The hadronic contribution to $\Delta\alpha_{\mathrm{QED}}$

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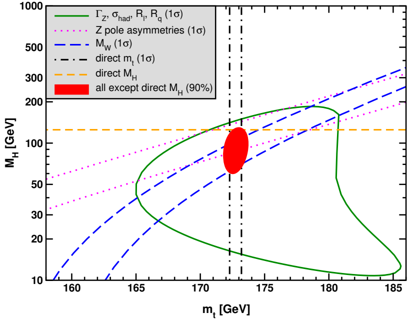


Relevant for many areas:

- Future experiments around M_Z pole energy (future e^+e^-- collider).
- $(g-2)_\mu$ puzzle
- Global fits of SM

$\alpha(M_Z)$ enters into global fits of SM. Excluding kinematic constraints from LHC, the SM predicts $M_H = 90^{+17}_{-16}$ GeV

If $\pm 10^{-4}$ in $\Delta\alpha^{(3)}_{\rm had}(2\ {\rm GeV}) \Rightarrow \mp 4.5$ GeV in M_H [Tanabashi et al. 2018]

Motivation. t-channel scattering to obtain $(g-2)_\mu$

Long standing discrepancy, $\Delta a_\mu(Exp - SM) \sim (28 \pm 8) \times 10^{-10}$

The hadronic leading contribution to $(g-2)_\mu$ is [Lautrup, Peterman, and Rafael 1972]

$$a_\mu^{\mathrm{HLO}} = \frac{\alpha}{\pi} \int_0^1 dx (1-x) \Delta\alpha_{\mathrm{had}}(Q^2), \quad Q^2 = \frac{x^2 m_\mu^2}{x-1} < 0$$

The integrand peaks at $x_{\mathrm{peak}} \approx 0.914$ ($Q^2 \approx -0.108\ \mathrm{GeV}^2$)

From here, if $(g-2)_\mu$ is solved by increasing $a_{\mu,\mathrm{SM}}$, we increase $\Delta\alpha_{\mathrm{had}}$ as well. This also lowers the estimated M_{H} from the global fit

MUonE experiment (CERN) measures $\Delta\alpha_{\mathrm{had}}(Q^2)$ [Abbiendi et al. 2017]

- It covers $0 < x < 0.93$ ($-0.143\ \mathrm{GeV}^2 < Q^2 < 0$)
- The region $0.93 < x < 1$ amounts to 13% of the integrand

LQCD can compute the range $Q^2 = -(0.14 - 4)\ \mathrm{GeV}^2$

Introduction

We parametrize the running of the QED coupling as

$$\alpha(Q^2) = \frac{\alpha}{1 - \Delta\alpha(Q^2)}$$

- In the Thomson limit we recover the fine structure constant, $\alpha^{-1}(Q^2 = 0) = \alpha^{-1} = 137.035999139(31)$ [Tanabashi et al. 2018]
- At the Z pole, $\hat{\alpha}^{(5)}(M_Z)^{-1} = 127.955(10)$

The uncertainty is dominated by hadronic loops at low energies
The hadronic contribution to $\Delta\alpha$ is

$$\Delta\alpha^{had}_{QED}(Q^2) = 4\pi\alpha\bar{\Pi}^{\gamma\gamma}(Q^2), \quad \bar{\Pi}^{\gamma\gamma}(Q^2) = \Pi^{\gamma\gamma}(Q^2) - \Pi^{\gamma\gamma}(0)$$

where the vacuum polarization function is defined as

$$\left(Q_\mu Q_\nu - \delta_{\mu\nu}Q^2\right)\Pi^{\gamma\gamma}(Q^2) = \Pi^{\gamma\gamma}_{\mu\nu}(Q^2) = \int d^4x e^{iQx} \langle j^\gamma_\mu(x) j^\gamma_\nu(0)\rangle$$

with the electromagnetic current

$$j^\gamma_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c$$

Time-momentum representation (TMR) method

[Bernecker and Meyer 2011; Francis et al. 2013]

$$\bar{\Pi}(Q^2) = \int_0^\infty dx_0\, G(x_0) \left[x_0^2 - \frac{4}{Q^2} sin^2\left(\frac{Qx_0}{2}\right)\right], \quad G(x_0) = -\frac{1}{3}\int d^3x \sum_{k=1}^{3} \langle j_k(x) j_k(0)\rangle$$

Any Q can be input to the kernel, but

- $Q \sim \pi/a$ suffer strong cut-off effects
- $Q << 1\ \mathrm{GeV}$ weights more the noise in the correlator

The bare correlators are, in the isospin basis,

$$G_{\mu\nu}(x) = G^{33}_{\mu\nu}(x) + \frac{1}{3} G^{88}_{\mu\nu}(x) + \frac{4}{9} C^{c,c}_{\mu\nu}(x)$$

where

$$G^{33}_{\mu\nu}(x) = \frac{1}{2} C^{\ell,\ell}_{\mu\nu}(x)$$

$$G^{88}_{\mu\nu}(x) = \frac{1}{6}\left[C^{\ell,\ell}_{\mu\nu}(x) + 2C^{s,s}_{\mu\nu}(x) + 2D^{\ell-s,\ell-s}_{\mu\nu}(x)\right]$$

The connected and disconnected Wick's contractions are

$$C^{f_1,f_2}_{\mu\nu}(x) = -\langle \mathrm{Tr}\left\{ D^{-1}_{f_1}(x,0)\gamma_\mu D^{-1}_{f_2}(0,x)\gamma_\nu \right\}\rangle$$

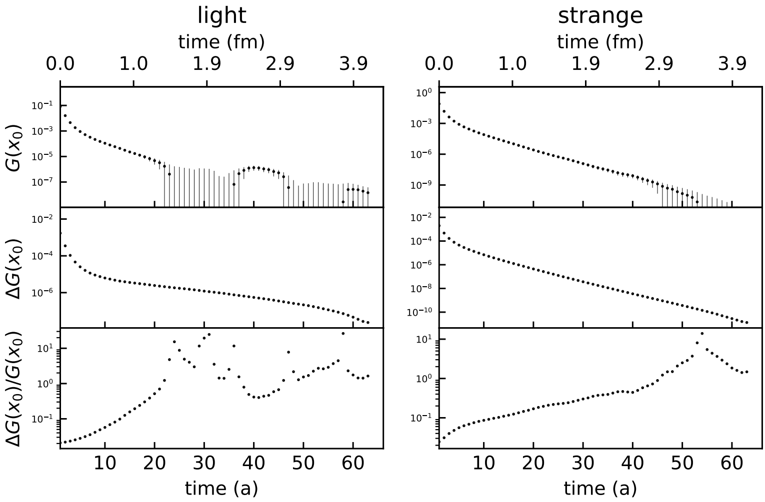
$$D^{f_1,f_2}_{\mu\nu}(x) = \langle \mathrm{Tr}\left\{ D^{-1}_{f_1}(x,x)\gamma_\mu \right\} \mathrm{Tr}\left\{ D^{-1}_{f_2}(0,0)\gamma_\nu \right\}\rangle$$

Set of $N_f = 2+1$ CLS ensembles [Bruno, Korzec, and Schaefer 2017; Bruno et al. 2015]

	T/a	L/a	t_0^{sym}/a^2	a [fm]	L [fm]	M_π,	M_K [MeV]	$M_\pi L$	τ_{int}(# conf.)
H101	96	32	2.860	0.086	2.8	415		5.8	1
H102	96	32			2.8	355	440	5.0	2
H105*	96	32			2.8	280	460	3.9	1
N101	128	48			4.1	280	460	5.8	0.5
C101*	96	48			4.1	220	470	4.6	0.5
B450§	64	32	3.659	0.076	2.4	415		5.1	2
S400	128	32			2.4	350	440	4.3	3
N401*	128	48			3.7	285	460	5.3	1
H200	96	32	5.164	0.064	2.1	420		4.4	1
N202	128	48			3.1	410		6.4	2
N203*	128	48			3.1	345	440	5.4	1
N200*	128	48			3.1	285	465	4.4	1
D200*	128	64			4.1	200	480	4.2	0.5
E250*§	192	96			6.2	130	490	4.1	0.5
N300	128	48	8.595	0.050	2.4	420		5.1	2
N302*	128	48			2.4	345	460	4.2	2
J303	192	64			3.2	260	475	4.2	0.5

* disconnected available. §PBC in time

Signal/noise problem



ensemble D200, local-local discretization

Exploit the spectral representation of the correlator in finite volume, $G(x_0, L)$,

$$G(x_0, L) = \sum_n |A_n|^2 e^{-m_n x_0} \xrightarrow{x_0 \to \infty} |A_0| e^{-m_0 x_0}$$

- Simplest option, fit long time behaviour with one state, [Della Morte et al. 2017]

$$G(x_0, L) = \begin{cases} \text{data}, & x_0 < x_0^{\text{cut}} \\ A_{\text{fit}} e^{-m_{\text{fit}} t}, & x_0 \geq x_0^{\text{cut}} \end{cases}$$

  where x_0^{cut} lies between 1.5 fm and 2.5 fm, depending on the noise

- Or better, apply the bounding method, [Gérardin et al. 2019]

$$0 \leq G(x_0^{\text{cut}}, L) e^{-E_{\text{eff}}(x_0^{\text{cut}})(x_0 - x_0^{\text{cut}})} \leq G(x_0, L) \leq G(x_0^{\text{cut}}, L) e^{-E_0(x_0 - x_0^{\text{cut}})},$$

  where both bounds converge at x_0^{cut}, and $E_{\text{eff}}(x_0) = -\partial \log G(x_0)/\partial x_0$, E_0 is the ground state

Add to the lattice data the difference between the modelling of the infinite-volume and finite-volume $\mathrm{I} = 1$ correlator. $x_{0i} = (M_\pi L/4)^2/M_\pi$

$x_0 \leq x_{0i}$, use scalar QED (NLO χPT) [Della Morte et al. 2017; Francis et al. 2013]

$$G(x_0,\infty) - G(x_0,L) = \frac{1}{3}\left(\int \frac{d^3\vec{k}}{(2\pi)^3} - \frac{1}{L^3}\sum_{\vec{k}}\right)\frac{\vec{k}^2}{\vec{k}^2 + m_\pi^2}e^{-2t\sqrt{\vec{k}^2+m_\pi^2}}$$

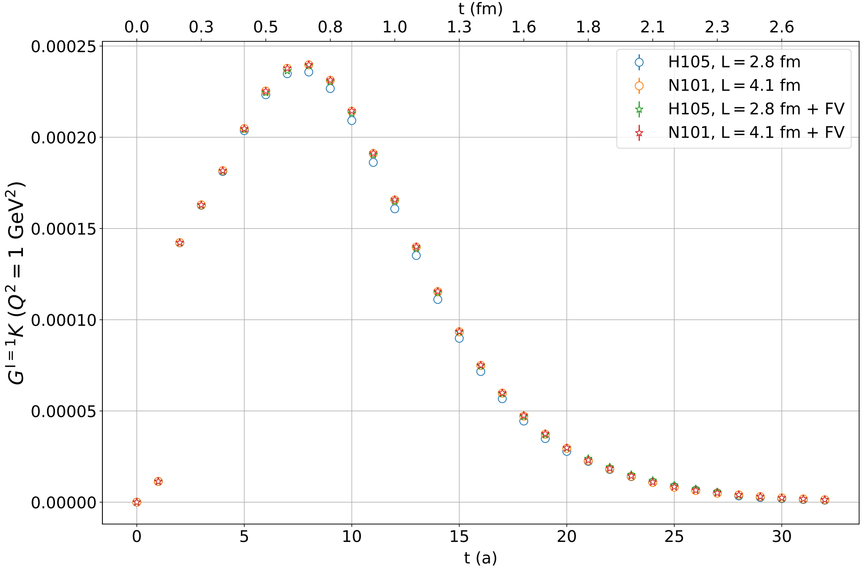
$x_0 > x_{0i}$, use Gounaris-Sakurai parameterization of $F_\pi(\omega)$ [Gounaris and Sakurai 1968]

$$G(x_0,\infty) = \int_{2m_\pi}^{\infty} dQ Q^2 \rho(Q) e^{-Q|x_0|}, \qquad \rho(Q) = \frac{1}{48\pi^2}\left(1 - 4\frac{m_\pi^2}{Q^2}\right)^{3/2}|F_\pi(Q)|^2$$

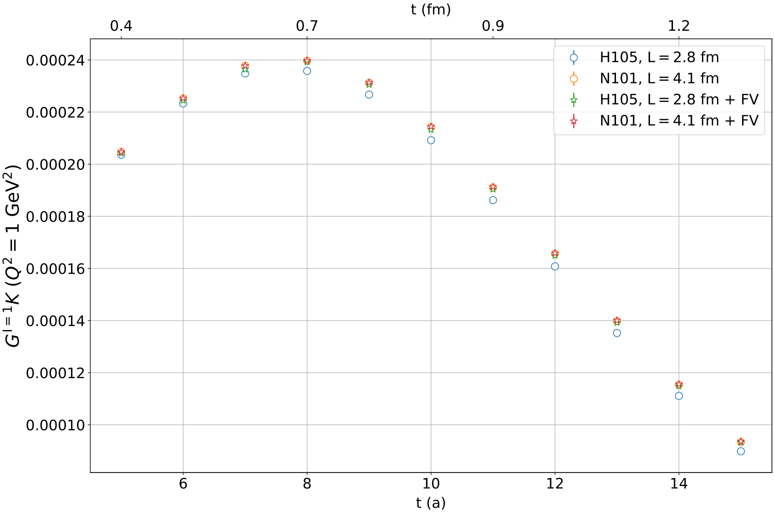
The finite-volume correlator from the Lüscher energies, ω_n and Lellouch-Lüscher amplitudes, A_n [Lellouch and Luscher 2001; Luscher 1991a; Meyer 2011]

$$G(x_0,L) = \sum_n |A_n|^2 e^{-\omega_n x_0}$$

Finite-volume correction. Compare H105 vs N101



Finite-volume correction. Compare H105 vs N101



Correction is $\sim 2\%$ for H105 and $\sim 2‰$ for N101. Bigger than the statistical uncertainty

Minimize

$$\chi^2 = \sum_{l \in \{\mathrm{CLS}\}} \left(v^T_{\mathrm{model}} - v^T_{l,\mathrm{data}}\right) C^{-1}_{l,\mathrm{data}} \left(v_{\mathrm{model}} - v_{l,\mathrm{data}}\right)$$

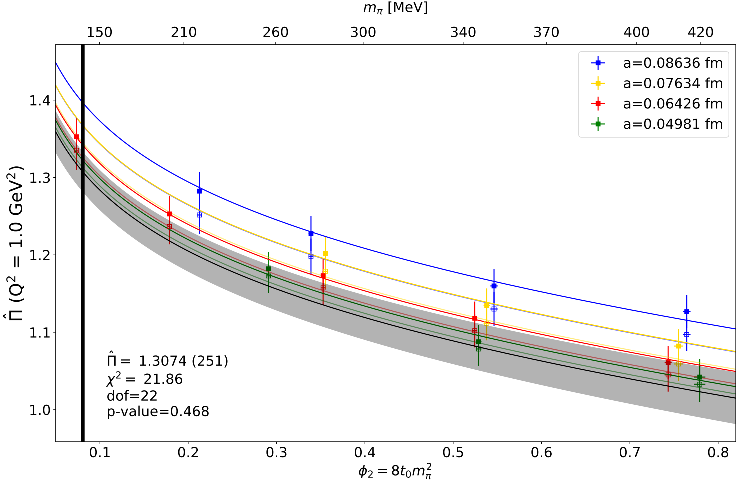
with

$$v_{\mathrm{model}} - v_{l,\mathrm{data}} = \begin{pmatrix} \phi_{2,l} \\ \phi_{4,l} \\ \bar{\Pi}^{33}_{ll}(a^2/t_0, \phi_{2,l}, \phi_{4,l}) \\ \bar{\Pi}^{33}_{lc}(a^2/t_0, \phi_{2,l}, \phi_{4,l}) \end{pmatrix} - \begin{pmatrix} 8t_0^{\mathrm{sym}} M_\pi^2 \\ 8t_0^{\mathrm{sym}} \left(M_K^2 + M_\pi^2/2\right) \\ \bar{\Pi}^{33}_{ll} \\ \bar{\Pi}^{33}_{lc} \end{pmatrix}_{\mathrm{data}}$$

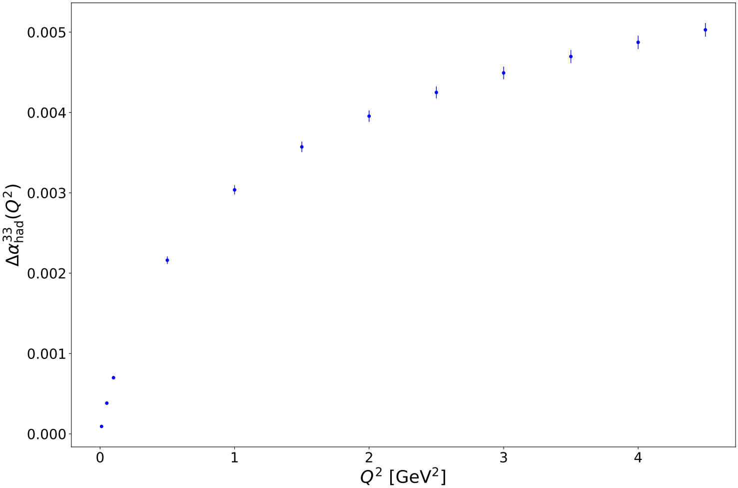
The model is

$$\bar{\Pi}\left(a^2/t_0^{\mathrm{sym}}, \phi_2, \phi_4\right) = \bar{\Pi}^{\mathrm{phy}} + \alpha_1 a^2/t_0^{\mathrm{sym}} + \alpha_2 \left(a^2/t_0^{\mathrm{sym}}\right)^{3/2} + \beta_1\left(\phi_2 - \phi_2^{\mathrm{phy}}\right) + \beta_2 log\left(\phi_2/\phi_2^{\mathrm{phy}}\right) + \gamma_1\left(\phi_4 - \phi_4^{\mathrm{phy}}\right)$$

Preliminary results. $\mathrm{I} = 1$ physical point



Preliminary results. $\mathrm{I} = 1$ physical point



At $Q^2 = 1\ \mathrm{GeV}^2$, $\Delta\alpha^{33}_{\mathrm{had}} = 0.003036(58)$

Summary and outlook

- We have computed the isovector component of $\Delta\alpha_{\mathrm{QED}}^{\mathrm{had}}$ in the energy range $(0.01 - 4.5)\ \mathrm{GeV}^2$.
- The statistical error at $Q^2 = 1\ \mathrm{GeV}^2$ is $\sim 2\%$.
- The inclusion of finite-volume effects is fundamental
- Treatment of the tail with single exponential

The next steps that we will take are:

- Investigate the systematics of the chiral and continuum extrapolation and the scale
- Analyze the isoscalar (including disconnected) and charm contributions
- Apply Bounding Method to long-time tail of light ensembles
- Perform a comparison with Phenomenology of $\Delta\alpha_{QED}$ including the four lightest flavors [Jegerlehner 2011]
- Add a new ensemble at $a \approx 0.050\ \mathrm{fm}$ and $M_\pi \approx 175\ \mathrm{MeV}$
- Add isospin breaking [Risch and Wittig 2018, 2019]

We $\mathcal{O}(a)$ improve and renormalize the electromagnetic current. To do that, we combine the u, d, s flavours in the isospin basis,

$$j_\mu^{a,l} = \bar{\psi}\gamma_\mu \frac{\lambda^a}{2}\psi,$$

with λ^a the Gell-Mann matrices, and the corresponding expression for the conserved current. The renormalization of the isovector ($\mathrm{I} = 1$, with $a = 3$) local current is

$$j_{\mu,R}^{3,\mathrm{l}} = Z_V \left(1 + 3\bar{b}_V am_q^{\mathrm{av}} + b_V am_{q,\ell}\right) j_{\mu,\mathrm{i}}^{3,\mathrm{l}}$$

The $\mathcal{O}(a)$-improved local current is

$$j_{\mu,\mathrm{i}}^{a,\mathrm{l}} = j_\mu^{a,\mathrm{l}} + ac_V^{\mathrm{l}}\tilde{\partial}_\nu T_{\nu\mu}^{a,l}$$

For the conserved current no renormalization is needed,

$$j_{\mu,R}^{a,\mathrm{c}} = j_\mu^{a,\mathrm{c}} + ac_V^{\mathrm{c}}\tilde{\partial}_\nu T_{\nu\mu}^{a,l}$$

The tensor current is

$$T_{\mu\nu}^{a,l} = -\bar{\psi}\left[\gamma_\mu, \gamma_\nu\right]\frac{\lambda^a}{2}\psi$$

Γ-method to estimate autocorrelations

Compute [De Palma et al. 2019; Wolff 2004]

$$\Gamma_{x_0 x_0'}(d) = \frac{1}{N-d} \sum_{\mathrm{i}=1}^{N-d} \left(G(\mathrm{i}, x_0) - \overline{G}(x_0) \right) \left(G(\mathrm{i}+\mathrm{d}, x_0') - \overline{G}(x_0') \right)$$

i runs over configurations. ($\Gamma_{x_0 x_0'}(0)$ is the covariance matrix). Add autocorrelations in window W,

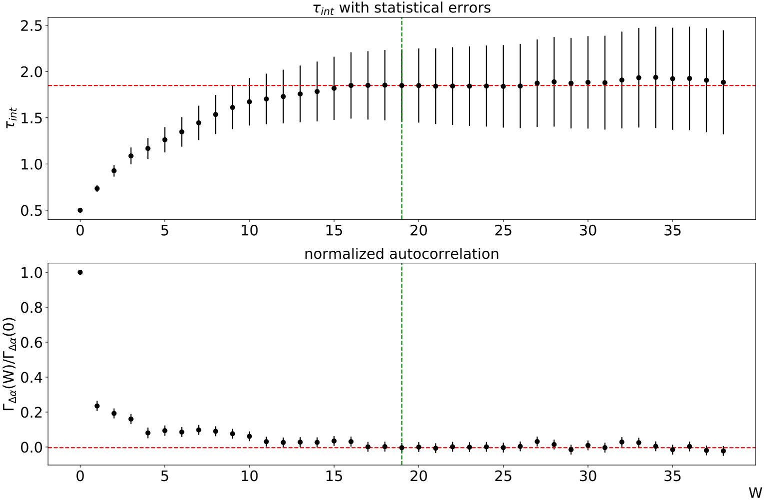
$$C_{\Delta\alpha}(W) = \Gamma_{\Delta\alpha}(0) + 2 \sum_{d=1}^{W} \Gamma_{\Delta\alpha}(d), \quad \Gamma_{\Delta\alpha}(d) = \sum_{x_0, x_0'} f_{x_0} f_{x_0'} \Gamma_{x_0 x_0'}(d), \qquad f_{x_0} = \frac{\partial \Delta\alpha}{\partial G(x_0)}$$

Obtain statistical uncertainty of $\Delta\alpha$

$$\sigma^2_{\Delta\alpha} = \frac{2\tau_{int,\Delta\alpha}}{N} \Gamma_{\Delta\alpha}(0), \qquad \tau_{int,\Delta\alpha}(W) = \frac{C_{\Delta\alpha}(W)}{2\Gamma_{\Delta\alpha}(0)}$$

$2\tau_{int,\Delta\alpha}$ is the bin size.

Autocorrelations



ensemble N300. Light flavor. $Q^2 = 4.5$ GeV^2

How to choose W

How to choose W? Consider the two sources of uncertainty in the autocorrelations:

- Finite statistics, $\propto \sqrt{\frac{W}{N}}$
- Truncation of Γ_F, $\propto exp(-W/\tau)$

τ dictates the decay of autocorrelations. Suppose $\tau = S\tau_{int}$, with S set by the user (reasonable ansatz $\sim 1 - 3$).

$W_{optimal}$ minimizes the total error. After W is determined, we need to check that we have included just the right amount of autocorrelations

The correlator in finite volume,

$$G(x_0, L) = \sum_n |A_n|^2 e^{-\omega_n x_0}$$

can be computed using the Lüscher method[2, 3] at long distances. First, solve numerically the following equation to obtain ω_n.

$$\delta_1(k) + \phi\left(\frac{kL}{2\pi}\right) = n\pi, \quad n = 1, 2, ...$$

$$\omega_n = 2\sqrt{m_\pi^2 + k^2}$$

where ϕ is a known function.

[2] Luscher 1991a.

[3] Luscher 1991b.

Then, the amplitudes are computed[4] as

$$|A_n|^2 = \frac{2k^5 |F_\pi(\omega_n)|^2}{3\pi\omega_n^2 \mathbb{L}(k)}$$

where the Lellouch-Lüscher factor $\mathbb{L}(k)$ is known[5],

$$\mathbb{L}(k) = \frac{kL}{2\pi}\phi'\left(\frac{kL}{2\pi}\right) + k\frac{\partial\delta_1(k)}{\partial k}$$

Both, $|F_\pi|$ and δ_1 are parametrized using the Gounaris-Sakurai model.

[4] Meyer 2011.

[5] Lellouch and Luscher 2001.

FSE. Gounaris-Sakurai model

The pion form factor F_π with its phase shift, δ_1 can be parametrized by the Gounaris-Sakurai model[6], which only depends on two parameters, the ρ meson mass, m_ρ and its decay width, Γ_ρ.

$$F_\pi(\omega) = \frac{f_0}{\frac{k^3}{\omega}(cot\delta_1(k) - i)}$$

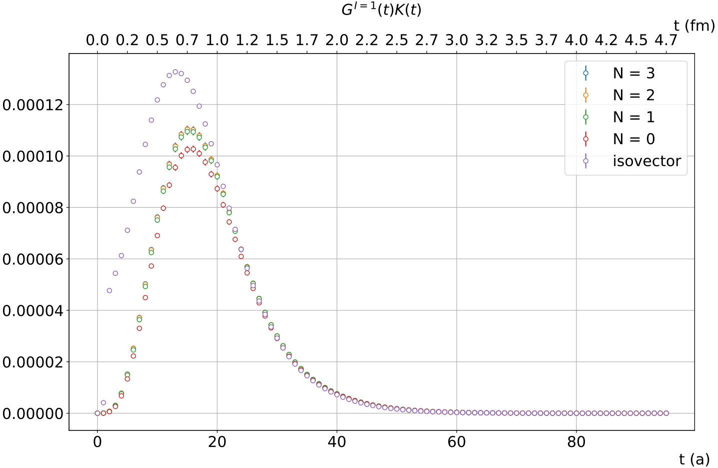
$$\frac{k^3}{\omega} cot\delta_1(k) = k^2 h(\omega) - k_\rho^2 h(m_\rho) + b(k^2 - k_\rho^2)$$

where f_0, b depend on m_ρ and Γ_ρ. All of them, f_0, b and h have a closed form.

[6] Gounaris and Sakurai 1968.

FSE. J303 long distance, finite volume

For the ensemble J303, at $Q^2 = 0.5$ GeV^2. $K = \frac{1}{\omega^2}\left(\omega^2 t^2 - 4sin^2\left(\frac{\omega t}{2}\right)\right)$



At $Q^2 = 1$ GeV^2

id	$\Delta\alpha \times 10^6$	FV $\times 10^6$	id	$\Delta\alpha \times 10^6$	FV $\times 10^6$
H101	2528.6(6.9)	7.3(0.2)	N202	2462.8(8.9)	3.1(0.1)
H102	2662.9(8.6)	18.4(0.6)	N203	2585.6(8.1)	10.1(0.3)
H105	2835.7(10.8)	56.0(3.5)	N200	2725.1(8.7)	28.6(0.9)
N101	2844.0(10.1)	5.1(0.2)	D200	2930.8(10.7)	31.7(1.0)
C101	3011.0(9.8)	19.1(0.9)	E250	3189.8(20.5)	31.2(1.2)
B450	2465.2(7.8)	16.4(0.4)	N300	2341.4(8.9)	17.7(2.6)
S400	2612.9(9.6)	40.3(1.6)	N302	2470.4(8.6)	47.7(1.1)
N401	2789.8(9.7)	10.4(0.4)	J303	2726.3(13.8)	36.9(0.8)