Large N Limits

- An important theoretical tool: some models simplify in the limit of a large number of degrees of freedom.
- One class of such large N limits is for theories where fields transform as vectors under O(N) symmetry with actions like

$$S_{\text{Wilson-Fisher}} = \int d^d x \left(\frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{g}{4}(\phi^i \phi^i)^2 \right)$$

- Describes magnets with O(N) symmetry near their second-order phase transitions.

- The O(N) vector model is solvable in the limit where N is sent to infinity while keeping gN fixed.
- Flow from the free d<4 scalar model in the UV to the Wilson-Fisher interacting one in the IR.
- For N=1 it describes the critical Ising model; for N=2 the superfluid transition; for N=3 the critical Heisenberg model.
- The 1/N expansion is generated using the Hubbard-Stratonovich auxiliary field.

$$S = \int d^d x \left(\frac{1}{2}(\partial\phi^i)^2 + \frac{1}{2}\sigma\phi^i\phi^i - \frac{\sigma^2}{4\lambda} \right)$$

- In d<4 the quadratic term may be ignored in the IR:

$$Z = \int D\phi D\sigma \, e^{-\int d^dx \left(\frac{1}{2}(\partial\phi^i)^2 + \frac{1}{2\sqrt{N}}\sigma\phi^i\phi^i\right)}$$

$$= \int D\sigma \, e^{\frac{1}{8N}\int d^dx d^dy \, \sigma(x)\sigma(y) \, \langle\phi^i\phi^i(x)\phi^j\phi^j(y)\rangle_0 + \mathcal{O}(\sigma^3)}$$

- Induced dynamics for the auxiliary field endows it with the propagator

$$\langle\sigma(p)\sigma(-p)\rangle = 2^{d+1}(4\pi)^{\frac{d-3}{2}}\Gamma\left(\frac{d-1}{2}\right)\sin(\frac{\pi d}{2})\,(p^2)^{2-\frac{d}{2}} \equiv \tilde{C}_\sigma(p^2)^{2-\frac{d}{2}}$$

$$\langle\sigma(x)\sigma(y)\rangle = \frac{2^{d+2}\Gamma\left(\frac{d-1}{2}\right)\sin(\frac{\pi d}{2})}{\pi^{\frac{3}{2}}\Gamma\left(\frac{d}{2}-2\right)}\frac{1}{|x-y|^4} \equiv \frac{C_\sigma}{|x-y|^4}$$

- The 1/N corrections to operator dimensions are calculated using this induced propagator. For example,

$$\Delta_\phi = \frac{d}{2} - 1 + \frac{1}{N}\eta_1 + \frac{1}{N^2}\eta_2 + \ldots$$

- For the leading correction need

$$\frac{1}{N}\int \frac{d^d q}{(2\pi)^d}\frac{1}{(p-q)^2}\frac{\tilde{C}_\sigma}{(q^2)^{\frac{d}{2}-2+\delta}}$$

- δ is the regulator later sent to 0.

$$\eta_1 = \frac{\tilde{C}_\sigma(d-4)}{(4\pi)^{\frac{d}{2}} d\Gamma(\frac{d}{2})} = \frac{2^{d-3}(d-4)\Gamma\left(\frac{d-1}{2}\right)\sin\left(\frac{\pi d}{2}\right)}{\pi^{\frac{3}{2}}\Gamma\left(\frac{d}{2}+1\right)}$$

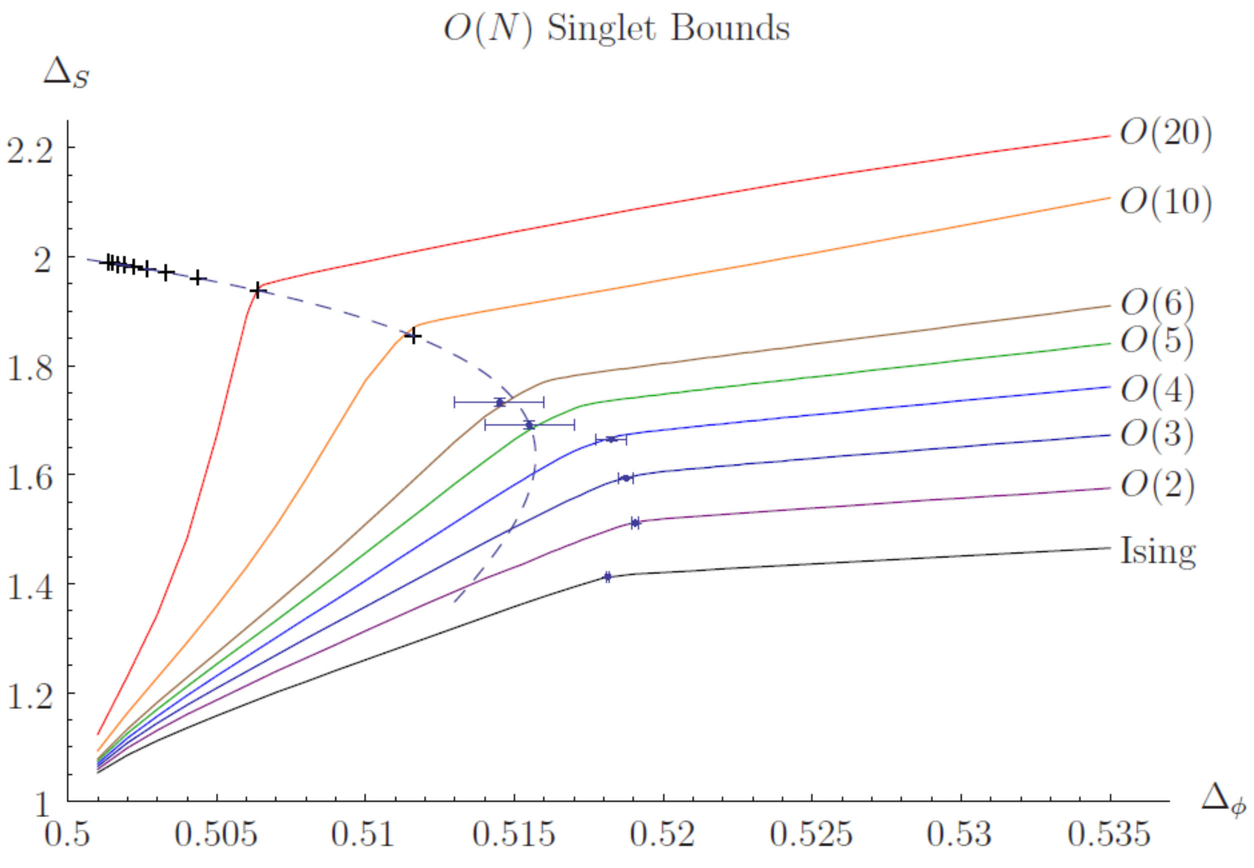
Operator Dimensions in d=3

- S is the O(N) singlet quadratic operator.
- T is the symmetric traceless tensor:

$$
\begin{aligned}
\Delta_\phi &= \frac{1}{2} + \frac{4}{3\pi^2}\frac{1}{N} - \frac{256}{27\pi^4}\frac{1}{N^2} \\
&\quad + \frac{32\left(-3188 + 3\pi^2(-61 + 108\log(2) - 3402\zeta(3))\right)}{243\pi^6}\frac{1}{N^3} + \mathcal{O}\left(\frac{1}{N^4}\right) \\
\Delta_S &= 2 - \frac{32}{3\pi^2}\frac{1}{N} + \frac{32(16 - 27\pi^2)}{27\pi^4}\frac{1}{N^2} + \mathcal{O}\left(\frac{1}{N^3}\right) \\
\Delta_T &= 1 + \frac{32}{3\pi^2}\frac{1}{N} - \frac{512}{27\pi^4}\frac{1}{N^2} + \mathcal{O}\left(\frac{1}{N^3}\right).
\end{aligned}
$$

Conformal Bootstrap Results

- From Kos, Poland, Simmons-Duffin, arxiv: 1307.6856



Interacting O(N) Model in d>4?

- Scalar large N model with $\frac{\lambda}{4}(\phi^i\phi^i)^2$ interaction has a good UV fixed point for 4<d<6. Parisi
- In $4+\epsilon$ dimensions $\beta_\lambda = \epsilon\lambda + \frac{N+8}{8\pi^2}\lambda^2 + \dots$
- So, the UV fixed point is at a negative coupling

$$\lambda_* = -\frac{8\pi^2}{N+8}\epsilon + O(\epsilon^2)$$

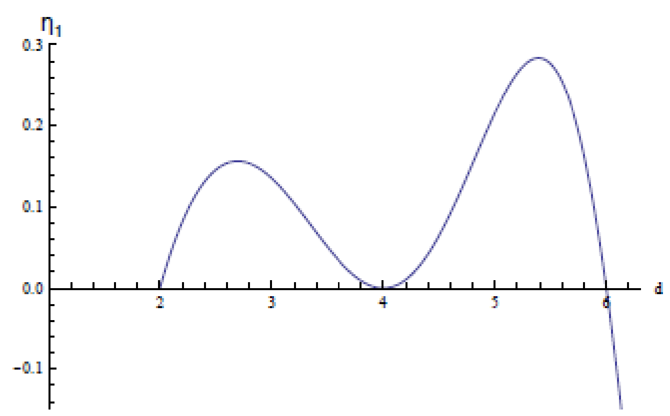
- A more complete definition of the O(N) model in 4<d<6 was proposed in my paper with L. Fei and S. Giombi, arXiv: 1404.1094

- In the O(N) model,

$$\Delta_\phi = \frac{d}{2} - 1 + \frac{1}{N}\eta_1 + \frac{1}{N^2}\eta_2 + \dots$$

$$\eta_1 = \frac{\tilde{C}_\sigma (d-4)}{(4\pi)^{\frac{d}{2}} d\Gamma(\frac{d}{2})} = \frac{2^{d-3}(d-4)\Gamma\left(\frac{d-1}{2}\right)\sin\left(\frac{\pi d}{2}\right)}{\pi^{\frac{3}{2}}\Gamma\left(\frac{d}{2}+1\right)}$$

- It is positive not only for 2<d< 4, but also for 4<d<6.



Perturbative IR Fixed Points

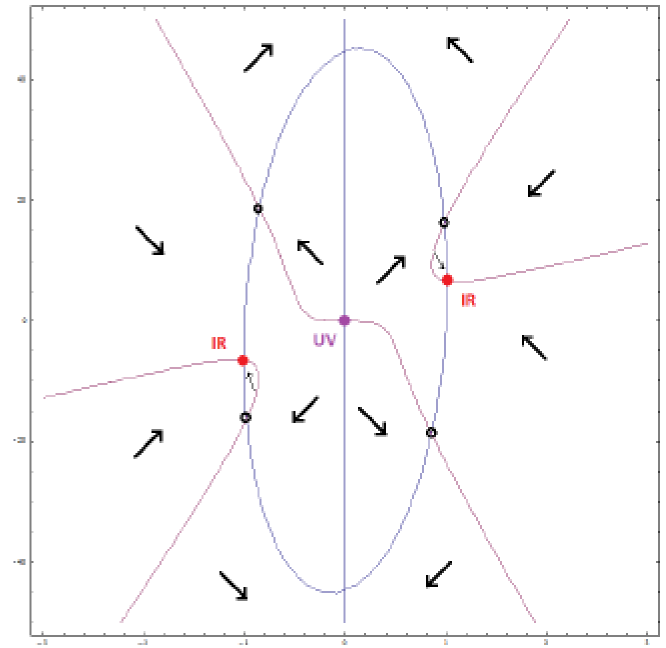
- Work in $d = 6 - \epsilon$ with O(N) symmetric cubic scalar theory $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{g_1}{2}\sigma(\phi^i\phi^i) + \frac{g_2}{6}\sigma^3$
- The beta functions Fei, Giombi, IK

$$\beta_1 = -\frac{\epsilon g_1}{2} + \frac{(N-8)g_1^3 - 12g_1^2 g_2 + g_1 g_2^2}{12(4\pi)^3}$$
$$\beta_2 = -\frac{\epsilon g_2}{2} + \frac{-4N g_1^3 + N g_1^2 g_2 - 3g_2^3}{4(4\pi)^3}$$

- For large N, the IR stable fixed point is at real couplings

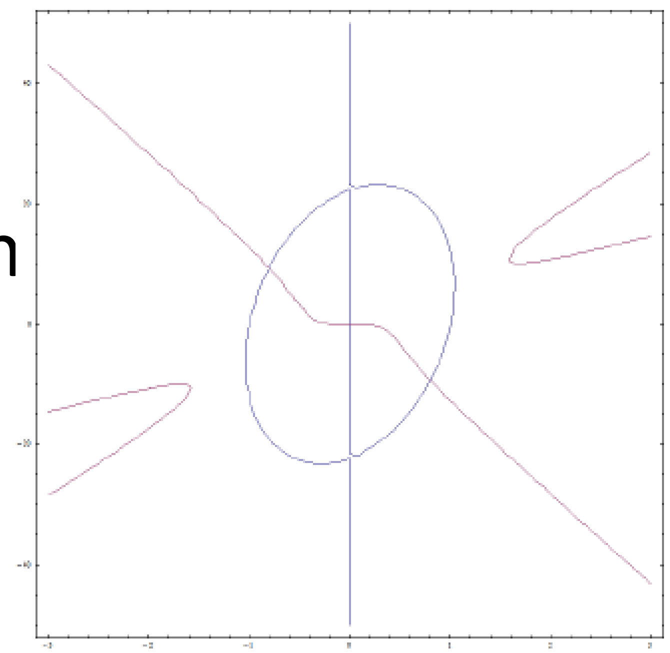
$$g_{1*} = \sqrt{\frac{6\epsilon(4\pi)^3}{N}} \qquad g_{2*} = 6g_{1*}$$

RG Flows



- Here is the flow pattern for N=2000

- The IR stable fixed points go off to complex couplings for N < 1039. Large N expansion breaks down very early!



- The dimension of sigma is $\Delta_\sigma = 2 - \frac{\epsilon}{2} + \frac{N g_1^2 + g_2^2}{12(4\pi)^3}$
- At the IR fixed point this is $2 + 40\frac{\epsilon}{N}$
- Agrees with the large N result for the O(N) model in d dimensions:

Petkou (1995) $$2 + \frac{4}{N}\frac{\Gamma(d)}{\Gamma(d/2-1)\Gamma(1-d/2)\Gamma(d/2)\Gamma(d/2+1)}$$

- For N=0, the fixed point at imaginary coupling may lead to a description of the Lee-Yang edge singularity in the Ising model. Michael Fisher (1978)
- For N=0, Δ_σ is below the unitarity bound $2 - \frac{\epsilon}{2}$
- For N>1039, the fixed point at real couplings is consistent with unitarity in $d = 6 - \epsilon$

Metastability

- Since the UV lagrangian is cubic, does the theory make sense non-perturbatively?
- In d=5 there is evidence from bootstrap that there is a theory with N=500 and higher. Li, Su
- When the CFT is studied on S^d or $R \times S^{d-1}$ the conformal coupling of scalar fields to curvature renders the perturbative vacuum meta-stable. In 6-ε dimensions, scaling dimensions have imaginary parts of order exp (- A N/ε).
- Work in progress with Giombi, Huang, Pufu, Tarnopolsky.

Large N Fermionic Models

- Solvable vector large N limit also applies to the Gross-Neveu CFT and conformal QED with or without the Chern-Simons Term.

The Gross-Neveu Model

$$S_{\rm GN} = -\int d^dx \left(\bar{\psi}_i \gamma^\mu \partial_\mu \psi^i + \frac{g}{2}(\bar{\psi}_i \psi^i)^2\right) \qquad i = 1, 2, \ldots \tilde{N}$$

- In 2 dimensions it has some similarities with the 4-dimensional QCD.
- It is asymptotically free and exhibits dynamical mass generation.
- Another 2-dimensional model with similar physics is the O(N) non-linear sigma model.
- In dimensions slightly above 2 both the O(N) and GN models have weakly coupled UV fixed points.

2+ ε expansion

- The beta function of the Gross-Neveu model and the critical value of the coupling are

$$\beta = \epsilon g - (N-2)\frac{g^2}{2\pi} + (N-2)\frac{g^3}{4\pi^2} + (N-2)(N-7)\frac{g^4}{32\pi^3} + \mathcal{O}(g^5)$$

$$g_* = \frac{2\pi}{N-2}\epsilon + \frac{2\pi}{(N-2)^2}\epsilon^2 + \frac{(N+1)\pi}{2(N-2)^3}\epsilon^3 + \mathcal{O}(\epsilon^4)\,,$$

- $N = \tilde{N}\mathrm{Tr}\mathbf{1}$ is the number of components of the Dirac fermions.
- The 2+ε expansion for scaling dimensions of simplest operators, like the fermion or fermion bilinear, have been developed. See a review by Moshe and Zinn-Justin.
- Similar expansions in the O(N) sigma model. Brezin, Zinn-Justin

The Gross-Neveu-Yukawa Model

- The GN model is in the same universality class as the GNY model Zinn-Justin; Hasenfratz, Hasenfratz, Jansen, Kuti, Shen

$$S_{\mathrm{GNY}} = \int d^d x \left(-\bar{\psi}_i (\not{\partial} + g_1 \sigma) \psi^i + \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{g_2}{24} \sigma^4 \right)$$

- Has an IR fixed point in 4-ε dimensions

$$g_{1\star} = \sqrt{\frac{16\pi^2 \epsilon}{N+6}},$$

$$g_{2\star} = 16\pi^2 \epsilon \frac{24N}{(N+6)\left((N-6) + \sqrt{N^2 + 132N + 36}\right)}$$

- Using the two ε expansions, we can study the Gross-Neveu CFT in the range 2<d<4.

1/N Expansion

- Both the Gross-Neveu and the scalar O(N) models have “double-trace” interactions

$$S_\lambda = S_{\mathrm{CFT}_0} + \lambda \int d^d x O(x)^2$$

$$\langle O(x) O(y) \rangle_0 = \frac{C_O}{|x-y|^{2\Delta_O}} = C_O \frac{(4\pi)^{d/2} \Gamma\left(d/2 - \Delta_O\right)}{4^{\Delta_O} \Gamma\left(\Delta_O\right)} \int \frac{d^d p}{(2\pi)^d} e^{ip(x-y)} (p^2)^{\Delta_O - d/2}$$

- Use the Hubbard-Stratonovich transformation

$$S_\lambda = S_{\mathrm{CFT}_0} + \int d^d x \sigma O - \frac{1}{4\lambda} \int d^d x \sigma^2$$

- Induced quadratic term for the auxiliary field

$$\begin{aligned} S[\sigma] &= -\frac{1}{2} \int d^d x d^d y \; \sigma(x) \sigma(y) \langle O(x) O(y) \rangle_0 - \frac{1}{4\lambda} \int d^d x \sigma^2 \\ &= -\frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \sigma(p) \sigma(-p) \left(C_O \frac{(4\pi)^{d/2} \Gamma\left(d/2 - \Delta_O\right)}{4^{\Delta_O} \Gamma\left(\Delta_O\right)} (p^2)^{\Delta_O - d/2} + \frac{1}{2\lambda} \right) \end{aligned}$$

- At the critical point the induced propagator is

$$G_\sigma(p) = \langle \sigma(p)\sigma(-p)\rangle = -\frac{4^{\Delta_O}\Gamma\left(\Delta_O\right)}{C_O(4\pi)^{d/2}\Gamma\left(d/2-\Delta_O\right)}(p^2)^{d/2-\Delta_O} \equiv \tilde{C}_\sigma(p^2)^{d/2-\Delta_O}$$

$$G_\sigma(x,y) = \frac{(d/2-\Delta_O)\sin\left((d/2-\Delta_O)\pi\right)\Gamma\left(d-\Delta_O\right)\Gamma\left(\Delta_O\right)}{\pi^{d+1}C_O|x-y|^{2(d-\Delta_O)}} \equiv \frac{C_\sigma}{|x-y|^{2(d-\Delta_O)}}$$

- The 1/N expansion is found using this induced propagator. In the GN model, Gracey

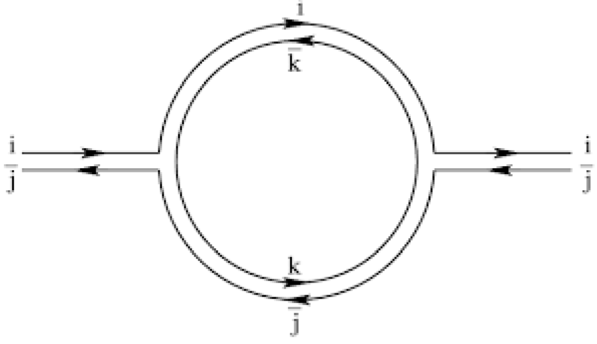
$$\Delta_\psi = \frac{d}{2} - \frac{1}{2} + \eta^{\rm GN} \qquad\qquad \eta^{\rm GN} = \eta_1^{\rm GN}/N + \eta_2^{\rm GN}/N^2 + \mathcal{O}(1/N^3)$$

$$\eta_1^{\rm GN} = \frac{\Gamma(d-1)(d-2)^2}{4\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2}+1)\Gamma(\frac{d}{2})^2}$$

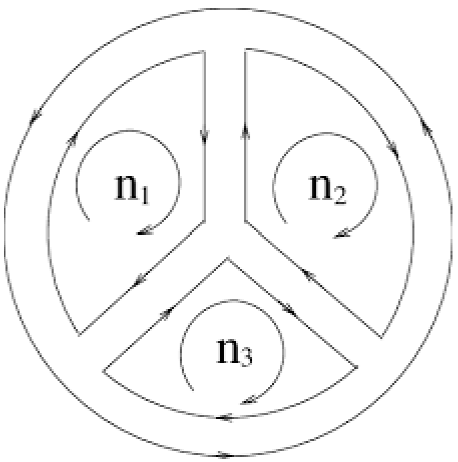
- This result agrees with the two ε expansions.

‘t Hooft Limit and Planar Graphs

- Another famous large N limit is for “planar” theories of N x N matrices with single-trace interactions.
- This has been explored widely in the context of large N QCD: SU(N) gauge theory coupled to matter.
- $g_{YM} N^{1/2}$ must be held fixed.
- The ‘t Hooft double line notation is very helpful:

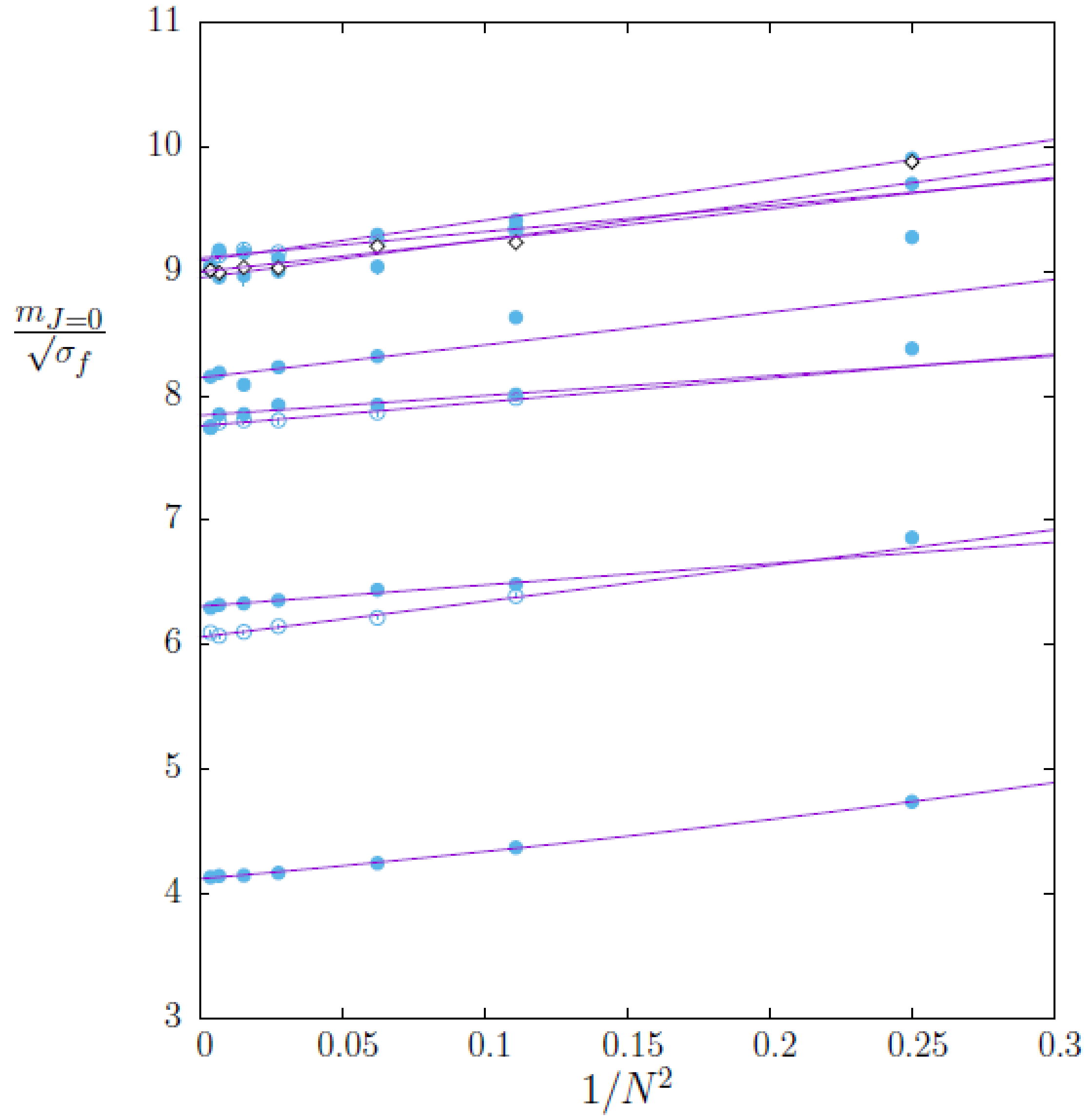


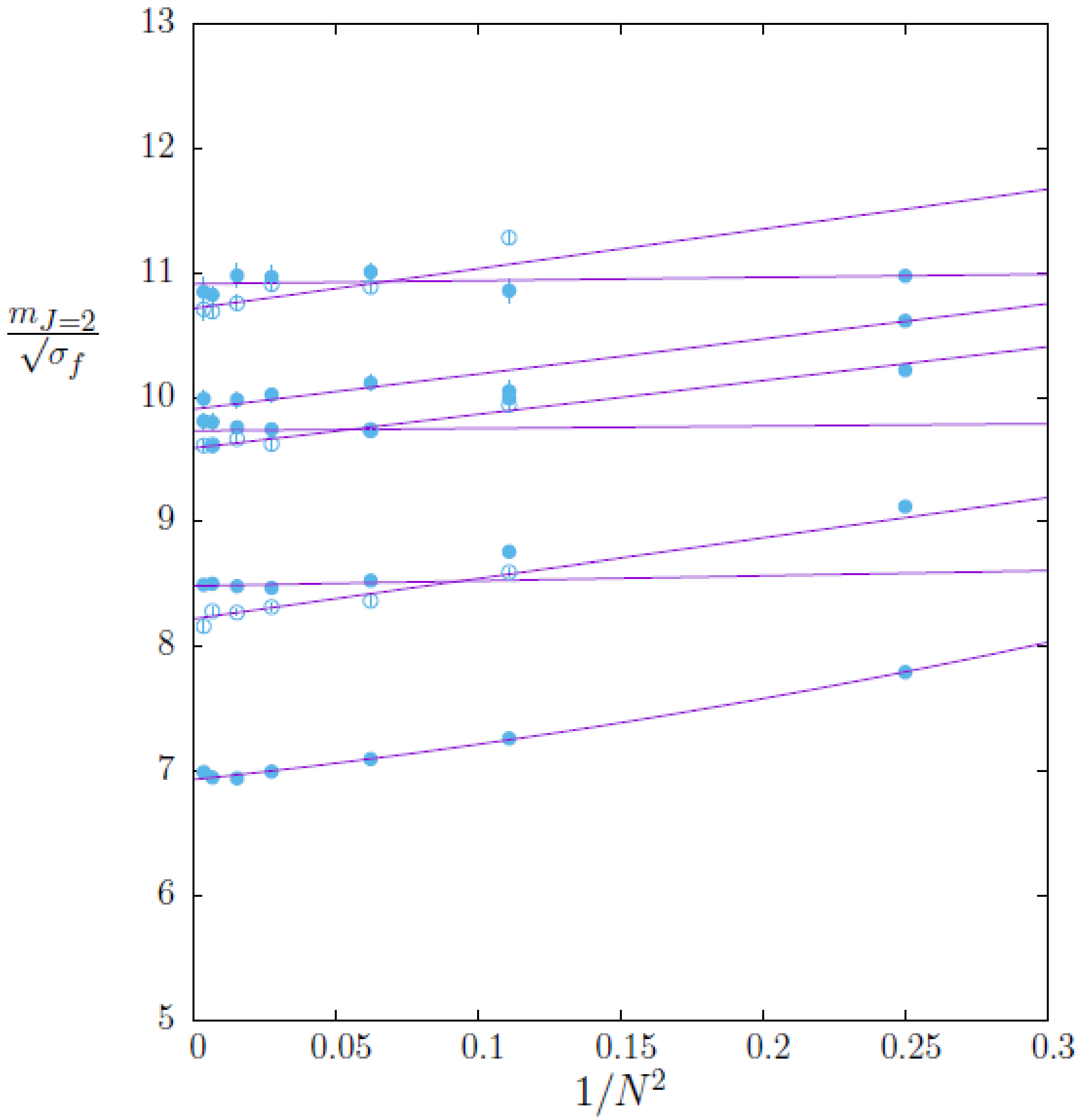
- Each vertex contributes factor ~N, each edge (propagator) ~1/N, each face (index loop)~N.
- The contribution to free energy of the Feynman graphs which can be drawn on a two-dimensional surfaces of genus g scales as $N^{2(1-g)}$



Glueballs in 3d SU(N) Theory

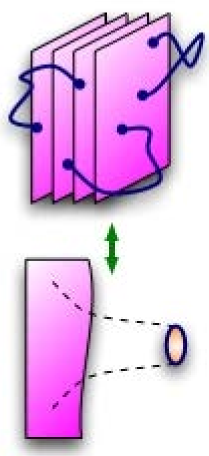
- For SU(N) the corrections are in powers of $1/N^2$
- Direct lattice evidence from Athenodorou,Teper, arXiv: 1609.03873




13
12
11
10
9
8
7
6
5
$\frac{m_{J=2}}{\sqrt{\sigma_f}}$
0
0.05
0.1
0.15
0.2
0.25
0.3
$1/N^2$

21 years of AdS/CFT Correspondence

- Starting in 1995 -- D-brane/black hole and D-brane/black brane correspondence. Polchinski; Strominger, Vafa; Callan, Maldacena; …
- A stack of N Dirichlet 3-branes realizes $\mathcal{N}$=4 supersymmetric SU(N) gauge theory in 4 dimensions. It also creates a curved RR charged background of type IIB theory of closed superstrings



$$ds^2 = \left(1 + \frac{L^4}{r^4}\right)^{-1/2} \left(-(dx^0)^2 + (dx^i)^2\right) + \left(1 + \frac{L^4}{r^4}\right)^{1/2} \left(dr^2 + r^2 d\Omega_5^2\right)$$

Large N is Important

- Matching the brane tensions gives $L^4 = g^2_{\rm YM} N \alpha'^2$

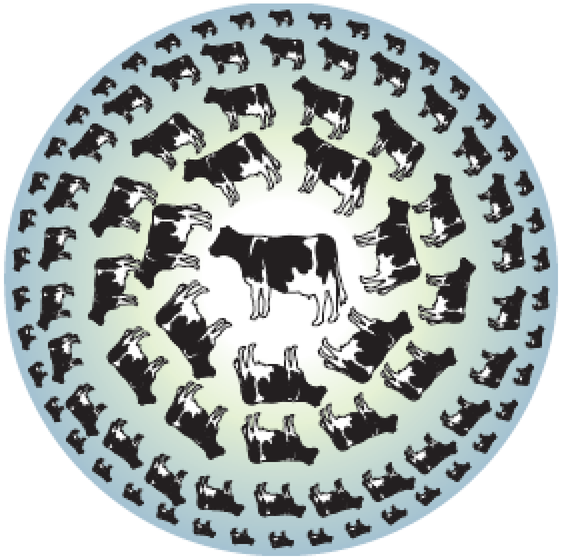
  Gubser, IK, Peet; IK; …
- The ‘t Hooft coupling makes a crucial appearance. In the large N limit, the effects of quantum gravity are suppressed by powers of $1/N^2$
- A serendipitous simplification for $g^2_{\rm YM} N \gg 1$:

  the background has a small curvature.
- This permitted calculation of two-point functions in strongly coupled gauge theory using classical gravitational absorption. IK
- In the r->0 limit, which corresponds to low energies, approaches $AdS_5 \times S^5$. Maldacena

The AdS/CFT Duality

Maldacena; Gubser, IK, Polyakov; Witten

- The low-energy limit taken directly in the geometry. Maldacena
- Relates conformal gauge theory in 4 dimensions to string theory on 5-d Anti-de Sitter space times a 5-d compact space. For the $\mathcal{N}$=4 SYM theory this compact space is a 5-d sphere.



- The geometrical symmetry of the AdS_5 space realizes the conformal symmetry of the gauge theory.
- Allows us to "solve" strongly coupled gauge theories, e.g. find operator dimensions $\Delta_{\pm} = 2 \pm \sqrt{4 + m^2 L^2}$

Some Tests of AdS/CFT

- String theory can make definite, testable predictions!
- The dimensions of unprotected operators, which are dual to massive string states, grow at strong coupling as $2\left(n g_{\mathrm{YM}}\sqrt{N}\right)^{1/2}$
- Verified for the Konishi operator dual to the lightest massive string state (n=1) using the exact integrability of the planar $\mathcal{N}$=4 SYM theory. Gromov, Kazakov, Vieira; …
- Similar successes for the dimensions of high-spin operators, which are dual to spinning strings in AdS space.