Topological effects in de Sitter QFT

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"The Mysterious Universe: Dark Matter - Dark Energy - Cosmic Magnetic Fields"

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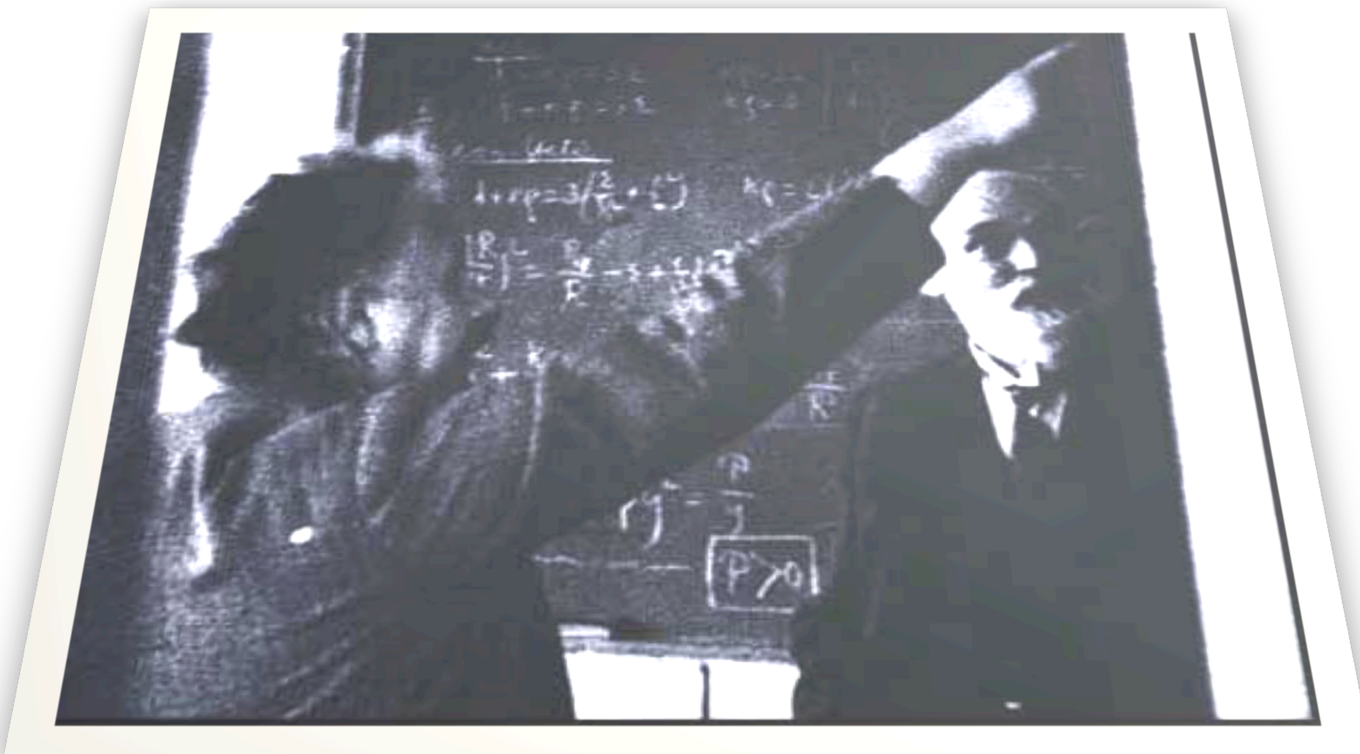
Plan of the talk

- A de Sitter survey
- Low mass bound states
- Surprising results about two-dimensional models

1917: The birth of cosmology as a science

1916-1919: the famous debate between Einstein, De Sitter, Weyl and Klein over the relativity of inertia.



A valuable paper

CHRISTIE'S

EINSTEIN, Albert.
Kosmologische Betrachtungen
zur allgemeinen Relativitätstheorie.
[Offprint from:]
Sitzungsberichte der Königlich
preussischen Akademie der Wissenschaften,
VI. Berlin, 1917.

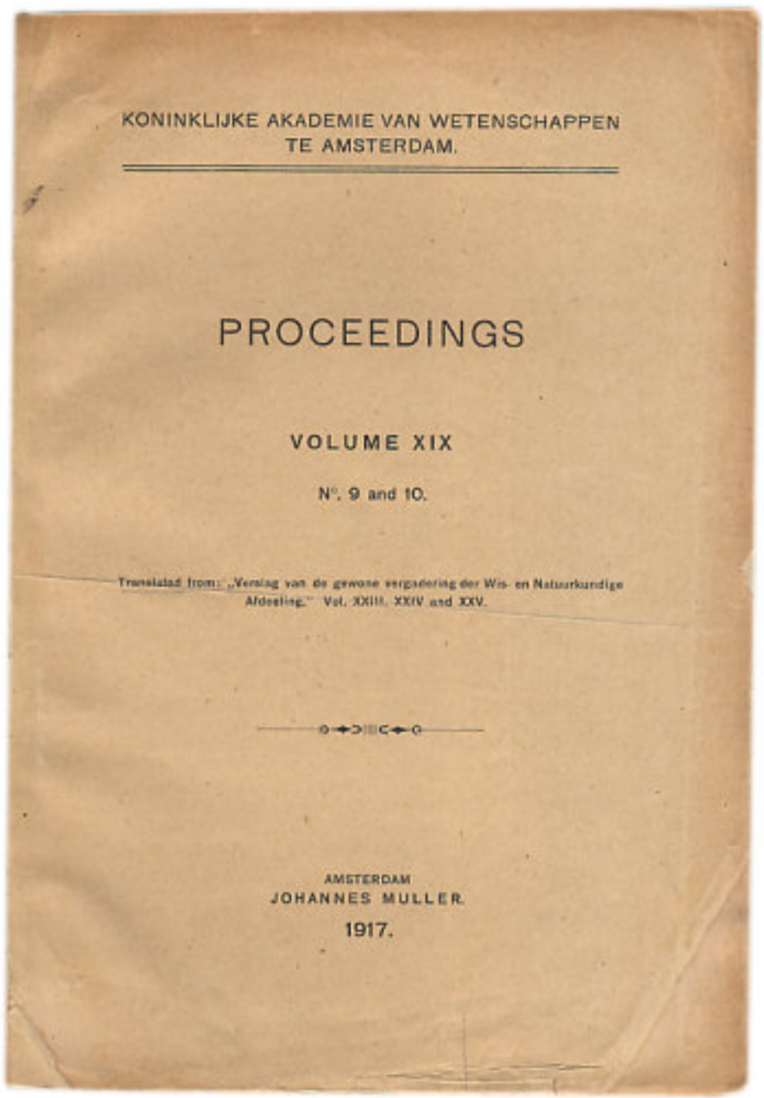
Lot 52 / Sale 1677 / 14 June 2006
New York, Rockefeller Plaza
Estimate
$1,000 - $1,500
Price Realized **$5,040**

The first modification of GR

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \qquad (1915)$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \qquad (1917)$$

A second valuable paper



The Manahattan Rare Book Company
1050 Second Ave, Gallery 50E
New York, NY 10022

DE SITTER, Willem.

On the relativity of inertia.

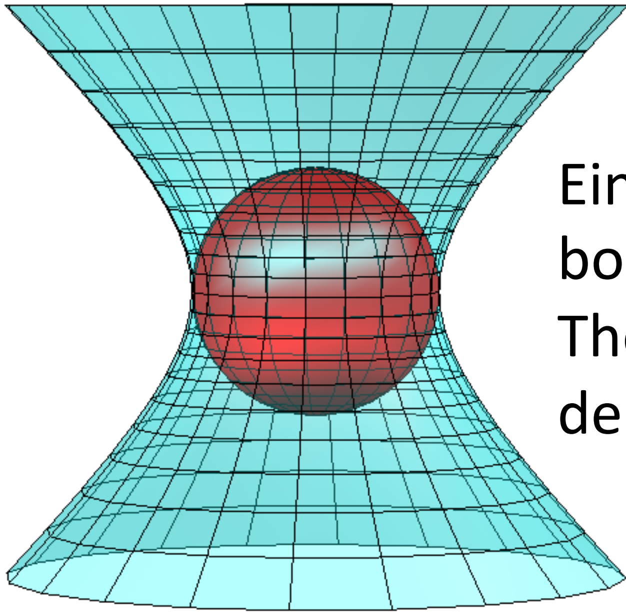
Remarks concerning Einstein's latest hypothesis. In Proceedings Koninklijke Akademie Van Wetenschappen Te Amsterdam, vol 19, no. 9 and 10, pp. 1217-1225.

First editions in English (translated from the German-language issue of the same journal) of two of cosmology's most important papers: Willem de Sitter's solution to Einstein's field equations, later to become known as the "De Sitter universe", providing a mathematical basis for an expanding universe.

Two complete issues. Octavo, original wrappers neatly rebacked.

$4500.

de Sitter's analogy



Einstein's static model does not solve the boundary condition problem in an invariant way. There is a sort of 'absolute space' remaining in it. de Sitter's alternative proposal:

$$(iX_0)^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = R^2$$

$$ds^2 = dX_0^2 - dX_1^2 - dX_2^2 - dX_3^2 - dX_4^2\Big|_{dS_4} =$$

$$= \sum_{\mu,\nu=0}^{3} \left(\eta_{\mu\nu} - \frac{X_\mu X_\nu}{R^2 - X_1^2 - X_2^2 - X_3^2 + X_0^2} \right) dX^\mu dX^\nu$$

Einstein was not happy!

- The de Sitter metric **is a vacuum solution of the new equations**; it is a counterexample to Einstein's requirements
- It is anti-Machian (but more 'relativistic' than Einstein's own solution)
- Einstein's strategy: try to kill it (*it is wrong, it is singular, it is not empty*)
- Einstein criticisms were all wrong but in the end he dismisses the de Sitter metric as unphysical because non globally static.

Lemaître's prophecy

"The history of science provides many instances of discoveries which have been made for reasons which are no longer considered satisfactory.

It may be that the discovery of the cosmological constant is such a case."

George E. Lemaître, article in the book "Albert Einstein: Philosopher–Scientist", 1949

Year: 1997

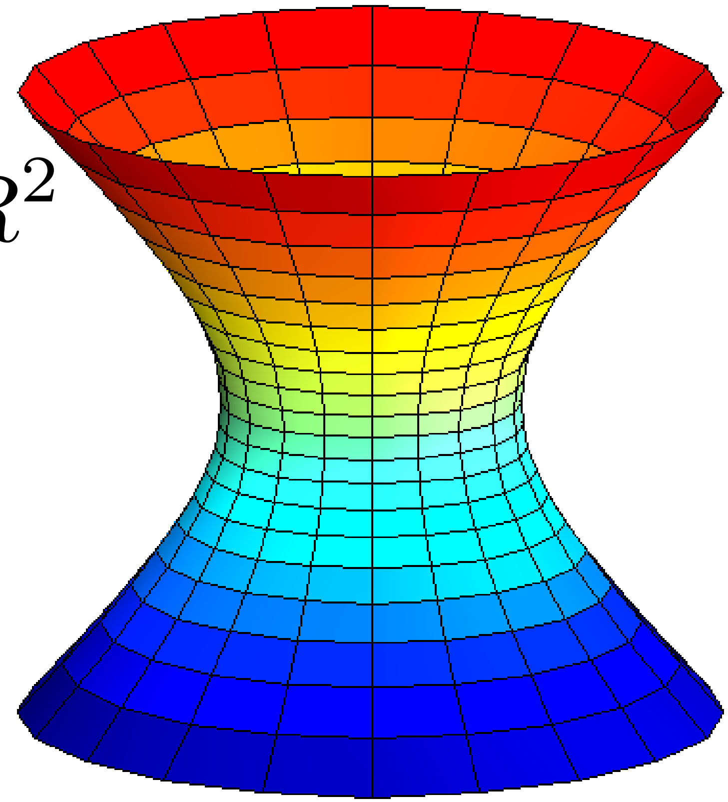
- A. G. Riess et al., “Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant”, Astronomical Journal 116, 1009 (1998).
- S. Perlmutter et al. “Discovery of a supernova explosion at half the age of the universe and its cosmological implications”, *Nature*, **391**, 51 (1998)

The de Sitter world

$$X_0^2 - X_1^2 - \ldots - X_d^2 = -R^2$$

$M^{(d+1)}$: $\eta_{\mu\nu}$ = diag(**1**,**-1**,……,**-1**)

$$SO_0(1,d)$$

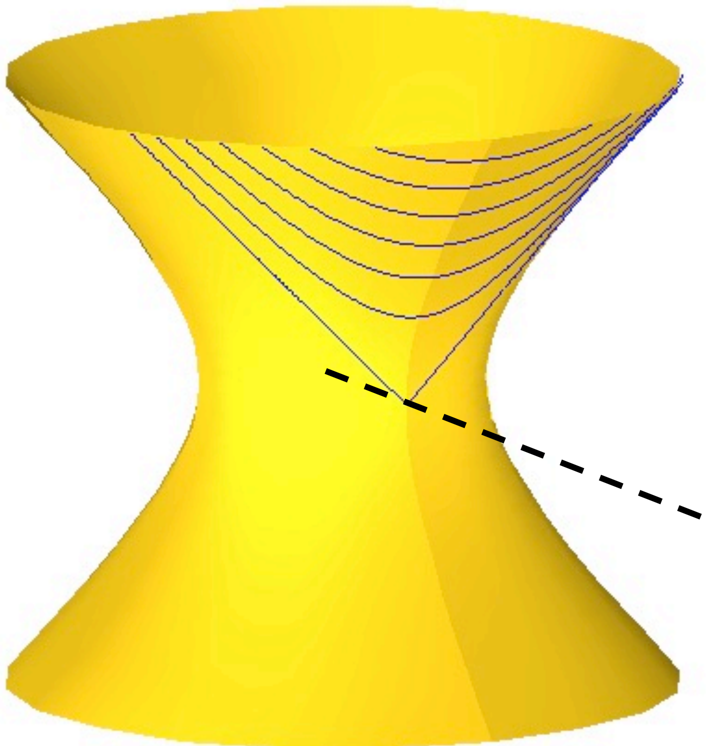


$$Complexification$$

$$Z_0^2 - Z_1^2 - \ldots - Z_d^2 = -R^2$$

$$SO_0^{(c)}(1,d)$$

Hyperbolic model (de Sitter 1917)

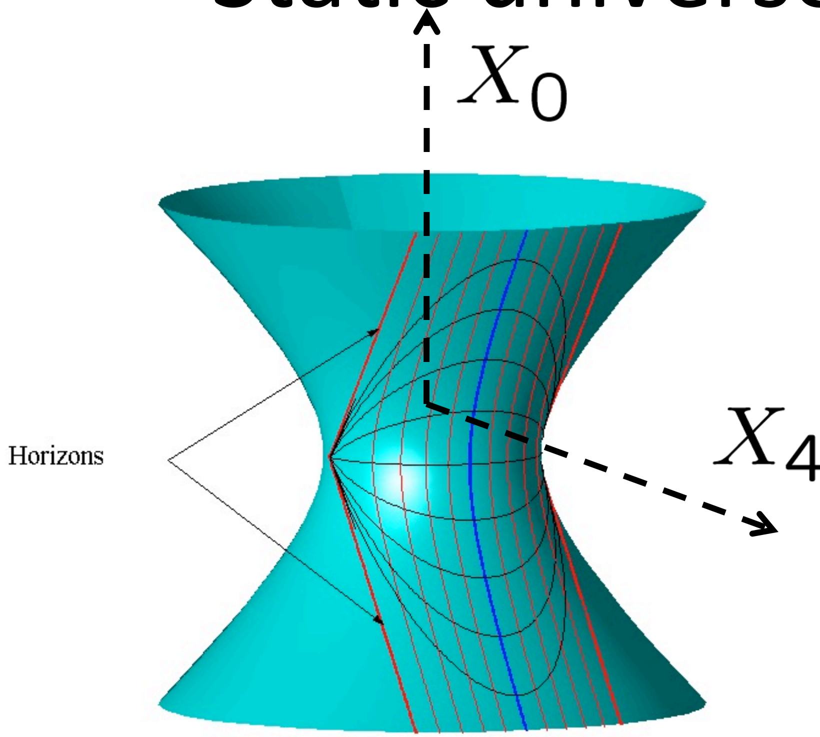


$$\begin{cases} X_0 &= R\sinh\frac{t}{R}\cosh\chi \\ X_1 &= R\sinh\frac{t}{R}\sinh\chi\sin\theta\sin\phi \\ X_2 &= R\sinh\frac{t}{R}\sinh\chi\sin\theta\cos\phi \\ X_3 &= R\sinh\frac{t}{R}\sinh\chi\cos\theta \\ X_4 &= R\cosh\frac{t}{R} \end{cases}$$

$$ds^2 = dX_0^2 - dX_1^2 - \ldots dX_4^2\Big|_{dS} =$$

$$= dt^2 - R^2\sinh^2\frac{t}{R}\left(d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right)$$

Querelle de“rest”
Static universe (de Sitter 1917)

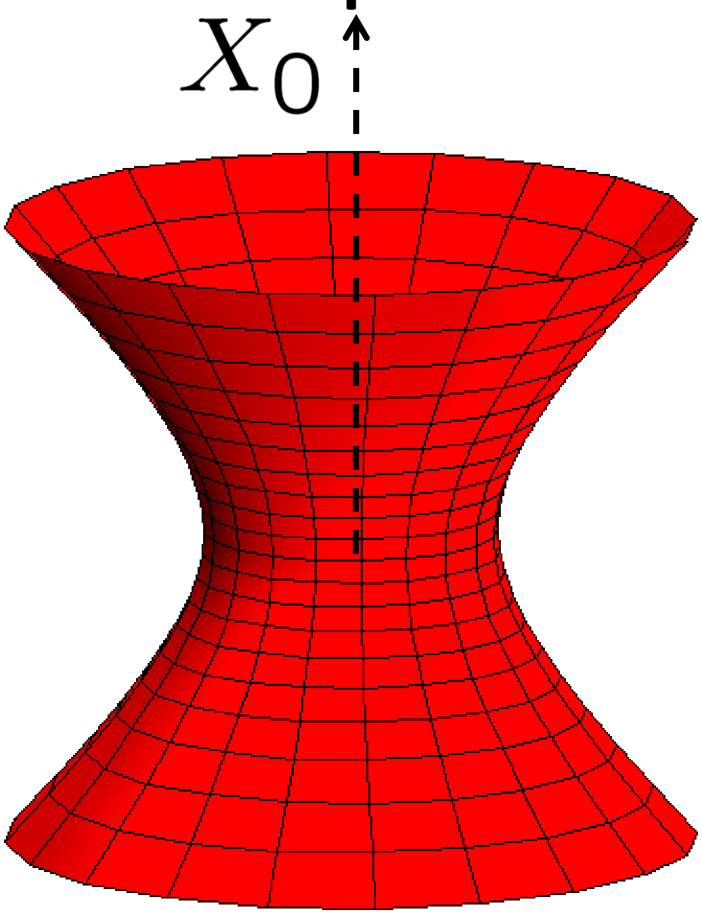


$$\begin{cases} X_0 &= \sqrt{R^2 - r^2}\,\sinh(t/R) \\ X_1 &= r\sin\theta\sin\phi \\ X_2 &= r\sin\theta\cos\phi \\ X_3 &= r\cos\theta \\ X_4 &= \sqrt{R^2 - \vec{r}^2}\,\cosh(t/R) \end{cases}$$

$$ds^2 = dX_0^2 - dX_1^2 - dX_2^2 - dX_3^2 - dX_4^2\Big|_{dS_4} =$$

$$= (1 - \frac{r^2}{R^2})dt^2 - \frac{1}{1-\frac{r^2}{R^2}}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Spherical model (Lanczos)



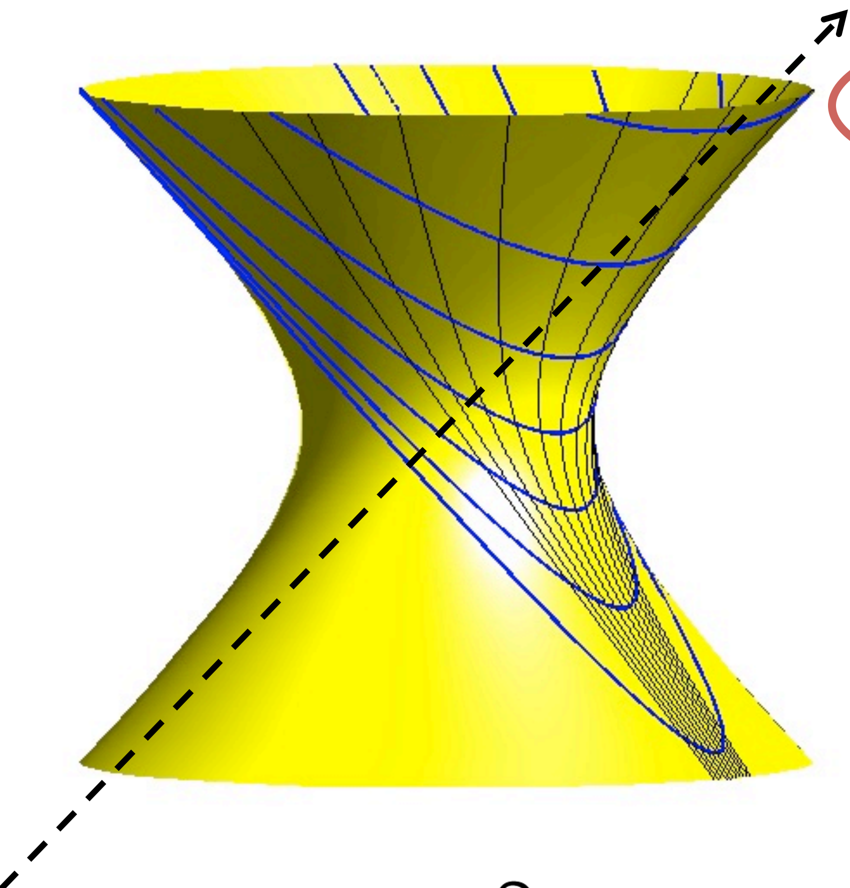
$$\begin{cases} X_0 &= R\sinh(t/R) \\ X_1 &= R\cosh(t/R)\sin\theta\sin\chi\sin\phi \\ X_2 &= R\cosh(t/R)\sin\theta\sin\chi\cos\phi \\ X_3 &= R\cosh(t/R)\sin\theta\cos\chi \\ X_4 &= R\cosh(t/R)\cos\theta \end{cases}$$

$$R = \sqrt{\frac{3}{\Lambda}}$$

$$ds^2 = dX_0^2 - dX_1^2 - \ldots dX_4^2\Big|_{dS} =$$

$$= dt^2 - R^2\cosh^2\frac{t}{R}\left(d\theta^2 + \sin^2\theta(d\chi^2 + \sin^2\chi d\phi^2)\right)$$

Flat model (Lemaître, 1924)



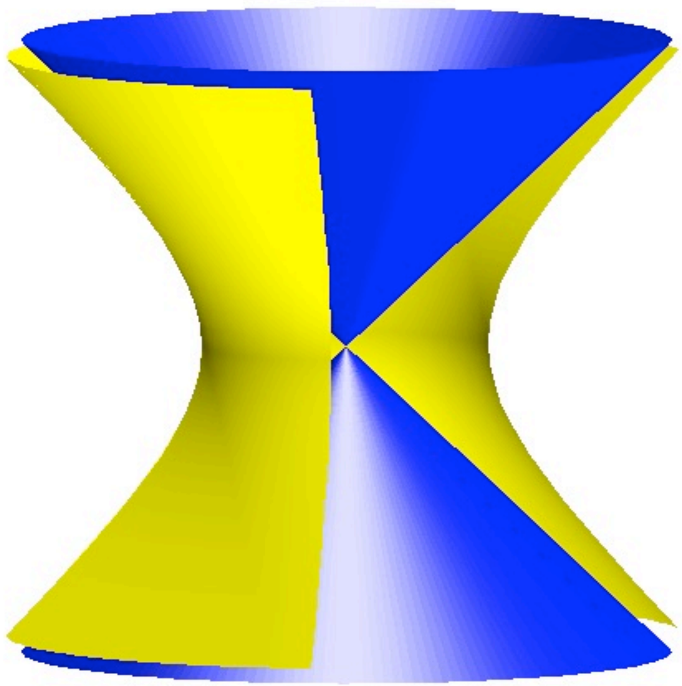
$$X_0 + X_4 = R \exp \frac{t}{R}$$

$$\begin{cases} X_0 &= R \sinh \frac{t}{R} + \frac{1}{2R} e^{\frac{t}{R}} |\vec{x}|^2 \\ X_1 &= \exp\left(\frac{t}{R}\right) x_1 \\ X_2 &= \exp\left(\frac{t}{R}\right) x_2 \\ X_3 &= \exp\left(\frac{t}{R}\right) x_3 \\ X_4 &= R \cosh \frac{t}{R} - \frac{1}{2R} e^{\frac{t}{R}} |\vec{x}|^2 \end{cases}$$

$$ds^2 = dX_0^2 - dX_1^2 - \ldots dX_4^2 \Big|_{dS} =$$
$$= dt^2 - \exp \frac{2t}{R} \left(dx_1^2 + dx_2^2 + dx_3^2 \right)$$

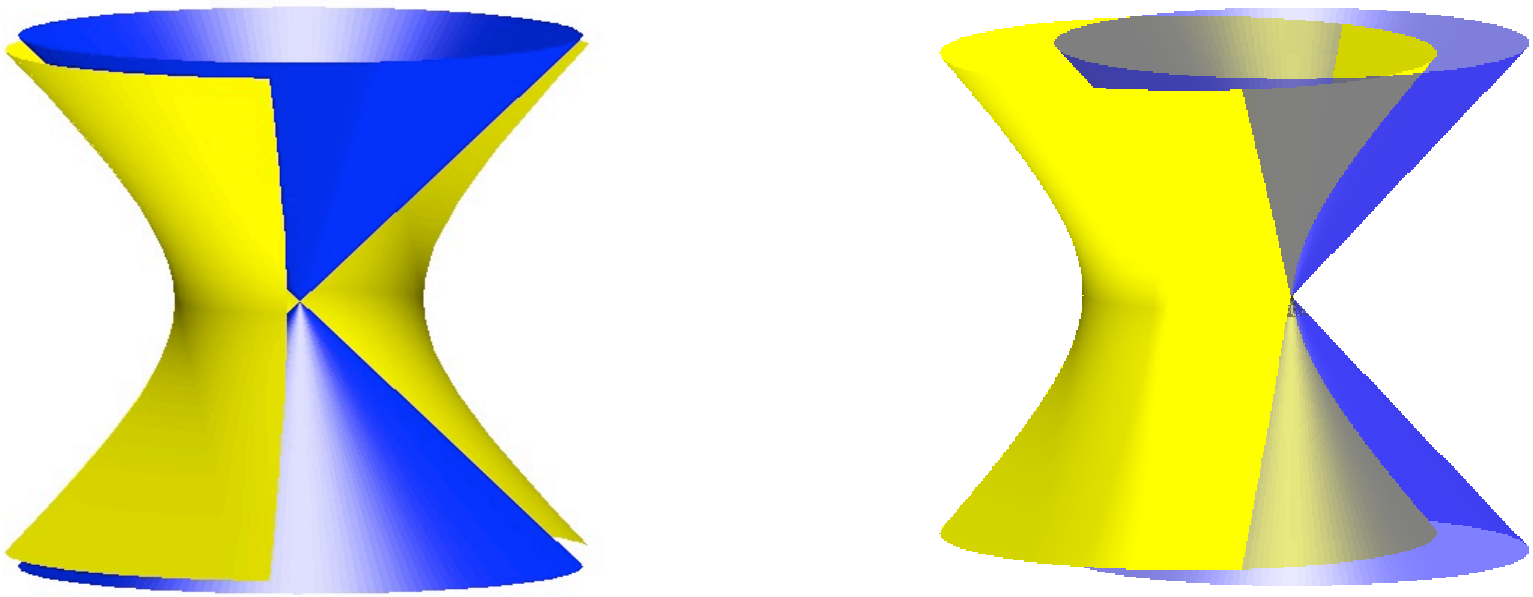
Asymtototic cone

$$\{\xi_0^2 - \xi_1^2 - \ldots - \xi_d^2 = 0\}$$



$M^{(d+1)}$: $\eta_{\mu\nu}$ = diag(**1,-1,……,-1**)

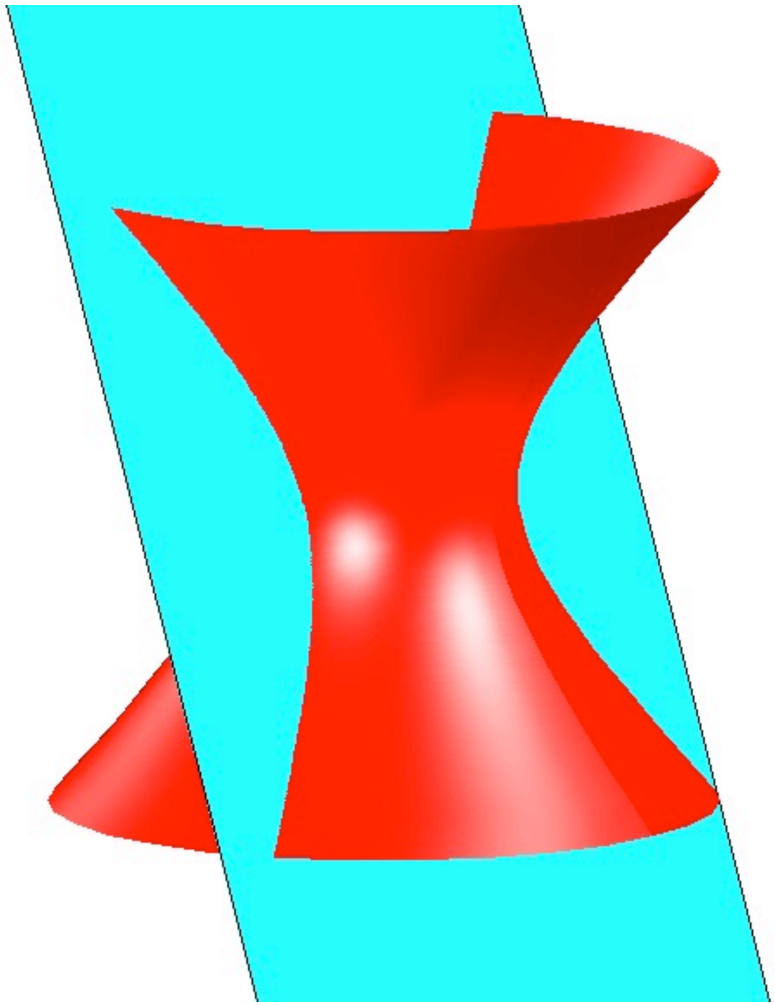
Causal structure



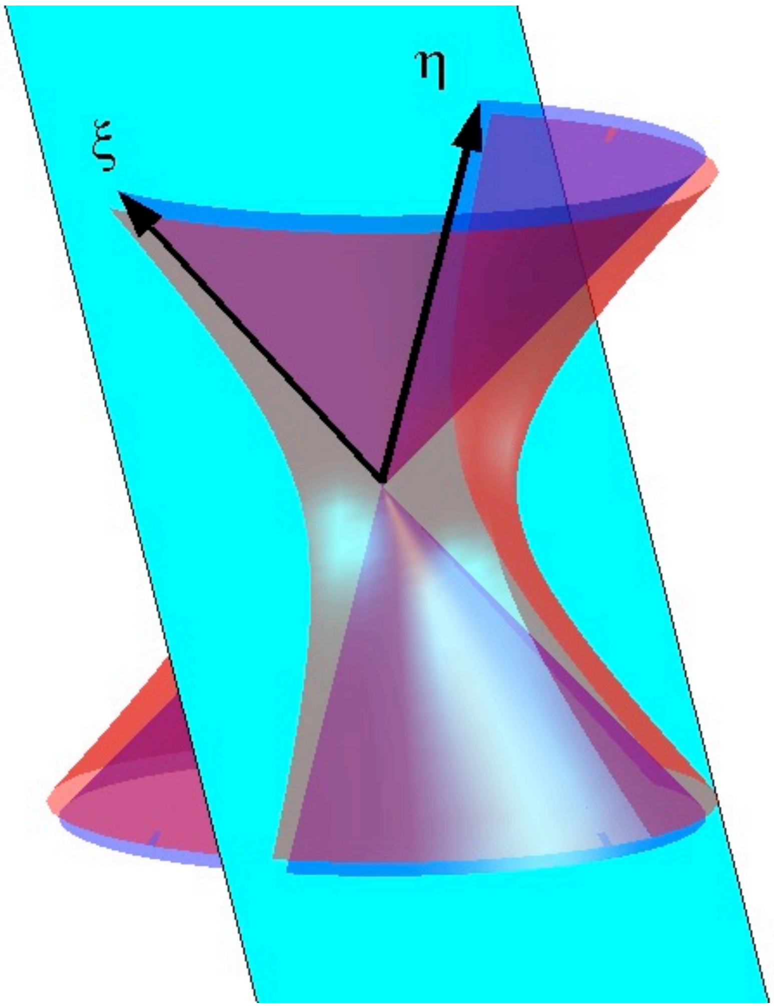
X, Y are spacelike separated iff $(X-Y)^2 < 0$ ($X - Y$ is outside the cone)

$$(X-Y)^2 = X^2 + Y^2 - 2X \cdot Y = -2R^2 - 2X \cdot Y$$

Geodesics



Geodesics



Geodesics: de Sitter

$$X_\mu(\tau) = \frac{R}{\sqrt{2\xi \cdot \eta}}\left(\xi_\mu e^{\frac{c\tau}{R}} - \eta_\mu e^{-\frac{c\tau}{R}}\right)$$

Minkowski

$$x_\mu(\tau) = x_\mu(0) + \frac{p_\mu \tau}{mc}$$

$$X_\mu(0) = \frac{R}{\sqrt{2\xi\cdot\eta}}(\xi_\mu - \eta_\mu)$$

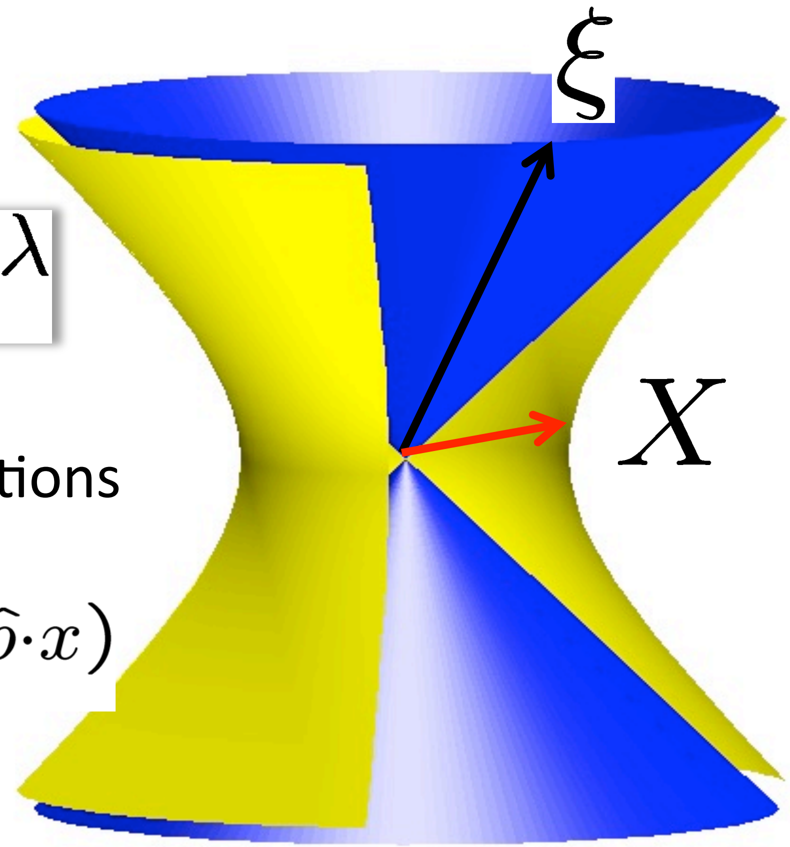
$$X(\tau) = X(0)e^{-\frac{c\tau}{R}} + \frac{kR\xi}{m}\sinh\frac{c\tau}{R}.$$

de Sitter plane waves

$$X \cdot \xi = X_0\xi_0 - X_1\xi_1 - \ldots - X_d\xi_d$$

$$\lambda \in \mathbf{C}, \quad \xi^2 = 0$$

$$\boxed{\psi_\lambda(X, \xi) = (X \cdot \xi)^\lambda}$$

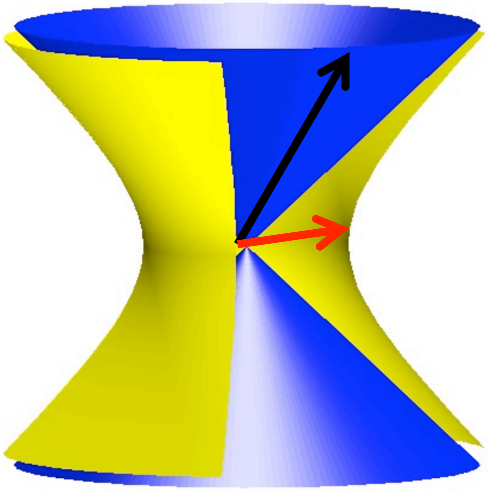


Plane waves are homogeneous functions

$$\psi(x, p) = e^{ip\cdot x} = e^{im(\widehat{p}\cdot x)}$$

de Sitter plane waves

$$\Box (X \cdot \xi)^{\lambda} = \lambda(\lambda + d - 1)(X \cdot \xi)^{\lambda}$$



Involution:

$$\lambda \longrightarrow \bar{\lambda} = -\lambda - (d - 1)$$

$$\lambda + \bar{\lambda} = -(d - 1)$$

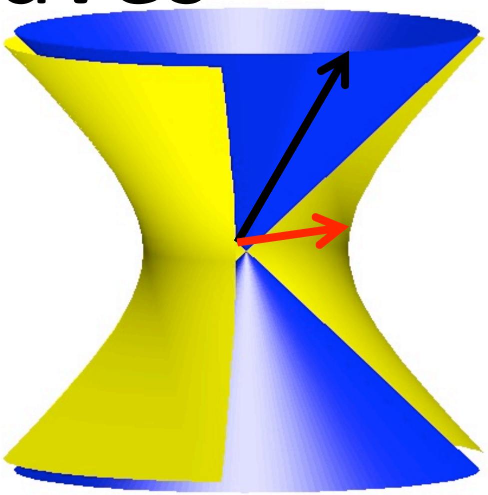
$$\Box (X \cdot \xi)^{-\lambda - d + 1} = (-\lambda - d + 1)(-\lambda)(X \cdot \xi)^{-\lambda - d + 1}$$

Scalar waves with (complex) squared mass:

$$m^2 = \lambda \bar{\lambda}$$

$$\left(\Box + \lambda\bar{\lambda}\right)(X \cdot \xi)^{\lambda} = 0, \qquad \left(\Box + \lambda\bar{\lambda}\right)(X \cdot \xi)^{\bar{\lambda}} = 0$$

Principal de Sitter waves



$$\lambda = -\tfrac{d-1}{2} + i\nu\,, \qquad \nu \in \mathbf{R}$$

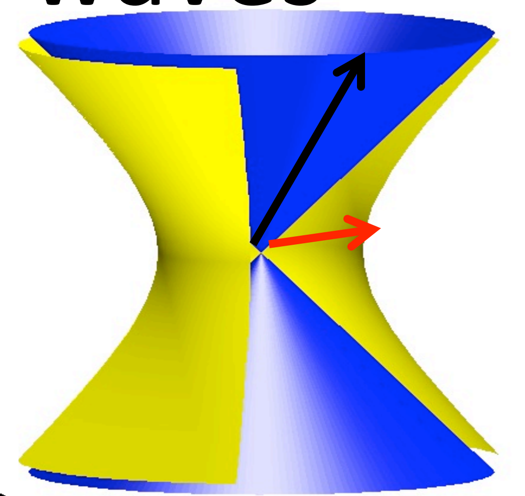
$$\psi_\lambda(X,\xi) = (X\cdot\xi)^{-\frac{d-1}{2}+i\nu}$$

$$\bar\lambda = -\lambda - (d-1) = -\tfrac{d-1}{2} - i\nu$$

$$\psi_{\bar\lambda}(X,\xi) = (X\cdot\xi)^{-\frac{d-1}{2}-i\nu} = \overline{\psi_\lambda(X,\xi)}$$

$$m^2 = \lambda\bar\lambda = |\lambda|^2 = \left(\tfrac{d-1}{2}\right)^2 + \nu^2 > \left(\tfrac{d-1}{2}\right)^2$$

Complementary de Sitter waves



$$\lambda = -\frac{d-1}{2} + \nu \,, \qquad \nu \in \mathbf{R}$$

$$\psi_\lambda(X,\xi) = (X \cdot \xi)^{-\frac{d-1}{2}+\nu}$$

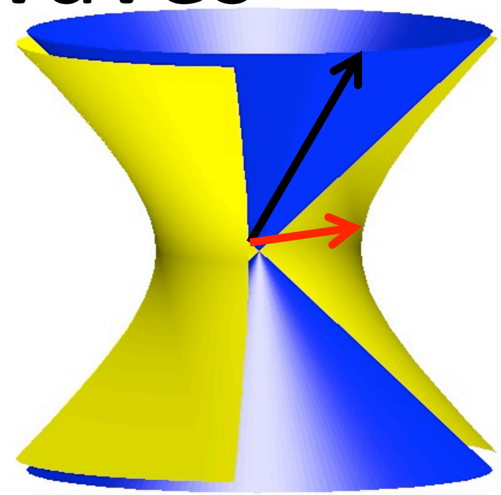
$$m^2 = \lambda\bar{\lambda} = \left(\frac{d-1}{2}\right)^2 - \nu^2 > 0$$

These waves do not oscillate!

$$\bar{\lambda} = -\lambda - (d-1) = -\frac{d-1}{2} - \nu$$

$$\psi_{\bar{\lambda}}(X,\xi) = (X \cdot \xi)^{-\frac{d-1}{2}-\nu} \neq \overline{\psi_\lambda(X,\xi)} = \psi_\lambda(X,\xi)$$

Tachyonic de Sitter waves

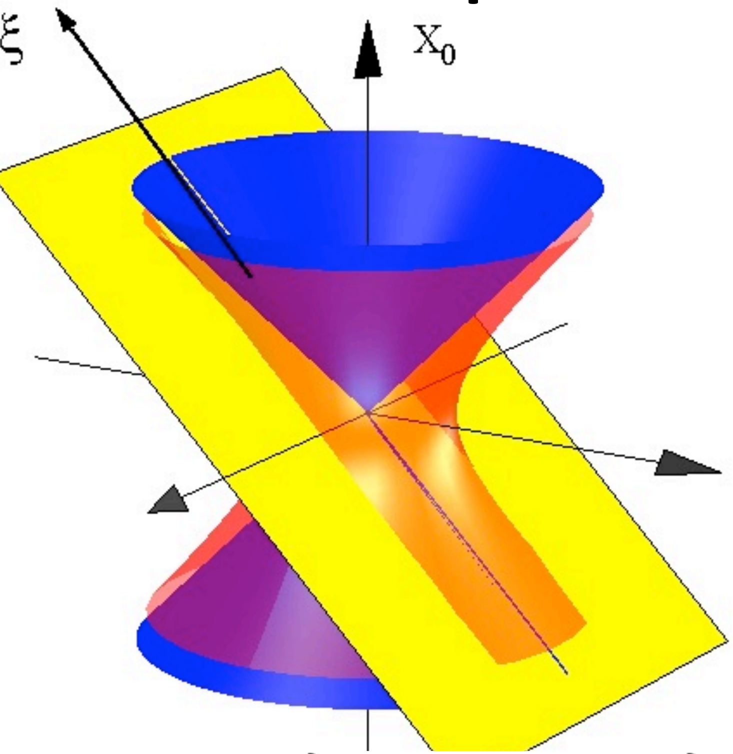


$$\lambda \geq 0 \qquad \lambda \leq -(d-1)$$

$$\psi_\lambda(X,\xi) = (X \cdot \xi)^\lambda$$

have real negative squared mass
$m^2 = \lambda\bar{\lambda} \leq 0$

The plane waves are irregular



$$\psi_\lambda(X,\xi) = (X\cdot\xi)^\lambda$$

$$X \in dS : (X\cdot\xi) = 0$$

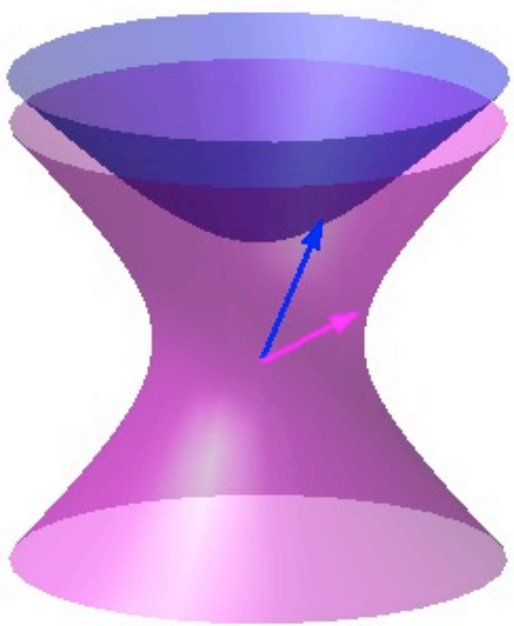
$$(X\cdot\xi)^\lambda \to |X\cdot\xi|^\lambda\,(a(\lambda)\theta(X\cdot\xi) + b(\lambda)\theta(-X\cdot\xi))$$

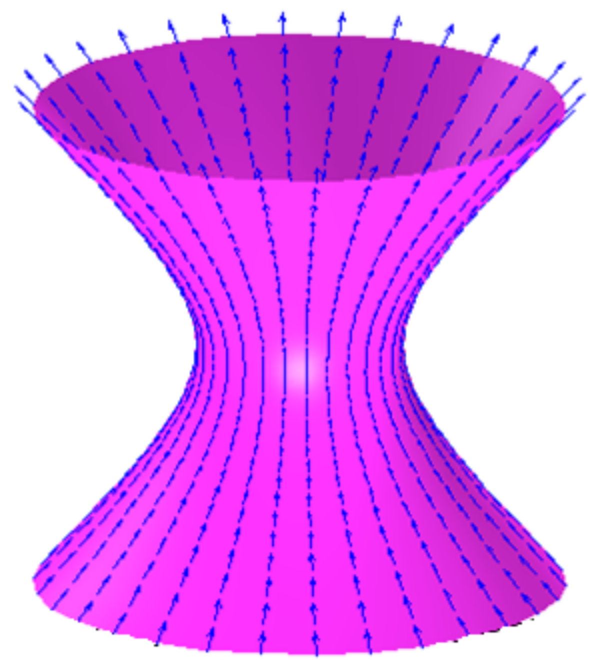
Geometry: de Sitter tubes

$Z = X + iY, \quad X^2 - Y^2 = -R^2 \quad X \cdot Y = 0$

$\mathcal{T}^+ = Y$ in the forward cone.

$\mathcal{T}^- = Y$ in the backward cone.

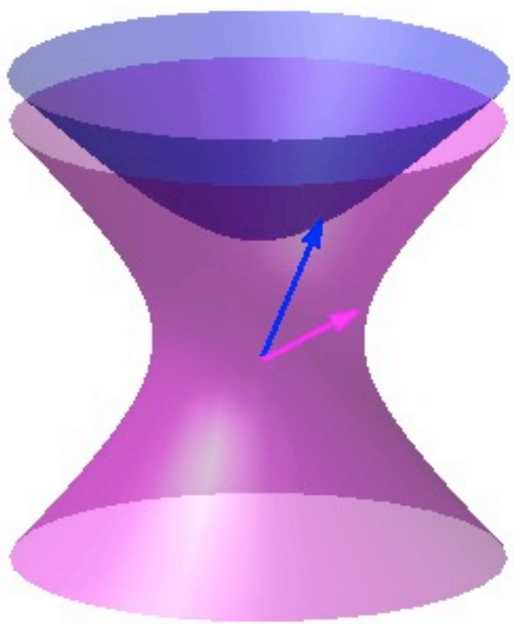


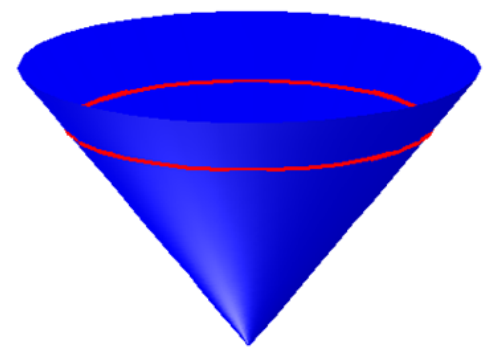


$$\boxed{\psi_\lambda^{\pm}(Z,\xi) = (Z\cdot\xi)^{\lambda}}$$

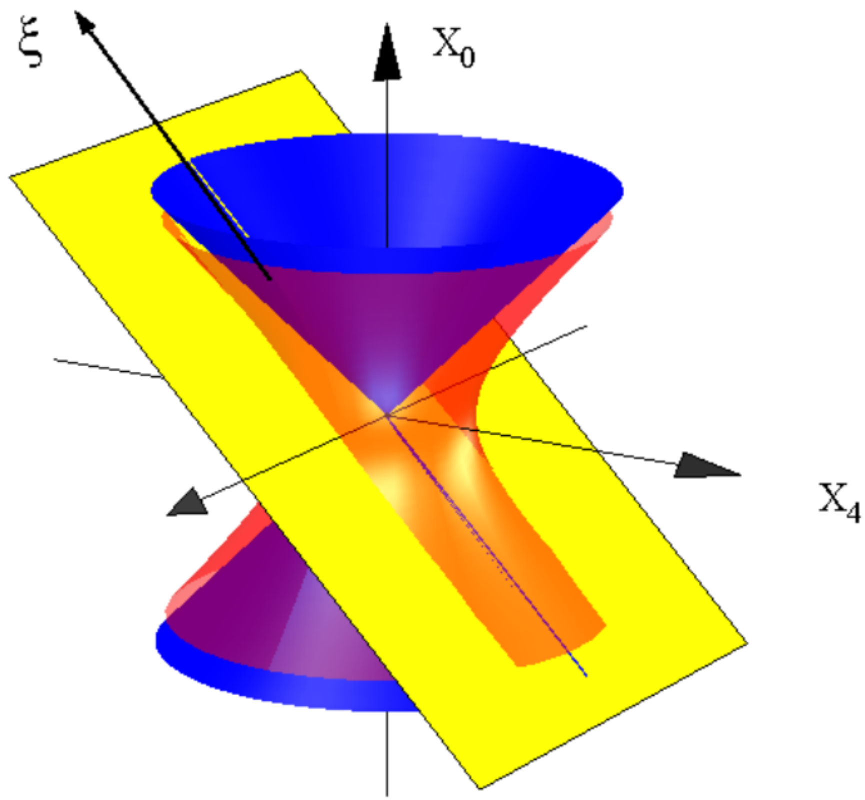
$\Im Z\cdot\xi$ is positive for $Z\in\mathcal{T}^{+}$

$\Im Z\cdot\xi$ is negative for $Z\in\mathcal{T}^{-}$





$$\boxed{\psi_\lambda^{\pm}(Z,\xi) = (Z\cdot\xi)^\lambda}$$



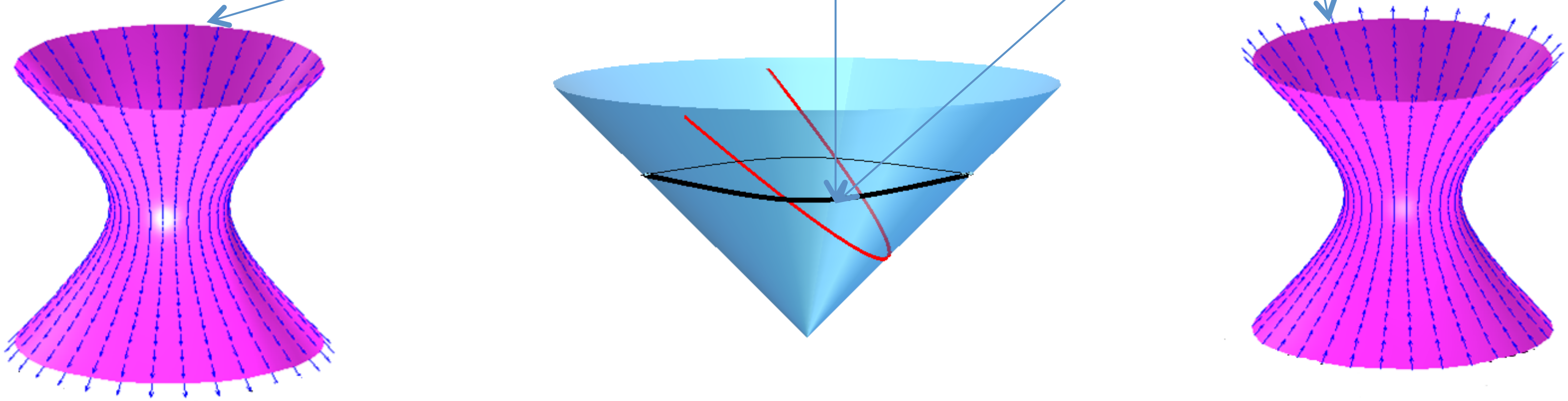
Boundary values on the reals:

$$(X\cdot\xi)^\lambda_\pm \to |X\cdot\xi|^\lambda \Big(\theta(X\cdot\xi) + e^{\pm i\pi\lambda}\theta(-X\cdot\xi)\Big)$$

Fourier complex representation

For $Z_1 \in \mathcal{T}^-$ e $Z_2 \in \mathcal{T}^+$

$$W_\lambda(Z_1, Z_2) = c(\lambda) \int_\gamma (Z_1 \cdot \xi)^{-\lambda-d+1} (\xi \cdot Z_2)^\lambda d\mu(\xi)$$

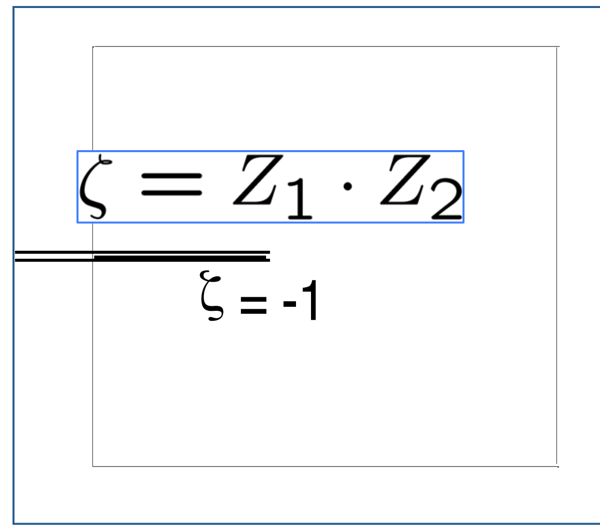


$$W(z_1 - z_2) = \int e^{-ip\cdot z_1} e^{ip\cdot z_2} \theta(p^0)\delta(p^2 - m^2) d^4p$$

$z_1 \in T^-$ $z_2 \in T^+$

$$W_\lambda(Z_1,\ Z_2) \;=\; \frac{\Gamma(-\lambda)\Gamma(\lambda+d-1)}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\ {}_2F_1\left(-\lambda,\ \lambda+d-1;\ \frac{d}{2};\ \frac{1-\zeta}{2}\right)\ .$$
$$\;=\; \frac{\Gamma\left(-\lambda\right)\Gamma\left(\lambda+d-1\right)}{2(2\pi)^{\frac{d}{2}}}\ (\zeta^2-1)^{-\frac{d-2}{4}}\,P^{-\frac{d-2}{2}}_{-\lambda-\frac{d}{2}}(\zeta)$$

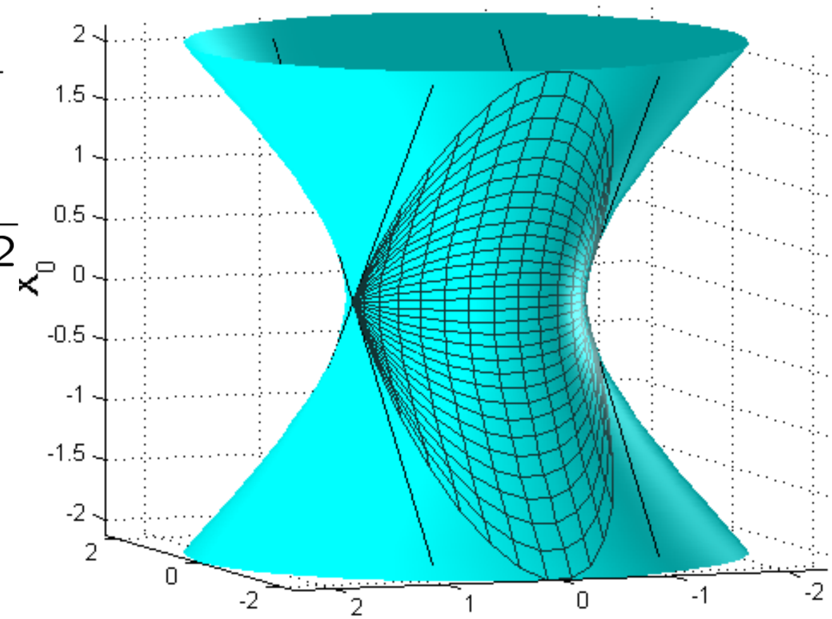
- *W* is invariant under the complex de Sitter group
- It is maximally analytic in the complex de Sitter manifold



- The inverse image of the cut reflects causality

Physical interpretation: temperature

$$X(t,\vec{r}) = \begin{cases} X_0 & = & R \sinh \frac{t}{R} \sqrt{R^2 - |\vec{r}|^2} \\ X_i & = & r_i \qquad (|\vec{r}|^2 \le R^2) \\ X_d & = & R \cosh \frac{t}{R} \sqrt{R^2 - |\vec{r}|^2} \end{cases}$$

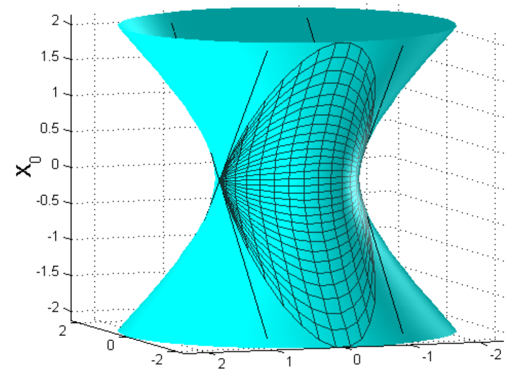


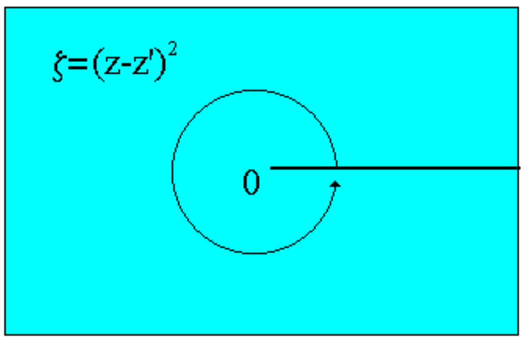
Time translations:

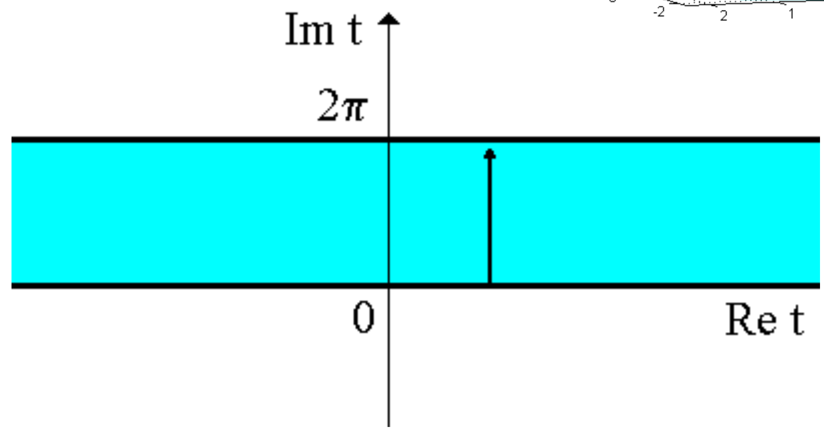
$$\alpha(s)X(t,\vec{r}) = X(t+s,\vec{r}) \equiv X(s)$$

Maximal analyticity ☛ KMS condition

$$W(X_1, X_2(t)) =$$
$$= \langle \Psi_0, \phi(X_1) U(t) \phi(X_2) U(t)^{-1} \Psi_0 \rangle$$







1) $W(X_1, X_2(t))$ is analytic in the strip $0 < Im\ t < 2\pi R$

2) For real t $W(X_1, X_2(t + 2\pi i)) = W(X_2(t) X_1)$

KMS condition at inverse temperature 2πR

Low mass «bound states»

Particles

- In quantum field theory a one "particle" state of mass m is a quantum state obtained by applying the smeared field to the vacuum

$$\Psi_f^{(1)} = \left[\int \phi_m(x) f(x) dx \right] \Psi_0$$

2-particle (Wick) states

- Two-particle states

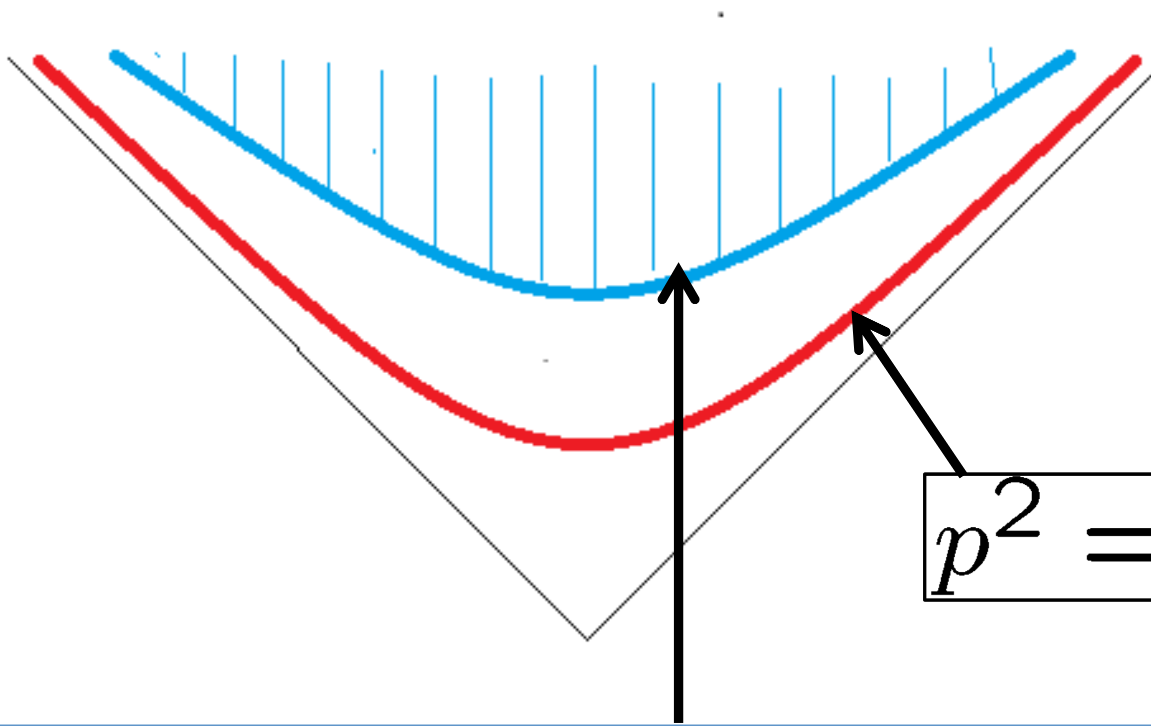
$$\left(\int :\phi(x_1)\phi(x_2): f(x_1,x_2)dx_1dx_2\right)\Psi_0$$

- Wick powers

$$\Psi_f = \left(\int :\phi^2:(x)f(x)\right)\Psi_0$$

$$\langle\Psi_0 :\phi^2:(x):\phi^2:(y)\Psi_0\rangle = (W_m(x,y))^2$$

Spectrum



$\boxed{p^2 = m^2}$

Continuum starting at $p^2 = 4m^2$

Kallen Lehmann expansions

$$W_m(x,\ y)^2 = \int_0^\infty \rho(\mu^2;\ m)\, W_\mu(x,\ y) d\mu^2$$

More generally

$$W_{m_1}(x,\ y) \ldots W_{m_1}(x,\ y) =$$
$$= \int_0^\infty \rho(\mu^2;\ m_1, \ldots, m_n)\, W_\mu(x,\ y) d\mu^2$$

KL weight

Evaluate the Mehler-Fock transform

$$h_d(\kappa,\nu,\lambda) = \int_1^\infty P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\kappa}(u)\, P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(u) P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\lambda}(u)(u^2-1)^{-\frac{d-2}{4}}\, du$$

which provides the Kallen-Lehmann weight

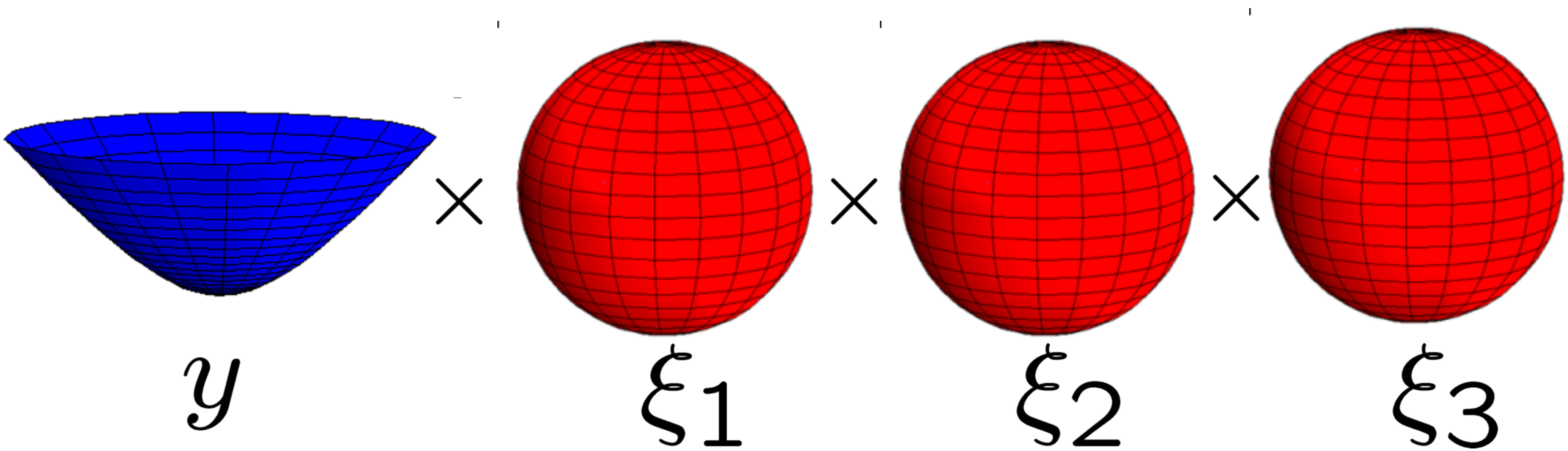
$$\rho(\kappa^2,\nu,\lambda) \;=\; \frac{\Gamma\left(\frac{d-1}{2}+i\nu\right)\Gamma\left(\frac{d-1}{2}-i\nu\right)\Gamma\left(\frac{d-1}{2}+i\lambda\right)\Gamma\left(\frac{d-1}{2}-i\lambda\right)}{2(2\pi)^{1+\frac{d}{2}}}\sinh(\pi\kappa)\; h_d(\kappa,\nu,\lambda),$$

One needs something like a vectorial Fourier transform adapted to the de Sitter geometry

Fourier-like representation

$$h_d(\kappa,\nu,\lambda) = \int_1^\infty P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\kappa}(u)\, P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(u) P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\lambda}(u)(u^2-1)^{-\frac{d-2}{4}}\, du =$$

$$\int_{H_d}\int_{S^3_{d-1}} (y\cdot\xi_1)^{-\frac{d-1}{2}-i\kappa}\,(y\cdot\xi_2)^{-\frac{d-1}{2}-i\nu}\,(y\cdot\xi_3)^{-\frac{d-1}{2}-i\lambda}\, dy\, d\Omega_1\, d\Omega_2\, d\Omega_3$$

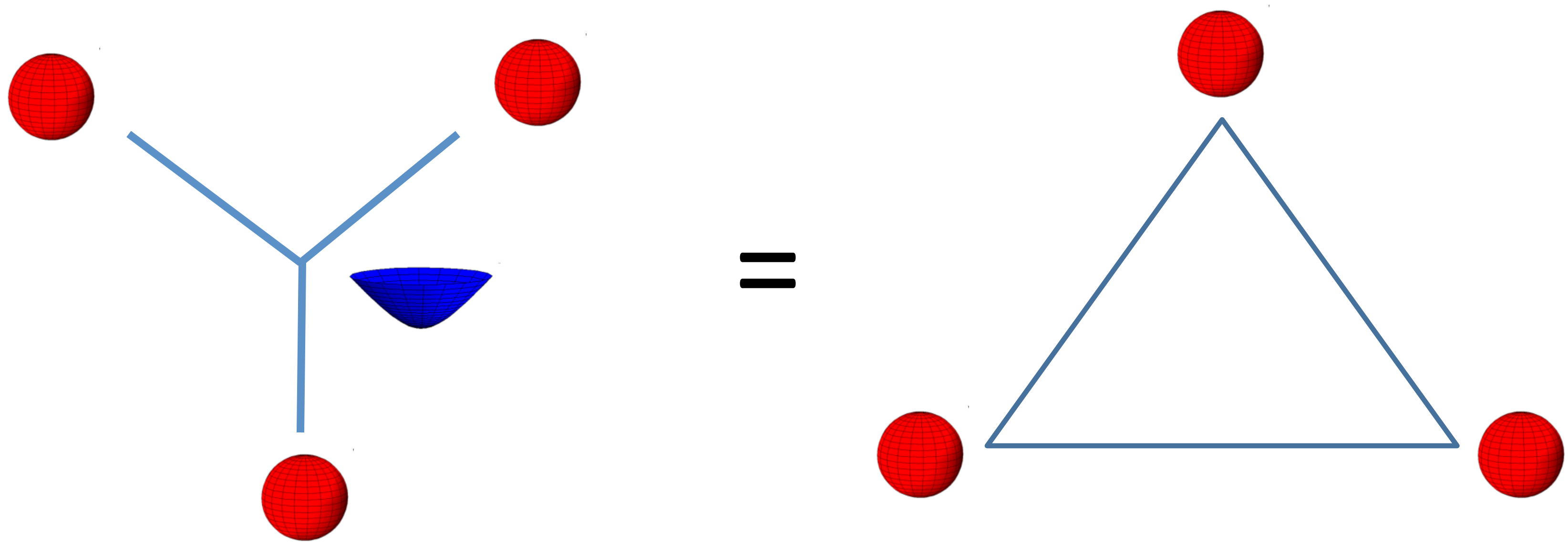


Star-triangle identity

1) Integral on the hyperboloid: a star-triangle relation

$$\int_H (y\cdot\xi_1)^{a_2+a_3}(y\cdot\xi_2)^{a_3+a_1}(y\cdot\xi_3)^{a_1+a_2}dy =$$

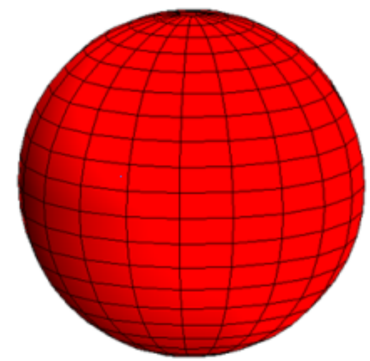
$$= \; c(a_1,a_2,a_3)(\xi_1\cdot\xi_2)^{a_3}(\xi_2\cdot\xi_3)^{a_1}(\xi_3\cdot\xi_1)^{a_2}$$



Integrate the triangle

$$J = \int_{S^3_{d-1}} (\xi_1 \cdot \xi_2)^{a_3} (\xi_2 \cdot \xi_3)^{a_1} (\xi_3 \cdot \xi_1)^{a_2} \, d\Omega_1 \, d\Omega_2 \, d\Omega_3,$$

$$= \int_{S^3_{d-1}} \Delta_{12}^{2a_3} \; \Delta_{23}^{2a_1} \; \Delta_{31}^{2a_2} \; d\Omega_1 \, d\Omega_2 \, d\Omega_3 =$$



$$= \int \rho(r_1, r_2, r_3) \, r_1^{2a_1} \, r_2^{2a_2} \, r_3^{2a_3} \, dr_1 dr_2 dr_3.$$

$$\rho(r_1, r_2, r_3) = \frac{4^a \omega_{d-}}{\omega} \int \rho(r_1, r_2, r_3) dr_1 dr_2 dr_3 = 1 + r_2^4 + r_3^4) - r_1^2 r_2^2 r_3^2]_+^{\frac{d-4}{2}}.$$

$J(a_1, a_2, a_3)$ are the moments of the probability density of three random points on a sphere making a triangle with sides r_1, r_2 and r_3.

Final result

$$\int_1^\infty P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\kappa}(u)\, P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(u) P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\lambda}(u)(u^2-1)^{-\frac{d-2}{4}}\, du =$$

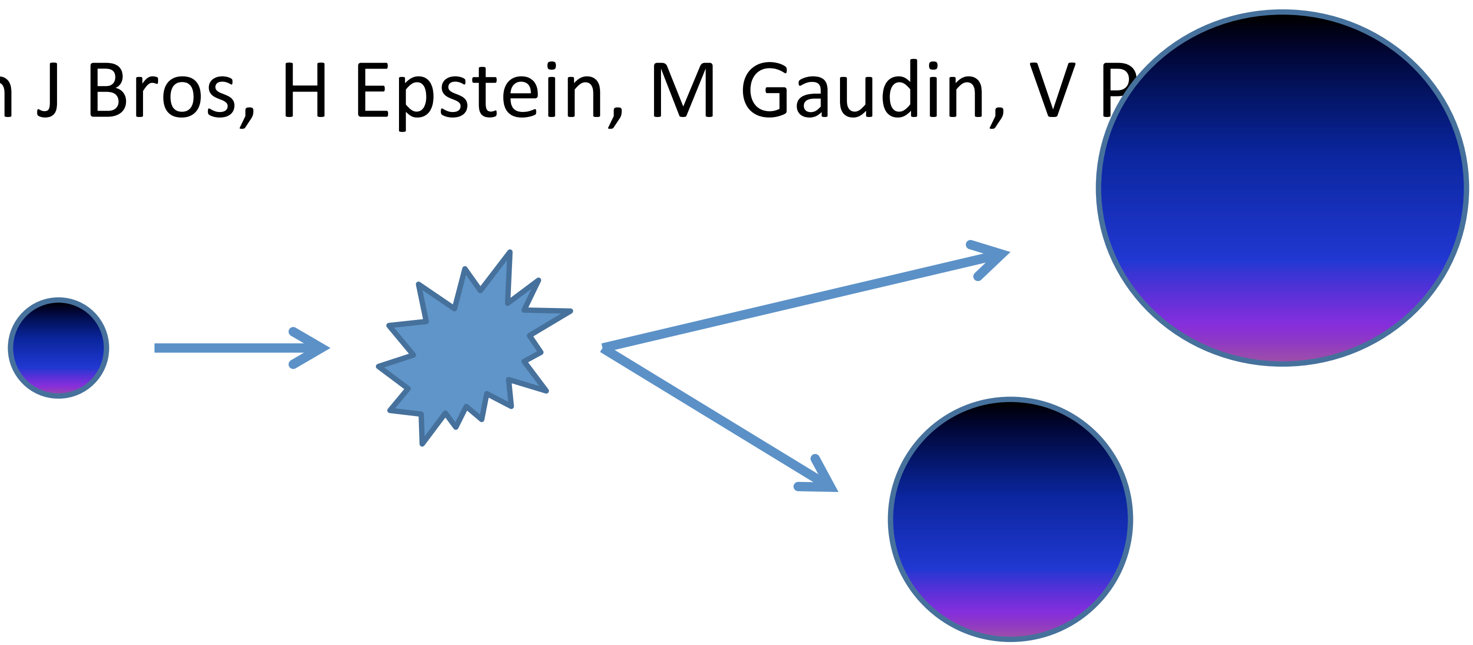
$$= \frac{\prod_{\epsilon,\epsilon',\epsilon''=\pm 1} \Gamma\left(\frac{d-1}{4} + \frac{i\epsilon\kappa + i\epsilon'\nu + i\epsilon''\lambda}{2}\right)}{\left[\prod_{\epsilon=\pm 1}\Gamma\left(\frac{d-1}{2} + i\epsilon\kappa\right)\right]\left[\prod_{\epsilon'=\pm 1}\Gamma\left(\frac{d-1}{2} + i\epsilon'\nu\right)\right]\left[\prod_{\epsilon''=\pm 1}\Gamma\left(\frac{d-1}{2} + i\epsilon''\lambda\right)\right]}$$

Application: particle decays

- There are no stable particle of mass

$$m^2 \geq \frac{9}{4R^2}$$

(with J Bros, H Epstein, M Gaudin, V P



Application: bound states

For principal fields

$$[w_{-\frac{d-1}{2}+i\nu}(\zeta)]^2 = \int_{-\infty}^{\infty} \kappa\, \rho(\kappa,\nu,\nu)\, w_{-\frac{d-1}{2}+i\kappa}(\zeta)\, d\kappa \ ,$$

$$\kappa\, \rho(\kappa,\nu,\nu) = \frac{\left|\Gamma\left(\mu+\frac{i\kappa}{2}\right)\right|^2 \prod_{\epsilon,\epsilon'=\pm 1} \Gamma\left(\mu+\frac{i\epsilon\kappa}{2}+i\epsilon'\nu\right)}{2^{d+2}\pi^{\frac{d+1}{2}} R^{d-2}\, \Gamma(i\kappa)\Gamma(-i\kappa)\Gamma\left(\frac{1}{2}+\mu+\frac{i\kappa}{2}\right)\Gamma\left(\frac{1}{2}+\mu-\frac{i\kappa}{2}\right)\Gamma(2\mu)}$$

$$\mu = (d-1)/4$$

Complementary fields

$$\left[w_{-\frac{d-1}{2}+i\nu}(\zeta)\right]^2 = \int_R \frac{\kappa \sinh(\pi\kappa)\left|\Gamma\left(\mu+\frac{i\kappa}{2}\right)\right|^4 \prod_{\epsilon,\epsilon'=\pm 1}\Gamma\left(\mu+\frac{i\epsilon\kappa}{2}+i\epsilon'\nu\right)}{2^{d+5}\pi^{d+\frac{5}{2}}\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d-1}{2}\right)R^{2d-4}} \times$$

$$\times \quad F\left(\frac{d-1}{2}+i\kappa,\ \frac{d-1}{2}-i\kappa\ ;\ \frac{d}{2}\ ;\ \frac{1-\zeta}{2}\right) d\kappa\ .$$

When $0 < \alpha = \mathrm{Im}\,\nu < (d-1)/4$ no pole reaches the real axis: the formula continues to hold. True in particular for the principal series and half of the complementary series, i.e for all the masses satisfying

$$m^2 > \frac{3(d-1)^2}{16R^2}.$$

When the threshold is attained the KL has to be replaced by a contour integral.

- For (d-1)/4 < α (d-1)/4 +1 we can extract the residues at the first pole
- A similar situation occurs when the successive poles.
- The KL rep. is modified by the appearance of a sum of discrete contributions
- Can be done down to m=0 (the limiting case excluded)

$$\left[w_{-\frac{d-1}{2}+i\nu}(\zeta)\right]^2 = \int_R \kappa\,\rho(\kappa,\nu,\nu)\,w_{-\frac{d-1}{2}+i\kappa}(\zeta)\,d\kappa + \\ + \sum_{k=0}^{N} A_k(\nu)\,w_{-\frac{d-1}{2}-2(\mu+i\nu+k)}(\zeta).$$

Conclusion

- In particular, in sharp contrast with the Minkowski case in d=4 in the spectrum of a dS scalar QFT of mass

$$m < \frac{3\sqrt{3}}{4R}$$

- there are also compounds in the two-particle Hilbert subspace of the Fock space of the theory with mass

$$m_2 = \frac{1}{R}\sqrt{4m^2R^2 + 3\sqrt{9-4m^2R^2} - 9}$$

- In general for

$$m < \frac{1}{2R}\sqrt{\frac{(d-1)^2(2n-1)}{n^2}}$$

- discrete states appear in the first n finite-particle subspaces of the Fock space of the theory.

Two-dimensional models

What tells us perturbation theory?

- There several approaches to nonperturbative QFT on flat space
- One is the study of exactly soluble two-dimensional models of QFT
- Why not to try and explore soluble (?) two-dimensional models in de Sitter?

Two historically important models

1) The Thirring Model (current-current)

$$i\gamma^\mu \partial_\mu \psi(x) = -gJ^\mu(x)\gamma_\mu \psi(x)$$
$$J^\mu(x) = \overline{\psi}(x)\gamma^\mu \psi(x) \qquad \partial_\mu J^\mu(x) = 0$$

2) Schwinger Model (2-DIM QED)

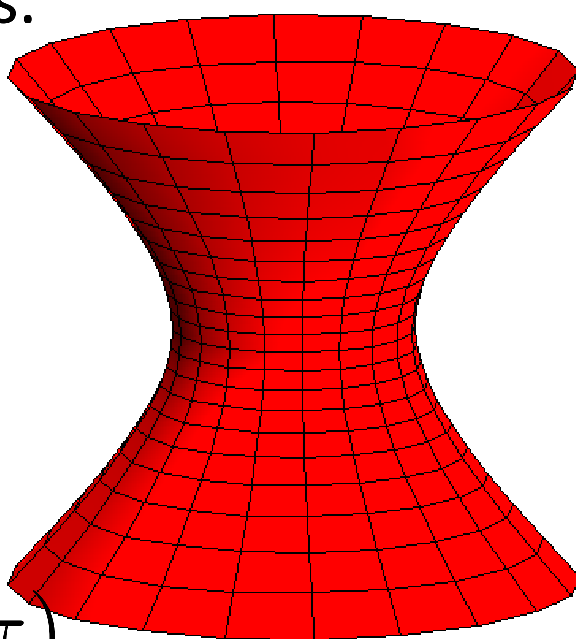
$$i\gamma^\mu \partial_\mu \psi(x) = -e\gamma^\mu A_\mu(x)\psi(x)$$
$$\partial^\nu F_{\mu\nu}(x) = -eJ_\mu(x) + \mathcal{A}_\mu(x)$$

3) Are their dS counterparts still soluble?
4) What can be learnt from the solutions?
5) Do they share the same difficulties of perturbation theory?

Two preliminary elementary questions

- What is going to replace the Dirac equation on the de Sitter manifold?
- What is the meaning of de Sitter covariance for spinor fields ?

Spin bundles on dS(2)



- There are two inequivalent spin bundles.
- Two boundary conditions:
- 1) Periodic (Ramond)

$$\psi(t,\theta) = \psi(t,\theta+2\pi)$$

- 2) Anti-periodic (Neveu-Schwarz)
- Well defined on the double covering

$$\psi(t,\theta) = -\psi(t,\theta+2\pi)$$

- In both cases observables are fully well defined on the cylinder (i.e. they are periodic)

Conformal transformation of the spinors

- Massless de Sitter Dirac (quantum) spinor field (either Ramond or Neveu-Schwarz)

$$\phi = (\cos t)^{\frac{1}{2}} \psi$$

- What about the de Sitter symmetry?
- There is a priori no reason to expect it. The spinors on the cylinder have less symmetries (space rotations + time translations)!

dS_2 : Homogeneous space $SO_0(1,2)/A$

$$g = \underbrace{\overset{k(\zeta)}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\zeta & \sin\zeta \\ 0 & -\sin\zeta & \cos\zeta \end{pmatrix}} \overset{n(\lambda)}{\begin{pmatrix} 1+\frac{\lambda^2}{2} & -\frac{\lambda^2}{2} & -\lambda \\ \frac{\lambda^2}{2} & 1-\frac{\lambda^2}{2} & \lambda \\ \lambda & -\lambda & 1 \end{pmatrix}}}_{x(\lambda,\zeta)=k(\zeta)n(\lambda)} \overset{a(\chi)}{\begin{pmatrix} \cosh\chi & \sinh\chi & 0 \\ \sinh\chi & \cosh\chi & 0 \\ 0 & 0 & 1 \end{pmatrix}}.$$

$SO_0(1,2)$ acts on $SO_0(1,2)/A$ by left multiplication $: x(\lambda', \zeta') = g x(\lambda, \zeta)$

$$\boxed{u = \tan\left(\frac{\zeta}{2} + \arctan\lambda\right), \qquad v = \cot\frac{\zeta}{2},}$$

$$g = k(\alpha)n(\mu)a(\kappa) : u \to u' = \frac{(e^\kappa u + \mu)\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - (e^\kappa u + \mu)\sin\frac{\alpha}{2}}, \;\; v \to v' = \frac{(e^\kappa v - \mu)\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} + (e^\kappa v - \mu)\sin\frac{\alpha}{2}}.$$

Iwasawa coordinate system

$$x(\lambda,\zeta)=\begin{pmatrix} \frac{\lambda^2}{2}+1 & -\frac{\lambda^2}{2} & \lambda \\ \frac{1}{2}\lambda^2\cos\zeta+\lambda\sin\zeta & \left(1-\frac{\lambda^2}{2}\right)\cos\zeta-\lambda\sin\zeta & \lambda\cos\zeta+\sin\zeta \\ \lambda\cos\zeta-\frac{1}{2}\lambda^2\sin\zeta & -\lambda\cos\zeta-\left(1-\frac{\lambda^2}{2}\right)\sin\zeta & \cos\zeta-\lambda\sin\zeta \end{pmatrix}$$

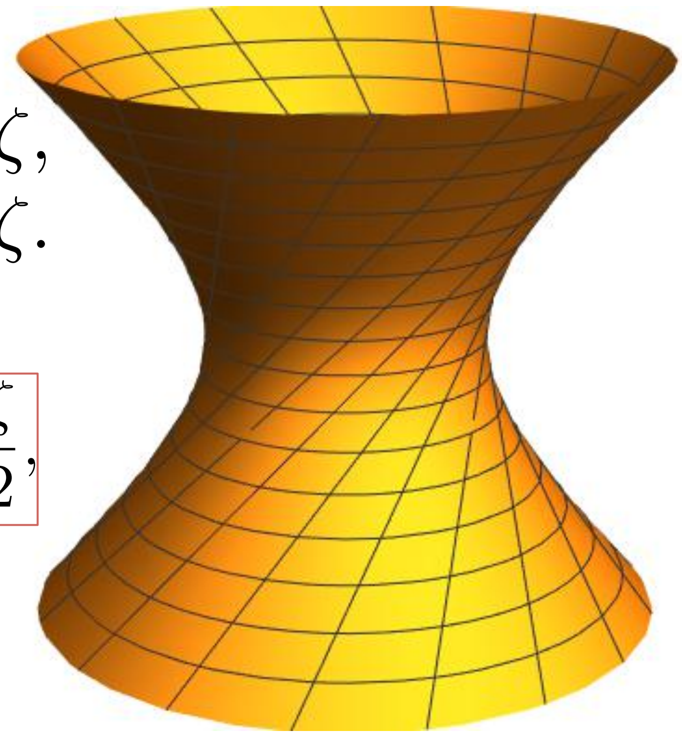
$$x(\lambda,\zeta)=\begin{cases} x^0=\lambda, \\ x^1=\lambda\cos\zeta+\sin\zeta, \\ x^2=\cos\zeta-\lambda\sin\zeta. \end{cases}$$

$$dS_2=\left\{x\in M_3:\ {x^0}^2-{x^1}^2-{x^2}^2=-1\right\}=G/A$$

$$dS_2^{(c)}=\left\{x\in C_3:\ {z^0}^2-{z^1}^2-{z^2}^2=-1\right\}=G^c/A^c.$$

Iwasawa de Sitter coordnates

$$x(\lambda,\zeta)=\begin{cases} x^0=\lambda, \\ x^1=\lambda\cos\zeta+\sin\zeta, \\ x^2=\cos\zeta-\lambda\sin\zeta. \end{cases}$$



$$\boxed{u=\tan\left(\frac{\zeta}{2}+\arctan\lambda\right), \quad v=\cot\frac{\zeta}{2},}$$

$$u=\frac{1-x^2}{x^1-x^0}, \quad v=\frac{1+x^2}{x^1-x^0}.$$

$$ds^2=\left(dX^{0^2}-dX^{1^2}-dX^{2^2}\right)\Big|_{dS_2}=-2d\lambda d\zeta-\left(\lambda^2+1\right)d\zeta^2$$

Double-covering group $\widetilde{G} = SL(2,R)$

$$\widetilde{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad ad - bc = 1$$

$$\widetilde{g} = \underbrace{\overbrace{\begin{pmatrix} \cos\frac{\zeta}{2} & \sin\frac{\zeta}{2} \\ -\sin\frac{\zeta}{2} & \cos\frac{\zeta}{2} \end{pmatrix}}^{\widetilde{k}(\zeta)} \overbrace{\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}}^{\widetilde{n}(\lambda)}}_{\widetilde{x}(\lambda,\zeta) = \widetilde{k}(\zeta)\widetilde{n}(\lambda)} \overbrace{\begin{pmatrix} e^{\frac{\chi}{2}} & 0 \\ 0 & e^{-\frac{\chi}{2}} \end{pmatrix}}^{\widetilde{a}(\chi)},$$

$\widetilde{G}$ acts on $\widetilde{G}/\widetilde{A}$ by left multiplication $: \widetilde{x}(\lambda', \zeta') = \widetilde{g}\widetilde{x}(\lambda, \zeta)$

$\widetilde{G}$ acts on dS_2 by similarity

$$x \to X = \begin{pmatrix} x^0 + x^1 & x^2 \\ x^2 & x^0 - x^1 \end{pmatrix}, \quad X' = \begin{pmatrix} x^{0\prime} + x^{1\prime} & x^{2\prime} \\ x^{2\prime} & x^{0\prime} - x^{1\prime} \end{pmatrix} = \tilde{g} X \tilde{g}^T$$

Maureer-Cartan metric

- The Maureer-Cartan form $dg\, g^{-1}$ gives to the symmetric space $SL(2,R)/\widetilde{A}$ a natural Lorentzian metric
- There exists a inner automorphism of SL(2,R) that leaves $\widetilde{A}$ invariant

$$g \to \mu(g) = -\gamma^2 g \gamma^2 \qquad \gamma^2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

- It may be used to construct a map from the coset space $SL(2,R)/\widetilde{A}$ into the group SL(2,R) and an induced Lorentzian metric on

$$g(\tilde{X}) = g\mu(g)^{-1} = -\tilde{X}\gamma^2\tilde{X}^{-1}\gamma^2.$$

$$ds^2 = \frac{1}{2}\mathrm{Tr}(dg\, g^{-1})^2 = -2d\lambda d\zeta - \left(\lambda^2 + 1\right) d\zeta^2$$

Maureer-Cartan metric

$$ds^2 = \frac{1}{2}\mathrm{Tr}(dg\, g^{-1})^2 = -2d\lambda d\zeta - \left(\lambda^2+1\right) d\zeta^2$$

1. The metric is invariant under the group left action
2. The curvature is constantand the Ricci tensor is proportional to the metric:

3. The map

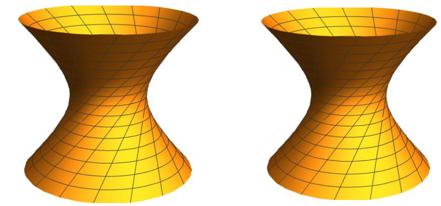
$$p : \tilde{x}(\lambda,\zeta) \to x(\lambda,\zeta) = \begin{cases} x^0 = \lambda \\ x^1 = \lambda\cos\zeta + \sin\zeta \\ x^2 = \cos\zeta - \lambda\sin\zeta \end{cases}$$

is a covering map.

$$ds^2 = \left(dX^{0^2} - dX^{1^2} - dX^{2^2}\right)\Big|_{dS_2} = -2d\lambda d\zeta - \left(\lambda^2+1\right) d\zeta^2$$

Double covering of de Sitter

- In conclusion: the symmetric space



$$\widetilde{dS_2} = SL(2,R)/\widetilde{A}$$

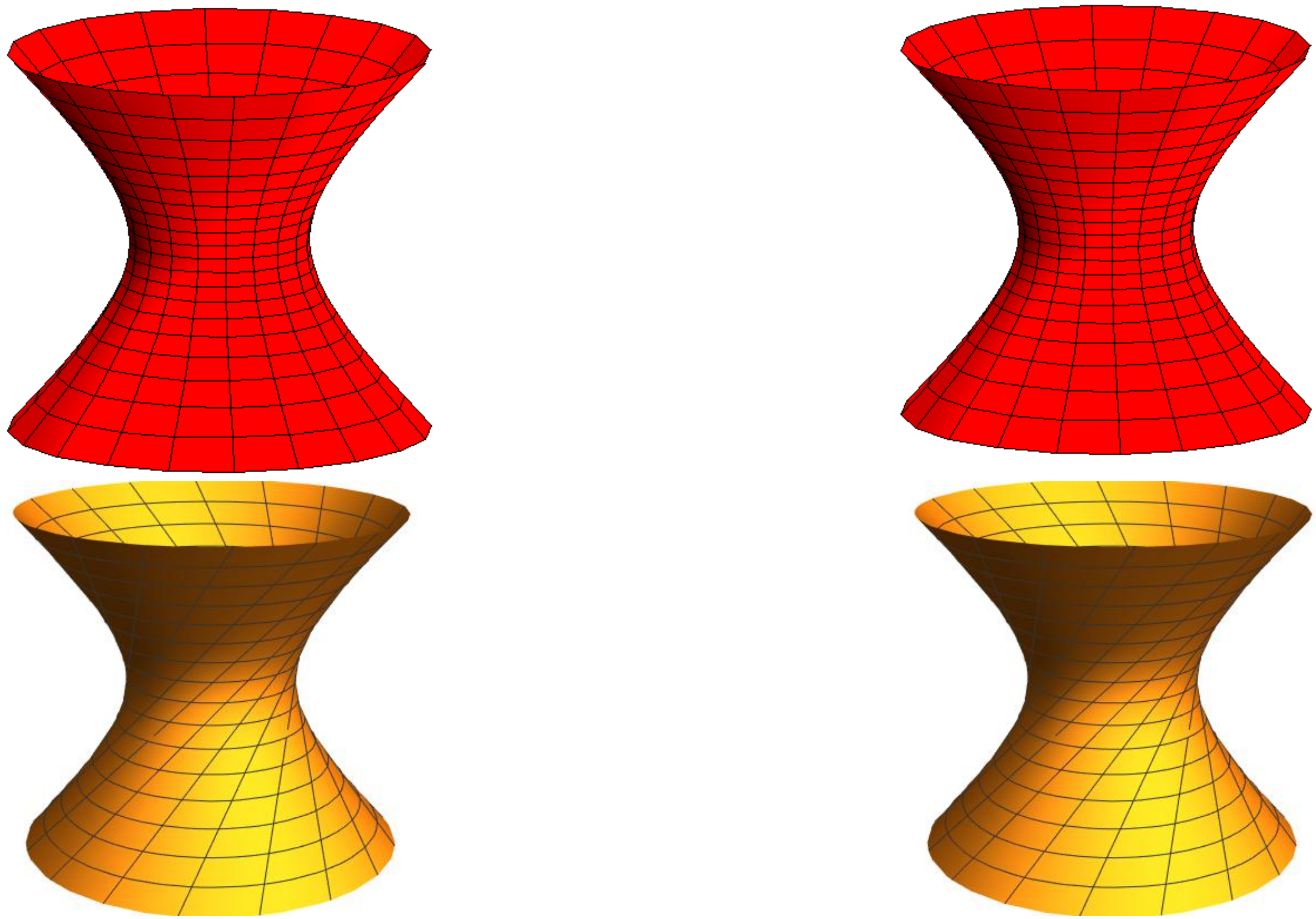
may be identified with the double covering of the two dimensional de Sitter universe.

- The spin group SL(2,R) acts effectivelyon the covering space as a group of spacetime transformations:

$$\tilde{X} \to g\tilde{X}$$

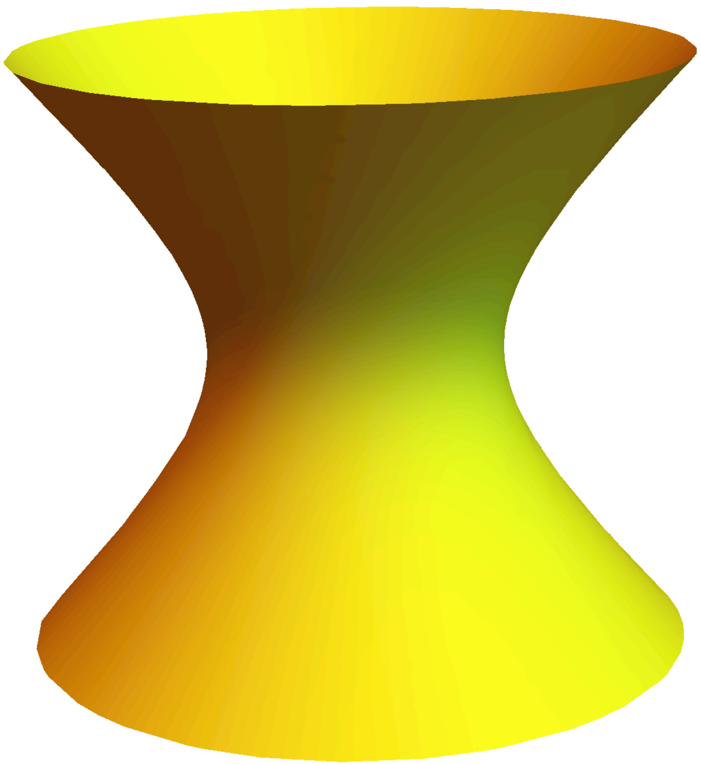
- We were not able to find the above identification in the (enormous) literature on the group SL(2,R).

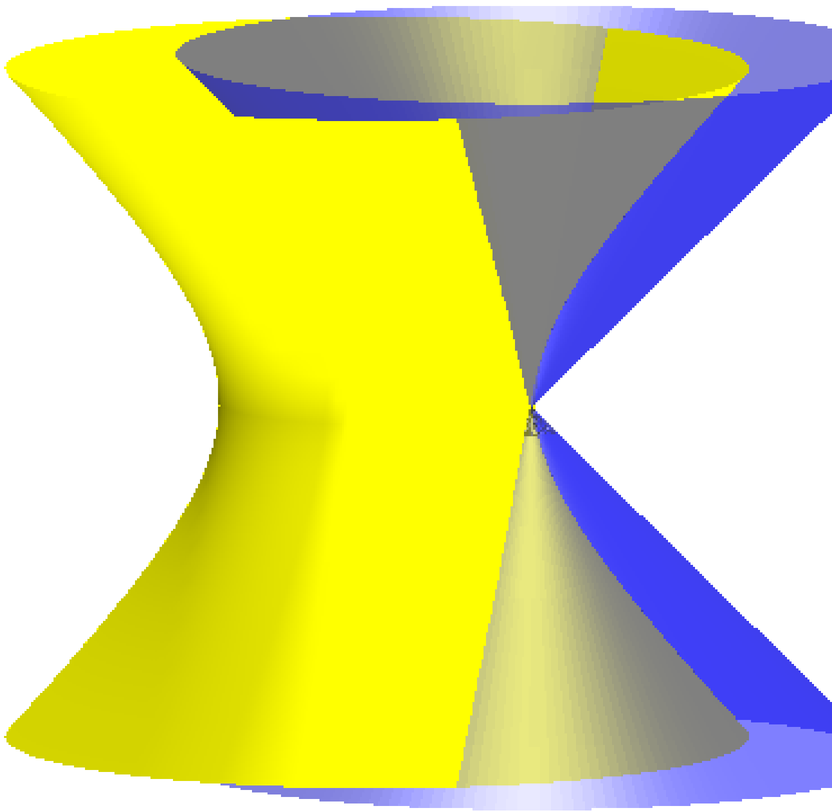
Spherical coordinates



$$\lambda = \sinh t, \quad \tan\theta = \tan(\zeta + \arctan\lambda).$$

Causality





Canonical quantization

$$\Box\phi - \lambda(\lambda+1)\phi = \frac{1}{(\cosh t)}\ \partial_t(\cosh t\ \partial_t\phi) - \frac{1}{\cosh^2 t}\ \partial_\theta^2\phi - \lambda(\lambda+1)\phi = 0.$$

$$\begin{array}{rl} \text{either} & \lambda = -\frac{1}{2} + i\rho, \quad \Im\rho = 0, \quad m = \sqrt{\frac{1}{4}+\rho^2} \geq \frac{1}{2}, \\ \text{or} & \Im\lambda = 0, \quad -1 < \Re\lambda < 0, \quad 0 < m < \frac{1}{2}. \end{array}$$

$$\begin{aligned} \phi_l(t,\theta) &= [a_l \mathbf{P}_\lambda^{-l}(i\sinh t) + b_l \mathbf{P}_\lambda^{-l}(-i\sinh t)]e^{il\theta} \\ \phi_l^*(t,\theta) &= [a_l^* \mathbf{P}_\lambda^{-l}(-i\sinh t) + b_l^* \mathbf{P}_\lambda^{-l}(i\sinh t)]e^{-il\theta} \end{aligned}$$

$$\begin{aligned} a_{-l} &= c_l(a_l \sin(\pi\lambda) + b_l \sin(\pi l)), \\ b_{-l} &= c_l(b_l \sin(\pi\lambda) + a_l \sin(\pi l)). \end{aligned}$$

$$(|a_l|^2 - |b_l|^2) = \frac{1}{8\pi}\Gamma(l-\lambda)\Gamma(1+\lambda+l)$$

Covariant Commutator

$$C(t,\theta,t',\theta') = \sum_{kl\in\mathbf{Z}} \frac{\gamma_l}{2k\pi}[\mathbf{P}_\lambda^{-l}(i\sinh t)\mathbf{P}_\lambda^{-l}(-i\sinh t')\exp i(l\theta - l\theta')$$
$$- \sum_{kl\in\mathbf{Z}} \frac{\gamma_l}{2k\pi}[\mathbf{P}_\lambda^{-l}(-i\sinh t)\mathbf{P}_\lambda^{-l}(i\sinh t')\exp i(l\theta - l\theta')$$

$$k = 1\ (dS_2) \quad , \quad k = 2\ (\widetilde{dS_2})$$

$$\gamma_l = \frac{1}{2}\Gamma(l-\lambda)\Gamma(1+\lambda+l)$$

Quantization

$$C(x, x') = W(x, x') - W(x', x). \qquad dS_2$$

$$C(\tilde{x}, \tilde{x}') = W(\tilde{x}, \tilde{x}') - W(\tilde{x}', \tilde{x}). \qquad \widetilde{dS_2}$$

Covariance

$$W(gx, gx') = W(x, x'). \qquad g \in SO_0(1,2)$$

$$W(\tilde{g}\tilde{x}, \tilde{g}'\tilde{x}') = W(\tilde{x}, \tilde{x}'). \qquad \tilde{g} \in SL(2, R)$$

Vacuum (Pure) states

$$W(x,x') = \sum_l \phi_l(x)\phi_l^*(x') =$$

$$\sum_l [a_l \mathbf{P}_\lambda^{-l}(i\sinh t) + b_l \mathbf{P}_\lambda^{-l}(-i\sinh t)][a_l^* \mathbf{P}_\lambda^{-l}(-i\sinh t') + b_l^* \mathbf{P}_\lambda^{-l}(i\sinh t')]e^{il\theta - il\theta'}$$

$$a_l = \sqrt{\frac{\gamma_l}{2\pi k}}\cosh\alpha_\epsilon, \qquad b_l = \sqrt{\frac{\gamma_l}{2\pi k}}\sinh\alpha_\epsilon e^{i\phi_\epsilon - il\pi}, \qquad \epsilon = 0,1.$$

$\epsilon = 0$ for $l \in \mathbf{Z}$ and $\epsilon = 1$ for $l \in \frac{1}{2} + \mathbf{Z}$

$$C(x,x') = W(x,x') - W(x',x). \qquad dS_2$$

$$W_{BD}(x,x') = W^{(0)}_{\alpha_0=0}(x,x') = \sum_{l\in\mathbf{Z}} \frac{\gamma_l}{2\pi}\mathbf{P}_\lambda^{-l}(i\sinh t)\mathbf{P}_\lambda^{-l}(-i\sinh t')e^{il\theta - il\theta'} =$$

$$= \frac{\Gamma(-\lambda)\Gamma(\lambda+1)}{4\pi} P_\lambda(\zeta), \quad (1)$$

Vacuum (Pure) states

$$W(x,x') = \sum_l \phi_l(x)\phi_l^*(x') =$$

$$\sum_l [a_l \mathbf{P}_\lambda^{-l}(i\sinh t) + b_l \mathbf{P}_\lambda^{-l}(-i\sinh t)][a_l^* \mathbf{P}_\lambda^{-l}(-i\sinh t') + b_l^* \mathbf{P}_\lambda^{-l}(i\sinh t')]e^{il\theta - il\theta'}$$

$$a_l = \sqrt{\frac{\gamma_l}{2\pi k}}\cosh\alpha_\epsilon, \qquad b_l = \sqrt{\frac{\gamma_l}{2\pi k}}\sinh\alpha_\epsilon e^{i\phi_\epsilon - il\pi}, \qquad \epsilon = 0, 1.$$

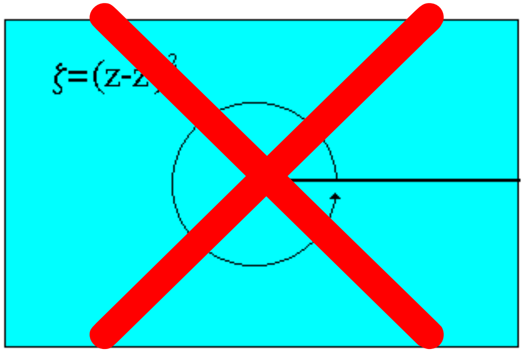
$$C(\tilde{x}, \tilde{x}') = W(\tilde{x}, \tilde{x}') - W(\tilde{x}', \tilde{x}). \qquad \widetilde{dS_2}$$

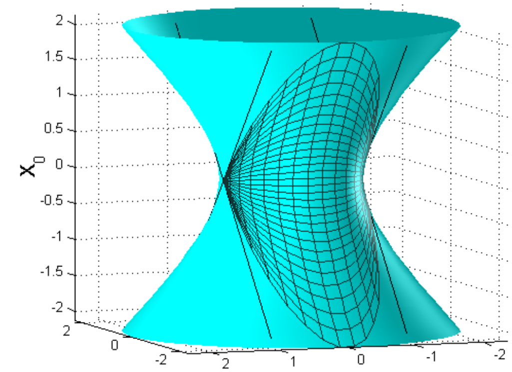
For $\lambda = -1/2 + i\nu$ there is only one solution $W_{i\nu}^{(\frac{1}{2})}(x, x')$

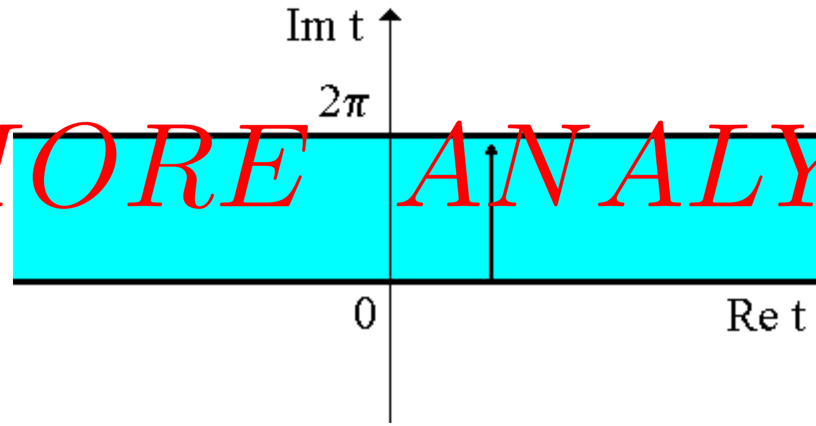
$$\coth 2\alpha = \cosh \pi\nu, \quad \phi = \frac{\pi}{2}.$$

$\alpha = 0$ is excluded $(m \to \infty)$

For $\lambda = -1/2 + \nu$ there no solution







$NO\quad MORE\quad ANALYTICITY$

No KMS condition, No temperature