Semiclassical Einstein equations and their fluctuations in cosmological spacetimes

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Motivations

- **Quantum matter - gravity interplay** should be described within:

  quantum gravity.

- However at least in some regimes it should be possible to analyze it in the semiclassical approximation:

$$G_{ab} = \langle T_{ab} \rangle_\omega$$

- It equates **classical** quantities with **probabilistic** ones.
- We could take into account fluctuations in the sense of probability theory.

Plan of the talk

- Global existence of solutions of semiclassical gravity in cosmology.
- Metric fluctuations in Starobinski model of inflation.

This talk is based on

- NP, D. Siemssen, J. Math. Phys. **56** 022303 (2015) .
- NP, D. Siemssen, Comm. Math. Phys. **334** 171-191 (2015).
- NP, CMP **305** 563-604 (2011).

Semiclassical Einstein equation in cosmology.

- Cosmological spacetimes

$$(M, g)\ , \qquad M = I \times \Sigma\ ,$$

- For flat cosmological spacetime.

$$g = a(\tau)^2 \left[-d\tau \otimes d\tau + dx^i \otimes dx^i\right]\ ,$$

a is the scale factor and τ conformal time.

- Semiclassical Einstein equation

$$G_{ab} = \langle T_{ab} \rangle_\omega$$

- Simpler equation

$$-R = \langle T \rangle_\omega\ , \qquad \nabla_a T^{ab} = 0\ , \qquad \rho(0) = cH^2(0)$$

- We look for existence and uniqueness of solutions of that system.

Considered matter

- Massive scalar quantum field conformally coupled to gravity.

$$-\Box\varphi + \frac{1}{6}R\varphi + m^2\varphi = 0$$

- The quantization on every FRW spacetime is very well under control.
- Assign to every spacetime

$$M \mapsto \mathcal{A}(M)$$

- The $*-$algebra generated by linear fields $\varphi(f)$, implementing:

$$\varphi^*(f) = \varphi(\overline{f}) \ , \qquad [\varphi(f), \varphi(h)] = i\Delta(f,h) \ , \qquad \varphi(Pf) = 0 \ .$$

- We include in the algebra the Wick powers. (by deforming the product and then by extending it trivially) Details.
- It can be tested on Hadamard states.
- We need a **rule** to prescribe a state on every FRW spacetime and than use the semiclassical equation to select the correct one.

Where we need to put initial conditions

$$-R = \langle T \rangle_\omega$$

- It was solved employing **very special initial conditions** at the singularity (Null big bang scenario). *[NP 2011]*.
- This was necessary to control the regularity of the state.
- However, the radiation content of the solution has negative energy density. $\Longrightarrow$ Physically not acceptable.
- Impose initial conditions at finite time, say $\tau = \tau_0$ (where the spacetime is regular).

State

- At $\tau = \tau_0$ we fix the state to be as close as possible to the vacuum.
- The **pure**, **homogeneous** and **isotropic** Gaussian state

$$\omega_2(x,y) := \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{\overline{\chi_k}(x_0)}{a(x_0)}\frac{\chi_k(y_0)}{a(y_0)} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} d\mathbf{k} \ ,$$

$$\chi_k''(\tau) + (m^2 a(\tau)^2 + k^2)\chi_k(\tau) = 0,$$

$$\overline{\chi_k}\frac{d}{d\tau}\chi_k - \frac{d}{d\tau}\overline{\chi_k}\chi_k = i \ .$$

- In order to be close to the vacuum at $\tau = \tau_0$ we fix

$$\chi_k(\tau_0) = \frac{1}{\sqrt{2k_0}}\ , \qquad \chi_k'(\tau_0) = -i\sqrt{\frac{k_0}{2}}\ , \qquad k_0 = \sqrt{k^2 + m^2 a_0^2}$$

- It is an **adiabatic state** of order 0 at $\tau = 0$. (Although it is not Hadamard it is sufficiently regular to construct T)

Point splitting regularization

Hadamard two-point function:

$$\mathcal{H} = \frac{U}{\sigma_\epsilon} + V \log\left(\frac{\sigma_\epsilon}{\lambda^2}\right) + W$$

Stress-Energy Tensor: *[Moretti 2003]*

$$T_{ab} := \partial_a\varphi\partial_b\varphi - \frac{1}{6}g_{ab}\left(\partial_c\varphi\partial^c\varphi + m^2\varphi^2\right) - \frac{1}{6}\nabla_{(a}\partial_{b)}\varphi^2$$
$$+\frac{1}{6}\left(R_{ab} - \frac{R}{6}g_{ab}\right)\varphi^2.$$

Expectation values:

$$\omega(T_{ab}) = \lim_{y\to x} D_{ab}\left[\omega_2(x,y) - \mathcal{H}(x,y)\right]$$

Components of $\langle T \rangle$

- In T there are three contributions, the state dependent part, the anomalous term and the renormalization freedom.

$$\langle T \rangle_\omega = T_{anomaly} + \alpha m^4 + \beta m^2 R + \gamma \Box R + m^2 \langle \varphi^2 \rangle_\omega \ .$$

$$T_{anomaly} = \frac{1}{360}\left(C_{ijkl} C^{ijkl} + R_{ij} R^{ij} - \frac{R^2}{3} + \Box R \right) + \frac{m^4}{4}$$

- α expresses a renorm. of the cosmological constant. $\Longrightarrow (\alpha m^4)$
- β expresses a renorm. of the Newton constant $\Longrightarrow (\beta m^2 R)$
- γ is a pure quantum freedom. $\Longrightarrow (\gamma R^2)$

- After fixing the renormalization freedom (α, β and $\gamma = -1/360$) we may rewrite the equation as a **Volterra-functional equation**.
- For simplicity, in this talk, we will assume that the anomalous part vanish and that $\Lambda = 0$.

$$a'(\tau) = a'_0 + m^2 \int_{\tau_0}^{\tau} \langle \varphi^2 \rangle_\omega a^3 \, d\eta$$

$$\langle \varphi^2 \rangle_\omega = \frac{1}{2\pi^2 a^2} \int_0^\infty \left[\overline{\chi}_k \chi_k - \frac{1}{\sqrt{k^2 + m^2 a^2}} \right] k^2 dk$$

$$\chi''_k(\tau) + (m^2 a(\tau)^2 + k^2)\chi_k(\tau) = 0,$$

- We fix the state so that it looks like the vacuum at $\tau = \tau_0$.
- We can control $\langle \varphi^2 \rangle_\omega$ w.r.to a' and its (first-)functional derivative on $C^0(\tau_0, \tau)$. const.

$$\| \langle \varphi^2 \rangle_\omega \|_\infty \leq c(\|a'\|_\infty, \tau - \tau_0) \,, \quad \| D\langle \varphi^2 \rangle_\omega \|_\infty \leq c(\|a'\|_\infty, \tau - \tau_0) \|\delta a'\|_\infty$$

- We get an estimate valid on every spacetime ($\forall a' \in C^0[\tau_0, \tau]$)

Local existence

Proposition

Fix a_0 and the state at τ_0. An **unique** *solution a_I* **exists** *in $I = [\tau_0, \tau_1)$ for some $\tau_1 > \tau_0$.*

Proved applying the Banach fixed point theorem to the Volterra like equation. The estimates permits to construct a contraction map.

Global solution

Proposition

Let a_I in $I = [\tau_0, \tau_1)$ be a solution then, if $a'(\tau_1)$ do not diverge and $a(\tau_1) > 0$ the solution can be extended further in a **unique** *way to a_J with $I \subset J$.*

- We can order all the solutions a_I. $a_I \leq a_J$ is $I \subset J \implies$ a maximal solution exists

Proposition

The maximal solutions is unique because of the unique extension.

Summarizing: fixing the initial condition, either the solution exists till infinity or a singularity is encountered. ($a = 0$, $a' = \infty$)

Other initial values

- Changing the initial values a_0, a_0' correspond to change the state.

$$\chi_{k,1} = A\chi_k + B\overline{\chi}_k$$

- If the state are sufficiently close to ω (B suff. reg.) we can still find solutions.
- The obtained solution is unfortunately only C^2.
- The employed estimates for $\langle \varphi^2 \rangle_\omega$ do not permit to control the global behavior from the initial condition.

Semiclassical Einstein equations and stochastic gravity

- Let's look again at

$$G_{ab} = \langle T_{ab} \rangle_\omega$$

- The right hand side is a **probabilistic quantity**.

- It is an effective equation, valid when **"fluctuations"** are negligible.

- There are regimes where this should be correct.

- If the variance of $\langle T_{ab} \rangle_\omega$ is not negligible, like for the **Brownian motion** $\Longrightarrow$ the equation could make sense as a stochastic one.
Fluct.

- The probabilistic distribution of $\langle \varphi^2 \rangle_\omega$ has been recently discussed by *[Fewster Ford Roman]*

Stochastic approach

- Einstein-Langevin equation *[Verdaguer]*

$$G_{ab}(x) = \langle T_{ab}(x) \rangle_\omega$$

- We interpret it as a **stochastic equation**.
- Study the **passive** influence of **matter** fluctuations on **curvature** fluctuations.
- It is not easy to compute the **probability distribution** for $\langle T_{ab} \rangle_\omega$.

$$\delta G_{ab} = G_{ab} - \langle G_{ab} \rangle \ , \qquad \delta T_{ab} = T_{ab} - \langle T_{ab} \rangle_\omega \qquad \delta G_{ab} = \delta T_{ab}$$

- The correlations of $T_{ab}(x)$ are **more complicated** than in *Wiener processes* or *Brownian motions*.
- We can equate their moments:

$$\langle G_{ab}(x) \rangle = \langle T_{ab}(x) \rangle_\omega$$
$$\langle \delta G_{ab}(x_1) \delta G_{cd}(x_2) \rangle = \langle \delta T_{ab}(x_1) \delta T_{cd}(x_2) \rangle_\omega$$
$$\ldots$$
$$\langle \delta G^n(x_1, \ldots, x_n) \rangle = \langle \delta T^n(x_1, \ldots, x_n) \rangle_\omega$$

Question

What is the impact of fluctuations?

- We shall analyze it in a simple case: Metric perturbations in the Starobinski model of inflation

- It is believed that quantum fluctuations seeds structure formation in the universe.
- *[Verdaguer]* obtains a **scale free spectrum** of the **metric fluctuations** (Bardeen potentials) considering a "linearized version" of ω_2^2 as a source.

Starobinski model of inflation

- Inflation is usually induced by a **classical** scalar field in a potential.

$$\mathcal{L} = \partial\Phi\partial\Phi + V(\Phi)$$

- The origin of this potential is unclear.
- It plays a crucial role in the standard analysis of fluctuations on the system.

- There is another model of inflation, which is more in the spirit of QFT on curved spacetime.
- This is the Starobinski model where inflation is driven by R^2 term in the effective action.
- We would like to derive the metric fluctuation for this system as induced by matter one.

Metric perturbation and fluctuation

- Usually linearized gauge invariant perturbations of the inflaton-gravity system are quantized.

- We cannot use this approach in the Starobinski model: We have already quantized the system and we have put ourself in a regime were the quantum nature of gravity is not necessary.
- There is however a way out, namely fluctuations of matter are naturally present in the stress tensor.
- We could evaluate their influence on the right hand side $\langle G \rangle$.

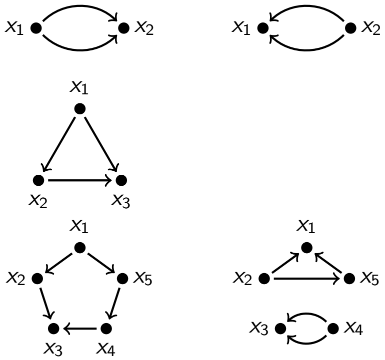
Graphical representation for $\delta\varphi^2$ in a Gaussian state

$$\langle \delta G(x_1)\delta G(x_2)\rangle = m^4\,\omega(\delta\varphi^2(x_1)\delta\varphi^2(x_2)) = \frac{m^4}{2}\omega_2^2(x_1,x_2) + \frac{m^4}{2}\omega_2^2(x_2,x_1)$$

$n = 2$:

$n = 3$:

$n = 5$:



$\dots$

$\dots$

All these graphs are well defined distributions:

$$\omega_2^2(x,y) \qquad \omega_2(x,y)\omega_2(y,z) \qquad \dots .$$

Comments

- Semiclassical Einstein equations **link** quantum matter fluctuations with curvature fluctuations
- δG is not Gaussian (3-point function do not vanish ...)

Perturbations around an inflationary spacetime

- We start with a de Sitter spacetime

$$\overline{g} = \frac{1}{(H\tau)^2}\left(-d\tau^2 + d\vec{x}^2\right)$$

- Let's add **Newtonian perturbations** $\overline{g} \to g = \overline{g} + \epsilon\tilde{g}$:

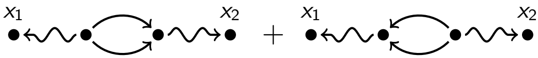
$$g = \frac{1}{(H\tau)^2}\left(-(1+2\Psi)d\tau^2 + (1-2\Psi)d\vec{x}^2\right)$$

- Linear perturbation of **scalar curvature**

$$\delta G = \overline{g}^{ab}\left(G_{ab} - \langle G_{ab}\rangle\right) = 6(H\tau)^4\left(\frac{\partial^2}{\partial\tau^2} - \frac{1}{3}\vec{\nabla}^2\right)\frac{\Psi}{(H\tau)^2}$$

- Inverting (with retarded propagator) we get

$$\langle\Psi(x_1)\Psi(x_2)\rangle = m^4 \left(\quad + \quad \right)$$



Power spectrum of Ψ

Is obtained computing the spatial Fourier transform of $\langle \Psi(x_1)\Psi(x_2)\rangle$

where the state is

$$\omega_2(x_1,x_2) = \frac{U}{\sigma_\epsilon} + \text{less singular term} = H^2\tau_1\tau_2\,\omega_{\mathbb{M}}(x_1,x_2) + \text{less singular term}$$

and the square of the two-point function

$$\widehat{\omega_2^2}(\tau_1,\tau_2,\vec{k}) = \frac{1}{16\pi^2}\int_k^\infty e^{-ip(\tau_1-\tau_2)}dp$$

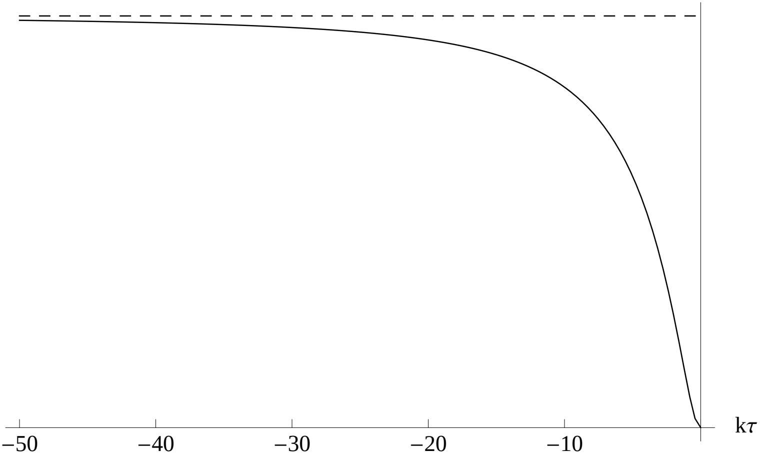
We can consider its contribution to the power spectrum

$$\langle \widehat{\Psi}(\tau,k)\widehat{\Psi}(\tau,k)\rangle \approx \frac{1}{k^3}\mathcal{P}_0(k\tau)$$

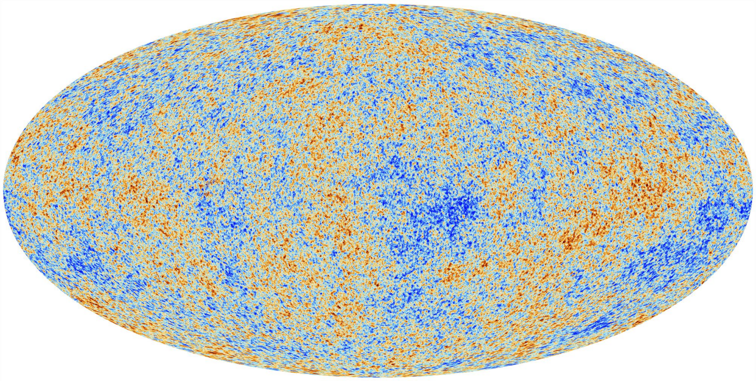
Non gaussianities arise naturally

$$\langle \widehat{\Psi}(\tau,k_1)\widehat{\Psi}(\tau,k_2)\widehat{\Psi}(\tau,k_3)\rangle \approx \frac{1}{k_1^2k_2^2k_3^2}\mathcal{B}_0(k_1\tau,k_2\tau,k_3\tau)$$

The rescaled power spectrum $\mathcal{P}_0(k\tau)$



CMB Temperature fluctuations



- CMB anisotropies observed by the Planck space telescope.
- Produced at the time of matter/radiation decoupling.
- Usually explained by inflation.

CMB Temperature fluctuations

$$\Theta(\tau,\vec{x},\vec{e}) = \frac{\delta T(\tau,\vec{x},\vec{e})}{T(\tau)} = \sum_{\ell,m} \Theta_{\ell m}(\tau,\vec{x})\,Y_{\ell m}(\vec{e})$$

$\Theta_{\ell m}(\tau_0,\vec{x}_0)$ are **statistically homogeneous random variables** with correlations

$$\langle \Theta_{\ell m}(\tau_0,\vec{x}_0)\Theta_{\ell' m'}(\tau_0,\vec{x}_0)^* \rangle = \delta_{\ell\ell'}\delta_{mm'}C_\ell$$

where, in terms of **Newtonian perturbations** Ψ,

$$C_\ell = 4\pi \int_0^\infty T_\ell(k)^2 \, \langle \widehat{\Psi}(\tau_1,k)\widehat{\Psi}(\tau_1,k)\rangle \, k^2 \, dk$$

In order to be coherent with observations, for small k it should be

$$\langle \widehat{\Psi}(\tau_1,k)\widehat{\Psi}(\tau_1,k)\rangle \approx C\left(\frac{k_0}{k}\right)^{3-\epsilon}$$

we shall compare this with results obtained in an inflationary model.

Summary

- Semiclassical Backreaction can be analyzed as a well posed problem in cosmology.
- Beyond mean field approximation it can be understood as a stochastic equation.

Thanks a lot for your attention!

Deformation

- **Deformation:** if we use $-2i\mathcal{H}$ in the place of Δ to construct a star product $\star_{\mathcal{H}}$.
- It is realized by
$$\alpha_{\mathcal{H}} : (\mathcal{A}, \star) \to (\mathcal{A}, \star_{\mathcal{H}})$$
such that
$$F \star_{\mathcal{H}} G = \alpha_{\mathcal{H}}(\alpha_{\mathcal{H}}^{-1}(F) \star \alpha_{\mathcal{H}}^{-1}(G))$$
- $\alpha_{\mathcal{H}}$ is an isomorphism.
- We can push forward states on $(\mathcal{A}, \star)$ to $(\mathcal{A}, \star_{\mathcal{H}})$. For ω_2
$$: \omega_2 : := \alpha_{\mathcal{H}}^{-1^*}(\omega_2) = (\omega_2 - \mathcal{H})_s$$
- The scheme incorporates the point splitting regularization

Extended algebra

We can now safely extend $(\mathcal{A}, \star_{\mathcal{H}})$ to local functionals.

- $\mathcal{A}_e$: a finite number of non vanishing func. der.
- $F \in \mathcal{A}_e$. $F^{(n)}(0)$: compact supp., symmetric, $\mathrm{WF}(F^{(n)}(0)) \cap \overline{V_+} \cup \overline{V_-} = \emptyset$.

$$(F \star_{\mathcal{H}} G)(\varphi) := \sum_n \frac{1}{n!} \langle F^{(n)}(\varphi), \mathcal{H}^{\otimes n} G^{(n)}(\varphi) \rangle$$

Hörmander criterion for multiplication of distributions holds.

- $(\mathcal{A}_e, \star_{\mathcal{H}})$ is the extended $*$-algebra of fields. *[Brunetti Dütsch Fredenhagen Hollands Wald]*
- They satisfy the axiom of a **Locally covariant theory**

Regularization freedom and ambiguity

- The product in $\mathcal{A}_e$ is constructed out of $\mathcal{H}$, there are **ambiguities** (**renormalization freedom**).
- The algebras for different $\mathcal{H}$ are isomorphic
- But the local fields are not invariant

$$\tilde{\varphi}^k(x) = \varphi^k(x) + \sum_{i=1}^{k-2} C_i(x)\varphi^i(x)$$

- $C_i(x)$: real polynomials of the metric. *[Hollands Wald]*
- With the correct scaling under rigid dilation

$$C_i \to \lambda^i C_i$$

Back.

Analysis of φ^2

How $\langle\varphi^2\rangle_{\omega_{1,0}}$ depends on H?

$$\langle\varphi^2(x)\rangle_{\omega_{1,0}} = \lim_{y\to x}\left[\omega_{1,0}(x,y) - \mathcal{H}(x,y)\right] + \alpha R + \beta m^2$$

Remind: Prescription for fixing the renormalization freedom:

- Minkowksi spacetime on Minkowksi vacuum, fixes β.
- α changes the value of H_c or $X_c \implies H_c$ **is a ren. constant**

We regularize on Minkowski spacetime the problem $-\Box_{\mathbb{M}} + (ma)^2$

$$\lim_{y\to x}\mathcal{H}(y,x) - \frac{1}{a(\tau_1)a(\tau_2)}\mathcal{H}_{\mathbb{M}}(y,x) = \frac{m^2}{8\pi^2}\log a + \alpha' R\ ,$$

Point splitting at fixed time, then it is enough to subtract

$$\mathcal{H}^0_{\mathbb{M}}(y,x) := \frac{1}{(4\pi)^2}\left(\frac{2}{\sigma_\epsilon} + m^2 a(\tau_x)^2\log\left(\frac{\sigma_\epsilon}{\lambda^2}\right)\right)$$

Other reg. scheme

Comparison with the first order adiabatic approximation

$$\mathcal{H}^0_{\mathbb{M}}(y,x) - \frac{1}{(2\pi)^3}\int \frac{e^{i\mathbf{k}(\mathbf{y}-\mathbf{x})}}{2\sqrt{\mathbf{k}^2+m^2a(\tau)^2}}\; d^3\mathbf{k}$$

is a continuous function

$$\langle\varphi^2\rangle_{\omega_{1,0}} :=$$

$$\frac{1}{2\pi^2a^2}\int_0^\infty k^2dk\left[\overline{\chi}_k\chi_k - \Theta(k-ma)\left(\frac{1}{2k}-\frac{m^2a^2}{4k^3}\right)\right] - \frac{m^2}{8\pi^2} + \alpha R,$$

Consider

$$\Phi(X) := 2\pi^2 a^2 \langle \varphi^2 \rangle_{\omega_{1,0}}$$

Proposition

On B_c

$$|\Phi(X)(t)| \leq C_1(a_0, t_0, c)t \ ,$$

$$|D\Phi(X, \delta)| \leq C_2(a_0, t_0, c)t_0\|\delta\| \ ,$$

where at fixed λ, $C_1(\lambda t_0, t_0, c)$ and $C_2(\lambda t_0, t_0, c)$ are bounded for $t_0 \to 0$.

Construction of the χ

$$\chi_k'' + (k^2 + m^2 a_0^2)\chi + m^2(a^2 - a_0^2)\chi = 0$$

Perturbative const. over the massless solution $\chi_k^0(\tau) = \frac{e^{-ik_0(\tau-\tau_0)}}{\sqrt{2k_0}}$

$$\chi_k = \sum_{n=0}^{\infty} \chi_k^n$$

$$\chi_k^n(\tau) = -\int_{\tau_0}^{\tau} \frac{\sin(k_0(\tau-\tau'))}{k_0} \left(a(\tau')^2 - a_0^2\right) m^2 \chi_k^{n-1}(\tau')\, d\tau' \;,$$

Proposition

The series converges absolutely on $[\tau_0, \tau]$, *and*

$$|\chi_k| \le \frac{1}{\sqrt{2k}} \exp\left(\frac{m^2 a^2 (\tau-\tau_0)}{k_0}\right) , \qquad |\chi_k| \le \frac{1}{\sqrt{2k}} \exp\left(m^2 a^2 (\tau-\tau_0)^2\right) .$$

Every χ_k^n is $O(m^{2n})$ Back.

Proof

$$\int_0^\infty k^2 dk \left[\overline{\chi}_k \chi_k - \frac{1}{\sqrt{k^2 + m^2 a^2}} \right],$$

- Small k do not create problems
- Expand χ_k in powers of m^2
- The large powers decay in k sufficiently rapidly
- $O(m^0)$ vanishes
- $O(m^2)$ and $O(m^4)$ can be directly analyzed

Back.

Proposition

If t_0 is sufficiently small, the map $\mathcal{T}$ is a contraction in B_c.

Proof

$$\|\mathcal{T}(y) - \mathcal{T}(x)\| \leq C\|y - x\| \qquad C < 1$$

$$x_\lambda = x + \lambda\delta \text{ with } \delta = (y - x)$$

$$\left|\frac{d\mathcal{T}}{dt}(y) - \frac{d\mathcal{T}}{dt}(x)\right| \leq \left|\int_0^1 \frac{d}{d\lambda}\frac{d\mathcal{T}}{dt}(x_\lambda)\, d\lambda\right| \leq \sup_{\lambda\in[0,1]}\left|D\frac{d\mathcal{T}}{dt}(x_\lambda, \delta)\right|$$

Theorem

$X_0 := H_0^{-1}$ the massless solution then the sequence

$$X_n := \mathcal{T}\left(X_{n-1}\right)$$

converges in B_c to the solution $X = H^{-1}$ of the semiclasscal Einstein equation.

Some comments

- The found solution can be shown to be C^1 only
- **But** there is always a smooth spacetime as close as you want to the solution
- The existence does not depend on the state, in the sense that other $\omega_{A,B}$ with B rapidly decreasing produce a solution.
- All these solutions show a phase typical of power law inflation which is then state independent.
- The found solution does not depend on a_0
- It can be be fixed a posteriori by $G_{00} = 8\pi\langle T_{00}\rangle$

Back.

Analysis of the fluctuations

The solution is meaningful provided the variance of ${T_\mu}^\mu$ is small

- The anomaly is a $C-$number

- The variance of $\langle \varphi^2 \rangle$

$$\Delta_\omega(\varphi^2) := \omega(\varphi^2 \star_H \varphi^2) - \omega(\varphi^2)\omega(\varphi^2)$$

  diverges: it is proportional to $\omega_2 \cdot \omega_2(x,x)$

When smeared the situation is better, consider the family centered in x_τ

$$f_{n_1,n_2}(\tau', \mathbf{x}) = \frac{n_1}{n_2^3} f\left(n_1(\tau' - \tau) + \tau, \frac{\mathbf{x}}{n_2} \right)$$

where

$$f(x_\tau) = 1\,, \qquad \int_M f \; d\mu(g) = 1\,, \qquad f \geq 0$$

We study the limit

$$\lim_{n_1\to\infty}\lim_{n_2\to\infty}\left[R(f_{n_1,n_2})+8\pi\langle T\rangle_\omega(f_{n_1,n_2})\right]=R(x_\tau)+8\pi\langle T\rangle_\omega(x_\tau)$$

Theorem

We have

$$\lim_{n_2\to\infty}\Delta_{\omega_{1,0}}(\varphi^2(f_{n_1,n_2}))=0\ .$$

- In a **weaker** sense, the solution we have found is meaningful also when H is very large.

back