The BEC-BCS crossover and the Dirac spectrum in QCD at nonzero isospin asymmetry

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Outline

1. QCD with isospin
2. The phase diagram of QCD with isospin on the lattice
3. Signatures of the BCS phase at high μ_I
4. Results

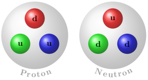
QCD WITH ISOSPIN

[Physical motivation, numerical advantages, analytical results for the phase diagram]

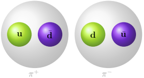
Motivation

A nonzero isospin density $n_\mathrm{I} = n_\mathrm{u} - n_\mathrm{d}$ describes an asymmetry between the densities of up and down quarks

- hence between the densities of protons and neutrons



- hence between the densities of π^+ and π^-



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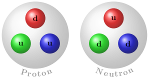

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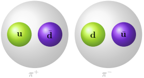
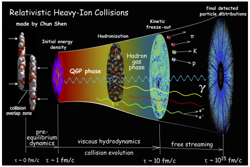
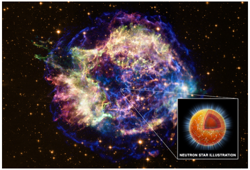


The $n_I < 0$ case is relevant for

- the initial state of heavy ion collisions
  - imbalance between produced charged pions



- structure of cold neutron stars
  - very low proton fraction



Isospin chemical potential

QCD with three flavors of fermions in the grand canonical ensemble.
Quark chemical potentials conjugated quantities to quark densities

$$\mu_u = \frac{\mu_B}{3} + \mu_I, \qquad \mu_d = \frac{\mu_B}{3} - \mu_I, \qquad \mu_s = \frac{\mu_B}{3} - \mu_S$$

- Consider zero baryon number and strangeness, but nonzero isospin

$$\mu_B = 0, \qquad \mu_S = 0, \qquad \mu_I = \mu_u = -\mu_d$$

- One can then define a pion chemical potential $\mu_\pi = \mu_u - \mu_d = 2\mu_I$ to which corresponds the isospin density $n_I = n_u - n_d$

- Systems with $n_I \neq 0$ can be simulated with standard Monte Carlo importance sampling techniques using $\mu_I \in \mathbb{R}$ that couples to $I_3 = \frac{\tau_3}{2}$!

Alford, Kapustin, Wilczek (1999)

QCD at finite isospin density

VOLUME 86, NUMBER 4 PHYSICAL REVIEW LETTERS 22 JANUARY 2001

QCD at Finite Isospin Density

D. T. Son[1,3] and M. A. Stephanov[2,3]
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(Received 31 May 2000)

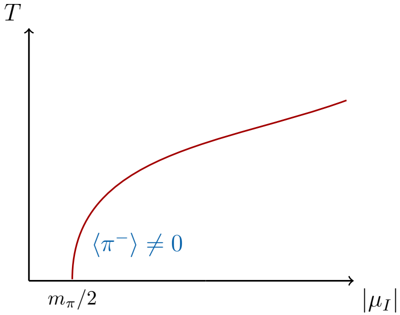
QCD at finite isospin chemical potential μ_I has no fermion sign problem and can be studied on the lattice. We solve this theory analytically in two limits: at low μ_I, where chiral perturbation theory is applicable, and at asymptotically high μ_I, where perturbative QCD works. At low isospin density the ground state is a pion condensate, whereas at high density it is a Fermi liquid with Cooper pairing. The pairs carry the same quantum numbers as the pion. This leads us to conjecture that the transition from hadron to quark matter is smooth, which passes several tests. Our results imply a nontrivial phase diagram in the space of temperature and chemical potentials of isospin and baryon number.

Son, Stephanov (2001)

Non trivial phase diagram drawn on the basis of analytical computations in

- the $n_I \to 0$ limit $\longleftarrow$ Chiral Perturbation Theory
- the $n_I \to \infty$ limit $\longleftarrow$ Perturbative QCD

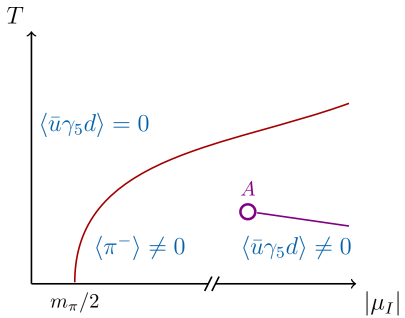
QCD at finite isospin density - The “analytical phase diagram”



In the limit $n_I \to 0$, i.e. $|\mu_I| \ll m_\rho$ χPT applies

- $\pi^\pm$ lightest hadrons coupling to μ_I: χPT describes their effective dynamics
- At $T = 0$, $\mu_I \geq \mu_{I,c} = m_\pi/2$, sufficient energy to create $\pi^\pm$
- A Bose-Einstein condensate (BEC) is formed
- Hadronic/BEC phase transition predicted, by χPT, to be second order ($\mathrm{O}(2)$ universality class)

QCD at finite isospin density - The "analytical phase diagram"



In the limit $n_I \to \infty$, i.e. $|\mu_I| \gg \Lambda_{QCD}$ p-QCD applies

- Attractive gluon interaction forms pseudoscalar $u - \bar{d}$ Cooper-pairs
- BEC/BCS phse transition expected to be analytic crossover (same symmetry breaking pattern)
- At asymptotically large μ_I, decoupling of the gluonic sector and first-order deconfinement phase transition

THE PHASE DIAGRAM OF QCD WITH ISOSPIN ON THE LATTICE

Brandt, Endrődi, Schmalzbauer (2018)

[Symmetry breaking patterns, Pion BEC, Pionic source λ, lattice setup]

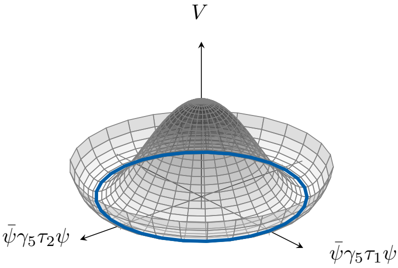
n_I-QCD on the lattice - Symmetry breaking patterns

- $SU_V(2) \times U_V(1)$ flavor symmetry group for QCD with light quark matrix

$$\mathcal{M}_{ud}\big|_{\mu_i=\lambda=0} = \gamma_\mu(\partial_\mu + iA_\mu)\,\mathbb{1} + m_{ud}\mathbb{1}, \qquad \psi = (u,d)^\top$$

- At $\mu_I \neq 0$ $\longrightarrow$ $\mathcal{M}_{ud} = \mathcal{M}_{ud}\big|_{\mu_i=\lambda=0} + \mu_I\gamma_4\tau_3$

$$SU_V(2) \times U_V(1) \longrightarrow U_{\tau_3}(1) \times U_V(1)$$



- **Spontaneous breaking** with pion condensate $\langle\bar\psi\gamma_5\tau_{1,2}\psi\rangle$
  $\longrightarrow$ Appearance of Goldstone mode

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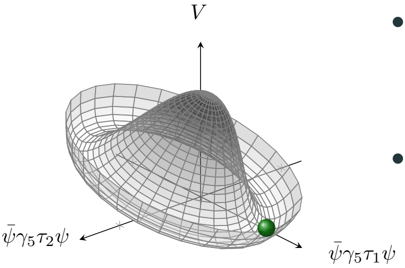
- At $\mu_I \neq 0$, $\lambda \neq 0$ $\longrightarrow$ $\mathcal{M}_{ud} = \mathcal{M}_{ud}\big|_{\mu_i=\lambda=0} + \mu_I\gamma_4\tau_3 + i\lambda\gamma_5\tau_2$

$$SU_V(2) \times U_V(1) \longrightarrow U_{\tau_3}(1) \times U_V(1) \longrightarrow \varnothing \times U_V(1)$$



- **Spontaneous breaking** with pion condensate $\langle\bar{\psi}\gamma_5\tau_{1,2}\psi\rangle$
  $\longrightarrow$ Appearance of Goldstone mode
- **Explicit breaking** via pionic source λ,
  $\longrightarrow$ pseudo-Goldstone boson
  (λ necessary trigger for spontaneous breaking at finite V and I.R. regulator)

n_I-QCD on the lattice - Setup

- Light quark matrix in the basis of up and down quarks $\psi = (u, d)^\top$

$$\mathcal{M}_{ud} = \gamma_\mu(\partial_\mu + iA_\mu)\,\mathbb{1} + m_{ud}\mathbb{1} + \mu_I\gamma_4\tau_3 + i\lambda\gamma_5\tau_2$$

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- Unphysical symmetry breaking term in $\mathcal{M}_{ud}$ couples to the charged pion field $\pi^\pm$

$$S_{ud} = S_{ud}(\lambda = 0) + \lambda\,\pi^\pm, \quad \pi^\pm \equiv \bar{\psi} i\gamma_5\tau_2\psi = \bar{u}\gamma_5 d - \bar{d}\gamma_5 u\,.$$

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- QCD with $N_f = 2 + 1$ improved dynamical staggered quarks with physical quark masses at various T, μ_I, λ

$$\mathcal{Z} = \int \mathcal{D}U_\mu\, e^{-\beta S_G^{\mathrm{Sym}}} (\det \mathcal{M}_{ud})^{1/4} (\det \mathcal{M}_s)^{1/4}\,, \qquad U_\mu = \exp(iaA_\mu)$$

n_I-QCD on the lattice - Breaking of $U_{\tau_3}(1)$ symmetry

- Spontaneous and explicit, by λ, breaking of the $\mathrm{U}_{\tau_3}(1)$ symmetry at $\mu_I \neq 0$ completely analogous to the spontaneous and explicit, by m_{ud}, breaking of the $SU_L(2) \otimes SU_R(2)$ chiral symmetry at $\mu_I = 0$

Pion condensation		**Chiral symmetry breaking**
$U_{\tau_3}(1) \to \varnothing$	$\xleftrightarrow{\text{breaking pattern}}$	$SU_L(2) \otimes SU_R(2) \to SU_V(2)$
1	$\xleftrightarrow{\#\text{ Goldstones}}$	3
$\langle \bar{\psi} \gamma_5 \tau_2 \psi \rangle$	$\xleftrightarrow{\text{condensates}}$	$\langle \bar{\psi} \psi \rangle$
$\lambda \to 0$	$\xleftrightarrow{\text{explicit breaking}}$	$m_{ud} \to 0$
$\rho^{\lvert \not{D}(\mu_I) + m_{ud} \rvert^2}(0)$	$\xleftrightarrow{\text{Banks-Casher}}$	$\rho^{(\not{D})}(0)$

- While in nature $m_{ud} > 0$, λ is unphysical: the limit $\lambda \to 0$ must be taken!

RESULTS

[Approximate order parameters,

phase diagram in the $\mu_I - T$ plane]

n_I-QCD on the lattice - Observables

- The pion condensate and quark condensate obtainable from $\mathcal{Z}$, via differentiation and measurable with noisy-estimator techniques

$$\langle \pi^{\pm} \rangle = \frac{T}{V} \frac{\partial \log \mathcal{Z}}{\partial \lambda} = \frac{T}{2V} \operatorname{tr} \frac{\lambda}{|\not{D}(\mu_I) + m_{ud}|^2 + \lambda^2}$$

$$\langle \bar{\psi}\psi \rangle = \frac{T}{V} \frac{\partial \log \mathcal{Z}}{\partial m_{ud}} = \frac{T}{2V} \operatorname{Re} \operatorname{tr} \frac{\not{D}(\mu_I) + m_{ud}}{|\not{D}(\mu_I) + m_{ud}|^2 + \lambda^2}$$

then becoming, after appropriate multiplicative/additive renormalization,

$$\Sigma_{\bar{\psi}\psi} = \frac{m_{ud}}{m_\pi^2 f_\pi^2} \left[\langle \bar{\psi}\psi \rangle_{T,\mu_I} - \langle \bar{\psi}\psi \rangle_{0,0} \right] + 1$$

$$\Sigma_\pi = \frac{m_{ud}}{m_\pi^2 f_\pi^2} \langle \pi^{\pm} \rangle_{T,\mu_I}$$

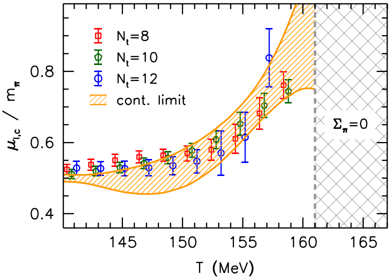
- The renormalized Polyakov loop $P_r(T, \mu_I) = Z \cdot \left\langle \frac{1}{V} \sum_{n_x, n_y, n_z} \operatorname{Tr} \prod_{n_t=0}^{N_t-1} U_t(n) \right\rangle$

with $Z = \left(\frac{P_\star}{P(T_\star, \mu_I = 0)} \right)^{T_\star/T}$, and $T_\star = 162$ MeV, hence $P_\star = 1$

n_I-QCD result - Continuum limit and the $\mu_I - T$ phase diagram

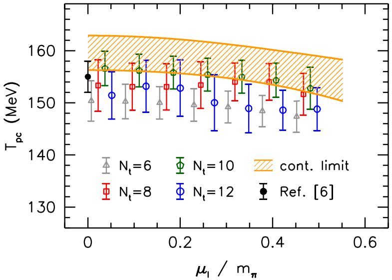
Pion condensate

- BEC phase boundary, $\mu_{I,c}(T)$, by onset of Σ_π
- $\mu_{I,c}(T, a)$, 4^{th} order polynomial in $(T - T_0)$ with $a-$dependent coefficients and $T_0 = 140$ MeV

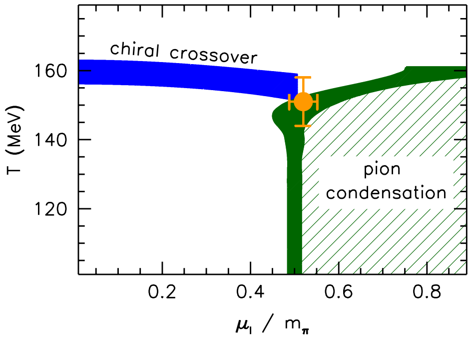


Quark condensate

- Chiral crossover $T_{pc}(\mu_I)$, by the inflection points of $\Sigma_{\bar{\psi}\psi}(T)$
- $T_{pc}(\mu_I, a)$, even-in-μ_I polynomial, including data up to $\mu_{I,c}(0) = m_\pi/2$



n_I-QCD result - Continuum limit and the $\mu_I - T$ phase diagram



- $T_{pc}(\mu_I = 0) = 159(4)$ MeV
- Small downward curvature $T_{pc}(\mu_I)$
- $\Sigma_\pi = 0$ up to $\mu_I = 120$ MeV, for $T \gtrsim 160$ MeV

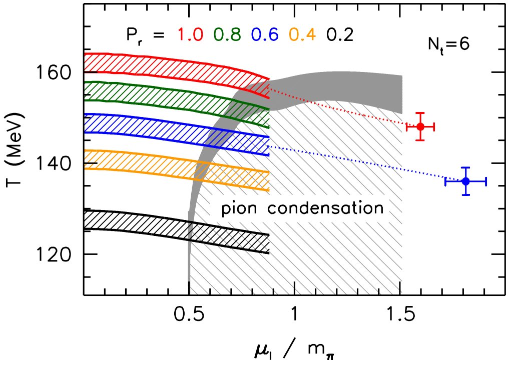
- BEC boundary at $\mu_{I,c} = m_\pi/2$ up to $T \approx 140$ MeV, very flat at larger μ_I
- Two "transition" lines meet at $\mu_{I,pt} = 70(5)$ MeV in a pseudo-triple point
- Chiral symmetry restoration and BEC boundary coincide for $\mu_I \geq \mu_{I,pt}$

SIGNATURES OF THE BCS PHASE AT HIGH μ_I

[Complex Dirac spectrum]

QCD at finite isospin density - The "numerical phase diagram"

- Prediction of a superfluid state of u and $\bar{d}$ Cooper pairs (BCS phase) at very high-μ_I and $T = 0$, plausibly connected via crossover to the BEC phase at $\mu_I \geq m_\pi/2$ 🔗 Son, Stephanov (2001) 🔗 Adhikari, Andersen, Kneschke (2018)



- Deconfinement transition connecting continuously to BEC-BCS crossover in the (T, μ_I) phase diagram
  - Large Polyakov loops P_r within BEC phase
  - Slow decrease of $T_c^{\mathrm{deconf.}}(\mu_I)$

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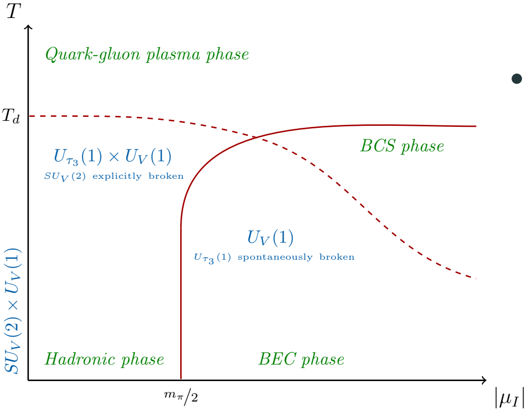

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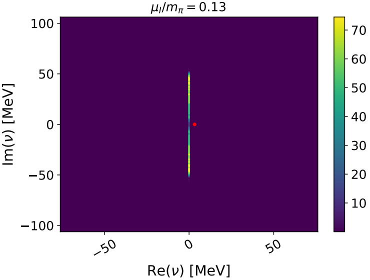
Signatures of the BCS phase at large μ_I from $\not{D}(\mu_I)$ spectrum

- For $|\mu_I| \gg \Lambda_{QCD}$ attractive channel between quarks near the Fermi surface lead to diquark pairing of the BCS type
- Banks-Casher relation extensible to the case of complex Dirac eigenvalues for QCD at $T = 0$, $\mu_I \neq 0$

🔗 Kanazawa, Wettig, Yamamoto (2013)

The BCS gap → $$\Delta^2 = \frac{2\pi^3}{3N_C}\,\rho(0)$$ ← The $\not{D}(\mu_I)$ spectral density

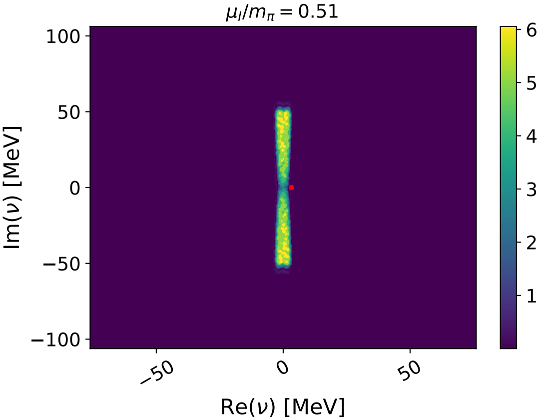


Complex spectrum of the Dirac operator

$$\underbrace{[\not{D}(\mu_I)]\,\psi_n = (\nu_n)\,\psi_n}_{\text{up sector},\mu_I} \xleftrightarrow[\text{chiral symmetry}]{\eta_5-\text{hermiticity}} \underbrace{\widetilde{\psi}_n^\dagger\,[\not{D}(-\mu_I)] = \widetilde{\psi}_n^\dagger(\nu_n^*)}_{\text{down sector},-\mu_I,\widetilde{\psi}_n=\gamma_5\psi_n}$$

- Complex eigenvalues $\nu_n \in \mathbb{C}$
- $[\not{D}(\mu_I), \not{D}^\dagger(\mu_I)] \neq 0$, so left and right eigenvectors of $\not{D}(\mu_I)$ do not coincide
- $\forall$ eigenvalue ν_n in the up sector, complex conjugate ν_n^* in the down sector
- Simulations at nonzero quark mass: instead of $\rho(0)$, we look at $\rho(m + i * 0)$ neglecting corrections at first.

Complex spectrum of $\not{D}(\mu_I)$ - Results, qualitatively



- Simulations are carried out away from the chiral limit $\rightarrow$ extract $\rho(m_{ud})$

- $\mu_I < m_\pi/2$: eigenvalues clustered along imaginary axis $\rightarrow \rho(m_{ud}) = 0$

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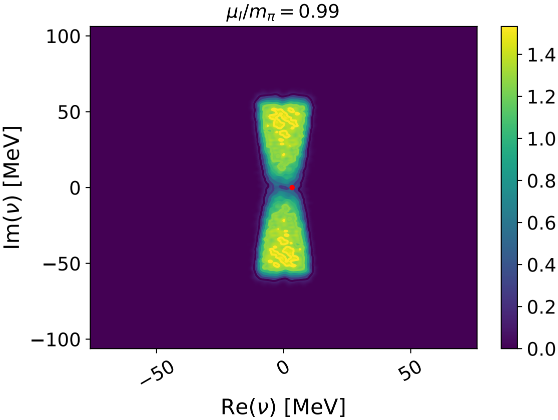

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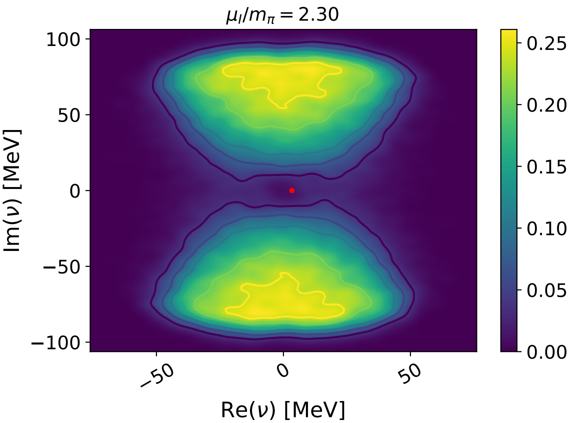

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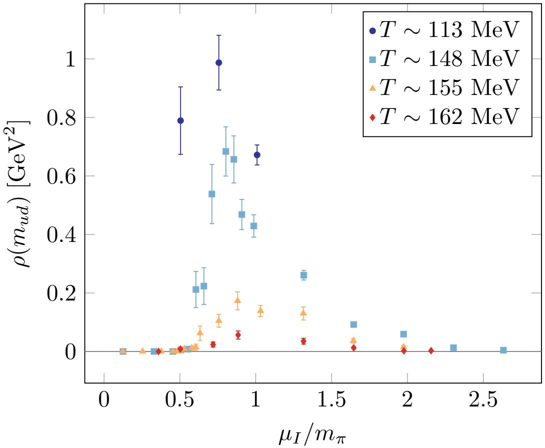
- $\mu_I > m_\pi/2$: spectrum 'wide' enough to include $m + i0 \rightarrow \rho(m_{ud}) \neq 0$
- Higher-μ_I: eigenvalues drifting away from the real axis $\rightarrow \rho(m_{ud}) \rightarrow 0$

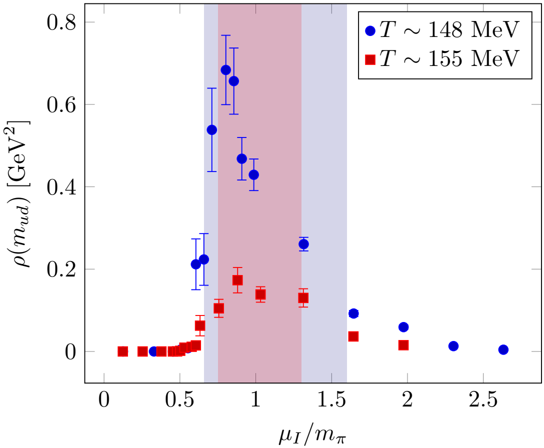
Complex spectrum of $\not{D}(\mu_I)$ - Results, quantitatively

$N_\sigma = 24$



Complex spectrum of $\not{D}(\mu_I)$ - Results, quantitatively

- Match μ_I- and T- dependence of $\rho(m_{ud})$ with the boundary of the BEC phase and with the deconfinement crossover



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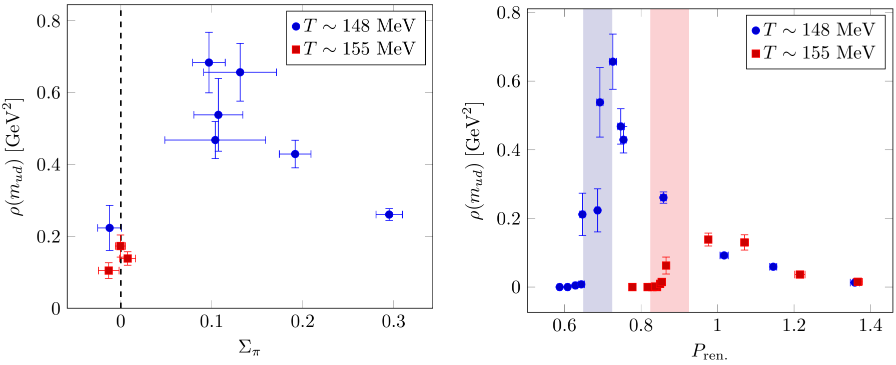
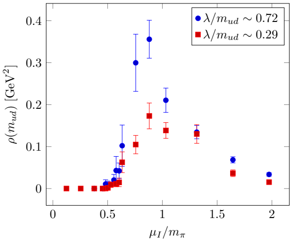
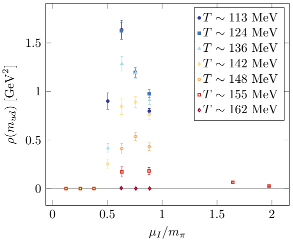
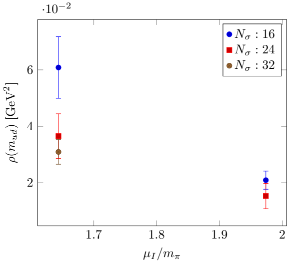


Complex spectrum of $\not{D}(\mu_I)$ - Conclusions, outlook

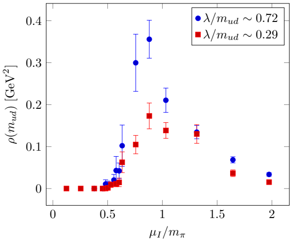


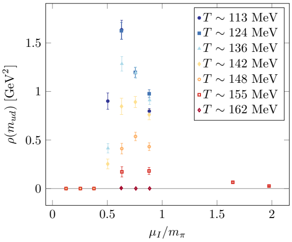




$a \to 0$

Complex spectrum of $\not{D}(\mu_I)$ - Conclusions, outlook



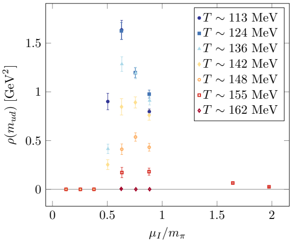


$V \to \infty$

$a \to 0$

Complex spectrum of $\not{D}(\mu_I)$ - Conclusions, outlook

$$\lambda \to 0$$



$$V \to \infty$$

$$a \to 0$$

Complex spectrum of $\not{D}(\mu_I)$ - Conclusions, outlook

$\lambda \to 0$

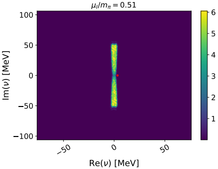
$T \to 0$

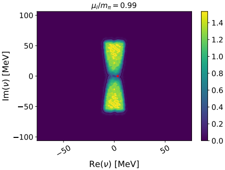
$V \to \infty$

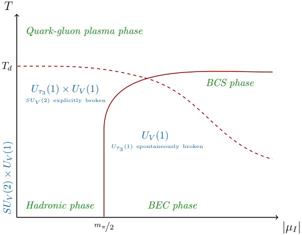
$a \to 0$

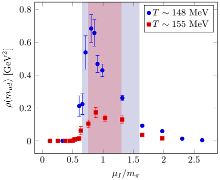
Complex spectrum of $\not{D}(\mu_I)$ - Conclusions, outlook

- Extrapolated spectral density $\rho(m_{ud})$ sensitive to the BEC boundary!



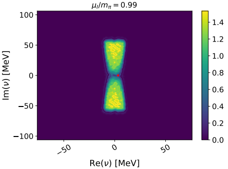






Complex spectrum of $\not{D}(\mu_I)$ - Conclusions, outlook

- Extrapolated spectral density $\rho(m_{ud})$ sensitive to the BEC boundary!



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  $\Delta \neq 0$ at high-μ_I?
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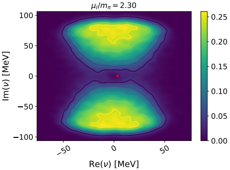

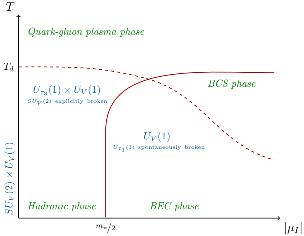

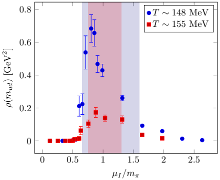

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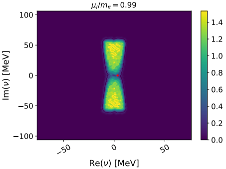

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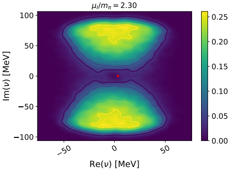
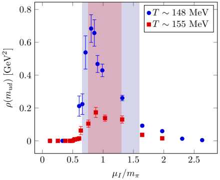


- Desired generalization of Banks-Casher relation away from $T = 0$ and $|\mu_I| \gg \Lambda_{QCD}$ limits



Thank you for your attention!

Chemical potential & positivity of the measure

PHYSICAL REVIEW D, VOLUME 59, 054502

Imaginary chemical potential and finite fermion density on the lattice

Mark Alford, Anton Kapustin, and Frank Wilczek
School of Natural Sciences, Institute for Advanced Study, Princeton, New Jersey 08540
(Received 7 August 1998; published 29 January 1999)

Standard lattice fermion algorithms run into the well-known sign problem with a real chemical potential. In this paper we investigate the possibility of using an *imaginary* chemical potential and argue that it has advantages over other methods, particularly for probing the physics at finite temperature as well as density. As a feasibility study, we present numerical results for the partition function of the two-dimensional Hubbard model with an imaginary chemical potential. We also note that systems with a net imbalance of isospin may be simulated using a real chemical potential that couples to I_3 without suffering from the sign problem.

Alford, Kapustin, Wilczek (1999)

Systems with $n_I \neq 0$ can be simulated with standard Monte Carlo importance sampling techniques using $\mu_I \in \mathbb{R}$ that couples to $I_3 = \frac{\tau_3}{2}$!

Chemical potential & positivity of the measure

Consider N_f flavors of fermions ψ, with $\mathcal{L}_\mathsf{F} = \bar{\psi}\, M(\phi)\, \psi$, and bosons ϕ. In the Euclidean formulation and after fermion fields are integrated out

$$\langle \mathcal{O} \rangle = \int \mathcal{D}\phi\ \mathcal{O}[\phi]\ \underbrace{\frac{(\det M(\phi))^{N_\mathsf{f}}\ e^{-S_\mathsf{B}(\phi)}}{\int \mathcal{D}\phi\ (\det M(\phi))^{N_\mathsf{f}}\ e^{-S_\mathsf{B}(\phi)}}}_{\text{pdf} \Leftrightarrow (\det M(\phi))^{N_\mathsf{f}} \in \mathbb{R}^+}$$

Once the theory is discretized on a lattice, one would like to estimate $\langle \mathcal{O} \rangle$ by employing importance sampling techniques.

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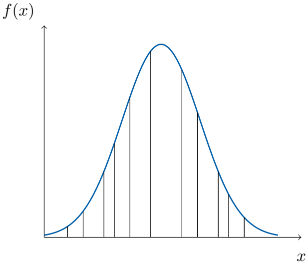

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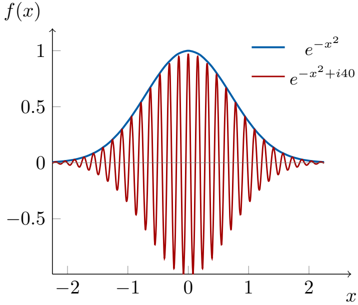

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$$M(\mu) = \begin{pmatrix} L(\mu) & 0 \\ 0 & L(-\mu) \end{pmatrix}, \quad \det M(\mu) = |\det L(\mu)|^2 \geq 0, \quad P = \begin{pmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{pmatrix}$$

Alford, Kapustin, Wilczek (1999)

n_I-QCD on the lattice - No sign problem

- In our partition function $\mathcal{Z} = \int \mathcal{D}U_\mu \, e^{-\beta S_G^{\text{Sym}}} (\det \mathcal{M}_{ud})^{1/4} (\det \mathcal{M}_s)^{1/4}$

$$\mathcal{M}_{ud} = \begin{pmatrix} \not{D}(\mu_I) + m_{ud} & \lambda\eta_5 \\ -\lambda\eta_5 & \not{D}(-\mu_I) + m_{ud} \end{pmatrix}, \quad \mathcal{M}_s = \not{D}(0) + m_s$$

- $\det \mathcal{M}_s \in \mathbb{R}^+$ due to the standard η_5-hermiticity relation $\eta_5 \mathcal{M}_s \eta_5 = \mathcal{M}_s^\dagger$ with $\eta_5 = \gamma_5^{\mathrm{S}} \otimes \gamma_5^{\mathrm{F}} = (-1)^{n_x+n_y+n_z+n_t}$ equivalent of γ_5 is the local staggered spin-flavor structure
- $\det \mathcal{M}_{ud} \in \mathbb{R}^+$ due to

$$\left.\begin{aligned} \not{D}(\mu_I)\eta_5 + \eta_5 \not{D}(\mu_I) &= 0 \\ \eta_5 \not{D}(\mu_I)\eta_5 &= \not{D}(-\mu_I)^\dagger \end{aligned}\right\} \Longrightarrow \tau_1 \eta_5 \, \mathcal{M}_{ud} \, \eta_5 \tau_1 = \mathcal{M}_{ud}^\dagger$$

  and

$$\mathcal{M}'_{ud} = B \mathcal{M}_{ud} B = \begin{pmatrix} \not{D}(\mu_I) + m_{ud} & \lambda \\ -\lambda & [\not{D}(\mu_I) + m_{ud}]^\dagger \end{pmatrix}, \quad B = \mathrm{diag}(1, \eta_5)$$

Signatures of the BCS phase from the complex Dirac spectrum

- Banks-Casher relation extensible to the case of complex Dirac eigenvalues for QCD at zero-temperature, nonzero isospin chemical potential
  - The necessary condition for the derivation is the positivity of the fermionic measure ($\rightarrow$ QCD inequalities $\rightarrow$ exclusion of symmetry breaking patterns)
  - For $|\mu_I| \gg \Lambda_{QCD}$ attractive channel between quarks near the Fermi surface lead to diquark pairing of the BCS type
- The density of the complex Dirac eigenvalues at the origin is proportional to the BCS gap squared

$$\Delta^2 = \frac{2\pi^3}{3N_C}\rho(0)$$

Kanazawa, Wettig, Yamamoto (2013)

  - Δ is the BCS gap
  - $\rho(\nu)$ is a 2d spectral density
- BC relations derived considering $\mathcal{Z}(M)$ as function of the quark mass matrix M
  - in the fundamental n_I-QCD theory. Suitable derivatives/limits yield $\rho(0)$
  - in the corresponding effective theory. Suitable derivatives/limits yield Δ^2

Complex spectrum of $\not{D}(\mu_I)$ - Measurement & analysis

Measurement

- Spectrum measured with SLEPc (Scalable Library for Eigenvalue Problem Computations), set up to obtain, via the Krylov-Schur method, $\sim$150 complex eigenvalues of $\not{D}(\mu_I)$ (the closest, in modulo to the origin).

Analysis

- Spectral density $\rho(\nu)$ extrapolated to m_{ud}, by
  - Using kernel density estimation (KDE) as a non-parametric way to estimate the multivariate probability density function from the measured spectrum.