57th International Winter Meeting on Nuclear Physics - Bormio, Italy

Breaking and restoration of rotational symmetry in the spectrum of $\alpha-$conjugate nuclei on the lattice

PRESENTATION SESSION

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Motivation

We investigate **rotational symmetry breaking** in the low-energy spectra of

light α-conjugate nuclei: ^{8}Be, ^{12}C, ^{16}O, ...

on a cubic lattice G.S. et al. EPJ A 54, 232 (2018). In particular, we aim at

- identifying lattice eigenstates in terms of SO(3) irreps

  $\Longrightarrow$ Phys. Lett. B 114, 147-151 (1982), PRL 103, 261001 (2009)
- exploring the dependence of physical observables on spacing and size

  $\Longrightarrow$ PRD 90, 034507 (2014), PRD 92, 014506 (2015)
- developing memory-saving and fast algorithms for the diagonalization of the lattice Hamiltonian $\Longrightarrow$ Phys. Lett. B 768, 337 (2017)
- testing techniques for the suppression of discretization artifacts

  $\Longrightarrow$ Lect. Notes in Phys. 788 (2010)

Applications

Nuclear Lattice EFT: ab initio nuclear structure PRL 104, 142501 (2010), PRL 112, 102501 (2014), PRL 117, 132501 (2016) and scattering Nature 528, 111-114 (2015)

The Hamiltonian of the system

The macroscopic α-cluster model[1] of B. Lu et al. PR D 90, 034507 (2014) is adopted
$\Longrightarrow$ nuclei are decomposed into M structureless α-particles

$$H = -\frac{\hbar^2}{2m_\alpha}\sum_{i=1}^{M}\nabla_i^2 + \sum_{i>j=1}^{M}\left[V_C(\mathbf{r}_{ij}) + V_{AB}(\mathbf{r}_{ij})\right] + \sum_{i>j>k=1}^{M} V_T(\mathbf{r}_{ij},\mathbf{r}_{ik},\mathbf{r}_{jk})$$

with $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$. The potentials are of the type

Coulomb[2]

$$\frac{4e^2}{4\pi\epsilon_0}\frac{1}{r_{ij}}\mathrm{erf}\left(\frac{\sqrt{3}r_{ij}}{2R_\alpha}\right)$$

with $R_\alpha = 1.44$ fm
rms radius of the ^{4}He
NB: Erf *adsorbs* the
singularity at $r = 0$

Ali-Bodmer[2]

$$V_a\,\mathrm{f}\,e^{-\eta_a^2 r_{ij}^2} + V_r e^{-\eta_r^2 r_{ij}^2}$$

with $\eta_r^{-1} = 1.89036$ fm,
$V_r = 353.508$ MeV
and $\eta_a^{-1} = 2.29358$ fm,
$V_a = -216.346$MeV,
auxiliary param. f $= 1$

Gaussian

$$V_0 e^{-\lambda(r_{ij}^2+r_{ik}^2+r_{jk}^2)}$$

with $\lambda = 0.00506$ fm^{-2},
$V_0 = -4.41$ MeV for ^{12}C [3]
s.t. $E_{g.s.} = -\Delta E_{Hoyle}$
and $V_0 = -11.91$ MeV for ^{16}O [4]
s.t. $E_{g.s.} = -\Delta E_{4\alpha}$

[1] G.S. et al. JP G 43, 8 (2016), [2] NP 80, 99-112 (1966) , [3] Z. Physik A 290, 93-105 (1979) , [4] G.S. (2017)

The lattice environment

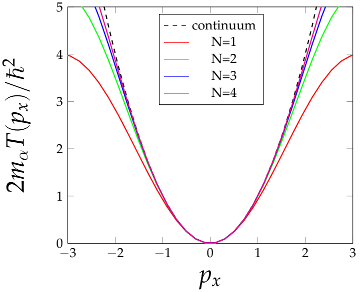
The configuration space in relative d.o.f. of an $M - body$ physical system into a cubic lattice reduces to

$$\mathbb{R}^{3M-3} \longrightarrow N^{3M-3}$$

where: $N \Longrightarrow$ number of points per dimension ($\equiv$ **lattice size**)
$a \Longrightarrow$ **lattice spacing** and $L \equiv Na$

Consequences: **discretization effects**

1. the action of differential operators is represented via finite differences: $\Longrightarrow$ Lect. Notes in Phys. 788 (2010)
2. breaking of Galiean invariance
3. breaking of continuous translational invariance (free-particle case)



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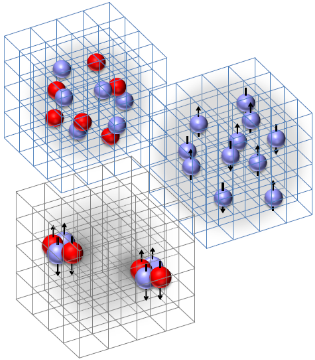
and **finite-volume effects** on physical observables



With periodic boundary conditions:

1. configuration space becomes isomorphic to a torus in $3M-3$-dimensions
2. lattice momenta become $\mathbf{p} = \hbar\frac{2\pi\mathbf{n}}{Na}$ where $\mathbf{n}$ is a vector of integers

Symmetries

On the lattice SO(3) symmetry reduces to the invariance under the cubic group $\mathcal{O}$.

Accordingly

«Only eight [five: A_1, A_2, E, T_1, T_2] different possibilities exist for rotational classification of states on a cubic lattice. So, the question arises: how do these correspond to the angular momentum states in the continuum? [...] To be sure of higher spin assigments and mass predictions it seems necessary to follow all the relevant irreps simultaneously to the continuum limit. »

R.C. Johnson, Phys. Lett. B 114, 147-151, (1982).

Integer spin irreps D^ℓ of SO(3) decompose into irreps of $\mathcal{O}$ as follows:

$$
\begin{aligned}
D^0 &= A_1 \\
D^1 &= T_1 \\
D^2 &= E \oplus T_2 \\
D^3 &= A_2 \oplus T_1 \oplus T_2 \\
D^4 &= A_1 \oplus E \oplus T_1 \oplus T_2 \\
D^5 &= E \oplus T_1 \oplus T_1 \oplus T_2 \\
D^6 &= A_1 \oplus A_2 \oplus E \oplus T_1 \oplus T_2 \oplus T_2
\end{aligned}
$$

Symmetries

Degenerate states belonging to the same $\mathscr{O}$ irrep can be labeled with the irreps I_z of the cyclic group $\mathscr{C}_4$, generated by an order-three element of $\mathscr{O}$ (e.g. $\mathcal{R}_z^{\pi/2}$):

$$\begin{array}{ccc} \mathrm{SO}(3) & \supset & \mathrm{SO}(2) \\ \downarrow & & \downarrow \\ l & & m, \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} \mathcal{O} & \supset & \mathcal{C}_4 \\ \downarrow & & \downarrow \\ \Gamma & & I_z, \end{array}$$

Conversely, the discrete symmetries of the Hamiltonian are preserved:

time reversal, parity, exchange symmetry

Applications

Within an iterative approach for the diagonalization of $\mathcal{H}$, the states belonging to an irrep Γ of a point group $\mathcal{G}$ can be extracted applying the projector

$$P_\Gamma = \sum_{g \in \mathcal{G}} \chi_\Gamma(g) D(g)$$

where $D(g)$ is a representation of dimension $3M - 3$ for the operation $g \in \mathcal{G}$

Finite volume energy corrections

LO finite volume energy corrections for relative two-body bosonic states with reduced mass μ, angular momentum ℓ and belonging to the Γ irrep of $\mathcal{O}$ are given by **PRL 107, 112011 (2011)**

$$\Delta E_B^{(\ell,\Gamma)} \equiv E_B^{(\ell,\Gamma)}(\infty) - E_B^{(\ell,\Gamma)}(L) = \beta\left(\frac{1}{\kappa_0 L}\right)|\gamma|^2 \frac{e^{-\kappa_0 L}}{\mu L} + \mathcal{O}\left(e^{-\sqrt{2}\kappa_0 N}\right)$$

with $\gamma \Rightarrow$ asymptotic normalization constant
$\kappa_0 \Rightarrow$ binding momentum and $\beta(x) \Rightarrow$ a polynomial

ℓ	Γ	$\beta(x)$
0	A_1^+	-3
1	T_1^-	$+3$
	T_2^+	$30x + 135x^2 + 315x^3 + 315x^4$
2	E^+	$-\frac{1}{2}(15 + 90x + 405x^2 + 945x^3 + 945x^4)$
	A_2^-	$315x^2 + 2835x^3 + 122285x^4 + 28350x^5 + 28350x^6$
3	T_2^-	$-\frac{1}{2}(105x + 945x^2 + 5355x^3 + 19530x^4 + 42525x^5 + 42525x^6)$
	T_1^-	$-\frac{1}{2}(14 + 105x + 735x^2 + 3465x^3 + 11340x^4 + 23625x^5 + 23625x^6)$

Although no analytic LO FVEC formula for the three-body case exists, results for zero-range potentials **PRL 114, 091602 (2015)** and the asymptotic ($\equiv$ large N) behaviour are available **Phys. Lett. B 779, 9-15 (2018)**.

Finite volume energy corrections

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with $\gamma \Rightarrow$ asymptotic normalization constant
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Multiplet averaging of the energies. $\Rightarrow$ the finite volume energy corrections assume an universal form, independent in magnitude on the SO(3) irreps

$$E_\infty(\ell_A^P) - E_L(\ell_A^P)\Big|^{LO} = (-1)^{\ell+1}3|\gamma|^2\frac{e^{-\kappa_0 L}}{\mu L} \quad \text{with} \quad E(\ell_A^P) \equiv \sum_{\Gamma\in\mathcal{O}}\frac{\chi^\Gamma(\mathbb{1})}{2\ell+1}E_B^{(\ell^P,\Gamma)}(L)$$

at LO, i.e. order $\exp(-\kappa_0 L)$.

where: $\Gamma \Rightarrow$ irrep of the cubic group
$\chi^\Gamma(\mathbb{1}) \Rightarrow$ character of Γ w.r.t. the identity conjugacy class ($\equiv \dim\Gamma$)
$P \Rightarrow$ eigenvalue of the inversion operator $\mathscr{P}$

Finite volume energy corrections including Coulomb interaction

Leading Coulomb corrections for the energies of states with $\ell = 0$ (A_1) describing two spinless singly-charged particles in a finite volume are given by

$$\Delta E^{(0,A_1)}_{B,\mathrm{QED}} \equiv E^{(0,A_1)}_{B,\mathrm{QED}}(\infty) - E^{(0,A_1)}_{B,\mathrm{QED}}(L) = \frac{\alpha}{\pi L}\mathcal{I} + \mathcal{O}(\alpha^2)$$

$\Longrightarrow$ **PRD 90, 074511 (2014)**

$$\mathcal{I} = \sum_{\mathbf{n}\neq 0}^{\Lambda_n} \frac{1}{|\mathbf{n}|^2} - 4\pi\Lambda_n = -8.9136$$

where: $\alpha = e^2/4\pi \Rightarrow$ fine structure constant

$\Lambda_n = N\Lambda/2\pi$ with $\Lambda \Rightarrow$ UV lattice momentum cutoff

$\mathbf{n} \Rightarrow$ three-vector of integers

As in the case without QED, FV corrections for the $\ell = 0$ state are negative

In presence of Coulomb interaction, the infinite-volume bound state energy $E \equiv -E^{(0,A_1)}_B(\infty) = -\kappa_0^2/2\mu$ and binding momentum κ_0 is modified into

$$E^{(0,A_1)}_{B,\mathrm{QED}}(\infty) = \frac{\kappa_0^2}{2\mu} - \frac{2\alpha\kappa_0}{1-\kappa_0 r_0}\left[\gamma_E + \log\left(\frac{\alpha\mu}{2\kappa_0}\right)\right]$$

where: $r_0 \Rightarrow$ effective range of strong interactions

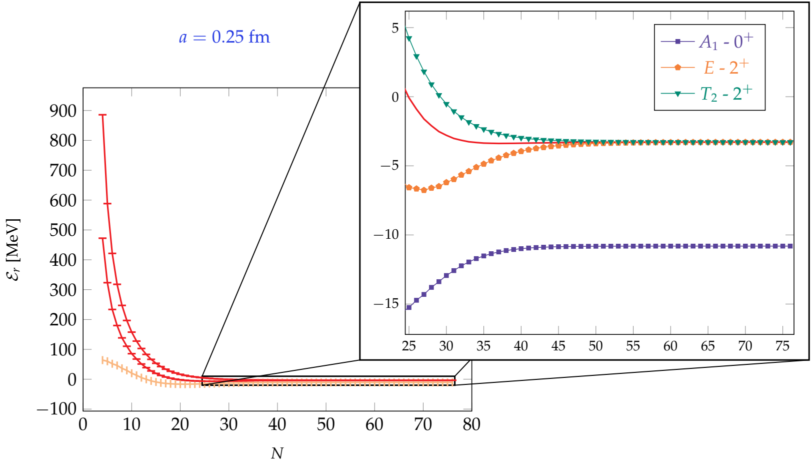
$\gamma_E \approx 0.57721 \Rightarrow$ Euler-Mascheroni constant

Remark: in absence of further forces there's no QED contribution at $\mathcal{O}(\alpha)$

Outlook: extension of the Coulomb FVEC formula to states with $\ell \geq 1$

The low-energy ^{8}Be spectrum

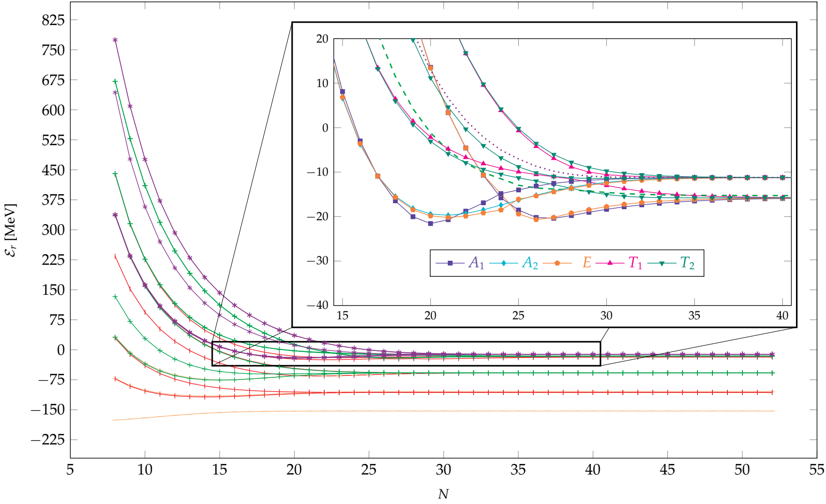
Increasing the parameter V_a of V_{AB} up to 130% of its eigenvalue ($f = 1.3$), the finite volume behaviour of the energies of the 0_1^+ and 2_1^+ bound states can be inspected:



♣ Remark: for $N \gtrsim 27$ the sign of the FVECs agrees with the $\Delta E_B^{(\ell,\Gamma)}$ formulas for $\ell = 0$ and 2, even if Coulomb corrections dominate outside the strong interaction region

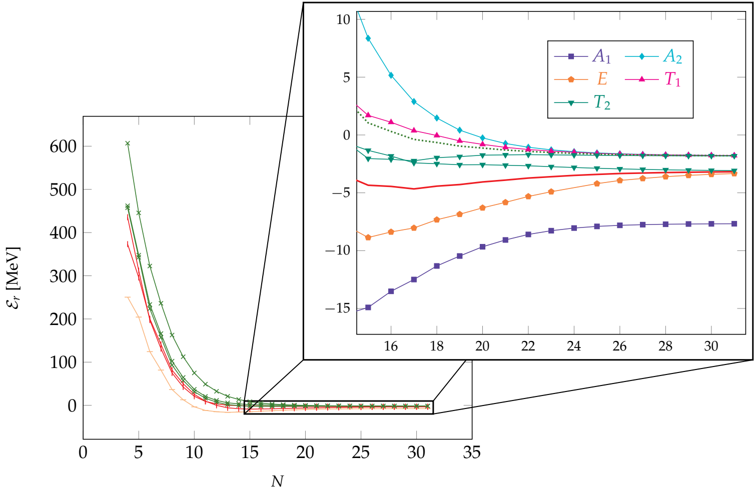
The low-energy ^{8}Be spectrum

The further increase of the parameter V_a of V_{AB} up to 250% ($f = 2.5$) permits to extend the FV analysis to the $\ell = 4$ and 6 states $\Longrightarrow$ the 4_2^+ and 6_1^+ multiplets (cf. magnification)



The low-energy ^{12}C spectrum

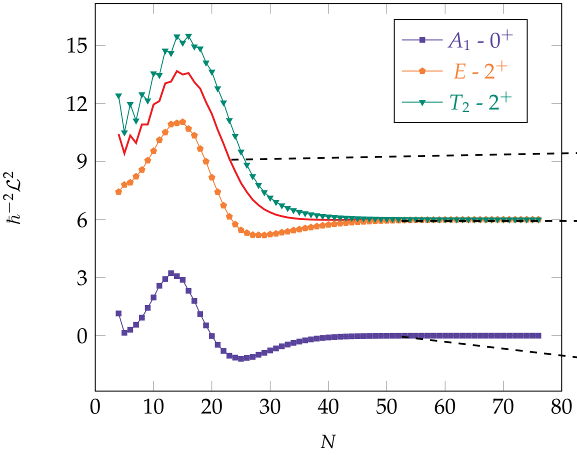
Since the nucleus is naturally bound, no artificial increase of the strength parameter V_0 of V_{AB} is necessary for the study of the lowest 0_1^+, 2_1^+ and 3_1^- states.



♣ Remark: the spacing is larger ($a = 0.5$ fm) $\Longrightarrow$ discretization effects: 10^{-2}-10^{-3} MeV.

The low-energy ^{8}Be spectrum: the 0_1^+ and 2_1^+ multiplets

Now we consider the average values of the squared total angular momentum operator $\mathcal{L}^2$, in relative coordinates. Fixing $a = 0.5$ fm, for the 0_1^+ and 2_1^+ states (f = 1.3) we find:



as for the energies, multiplet averaging suppresses FV effects

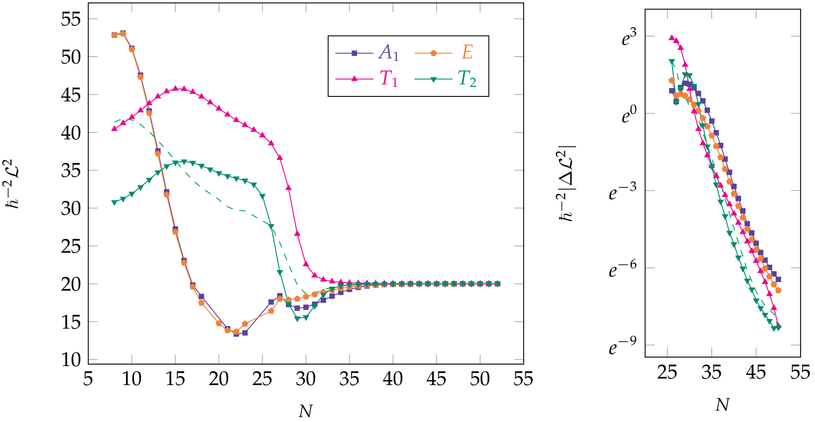
discretization effects $\approx 10^{-5}\hbar^2$

discretization effects $\approx 10^{-4}\hbar^2$

♣ Remark: The average values of $\mathcal{L}^2$ for the $0_{A_1}^+$, 2_E^+ and $2_{T_2}^+$ states smoothly converge to the eigenvalues equal to 0 and $6\hbar^2$, modulo residual discretization errors.

The low-energy ^{8}Be spectrum: the 4_2^+ multiplet

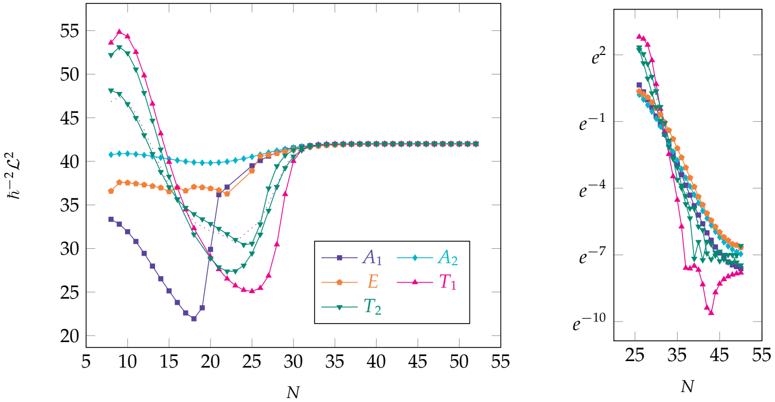
In the f = 2.5 case, by fixing $a = 0.25$ fm discretization effects for the 4_2^+ multiplets reduce to $\approx 10^{-4}\hbar^2$. Multiplet-averaging enhances convergence.



Remark: for $L \gtrsim 25$ $|\Delta\mathcal{L}^2| \propto \exp(m_\kappa L)$ with $m_\kappa < 0$

The low-energy ^{8}Be spectrum: the 6_1^+ multiplet

In the f = 2.5 case, fixing $a = 0.25$ fm residual discretization effects for the 6_1^+ multiplets amount to $\approx 10^{-4}\hbar^2$. Multiplet-averaging enhances convergence.

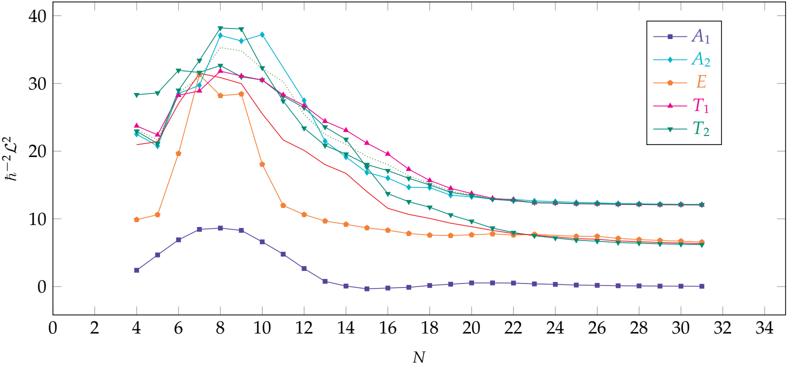


Remark: for $L \gtrsim 25$ $\quad |\Delta\mathcal{L}^2| \propto \exp(m_\kappa L)$ with $m_\kappa < 0$

The low-energy ^{12}C spectrum

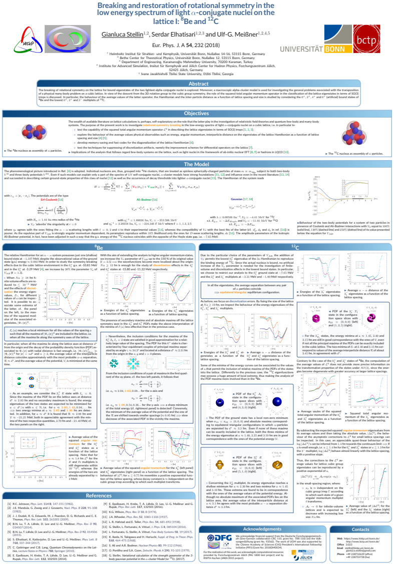
As a consequence of the isotropy of the potentials, the nucleus has an equilateral triangular equilibrium configuration, i.e. $\langle r_{12} \rangle = \langle r_{23} \rangle = \langle r_{13} \rangle \equiv \mathcal{R}$.

Restoring the V_a parameter of the Ali-Bodmer potential to its default value ($f = 1.0$) and fixing the spacing to $a = 0.50$ fm, we compute the average values of $\mathcal{L}^2$ on the 0_1^+, 2_1^+ and 3_1^- multiplets of states.



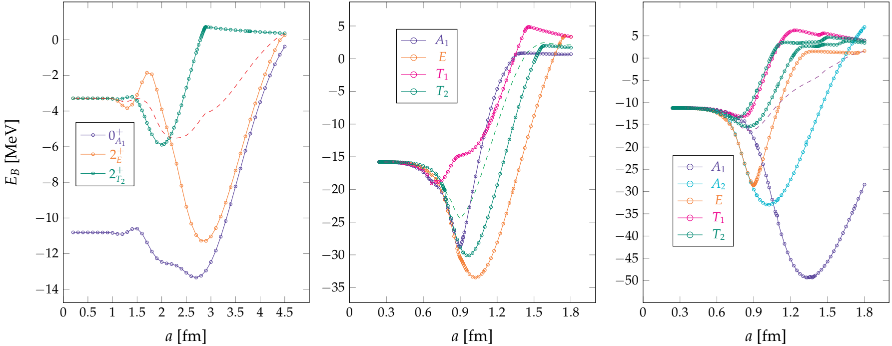
Remark: residual discretization errors are sensibly larger ($\approx 10^{-1} - 10^{-2}\hbar^2$).

Discretization effects: a cover story...



Discretization effects on energy

Unlike finite-volume effects, the dominant behaviour of dicretization corrections on energy, $\Delta E_B(a)$, is unknown.



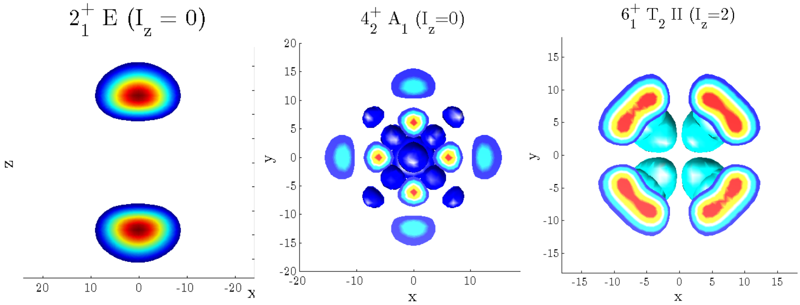
Nevertheless: some extrema of $E_B(a)$ can be associated to the maxima of the probability density function corresponding to the given energy eigenstate.

NB: If the primary maxima of the pdf lie at distance d^* w.r.t. the origin, the most probable $\alpha - \alpha$ separation $\mathcal{R}^*$ is given by d^*

Discretization effects on energy

If the all pdf maxima are absolute and lie along the coordinate axes, $\exists$ a value of a s.t. all the maxima of the pdf are included in the cubic lattice.

In particular: for $a = d^* \Longrightarrow E_B(a)$ is minimized and if $|\Psi_B^{\text{Max}}|^2 \gg |\Psi_B(\mathbf{r})|^2$ where $|\mathbf{r}| = nd^*$ and $n \geq 2 \Longrightarrow \langle \mathcal{R} \rangle \approx d^*$ and $\langle V \rangle$ is approximately minimized



✓ conditions fulfilled ✗ secondary maxima ✗ maxima off the axes

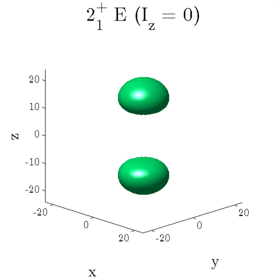
Discretization on ^{8}Be: the 2_1^+ E states

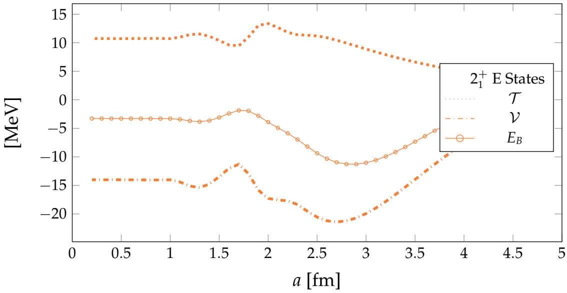
$I_z = 0$ Pdf: two principal maxima along the z axis, located at a distance $d^* = 2.83$ fm from the origin.

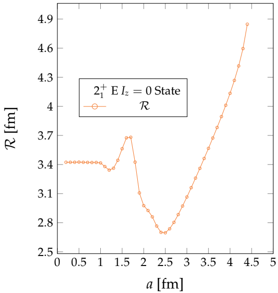
$\Longrightarrow$ $E_B(a)$ minima are, then, predicted to lie at

$$a = \frac{d^*}{n} \text{ with } n \geq 1, \text{ i.e. } a \approx 2.83, 1.42, 0.94, \ldots$$

In practice: two E_B minima at $a \approx 1.36$ and 2.85 fm are observed







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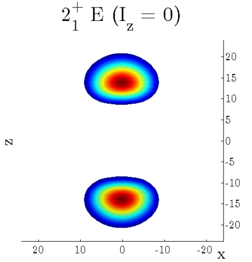

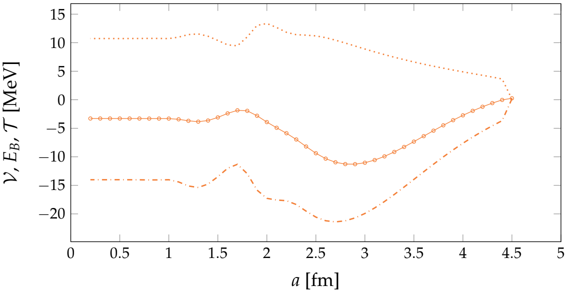

In addition:

$\mathcal{V} \approx -21.21$ MeV @ $a = d^*$
$\mathcal{V}^{min} \approx -21.40$ MeV @ $a \approx 2.70$ fm
and
$\mathcal{R} \approx 2.88$ fm @ $a = d^*$
$\mathcal{R}^{min} \approx 2.70$ fm @ $a \approx 2.50$ fm

Discretization on ^{8}Be: the 2_1^+ E states

$I_z = 2$ Pdf: 4 principal maxima on the x and y axes, located at a distance $d^* = 2.83$ fm from the origin.

$\Longrightarrow$ $E_B(a)$ minima are, then, predicted to lie at

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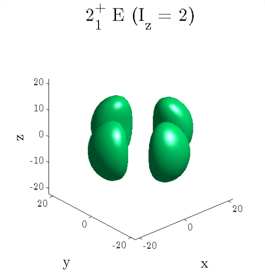
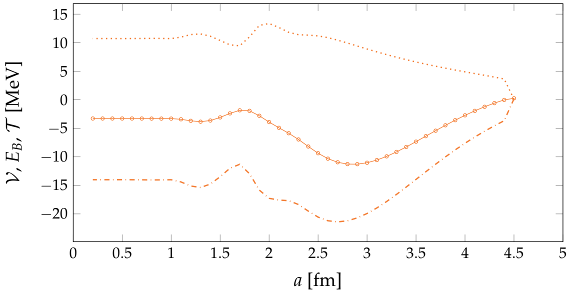


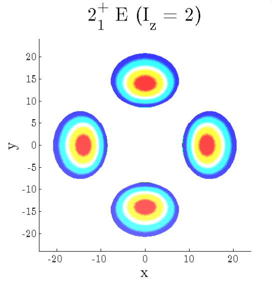


Still:

$\mathcal{V} \approx -21.21$ MeV @ $a = d^*$

$\mathcal{V}^{min} \approx -21.40$ MeV @ $a \approx 2.70$ fm

Discretization on ^{8}Be: the 2_1^+ E states

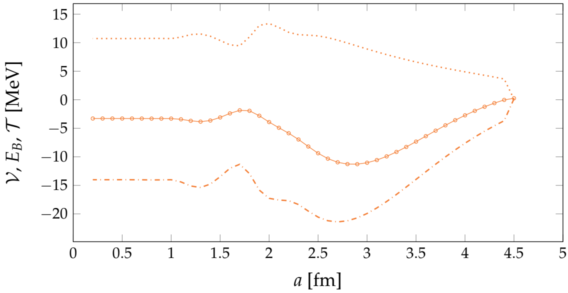


$I_z = 2$ Pdf: 4 principal maxima on the x and y axes, located at a distance $d^* = 2.83$ fm from the origin.

$\Longrightarrow$ $E_B(a)$ minima are, then, predicted to lie at

$$a = \frac{d^*}{n} \text{ with } n \geq 1, \text{ i.e. } a \approx 2.83, 1.42, 0.94, ...$$

In practice: two E_B minima at $a \approx 1.36$ and 2.85 fm are observed



Still:

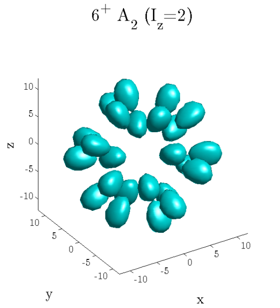
$\mathcal{V} \approx -21.21$ MeV @ $a = d^*$
$\mathcal{V}^{min} \approx -21.40$ MeV @ $a \approx 2.70$ fm

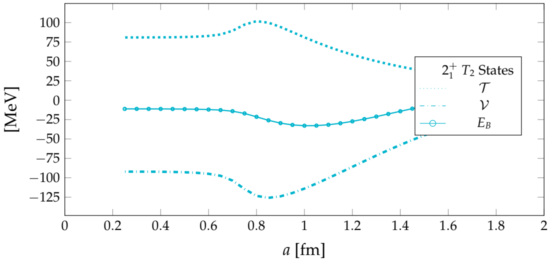
Discretization on ^{8}Be: the 6_1^+ A_2 state

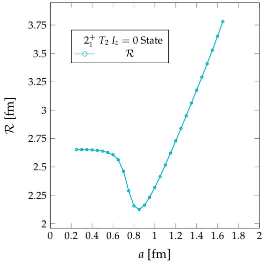
$\boxed{I_z = 2 \text{ Pdf}}$: four equidistant couples of principal maxima separated by an angle $\gamma \approx 34.2°$ and located at a distance $d^* \approx 2.31$ fm from the origin in the x, y and $z = 0$ planes.

$\Downarrow$

The 24 maxima cannot be included on the lattice







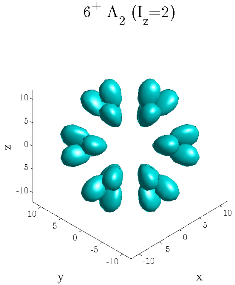
Discretization on ^{8}Be: the 6_1^+ A_2 state

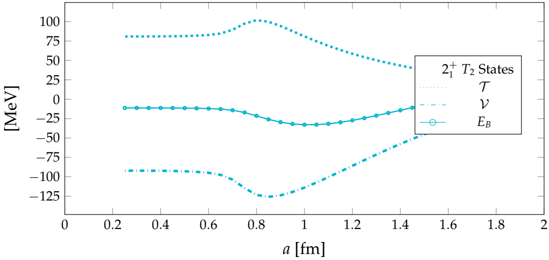
Considering the inclusion conditions of a couple of maxima in the 1st quadrant of the xy plane ($n \geq 1$):

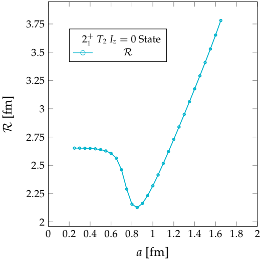
$$a_x = \frac{d^*}{n} \cos\left(\frac{\pi}{4} - \frac{\gamma}{2}\right), \text{ i.e } a_y \approx 2.04, 1.02, 0.68...$$

$$a_y = \frac{d^*}{n} \sin\left(\frac{\pi}{4} - \frac{\gamma}{2}\right), \text{ i.e } a_y \approx 1.08, 0.54, 0.36...$$

In practice: an E_B minimum at $a \approx 1.03$ fm is observed !







Discretization on ^{8}Be: the 6_1^+ A_2 state

Considering the inclusion conditions of a couple of maxima in the 1st quadrant of the xy plane ($n \geq 1$):

$$a_x = \frac{d^*}{n} \cos\left(\frac{\pi}{4} - \frac{\gamma}{2}\right), \text{ i.e } a_y \approx 2.04, 1.02, 0.68...$$

$$a_y = \frac{d^*}{n} \sin\left(\frac{\pi}{4} - \frac{\gamma}{2}\right), \text{ i.e } a_y \approx 1.08, 0.54, 0.36...$$

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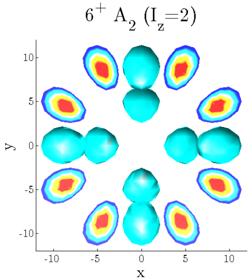
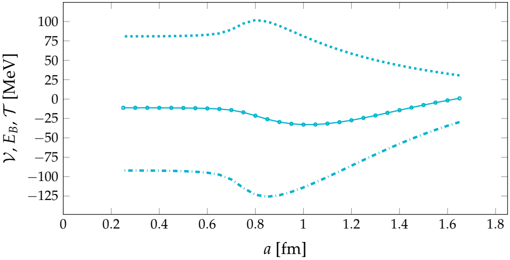




In addition:

$\mathcal{V} \approx 0.0$ MeV @ $a = d^*$ (unbound)

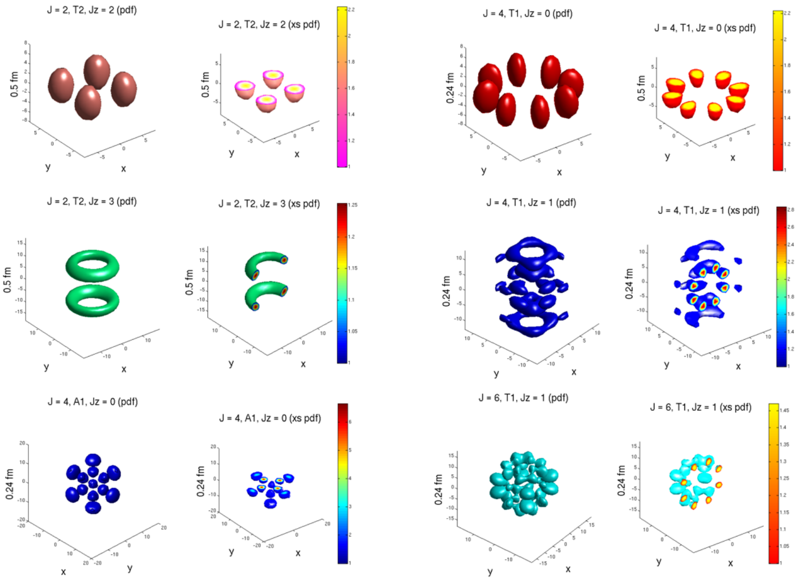
$\mathcal{V}^{min} \approx -125.85$ MeV @ $a \approx 0.85$ fm

and

$\mathcal{R} \gg \mathcal{R}^{min}$ @ $a = d^*$

$\mathcal{R}^{min} \approx 2.13$ fm @ $a \approx 0.85$ fm

Other low-energy ^{8}Be wavefunctions



Conclusions & Outlook

The macroscopic α-cluster model in PRD 90, 034507 (2014) has been applied to the ^{8}Be and ^{12}C on the lattice. A fully-parallel method based on the Lanczos iteration has been adopted for the diagonalization of the Hamiltonian, allowing for

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Perspectives and hints

- Extension of the analysis to the ^{16}O $\Rightarrow$ usage of the existing *exact* GPU codes for small volumes (memory issues!) and benchmarks as well as Metropolis - Monte Carlo wordline or auxiliary field algorithms for large volumes (under development);
- Derivation of an analytical formula for the leading order FV energy corrections for bound states with $\ell \geq 1$ in presence of a Coulomb-type potential.

Thanks for your attention!

Grazie per l'attenzione!

Rotational Symmetry

On the lattice 3-dim rotational symmetry reduces to a subgroup of SO(3), the cubic group $\mathcal{O}$. A process of descent in symmetry takes place: α = x; y; z

$$\text{continuum}, \infty - \text{volume}: \ SO(3) \Longrightarrow [H, L^2] = 0, [H, L_\alpha] = 0$$

$$\Downarrow$$

$$\text{continuum, finite volume}: \ \mathcal{O} \subset SO(3) \Longrightarrow [H, L^2] = 0, [H, L_\alpha] \neq 0$$

$$\Downarrow$$

$$\text{discrete, finite volume}: \ \mathcal{O} \subset SO(3) \Longrightarrow [\mathcal{H}, \mathcal{L}^2] \neq 0, [\mathcal{H}, \mathcal{L}_\alpha] \neq 0$$

Accordingly

«Only eight [five: A_1, A_2, E, T_1, T_2] different possibilities exist for rotational classification of states on a cubic lattice. So, the question arises: how do these correspond to the angular momentum states in the continuum? [...] To be sure of higher spin assigments and mass predictions it seems necessary to follow all the relevant irreps simultaneously to the continuum limit. »

R.C. Johnson, Phys. Lett. B 114, 147-151, (1982).

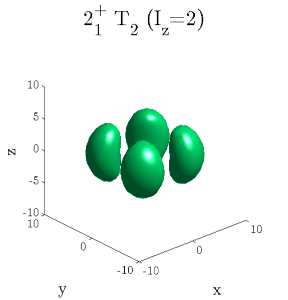
Discretization on ^{8}Be: the 2_1^+ T_2 states

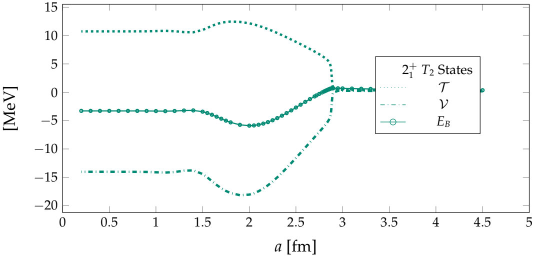
$\boxed{I_z = 2 \text{ Pdf}}$: four principal maxima in the intersection betw. the $z = 0$ plane and the $x = \pm y$ planes, s.t. $d^* = 2.83$ fm.

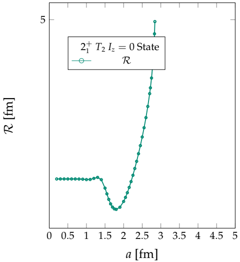
$\Longrightarrow$ $E_B(a)$ minima are, then, predicted to lie at

$$a = \frac{\sqrt{2}}{2}\frac{d^*}{n} \text{ with } n \geq 1, \text{ i.e. } a \approx 2.02, 1.01, 0.67, \ldots$$

In practice: two E_B minima at $a \approx 1.05$ and 2.02 fm are observed







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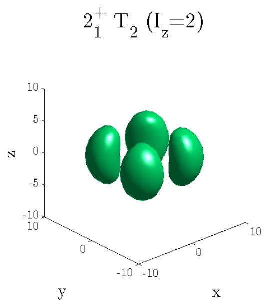

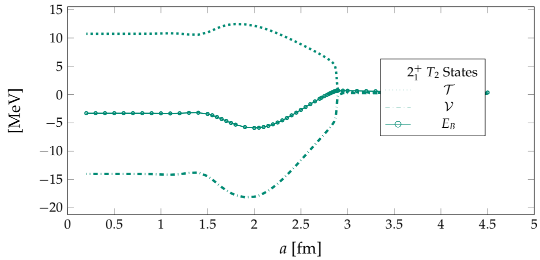

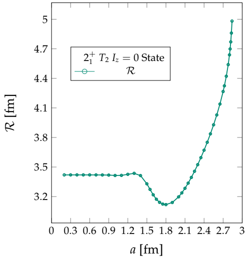

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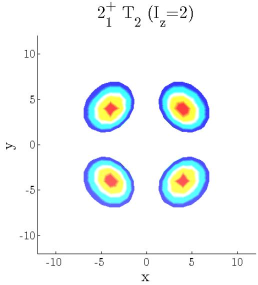

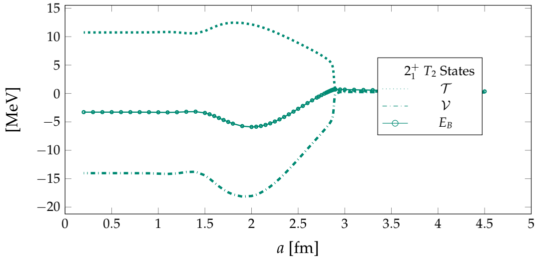

In addition:

$\mathcal{V} \approx -5.43$ MeV @ $a = d^*$

$\mathcal{V}^{min} \approx -18.05$ MeV @ $a \approx 1.15$ fm

and

$\mathcal{R} \approx 4.86$ fm @ $a = d^*$

$\mathcal{R}^{min} \approx 3.11$ fm @ $a \approx 1.78$ fm

Discretization on ^{8}Be: the 2_1^+ T_2 states

$I_z = 1, 3$ Pdf: 2 circles of principal maxima about the z axis,
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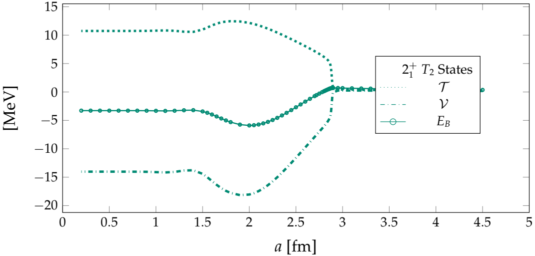

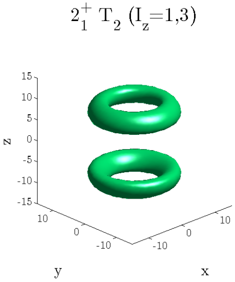

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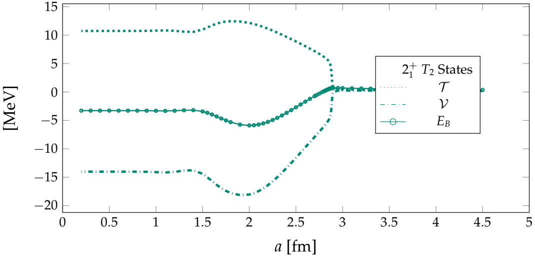

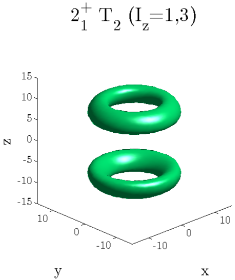

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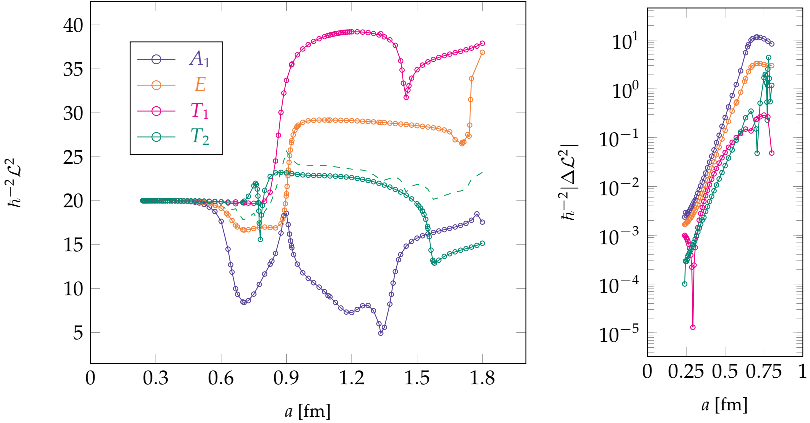
The low-energy ^{8}Be spectrum: the 4_2^+ multiplet

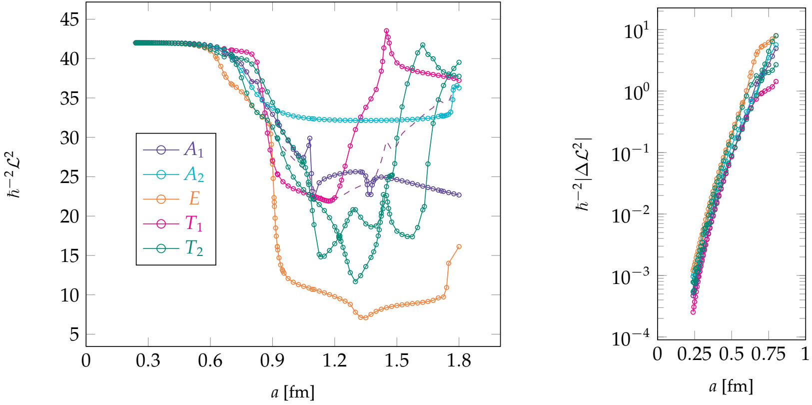
In the f = 2.5 case, fixing $Na \geq 12$ fm residual finite volume effects for the 4_2^+ multiplets amount to $\approx 10^{-3}\hbar^2$. Multiplet-averaging evens the spikes.



Remark: for $a \lesssim 0.80$ fm $\quad |\Delta\mathcal{L}^2| \propto \exp(c_\kappa a) \quad$ with $c_\kappa > 0$

The low-energy ^{8}Be spectrum: the 6_1^+ multiplet

In the f = 2.5 case, fixing $Na \geq 12$ fm residual finite volume effects for the 6_1^+ multiplets amount to $\approx 10^{-4}\hbar^2$. Multiplet-averaging evens the spikes.



Remark: for $a \lesssim 0.80$ fm $\quad |\Delta\mathcal{L}^2| \propto \exp(c_\kappa a)$ with $c_\kappa > 0$