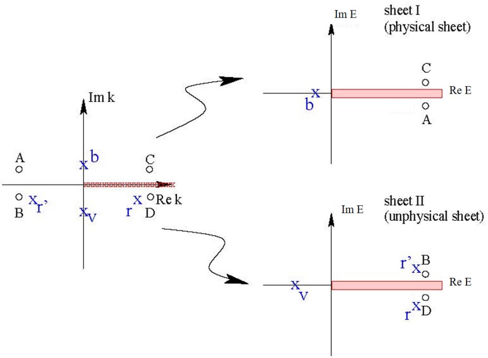

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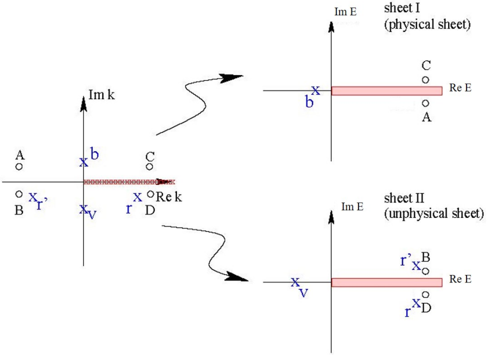

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Incoming packet: $\Phi_{in}(r,t) \equiv -\int_0^\infty dE\, A(E) e^{-ikr-iEt}$ (similar outgoing)

Scattering wave $\equiv$ outcoming "with interaction"-"without interaction"

$$\Phi_{sc}(r,t) = \int_0^\infty dE\, A(E)[S(E)-1]e^{ikr-iEt} = 2\pi \int_0^\infty dE\, A(E) e^{-ikr-iEt} G(r,E)$$

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Effect not influenced by Cause if $t' > t \Longrightarrow g(\tau) = 0$ for $\tau = t - t' < 0$

Thus: $$G(r,E) = \frac{1}{2\pi}\int_{0 \leftarrow zero!!}^{\infty} d\tau\, g(r,\tau) e^{iE\tau}.$$

Converges for $E = E_R + iE_I$, with $E_I > 0$, due to $e^{-E_I \tau}$ suppression

(if $g(r,\tau)$ well-behaved) Thus $G(r,E)$ is analytic in the upper half complex E-plane.

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Since the coefficients of the Schrödinguer eq. are real:

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This defines the S-matrix in the lower half of the E-complex plane. Hence:

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We can have singularities on the real axis of the FIRST SHEET:

- "Physical cut": Already seen. From threshold to ∞, that gives access to sheet II (also has Schwarz Symmetry)
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Resonances: conjugated pairs of poles on sheet II

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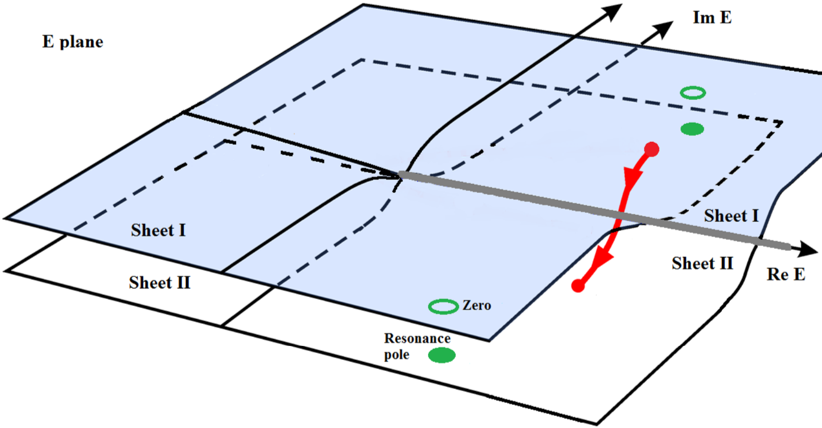
Sheets, cuts and poles



SHEETS, CUTS AND POLES

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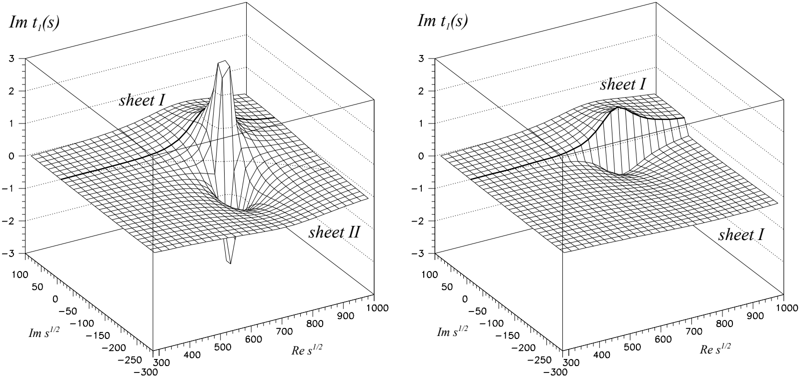

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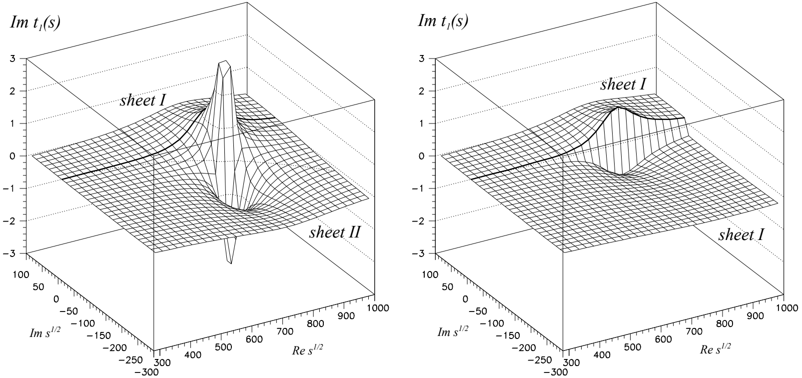

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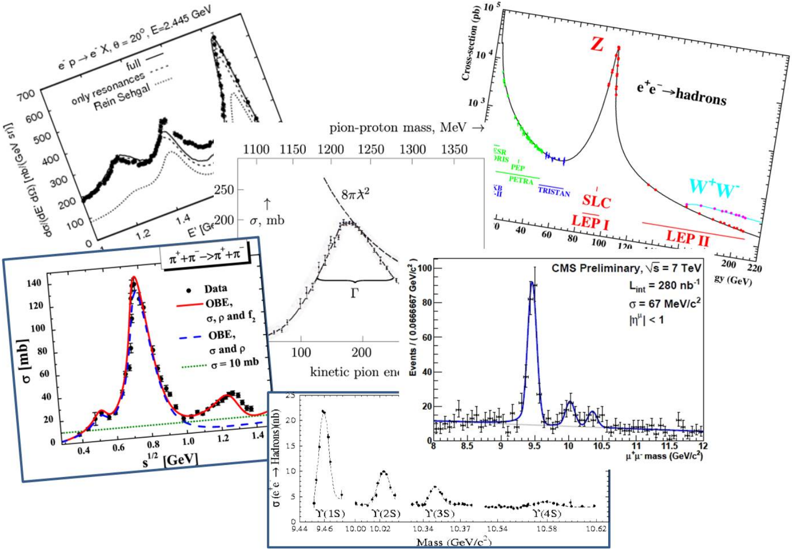
Resonances as Poles

When those poles are well isolated, the bumps become clearly visible:



Resonances as poles

Intuitively for a bound state at rest whose energy is just the mass $E = M$, its time evolution is ($\hbar = 1$):

$$\Psi(t) = \Psi(0)e^{-iMt} \longrightarrow |\Psi(t)|^2 = |\Psi(0)|^2,$$

i.e, the state does not disappear.

But if we allow an imaginary part $\boxed{E = M - i\Gamma/2}$, then

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Analytic properties now derived perturbatively from Feynman diagrams or in few cases from axiomatic QFT. Most inherited from NRQM:

- $(E, \vec{p}) \longrightarrow (s, t)$ Mandelstam variables
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- Schwartz reflection: $T(s^*, t) = T^*(s, t)$
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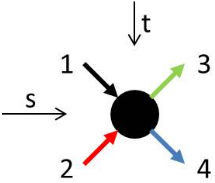

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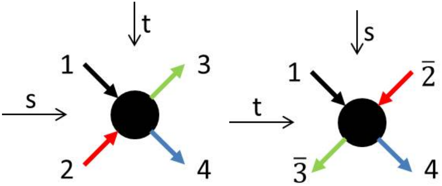

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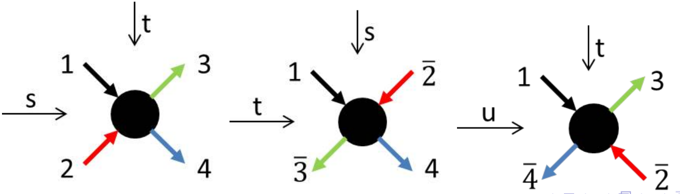
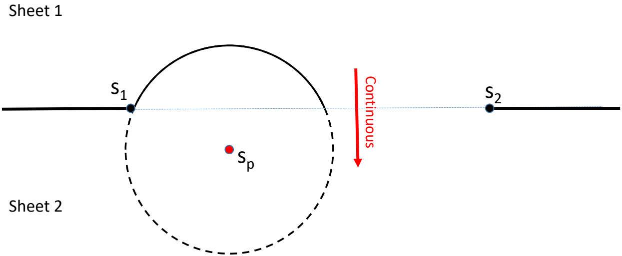


POLES AND RELATIVISTIC RESONANCES

Relativistic partial-wave amplitudes still have a physical cut giving access to two sheets

Assume a pole at:
$s_P = M^2 - i\gamma$



Define;
$g(s) \equiv (s - s_P)t(s)$,
which is regular

Expand around s_P: $g(s) \simeq g(s_P) + (s - s_P)g'(s) + \ldots$,
which converges in a circle up to the nearest singularity (a cut, another pole..) including some part of the real axis, where we see

$$t_\ell(s) \sim \frac{-g(s)}{M^2 - s - i\gamma} \longleftarrow \text{a bump around } M^2!! \quad \text{if g(s) varies slowly around } M^2$$

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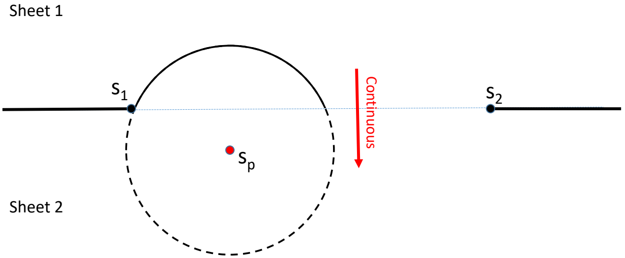

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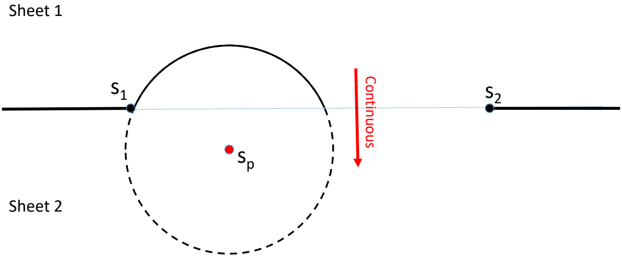

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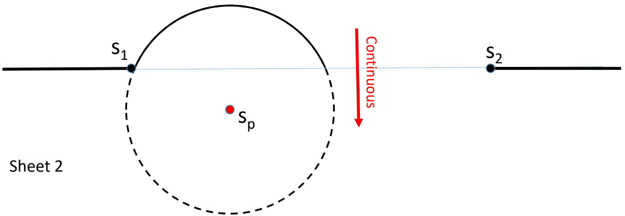

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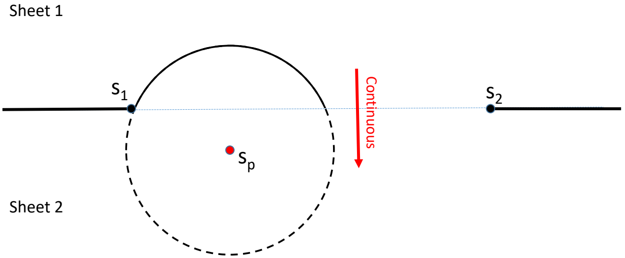

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Relativistic Breit Wigner Formula

If the pole is near the real axis. i.e, if γ is small, we can approximate $g(s) \simeq g(s_P) \equiv g$ for s near M^2. Defining $\Gamma \equiv \gamma/M$

$$t_\ell(s) \simeq \frac{-g}{M^2 - s - iM\Gamma}$$

Relativistic Breit-Wigner formula

in the real axis:

$$|t_\ell(s)|^2 \simeq \frac{g^2}{(M^2-s)^2 + M^2\Gamma^2}$$

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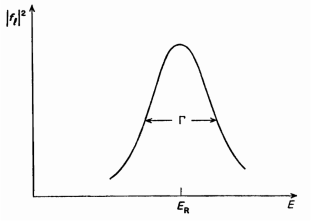

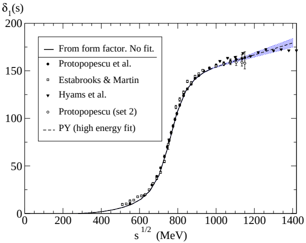

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BW-formula is an **approximation**, only valid for:

narrow resonances, well-isolated from other singularities

Unfortunately, very often used well beyond this approximation

BW resonances ”easier” to identify. But complications arise if:

- multiple channels (several thresholds)
- thresholds nearby (difficulty for ”molecular” states)
- overlapping resonances (several poles nearby)
- very wide resonances (poles deep in complex plane)
- there are backgrounds ($g(s)$ is not slowly varying)

It is important then to implement correctly the amplitude analytic properties and perform sensible analytic continuations to the complex plane. For this, dispersion relations and/or models with good analytic properties (cuts, sheets, etc...) are essential.

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- thresholds nearby (difficulty for "molecular" states)
- overlapping resonances (several poles nearby)
- very wide resonances (poles deep in complex plane)
- there are backgrounds ($g(s)$ is not slowly varying)

It is important then to implement correctly the amplitude analytic properties and perform sensible analytic continuations to the complex plane. For this, dispersion relations and/or models with good analytic properties (cuts, sheets, etc...) are essential.

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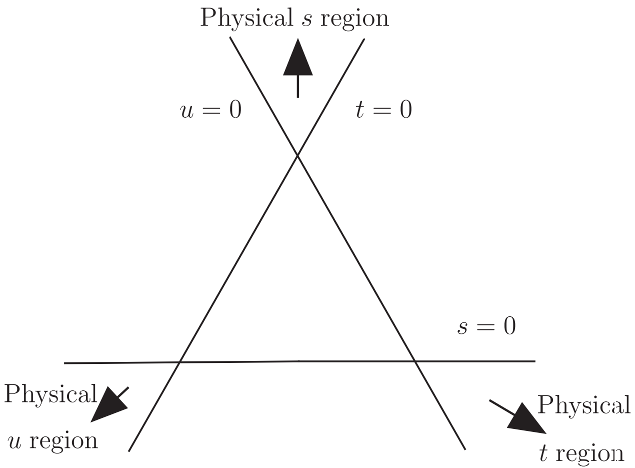
ANALYTICITY IN RELATIVISTIC SCATTERING

Physical Regions in Mandelstam plane:



ANALYTICITY IN RELATIVISTIC SCATTERING

Analyticity properties follow from crossing symmetry and the

Mandelstamm Hypothesis

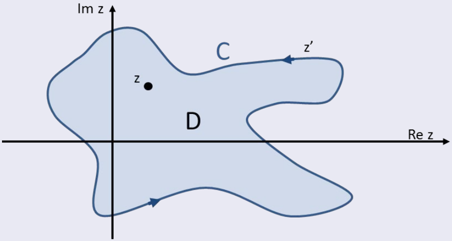
There is a unique analytic function that satisfies:

$$T(s,t,u) = \begin{cases} T_{12\to 34}(s,t,u), & s \geq 4m^2, \quad t \leq 0, \quad u \leq 0, \\ T_{1\bar{3}\to\bar{2}4}(t,s,u), & t \geq 4m^2, \quad s \leq 0, \quad u \leq 0, \\ T_{1\bar{4}\to 3\bar{2}}(u,t,s), & u \geq 4m^2, \quad s \leq 0, \quad t \leq 0. \end{cases}$$

+ “Minimal set of sigularities demanded by Physics” like cuts due to thresholds

Dispersion Relations

Cauchy's Integral Formula:



Let D be a domain of the complex plane where the function $f(z)$ is analytic (holomorphic) and let C be the closed curve* defined by its boundary. Then, for any $z \in D$

$$f(z) = \oint_C \frac{f(z')}{z' - z} dz'$$

* rectifiable, taken counter clock-wise, and with winding number 1

Dispersion Relations

Dispersion Relation ≡ Cauchy's Integral Formula applied to amplitudes.

But the Formula only applies to functions of one variable. Two options:

- Fix one variable → "Fixed-t" Dispersion Relations for $f(s) = T(s, t_0)$
  Particular case: Forward Dispersion Relations
- Integrate the previous ones to obtain "partial wave Dispersion Relations" for $t_J(s)$.
  Particular cases: Roy eqs., Roy-Steiner eqs, GKPY eqs., Inverse Amplitude Method

Interest of Dispersion Relations:

- To constrain data analyses
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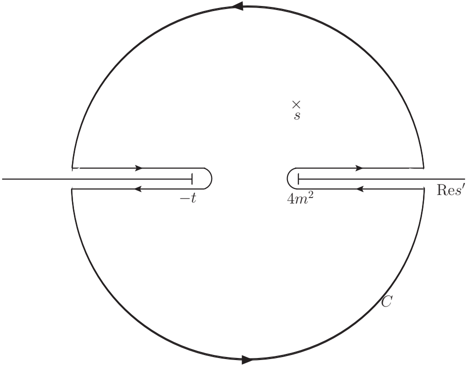

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Assume the integral on the circular parts of C vanish if radius sent to ∞.

Above and below the real axis the amplitude is conjugated (Schwartz reflection)

$$T(s,t,u) = \underbrace{\frac{1}{\pi}\int_{4m^2}^{\infty} ds' \frac{\mathrm{Im}\, T(s',t,u')}{s'-s}}_{\text{Right cut}} + \underbrace{\frac{1}{\pi}\int_{-\infty}^{-t} ds' \frac{\mathrm{Im}\, T(s',t,u')}{s'-s}}_{\text{Left cut}}$$

Provides T anywhere in the complex plane except the real axis

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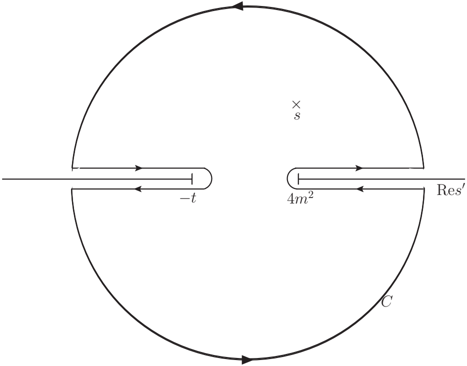

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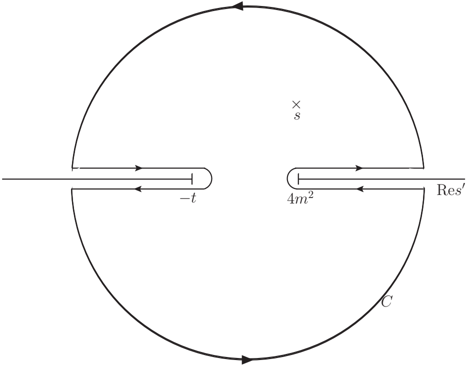

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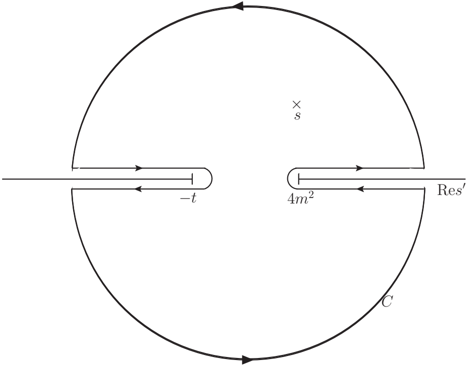

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When calculating T on the real axis, the $(s-s')$ denominator diverges

But recall that: $\frac{1}{s'-s-i\epsilon} = PV\frac{1}{s'-s} + i\pi\delta(s'-s)$, (PV ≡principal value)

Thus, on the real axis:

$$\mathrm{Re}\, T(s,t,u) = \frac{1}{\pi} PV \int_{4m^2}^{\infty} ds' \frac{\mathrm{Im}\, T(s',t,u')}{s'-s} + \frac{1}{\pi}\int_{-\infty}^{-t} ds' \frac{\mathrm{Im}\, T(s',t,u')}{s'-s}.$$

For physical values of s dispersion relations provide Re T from Im T.

(sometimes you may see a $-i\epsilon$ instead of the PV and the real part)

DATA SHOULD SATISFY THIS.

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Fixed-t Dispersion Relations: Subtractions

If $T \not\to 0$ or does it very slowly at ∞, the C circular part $\not\to 0$.

By subtracting T at other point s_0:

$$T(s,t) - T(s_0,t) = \frac{1}{2\pi i}(s-s_0)\oint ds' \frac{T(s',t)}{(s'-s)(s'-s_0)},$$

converges if $T(s,t,u)/s \to 0$ at ∞ faster than $1/s$.

If the circular contribution now cancels, the "once subtracted" dispersion relation reads:

$$T(s,t) = \overbrace{T(s_0,t)}^{\text{Subtraction constant}} + \frac{s-s_0}{\pi}\int_{4m^2}^{\infty} ds' \frac{T(s',t)}{(s'-s)(s'-s_0)} + \frac{s-s_0}{\pi}\int_{-\infty}^{-t} ds' \frac{T(s',t)}{(s'-s)(s'-s_0)}$$

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If that is not enough... make more subtractions, typically at the same point.

$$T(s,t) = \underbrace{T(s_0,t) + (s-s_0)\frac{\partial T(s_0,t)}{\partial s_0}}_{\text{Two subtraction constants}} + \frac{(s-s_0)^2}{2\pi i}\oint ds' \frac{T(s',t)}{(s'-s)(s'-s_0)^2},$$

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FORWARD DISPERSION RELATIONS

The problem now is to describe the input to the DR≡the data.

Most often $ImT(s,t)$ is not known in the whole energy region, nor on the left cut.

This why the most popular fixed-t DR are "Forward", $t = 0$. There are two reasons:

1. At high energy $ImT(s,0) \sim \sigma_{tot}$, which is much easier to measure.
2. The most relevant reactions $\pi\pi, K\pi, p\pi\ldots$ have crossing symmetries that allow to re-write the left cut into in terms of the physical cut.

But they also have a drawback. It is not possible to continue analytically to the second sheet. the relation $S^{II} = 1/S^{I}$ was only valid for partial waves. Still, they are very powerful to constrain the data parameterizations

→Examples: $\pi\pi$, $K\pi$

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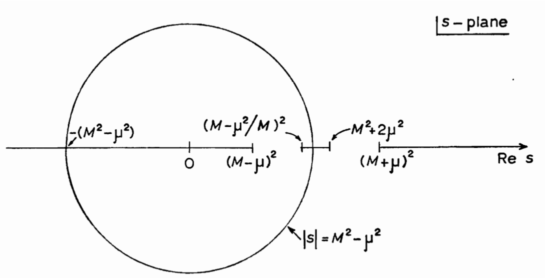
PARTIAL WAVE DISPERSION RELATIONS

Recall the definition of scattering Partial wave:

$$t_\ell(s) = \frac{1}{32K\pi}\int_{-1}^{1} T(s, t(\cos\theta))P_\ell(\cos\theta)d\cos\theta \quad (K = 1 \text{ or } K = 2 \text{ if particles identical})$$

Their analytic structure is:

- Right cut
- Left cut from $-\infty$ to 0
- Circular cut if $m \neq M$
- other cuts if bound states



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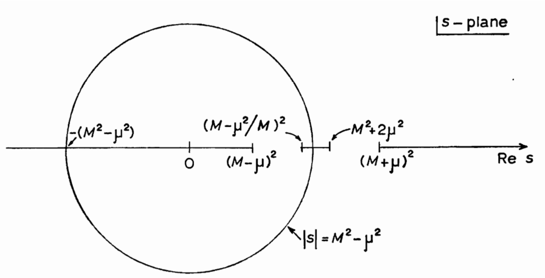

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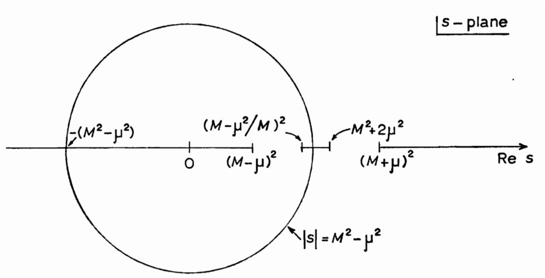

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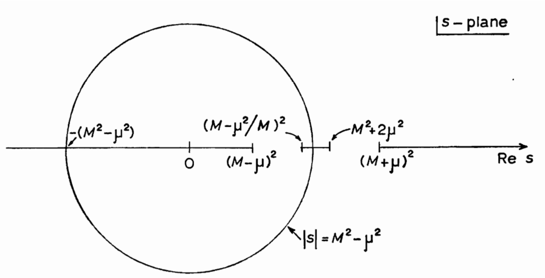

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Dispersion relations can be written as before but with contributions from all these singularities.

$$t_\ell(s) = C_0 + C_1 s + \frac{s^2}{\pi}\int_{(M_1+M_2)^2}^{\infty} \frac{\text{Im}\, t_\ell(s')ds'}{s'^2(s'-s-i\epsilon)} + \underbrace{LC(s)}_{\text{Left cut}} + \underbrace{CC(s)}_{\text{Circular cut}} + \underbrace{P(s)}_{\text{bound-state poles}}$$

Only right and left cuts if particles identical i.e, $\pi\pi$ scattering.

So far all DR were formulated on the first sheet, just providing constraints.

The additional interest of partial-wave DR is that they allow for a continuation to the second sheet. For an elastic partial wave, the S-matrix is just a number and we saw that $S^{II} = 1/S^{I}$. We can look for poles on sheet-II (resonances) as zeros on sheet-I.

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ROY AND ROY-STEINER DISPERSION RELATIONS

For $\pi\pi$ scattering the problem is the left cut.

Rigorous Solution: rewrite the left cut in t-channel partial basis using crossing symmetry. Infinite t-channel waves needed. All crossed amplitudes $\pi\pi$ again. Roy Eqs.$\equiv$ system of ∞ coupled pw-Dispersion relations. Truncation possible at low energies. Solve numerically the equations.

You can use ChPT for subtraction constants. No closed-form solution. Weak connection with QCD parameters.

The most rigorous way to extract resonance poles

But limited to low energies $\leq$ 1 GeV:
$f_0(500)$, $K_0^*(800)$, $\rho(770)$, $K^*(892)$, $f_0(980)$

Roy-Steiner eqs. Similar but for $K\pi$, $N\pi$ or $\gamma\gamma \to \pi\pi$. Even more amplitudes coupled.

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UNITARITY

For physical values of s, the S-matrix is unitary: $SS^\dagger = S^\dagger S = \mathbb{I}$, which for the T-matrix or amplitude $S_{fi} \equiv \delta_{fi} + i(2\pi)^4\delta^4(p_f - p_i)T_{fi}$, means:

$$T_{fi} - T^\dagger_{fi} = i(2\pi)^4 \underbrace{\sum_n}_{\text{sum over intermediate states n}} \delta^4(p_n - p_i)T^\dagger_{fn}T_{ni}$$

Where we have used $\mathbb{I} = \sum_n |n\rangle\langle n|$, with $|n\rangle$ physically accessible="open states"

$$\text{For } f = i: \quad 2\text{Im } T_{ii} = (2\pi)^4 \sum_n \delta^4(p_n - p_i)|T_{ni}|^2$$

For two-body states, the angles of the 3-momenta can be integrated out by projecting into partial waves.

UNITARITY FOR PARTIAL WAVES

Let us assume all states are two-body states. Then we find

Using the usual relations for Legendre polynomials and Spherical Harmonics:

$\int_{-1}^{1} P_\ell(x)P_{\ell'}(x)dx = \frac{2\delta_{\ell\ell'}}{2\ell+1}, \quad P_\ell(\hat{p}\cdot\hat{k}) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\hat{p})Y_{\ell m}(\hat{k}), \quad \int d\Omega_{\vec{k}} Y^*_{\ell m}(\hat{k})Y_{\ell' m'}(\hat{k}) = \delta_{\ell\ell'}\delta_{mm'}$

The coupled channel partial-wave unitarity relation:

$$\text{Im}\, t_\ell^{fi}(s) = \sum_n \sigma(s) t_\ell^{fn}(s) t_\ell^{ni}(s)^* \qquad \sigma(s) = \frac{2p_n}{\sqrt{s}} \sim \text{Phase space}$$

in matrix form: $\text{Im}\, T(s) = T(s)\Sigma T(s)^*$ with $\Sigma(s) = \begin{pmatrix} \sigma_1(s) & 0 & \cdots \\ 0 & \sigma_2(s) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$

And in the elastic case $f=i$ and no other n open:

$$\text{Im}\, t_\ell(s) = \sigma(s)|t_\ell(s)|^2 \qquad \text{Elastic unitarity condition}$$

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Thus: $|t_\ell| = \dfrac{\sin(\delta_\ell)}{\sigma} \Rightarrow \quad \boxed{t_\ell = \dfrac{e^{i\delta_\ell}\sin(\delta_\ell)}{\sigma}}$

which implies the following bounds:

$$\boxed{|t_\ell| \leq \frac{1}{\sigma}} \xrightarrow{s>>m_i^2} 1 \qquad\qquad \boxed{\text{Re } t_\ell \leq \frac{1}{2\sigma}} \xrightarrow{s>>m_i^2} \frac{1}{2}$$

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But from the elastic unitarity condition: $\text{Im } t_\ell = \sigma |t_\ell|^2 \longrightarrow \sin(\delta_\ell) = \sigma |t_\ell|$

Thus: $|t_\ell| = \dfrac{\sin(\delta_\ell)}{\sigma} \Rightarrow$ $\boxed{t_\ell = \dfrac{e^{i\delta_\ell}\sin(\delta_\ell)}{\sigma}}$

which implies the following bounds:

$$\boxed{|t_\ell| \leq \frac{1}{\sigma}} \xrightarrow{s>>m_i^2} 1 \qquad\qquad \boxed{\text{Re } t_\ell \leq \frac{1}{2\sigma}} \xrightarrow{s>>m_i^2} \frac{1}{2}$$

It can be shown that these bounds also hold in the inelastic case
A theory is said to be Strongly Interacting when these bounds are saturated. Typical resonant behavior. Unitarity essential for resonances

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Within this approximation/model the pole residue $g \simeq -M\Gamma/\sigma(M^2)$.

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No perturbation theory (series expansion in λ) satisfies unitarity exactly.

Assume the calculation is done to $O(\lambda^n)$. Then $\underbrace{\text{Im}\, t_\ell}_{O(\lambda^n)} = \sigma \underbrace{|t_\ell|^2}_{O(\lambda^{2n})}$

> Unitarity is only satisfied perturbatively within perturbation theory.

- For QED: Not an issue, since $\lambda = \alpha \simeq \frac{1}{137}$.
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Nevetheless, ChPT satisfies unitarity perturbatively. Namely, if $t(s) = t_2(s) + t_4(s) + t_6(s)...$ with $t_n = O(p^n)$:

$$\text{Im } t_2(s) = 0, \qquad \longrightarrow t_2 \text{ is real!}$$
$$\text{Im } t_4(s) = \sigma(s) t_2(s)^2,$$
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Similarly, in matrix form, when various coupled channels are open:

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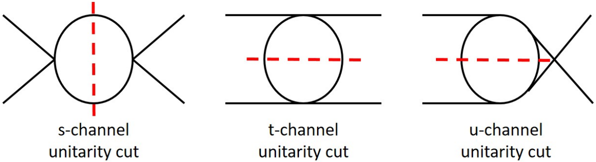

UNITARITY FOR THE INVERSE AMPLITUDE

Recall now that the inverse of a complex number z is $\dfrac{1}{z} = \dfrac{\bar{z}}{z\bar{z}} = \dfrac{\bar{z}}{|z|^2}$

Therefore: $\text{Im}\,\dfrac{1}{z} = \dfrac{\text{Im}\,\bar{z}}{|z|^2} = -\dfrac{\text{Im}\,z}{|z|^2}$

Thus, we can recast the elastic unitarity condition:

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Thus elastic unitarity fixes the imaginary part of the inverse amplitude.

For physical s, any elastic pw satisfies:

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Therefore: $\text{Im}\,\dfrac{1}{z} = \dfrac{\text{Im}\,\bar{z}}{|z|^2} = -\dfrac{\text{Im}\,z}{|z|^2}$

Thus, we can recast the elastic unitarity condition:

$$\text{Im}\,t_\ell = \sigma|t_\ell|^2 \Rightarrow \frac{\text{Im}\,t_\ell}{|t_\ell|^2} = \sigma \Longrightarrow \boxed{\text{Im}\,\frac{1}{t(s)} = -\sigma(s)}$$

Thus elastic unitarity fixes the imaginary part of the inverse amplitude.

For physical s, any elastic pw satisfies:

$$\boxed{t(s) = \frac{1}{\text{Re}\,t(s)^{-1} - i\sigma(s)}}$$

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We can repeat the argument in matrix form for the coupled channel case. For an energy s at which we have n open two-body states and an $n \times n$ T-matrix:

$$\text{Im } T_\ell = T \Sigma T^* \Longrightarrow \boxed{\text{Im } T(s)^{-1} = -\Sigma(s)}$$

Thus coupled channel two-body unitarity fixes the imaginary part of the inverse T-matrix.

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One of the simplest methods. Frequently used in experimental analysis.

K-matrix method

$K(s) = \text{Re } T(s)^{-1}$ =simple arbitrary analytic function on the real axis

- Various thresholds/cuts and resonances allowed
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Relatively simple improve on K-matrix approach.

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