Soft gluon and Coulomb resummation for top-quark pair production at hadron colliders

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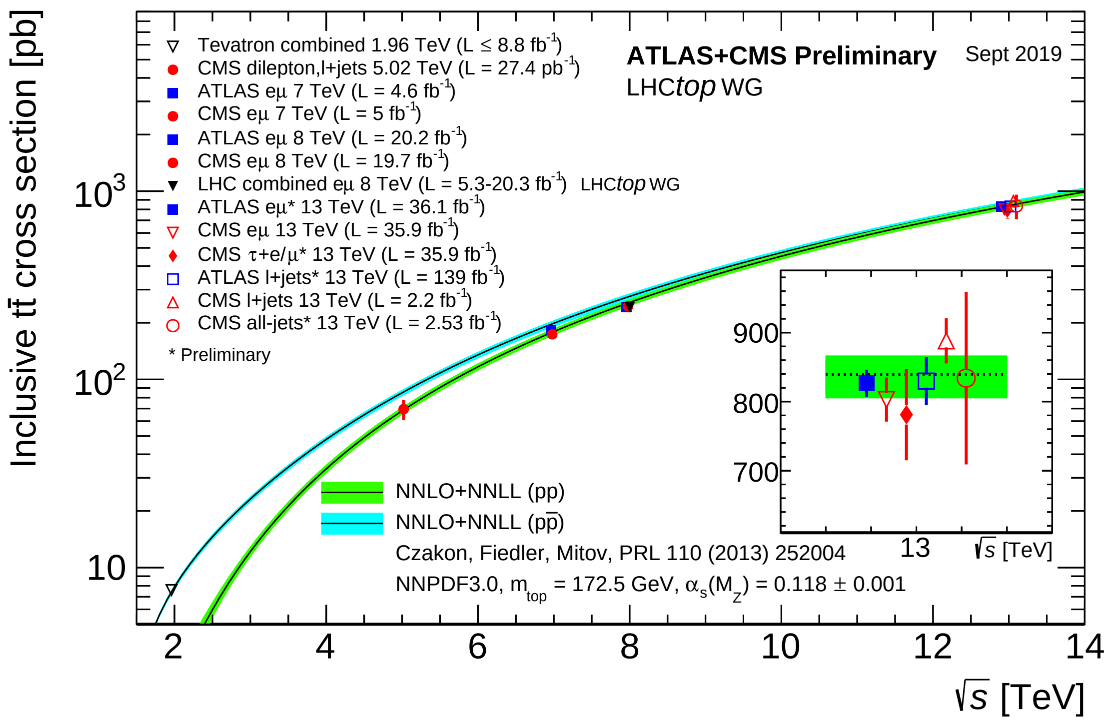
$t\bar{t}$ **production at LHC** test of QCD and nature of top-quark:

- Experimental precision $\Delta\sigma_{t\bar{t}} \sim 3-4\%$ comparable to uncertainty of NNLO+NNLL prediction (Bärnreuther/Czakon/Fiedler/Mitov 12–13)



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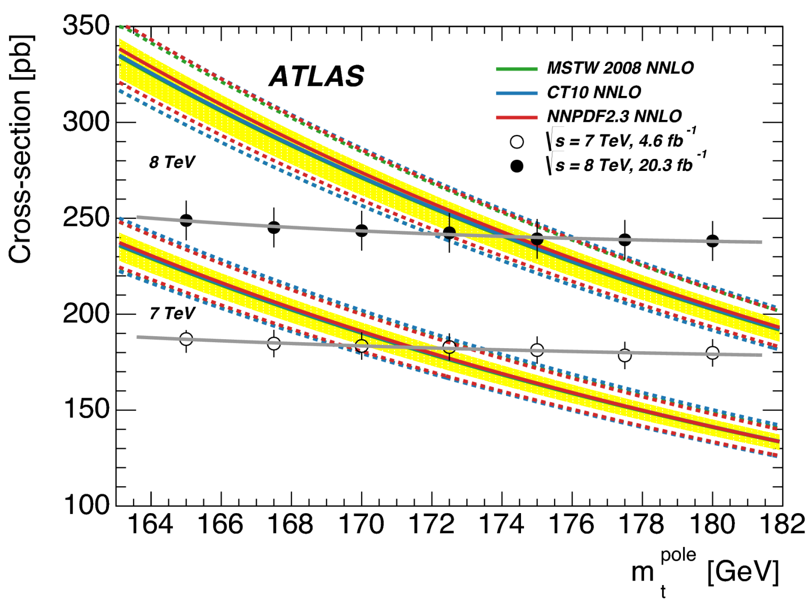
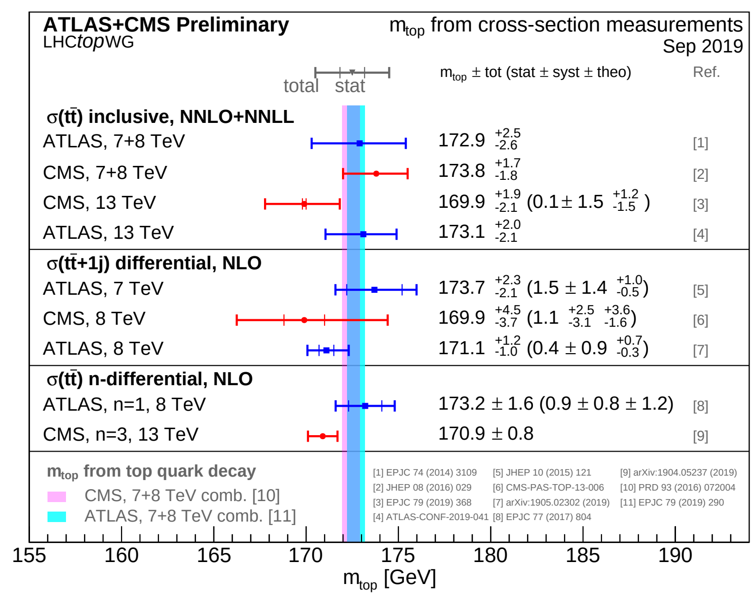
- Sensitivity to m_t, α_s, PDFs
  - pole mass from σ_{tt} measurement with $\Delta m_t \approx \pm 2\text{GeV}$
  - determination of $\alpha_s(M_Z) = 0.1177^{+0.0034}_{-0.0036}$
    (Klijnsma/Bethke/Dissertori/Salam 17)





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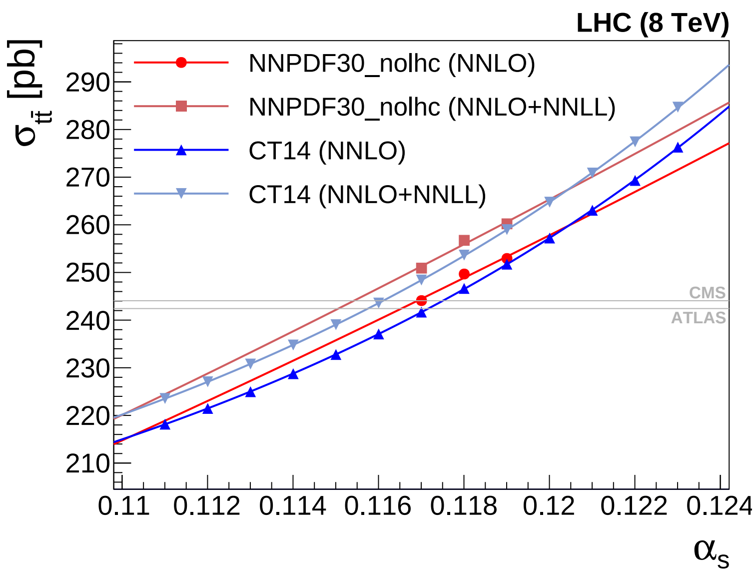
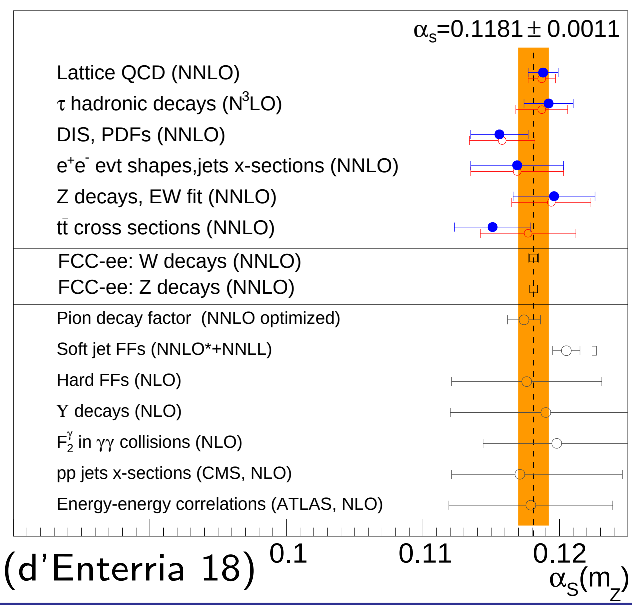




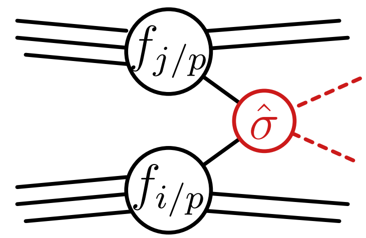
(d'Enterria 18)

Parton Model/QCD Factorization

Cross section for hadronic collision $pp \to X$:

$$\sigma_{pp\to X}(s) = \int \mathrm{d}x_1 \mathrm{d}x_2 f_{i/p}(x_1,\mu_f) f_{j/p}(x_2,\mu_f) \hat{\sigma}_{ij\to X}(\underbrace{x_1 x_2 s}_{\hat{s}},\mu_f)$$

Parton distribution functions $f_{i/p}$ nonperturbative; include dependence on quark masses, $\Lambda_{\rm QCD}$; fitted to experiments (MMHT, CTEQ, NNPDF, ABMP,...)



This talk: focus on partonic cross sections $\hat{\sigma}_{ij\to X}$:

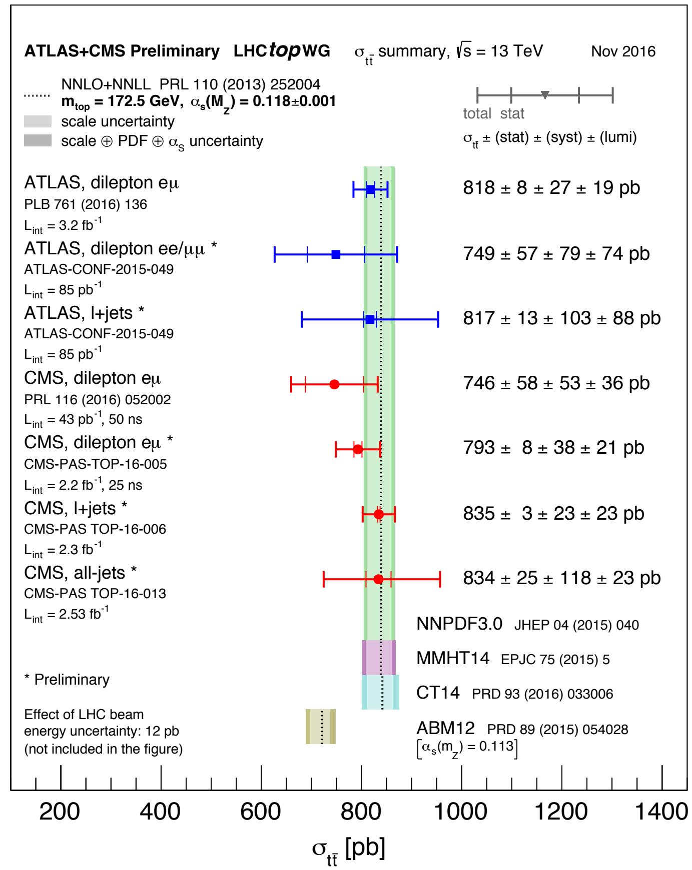
- NNLO QCD total cross section (Bärnreuther/Czakon/Fiedler/Mitov 12–13)
  (implemented in `top++` (Czakon/Mitov); `Hathor` (Moch/Uwer); `topixs` (Beneke et al.))
- NNLO differential cross section (Czakon et al. 16; Catani et al. 19)
- further higher-order corrections
  from soft-gluon and/or Coulomb resummation

Total cross section

Fixed-order prediction in QCD (Bärnreuther/Czakon/Fiedler/Mitov 12–13)

$$\sigma_{t\bar t}^{\rm NNLO}(13{\rm TeV}) = \begin{cases} 802.85^{+28.12+18.94}_{-44.97-20.87}\,{\rm pb} & \text{MMHT2014} \\ 805.14^{+28.28+26.05}_{-45.29-25.71}\,{\rm pb} & \text{CT14} \\ 794.00^{+28.18+12.81}_{-45.13-12.81}\,{\rm pb} & \text{NNPDF3.1} \\ 785.02\underbrace{^{+26.50}_{-42.68}}_{\rm scale}\underbrace{^{+19.37}_{-19.37}}_{{\rm PDF}+\alpha_s}{\rm pb} & \text{ABMP16} \end{cases}$$

$m_t = 173.3\,{\rm GeV},\ \alpha_s(M_Z) = 0.118 \pm 0.001;$
ABPM16: $m_t = 170.4\,{\rm GeV},\ \alpha_s(M_Z) = 0.1147 \pm 0.0008$

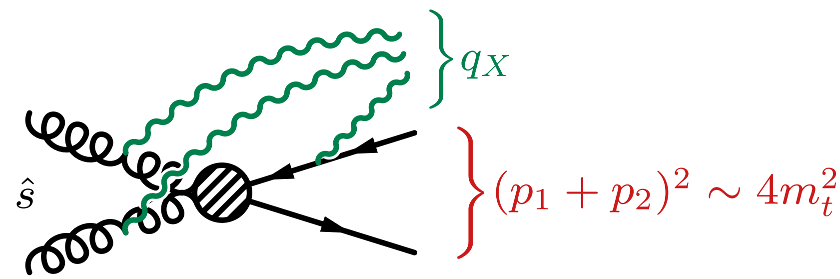


- $\sigma_{t\bar t}$ included in PDF fits
- Scale uncertainty $\sim 5\% \gtrsim$ PDF$+\alpha_s$ uncertainty
- Experimental uncertainty reaches $\sim 3-4\%$

Beyond fixed order

- Total partonic cross section $\hat{\sigma}$ function of $\rho = 4m_t^2/\hat{s}$, $\ln(m_t/\mu_{f/r})$
- Consider threshold limit $\beta = \sqrt{1-\rho} \to 0$ for $t\bar{t}$ hadroproduction

  ($s \gg 4m_t^2$ but contribution from $\rho \to 1$ enhanced by behaviour of PDFs)
- real-gluon emission near threshold necessarily soft: $q_X \sim M\beta^2$

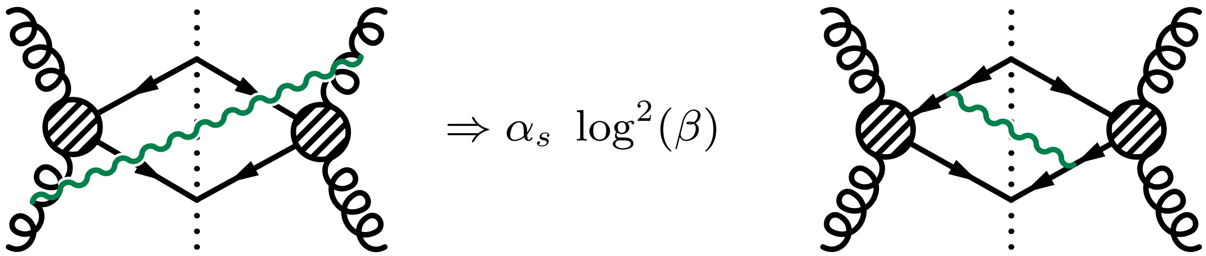


- structure of soft-gluon emission universal

  $\Rightarrow$ can predict threshold corrections at higher orders

  (Sterman 87; Catani, Trentadue 89, Kidonakis,Sterman 97, Bonciani et.al. 98, ...)

Threshold logarithms $\sim \alpha_s \log^2(1 - \frac{4m_t^2}{\hat{s}})$



$\Rightarrow \alpha_s \; \log^2(\beta)$ $\Rightarrow \alpha_s \; \log(\beta)$

remnants of cancellation of soft/collinear divergences between real and virtual corrections

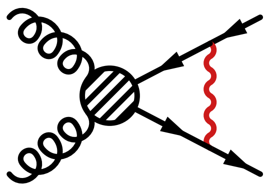
Resummation of threshold logarithms to all orders

$$\hat{\sigma}_{pp'} \propto \sigma^{(0)} \exp\Big[\underbrace{\ln\beta\, g_0(\alpha_s \ln\beta)}_{\text{(LL)}} + \underbrace{g_1(\alpha_s \ln\beta)}_{\text{(NLL)}} + \underbrace{\alpha_s g_2(\alpha_s \ln\beta)}_{\text{(NNLL)}} + \underbrace{\alpha_s^2 g_3(\alpha_s \ln\beta)}_{\text{(N}^3\text{LL)}} + \dots\Big]$$

Ingredients for NNLL resummation for $t\bar{t}$ production known

(Beneke/Falgari/CS; Czakon/Mitov/Sterman 09)

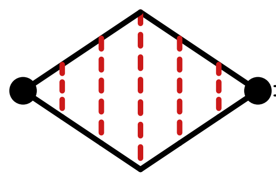
Coulomb gluon corrections from "potential" momenta $q \sim m(\beta^2, \beta)$



$$\sim \alpha_s \int dq^0 d^3\vec{q}\, \frac{1}{E_2 - \frac{\vec{p}_2^{\,2}}{2m_t}} \frac{1}{\vec{q}^{\,2}} \frac{1}{E_1 - \frac{\vec{p}_1^{\,2}}{2m_t}} \sim \frac{\alpha_s}{\beta}$$

Resummation of $\frac{\alpha_s}{\beta}$ corrections: (Fadin, Khoze 87; Peskin, Strassler 90)

- Green's function for non.rel. Schrödinger equation

$\mathrm{Im} G_{\mathrm{C}}^{R}(0,0;E) =$ 

$$= \begin{cases} \frac{m_t^2 \pi D_R \alpha_s}{2\pi} \left(e^{\pi D_R \alpha_s \sqrt{\frac{m_t}{E}}} - 1 \right)^{-1} & E > 0 \\ 2 \sum_{n=1}^{\infty} \delta(E - E_n) |\psi_n(0)|^2 & E < 0 \end{cases}$$

  sums all (α_s/β) corrections

  Colour factor for singlet and octet states

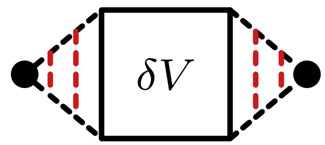
$$D_1 = -C_F = -4/3, \qquad D_8 = \frac{C_A}{2} - C_F = 1/6$$

- Bound-state poles at $E_n = -\frac{\alpha_s^2 D_R^2 m_t}{4n^2}$ smeared out by finite Γ_t.

- Non-relativistic perturbation theory with insertions of higher-order potentials δV with radiative/kinematic corrections

- NLO potential function sums all terms $\alpha_s(\alpha_s/\beta)^n$

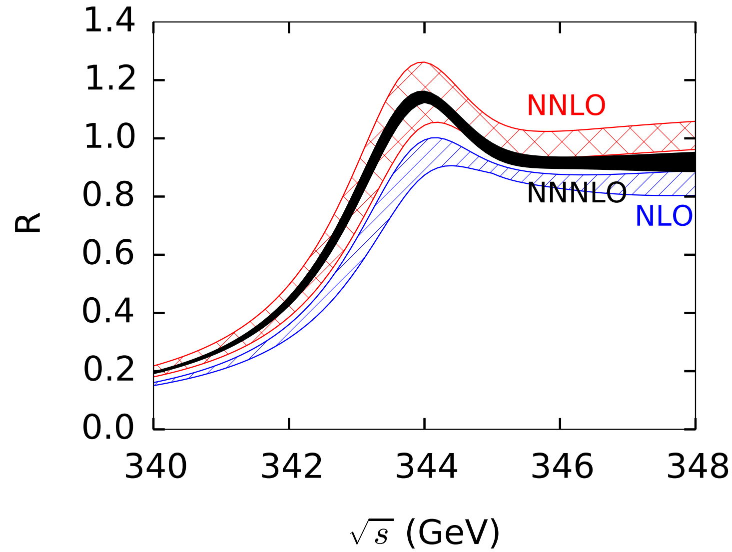
$$\delta G_R^{(1)}(0,0,E) =$$



$$= \int d^3z\, G_R^{(0)}(0,\vec{z},E)\,(i\delta V^R(\vec{z}))\, iG_R^{(0)}(\vec{z},0,E)$$

- N^{3}LO Green function for colour singlet known from $e^-e^+ \to t\bar{t}$

  (Beneke et al. 15; NNLL $\ln\beta$ summation: Hoang/Stahlhofen 13)



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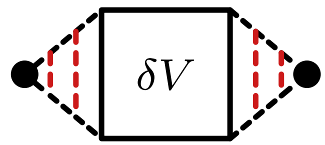
$\delta G_R^{(1)}(0,0,E) =$ 

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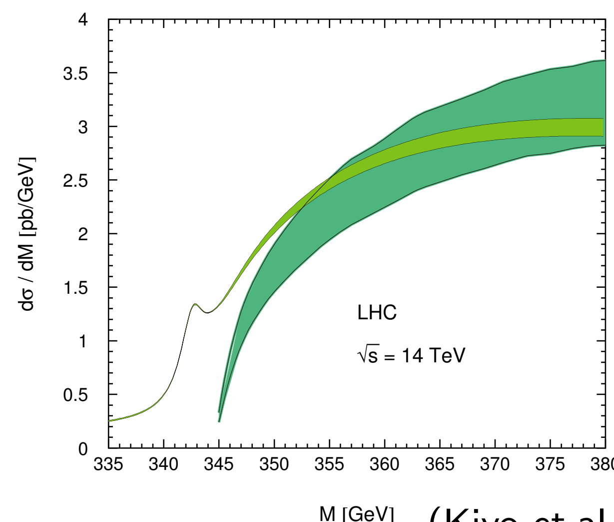
- Hadron colliders: Coulomb potential for octet repulsive

$\Rightarrow$ small effect for total cross section

- Effect on $M_{t\bar{t}}$ spectrum below nominal threshold $M_{t\bar{t}} \sim 2m_t$

  (Hagiwara et al.; Kiyo et al. 08)



(Kiyo et.al. 08)

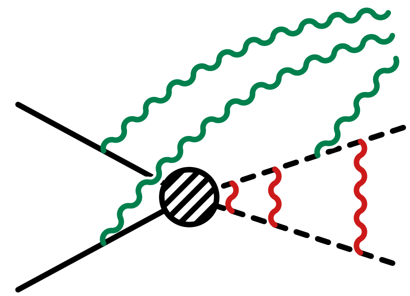
Combination of Coulomb- and soft effects?

Heavy particles nonrelativistic near threshold:

$$E \sim m\beta^2 \,, \quad |\vec{p}| \sim m\beta$$

soft gluon momenta of same order: $q_s \sim m\beta^2 \sim E$

$\Rightarrow$ heavy particles "feel" soft radiation



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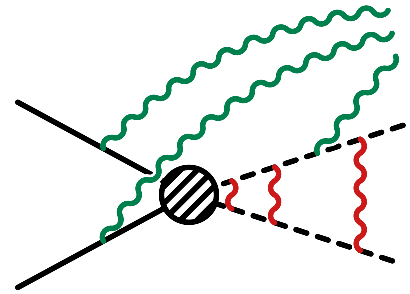

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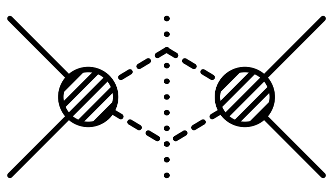
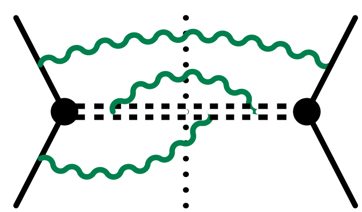
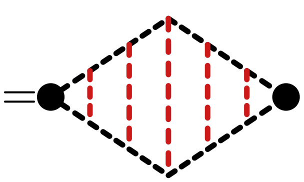
Factorization of cross section (Beneke, Falgari, CS 09/10)

$$\Rightarrow \hat{\sigma}_{t\bar{t}}|_{\hat{s}\to 4m_t^2} = \sum_{R_\alpha} H_{R_\alpha}(M,\mu) \int d\omega \; J_{R_\alpha}(\sqrt{\hat{s}} - 2M - \frac{\omega}{2}) W^{R_\alpha}(\omega,\mu)$$

Hard, soft and Coulomb functions:

$H_i =$  , ${W_i}^R =$  , $J^R =$ 

Soft radiation "sees" only total colour charge R of heavy particles

(Singlet, octet,...)

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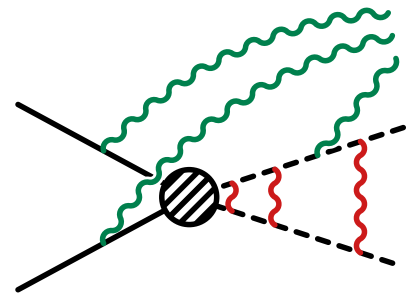

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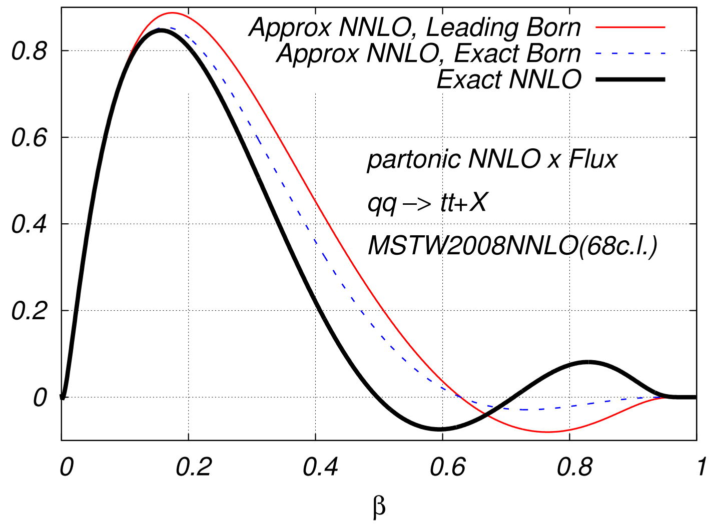
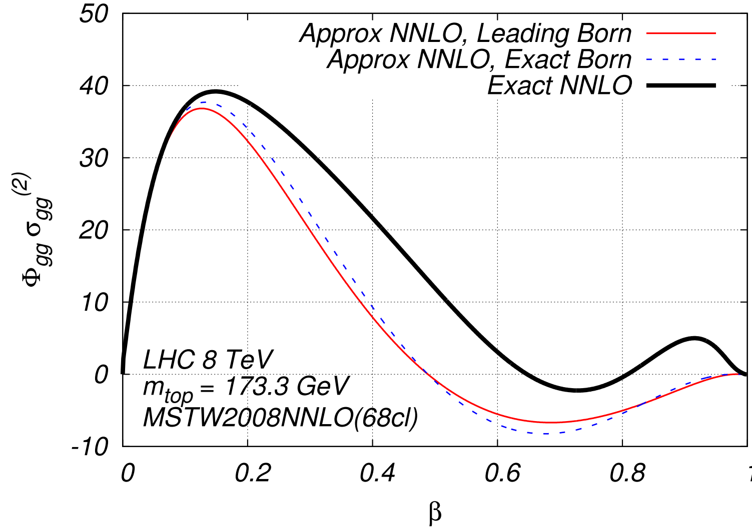
- derived using soft-collinear and non-relativistic effective field theories
- soft-gluon resummation using evolution equations for soft function W and hard function H.
- Coulomb resummation in potential function $J = 2\mathrm{Im}\, G_C$

- Top-pair production dominated by $\beta \sim 0.6$
  $\Rightarrow$ justification of threshold approximation?





$\frac{d\sigma}{d\beta} = \frac{8\beta m_t^2}{s(1-\beta^2)^2} L(\beta,\mu_f)\hat{\sigma}$, (Bärnreuther/Czakon/Mitov 12; Czakon/Fiedler/Mitov 13)

$\Rightarrow$ threshold corrections give estimate of higher-order corrections

$\Rightarrow$ careful estimate of uncertainties necessary

- resummation not mandatory for $t\bar{t}$ production at LHC

$\Rightarrow$ compare resummed results to fixed-order expansions

Threshold definitions for different variables

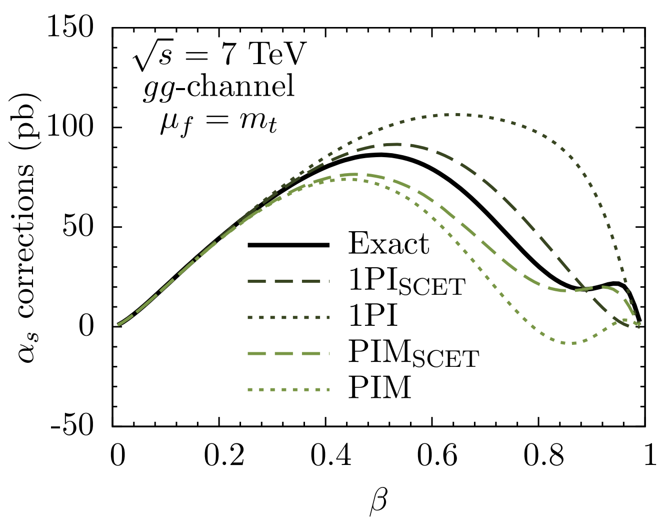
Pair invariant mass cross sections (Kidonakis,Sterman 97, Ahrens et al. 10)

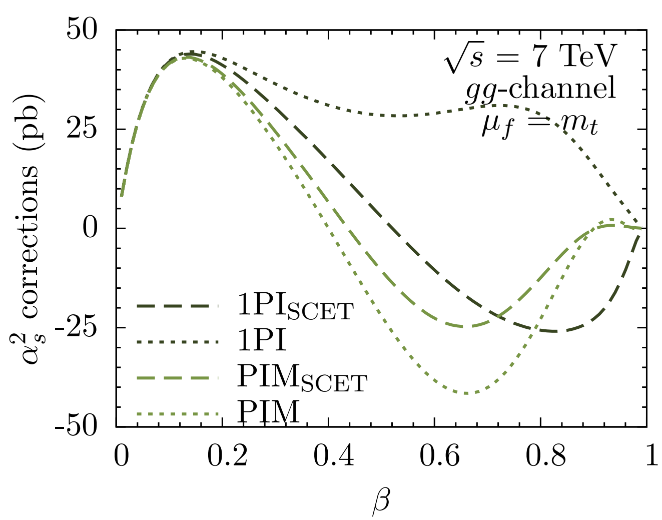
$$\frac{d\hat\sigma(t\bar t)}{dM_{t\bar t}} \Rightarrow \left[\frac{\log^n(1-z)}{1-z}\right]_+ \,,\; z=\frac{M_{t\bar t}^2}{\hat s}\,, \qquad \mathrm{PIM_{SCET}}: \log\left(\frac{1-z}{\sqrt{z}}\right)$$

One particle inclusive cross sections: (Laenen et al. 98, Ahrens et al. 11)

$$\frac{d\hat\sigma(t+X)}{ds_4} \Rightarrow \left[\frac{\log^n(s_4/m^2)}{s_4}\right]_+ \;;\; s_4=p_X^2-m_t^2\,, \qquad \mathrm{1PI_{SCET}}: \log\left(s_4/\sqrt{m^2+s_4}\right)$$

Approximations differ by higher-order terms in β: (Ahrens et al. 11)





NNLL corrections: (13TeV, MMHT2014)

reduced scale uncertainty, estimate of resum. uncertainty?

$$\sigma_{t\bar{t}}^{\text{NNLO}} = 802.8^{+28.1(3.5\%)}_{-44.9(5.6\%)}\text{pb} \Rightarrow \begin{cases} \text{NNLL(top++)}: & 821.4^{+20.3(2.5\%)}_{-29.6(3.6\%)}\text{pb} \\ \text{NNLL(topixs)}: & 807.1 \underbrace{{}^{+15.6(1.9\%)}_{-36.8(4.6\%)}}_{\text{scale}} \underbrace{{}^{+19.2(2.5\%)}_{-12.9(1.8\%)}}_{\text{resum}} \text{pb} \end{cases}$$

`top++`: Mellin-space resummation of threshold logarithms

(Czakon/Mitov/Sterman 09/Cacciari et al. 11)

`topixs:` momentum-space resummation of threshold logs combined with Coulomb corrections α_s/β (Beneke/Falgari/(Klein)/CS 09/11)

Main numerical differences:

- α_s^2 hard coefficient in `top++`: (NNLL'): $\Delta\sigma \approx 9\text{pb}$
- bound-state effects in `topixs`: $\Delta\sigma_{\text{BS}} \approx 3\text{pb}$

NNLL corrections: (13TeV, MMHT2014)

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Soft-gluon resummation for differential distributions

- p_T, $M_{t\bar{t}}$ from 1PI/PIM (Kidonakis10 ; Ahrens et al. 10/11)
- low p_T (Zhu et al 13; Catani et al 14/18)
- boosted tops (Ferroglia et al. 12; Czakon et al. 18)

Prospects of N³LL resummation (Piclum/CS 18)

- Input to N³LL resummation formula:
  - two-loop hard function (Bärnreuther/Czakon/Fiedler 13)
  - two-loop soft functions (Belitzky 98; Becher et al. 07; Czakon/Fiedler 13)
  - N³LO potential corrections

    (colour singlet: Beneke et al. 15; incomplete colour octet)
  - anomalous dimensions (4-loop γ_{cusp} (Moch et al. 17/18);

    3-loop collinear anomalous dimensions (Moch/Vermaseren/Vogt 04/05)

    missing: 3-loop massive soft anomalous dimension)
- Interplay of Coulomb and power-supressed corrections
  - "next-to-eikonal' corrections (Krämer et al. 98; Laenen et al. 10)

$$\frac{\alpha_s^2}{\beta^2} \times \alpha_s \beta^2 \log\beta \sim \alpha_s^3 \log\beta$$

  - P-wave production channels: $\beta^2 \times \{\frac{\alpha_s^3}{\beta^3}, \frac{\alpha_s^2}{\beta^2} \times \alpha_s \ln^{2,1}\beta\}$

Expansion to N³LO

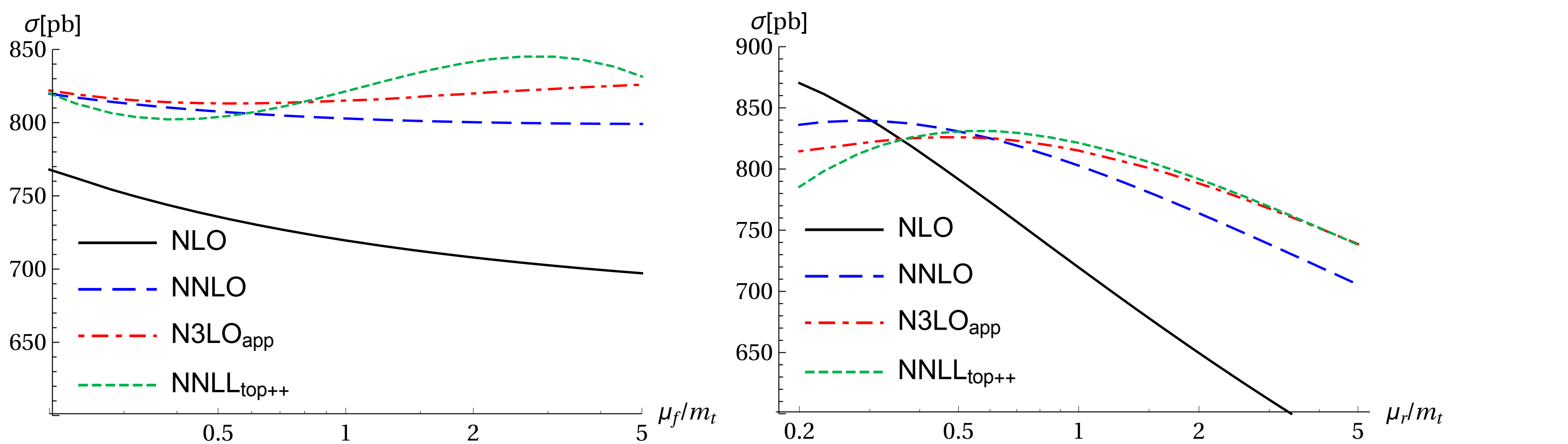
Moderate correction $+1.6\%$ relative to NNLO (Piclum/CS 18)

$$\sigma_{t\bar{t}}^{\mathrm{N^3LO_{app}}}(13\mathrm{TeV}) = 815.70^{+19.88(2.4\%)}_{-27.12(3.3\%)}(\mathrm{scale})^{+9.49(1.2\%)}_{-6.27(0.8\%)}(\mathrm{approx})\mathrm{pb},$$

- Reduction of scale uncertainty to $\sim 3\%$
- Estimate of systematic uncertainty of approx:

$$\Delta\sigma_{t\bar{t}}^{\mathrm{N^3LO_{app}}}(\mathrm{approx.}) = \underbrace{{}^{+7.87}_{-6.24}}_{C^{(3)}} \underbrace{{}^{+5.3}_{-0.0}}_{\text{kin.ambiguity}} \pm \underbrace{0.11}_{3-\text{loop soft-an.dim}} \pm \underbrace{0.60}_{\text{Coulomb octet}} \mathrm{pb},$$

- Available in latest version of `topixs`



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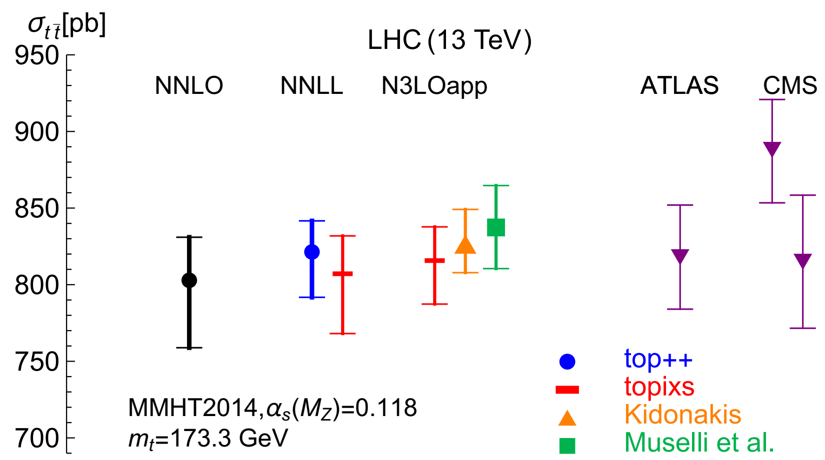
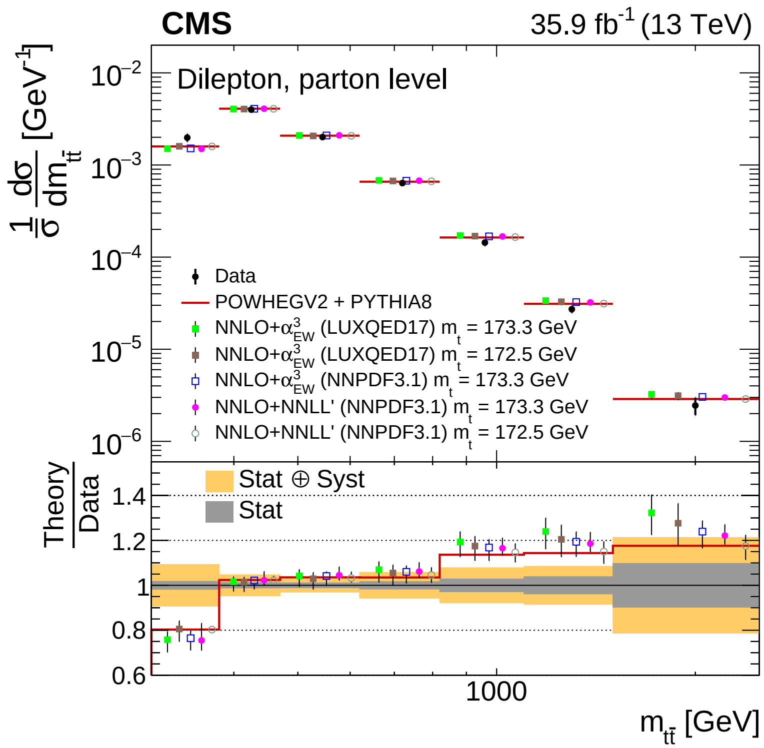
Other $\mathrm{N^3LO_{approx}}$ results:

- NNLL in one-particle inclusive kinematics: (Kidonakis 14)
- Including subleading collinear; $\beta \to 1$ terms (Muselli et al. 15)

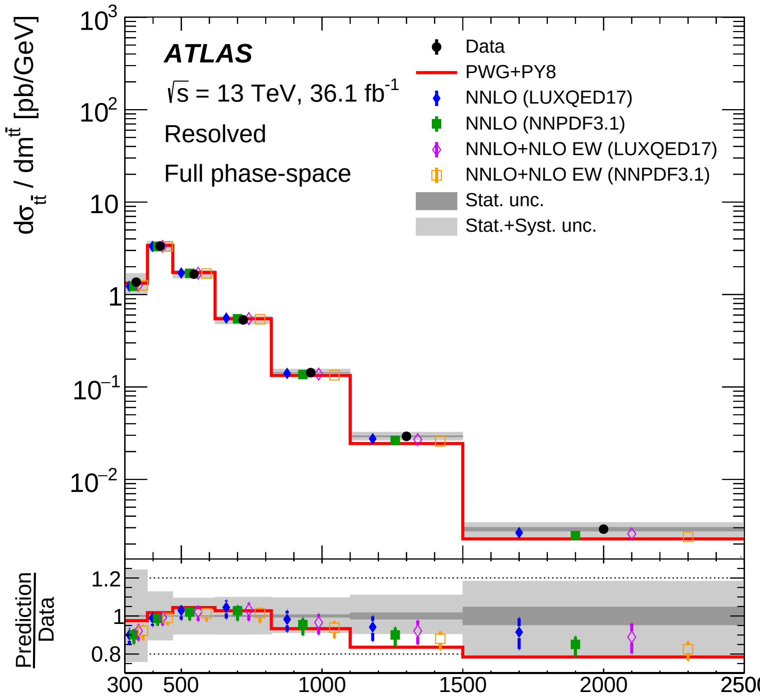


- Measurements of $M_{t\bar{t}}$ spectrum at LHC with $\approx 10\%$ accuracy
- Some hints for discrepancy to theory for $M_{t\bar{t}} \lesssim 400\,\mathrm{GeV}$

$\Rightarrow$ Coulomb effects'?



(arXiv:1811.06625)



(arXiv:1908.07305)

Different factorization approaches for invariant mass spectrum:

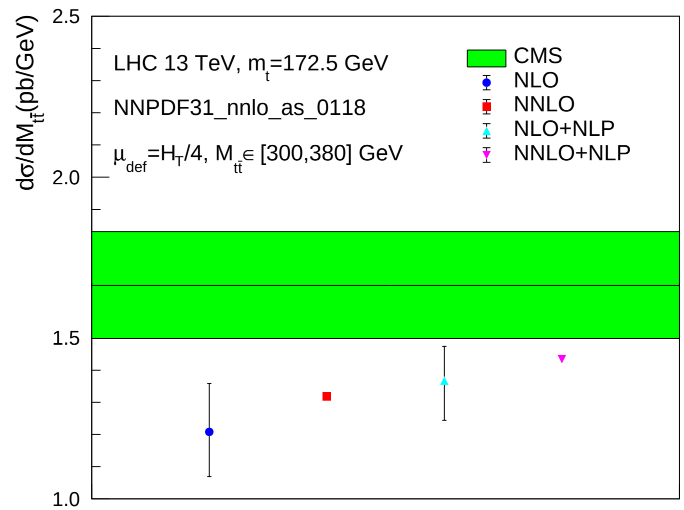
- Soft/potential/hard factorization for $\hat{s} \sim 4m_t^2$: (Beneke, Falgari, CS 10)

$$\frac{d\hat{\sigma}_{t\bar{t}}(\hat{s},\mu)}{dM_{t\bar{t}}} = \sum_{R=1,8} 2H_R(m_t,\mu) J_R(M_{t\bar{t}} - 2m_t)\, W^R(2(\sqrt{\hat{s}} - M_{t\bar{t}}),\mu)$$

- Potential/hard fact. for $M_{t\bar{t}}^2 \sim 4m_t^2$ (not $\hat{s} \to M_{t\bar{t}}^2$) (Ju et al. 19)

$$\frac{d\hat{\sigma}_{t\bar{t}}(\hat{s},\mu)}{dM_{t\bar{t}}} = \sum_{R=1,8} 2H_R(\hat{s},m_t,M_{t\bar{t}},\ldots,\mu) J_R(M_{t\bar{t}} - 2m_t)$$

- NLO Coulomb function matched to fixed-order NLO/NNLO QCD
- no soft-gluon resummation



(Ju et al. 19)

- **Experimental accuracy** of $\sigma_{t\bar{t}}$ comparable to NNLO prediction
- **Soft-gluon and Coulomb corrections**
  - enhanced in threshold limit $\hat{s} \to 4m_t^2$
    (other threshold definitions for differential observables)
  - different methods needed for b,c
    ($m \to 0$ limit for production at LHC; bound state effects for quarkonium production)
- **Predictions for total cross section:**
  - NNLL resummation moderate effect;
    reduction of scale ambiguity to 3%
  - N^3LO expansion of partial N^3LL complementary to NNLL resummation (includes input beyond NNLL)
- **Coulomb effects** small for total cross section;
  relevant for $M_{t\bar{t}} \sim 2m_t$?
- Astrophysics applications?
  (Sommerfeld enhancement in dark matter annihilation...)

Factorization scale dependence of H, W cancels against PDFs:

$$\frac{d\sigma}{d\mu} = \frac{d}{d\mu}\left(f_1 \otimes f_2 \otimes H \otimes W \otimes J\right) = 0$$

- $\frac{df_i}{d\mu} \Rightarrow$ Altarelli-Parisi equation (3-loop: Moch/Vermaseren/Vogt 04/05)
- $\frac{dH}{d\mu} \Rightarrow$ IR singularities (2-loop: Becher, Neubert; Ferroglia et.al. 09)

$\Rightarrow$ RGE for soft function (NNLL: Beneke/Falgari/CS; Czakon/Mitov/Sterman 09)

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Resummation:

Mellin-space approach:

(Sterman 87; Catani, Trentadue 89; Korchemsky, Marchesini 92)

Solve RGE using Mellin transform: ($\rho = 4M^2/\hat{s}$)

$$\sigma^N = \int_0^1 d\rho\rho^{N-1}\hat{\sigma}(4M^2/\rho)\,, \qquad \int_0^1 d\rho\rho^N \beta \log^n \beta \propto \ln^n N + \ldots$$

Numerical inverse transformation

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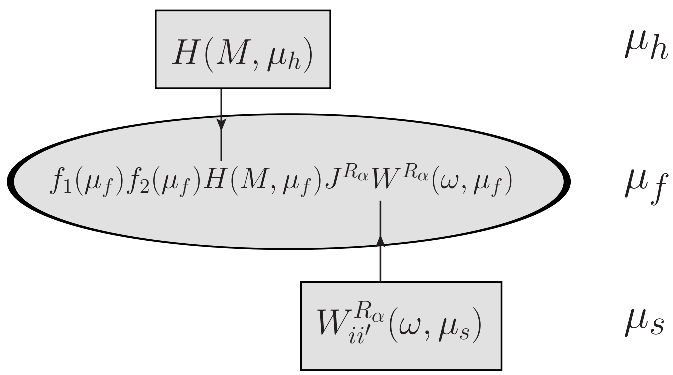
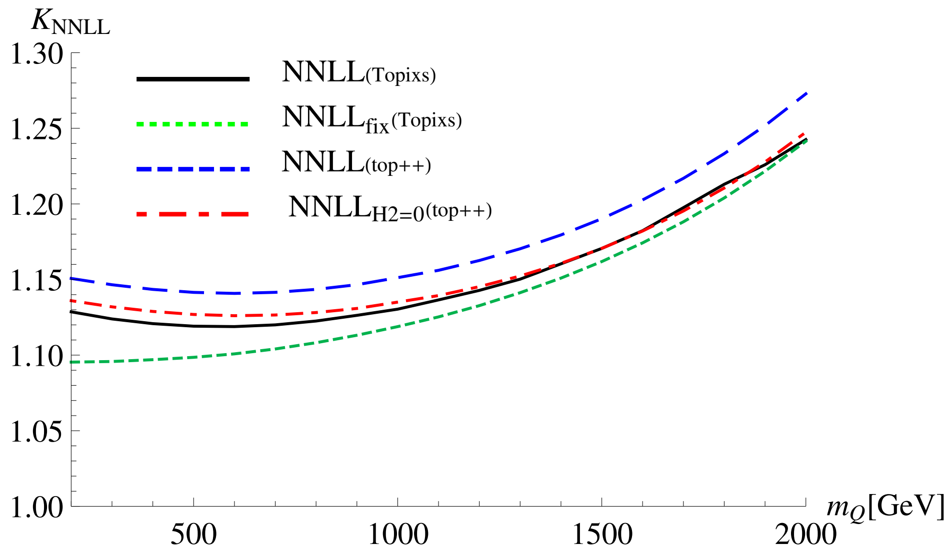
$\Rightarrow$ RGE for soft function (NNLL: Beneke/Falgari/CS; Czakon/Mitov/Sterman 09)

Resummation:

- Momentum-space solution to RGE (Becher, Neubert, Pecjak 07)
- evolve hard function from $\mu_h \sim 2M$ to μ_f
- evolve soft function from $\mu_s \sim M\beta^2$ to μ_f



Heavy Quarks as test case for resummation methods



($K_{\rm NNLL} = \sigma^{\rm NNLL}/\sigma^{\rm NLO}$,

LHC $\sqrt{s} = 8$ TeV)

$\Rightarrow$ resummation methods agree well for larger masses

- differences at m_t: estimate of resummation ambiguities
- main difference: treatment of $H_2 \Rightarrow \alpha_s^3 \log \beta^2$ terms (NNLL')

Potential function

General form of potential ($R=1,8,\ S=1,3$)

$$V^{R,S}(\mathbf{p},\mathbf{p}') = \frac{4\pi\alpha_s D_R}{\mathbf{q}^2}\left[\mathcal{V}_C^R - \mathcal{V}_{1/m}^R\,\frac{\pi^2\,|\mathbf{q}|}{m_t} + \mathcal{V}_{1/m^2}^{R,S}\,\frac{\mathbf{q}^2}{m_t^2} + \mathcal{V}_p^R\,\frac{\mathbf{p}^2+\mathbf{p}'^2}{2m_t^2}\right] + \frac{\pi\alpha_s}{m_t^2}\nu_{\rm ann}^{R,S},$$

Known for colour singlet and octet:

- One-loop spin-dependent $\mathcal{V}_{1/m^2}^{R,S}$

  (Wüster 03; colour-singlet: Beneke/Kiyo/Schuller 13, colour octet: Piclum/CS 18)

Unknown for octet

- Two-loop $\mathcal{V}_{1/m}^R$ (singlet: Kniehl et al. 01)
- (ultra)-soft corrections (singlet: Beneke/Kiyo 08)
  with chromoelectric vertex $\psi^\dagger \vec{x}\cdot\vec{E}_{us}\psi'^\dagger$

Unknown contributions at $\mathcal{O}(\alpha_s^3)$: $\alpha_s^3(\delta c_{J,3}^{(2,0)}\ln^2\beta + \delta c_{J,3}^{(1,0)}\ln\beta + \dots)$

Estimate $\delta c_{J,3}^{(i,0)}$ for octet by naive replacement $C_F \to (C_F - C_A/2)$