AN INTRODUCTION TO QUANTUM FIELD THEORY ON CURVED SPACETIMES

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1. Introduction

The problem of incorporating gravity into the framework of quantum physics is still largely open. Due to the rareness of observational hints for a solution, rather speculative ideas dominate the discussion, and it appears difficult to judge whether any of them will lead to a satisfactory answer. Cosmology might be the rather unique field of physics where both fundamental theories, quantum field theory and general relativity, are both needed for a better understanding; moreover, the increasing amount of observational data make a consistent framework urgent.

To reach such a consistent theory which contains the quantum field theory of particle physics and Einstein's theory of gravitation as limiting cases, one may proceed in the following way: Standard quantum field theory just ignores effects of gravity. This is justified in many cases due to the weakness of gravitational interactions at the presently accessible scales. In a first step beyond this approximation one may consider an external gravitational field which is not influenced by the quantum fields. Here one may think of sources of gravitational fields which are not influenced by the quantum fields under consideration, as high energy experiments in the gravitational field of the earth or quantum fields in the gravitational field of dark matter and dark energy. This approach amounts to the treatment of quantum field theory on curved spacetimes.

As a little step beyond this approximation one may also study perturbative gravity around a classical background. Up to problems of renormalizability and a somewhat tricky characterization of observables, this can also be subsumed in this framework [1]. In lowest nontrivial order, one thus reproduces cosmological perturbation theory [2].

Full quantum gravity might be rather remote from the perturbative approach; on the other hand, a satisfactory perturbative formulation might suggest features of the full theory; moreover, one may hope that its predictions can be tested in cosmological observations.

In the following we want to describe the crucial steps for the treatment of quantum field theory on manifolds with a Lorentzian metric [1]. Up to the mid-nineties, mainly free fields were discussed, and one looked for states generalizing the vacuum, and tried to understand the meaning of particles in such a background. A highlight was Hawking's discovery of radiation of black holes [5] which up to the present day motivated a

Talk at the MITP workshop "The mysterious universe", Mainz, may 20 -june 8, 2019.

[1]For recent reviews see [3, 4]

huge number of papers which try to use his finding for guessing structural properties of quantum gravity.

The discussion on the choice of a vacuum state ended essentially with the conclusion that instead one has to select a class of states, the so-called Hadamard states which have locally the same singularities as the vacuum on Minkowski space (see Kay in [6] and Haag, Narnhofer and Stein in [7]). Unfortunately, a precise characterization of Hadamard states, first achieved in [8], looked rather cumbersome.

A breakthrough was then the observation of Radzikowski [9] that the Hadamard states can be directly characterized in terms of their wave front sets, a concept of microlocal analysis [10]. The condition might be interpreted as a local version of the positive energy condition which is crucial for quantum field theory on Minkowski space.

The result of Radzikowski immediately opened the way for a perturbative approach to quantum field theory on curved spacetimes. Based on the concepts of causal perturbation theory (Stückelberg, Bogoliubov, Epstein-Glaser [11]) and of algebraic quantum field theory (Haag-Kastler [12, 13]), the renormalized perturbation series for generic quantum field theories on globally hyperbolic spacetimes could be constructed [14], and also the principle of general covariance could be incorporated [15, 16, 20]. What is still missing is a detailed interpretation of the state space, since the standard interpretation in terms of multiparticle configurations is meaningful only in special situations.

2. Free scalar field

We consider a scalar field ϕ on a globally hyperbolic 4d spacetime M, i.e. a 4d Lorentzian manifold with a Cauchy surface. On such a spacetime, the initial value problem for the Klein-Gordon equation is well posed, and there exist unique advanced and retarded Green's functions Δ_A, Δ_R [17]. A special role is played by the so-called causal propagator

$$\Delta = \Delta_R - \Delta_A \ .$$

Observables may be understood as functionals on the space of classical field configurations which may be identified with smooth real valued functions on the manifold. *Regular* observables $F \in \mathcal{F}_0$ are defined as functionals of the form

$$F[\phi] = \sum_{n=0}^{N} \int f_n(x_1, \ldots, x_n)\phi(x_1)\ldots\phi(x_n)$$

with compactly supported smooth densities f_n on M^n.

The causal propagator induces a noncommutative associative product on $\mathcal{F}_0$ by

$$F \star G = \sum_{n=0}^{\infty} \frac{i^n}{2^n n!} \langle F^{(n)}, \Delta^{\otimes n} G^{(n)} \rangle$$

with the functional derivatives

$$\langle F^{(n)}[\phi], \psi^{\otimes n} \rangle = \frac{d^n}{d\lambda^n} F[\phi + \lambda\psi]|_{\lambda=0} \ .$$

With respect to complex conjugation, $\mathcal{F}_0$ becomes an involutive algebra $\mathcal{A}_0$,

$$F^*[\phi] = \overline{F[\phi]}$$

such that

$$(F \star G)^* = G^* \star F^* \ .$$

Functionals F which vanish on solutions of the Klein-Gordon equation form an ideal. The quotient $\mathcal{A}_0$ is the on shell algebra of regular observables.

There are, however, many important observables which are not represented by regular functionals, as e.g. the energy momentum tensor. As a simple example we may consider functionals of the form

$$F[\phi] = \int f(x)\phi(x)^2$$

For them the $\star$-product is ill defined since it would require to square the causal propagator which is not a meaningful operation. Actually, this problem occurs in the same way in Minkowski space where it is circumvented by replacing these functionals by their normal ordered versions.

To find a generalization of normal ordering on generic spacetimes one can proceed in the following way. The problem of singularities of distributions can be analyzed in terms of their *wave front sets*. Roughly speaking, the wave front set is a local version of the support in momentum space. More precisely, one multiplies the distribution t with a testfunction χ with sufficiently small support and studies in some chart the decay properties of its Fourier transform. In this way one obtains for each point x a set Σ_x of directions in momentum space in which the Fourier transform of χt does not decay fast for all testfunctions χ with $\chi(x) \neq 0$. The collection

$$\mathrm{WF}(t) = \{(x,k)|x \in M, k \neq 0, \text{ the direction of } k \text{ belongs to } \Sigma_x \ \}$$

is called the wave front set of the distribution t. It can be identified with a subset of the cotangent bundle T^*M and is independent of the choice of a chart.

We will use two crucial facts on wave front sets [10]:

- The product of two distributions t and s is a well defined distribution, if it is not possible to find $(x,k) \in \mathrm{WF}(t)$ and $(x,k') \in \mathrm{WF}(s)$ such that $k + k' = 0$.
- Let P be a real differential operator with principal symbol $\sigma(P)$ and let t be a distribution with Pt smooth. Then the wave front set of t is invariant under the Hamiltonian flow generated by $\sigma(P)$ and contained in its zero set.

The causal propagator Δ is a bisolution of the Klein Gordon operator whose principal symbol is the inverse metric. Therefore the wave front set of Δ is invariant under the geodesic flow and contains only lightlike momenta. Since Δ vanishes for spacelike separated points, we can determine its wave front set to be

$$\begin{gathered}\mathrm{WF}(\Delta) = \{(x,x';k,k') \in T^*M^2 | \exists \text{ a lightlike geodesics connecting } x \text{ and } x', \\ k,k' \text{ are coparallel to the geodesics,} \\ \text{and the parallel transport of } k \text{ along the geodesics is } -k'\}\end{gathered}$$

Normal ordering in Minkowski space is related to the separation of positive and negative energies,

$$i\Delta = \Delta_+ - \Delta_-$$

where Δ_+ is the Wightman 2-point function in Minkowski space and Δ_- its complex conjugate.

A similar splitting can be performed with respect to wave front sets. A *Hadamard function* H is defined as a symmetric real bisolution of the Klein Gordon equation with the property

$$\mathrm{WF}(H + \frac{i}{2}\Delta) = \{(x, x'; k, k') \in \mathrm{WF}(\Delta) | k \in \overline{V_+}\}$$

where $\overline{V_+}$ denotes the closed forward light cone in momentum space. Hadamard functions in 4d possess an asymptotic expansion

$$H = \frac{u}{\sigma} + \sum_{n=0}^{\infty} v_n \sigma^n \log \mu^2 \sigma + w$$

where σ is the squared geodesic distance (equipped with the appropriate sign), u and v_n are smooth functions depending only on the metric and its derivatives on the geodesics connecting the arguments. μ is an arbitrary scale with the dimension of a mass and w is a smooth function which depends on the choice of H.

The positivity condition on the wave front set (the *microlocal spectrum condition*) is weaker than the positive energy condition on the Wightman function. In Minkowski space, e.g., the microlocal spectrum condition is satisfied not only for the vacuum, but also for the 2-point functions of KMS states. Different Hadamard functions can differ only by a smooth, real and symmetric bisolution.

Normal ordering with respect to a given Hadamard function H is defined by

$$:F:_H = \sum_{n=0}^{\infty} \frac{1}{2^n n!} \langle F^{(2n)}, H^{\otimes n} \rangle \ .$$

The product of normal ordered functionals is

$$:F:_H \star :G:_H = :F \star_H G: \tag{1}$$

with the product $\star_H$ (the *Wick product* in the sense of deformation quantization)

$$F \star_H G = \sum_{n=0}^{\infty} \frac{1}{n!} \langle F^{(n)}, (H + \frac{i}{2}\Delta)^{\otimes n} G^{(n)} \rangle \ .$$

In contrast to the product $\star$ (the *Weyl-Moyal product*) the Wick product is well defined also on more singular functionals F (the *microcausal functionals*) where the densities f_n might be distributions with wave front sets satisfying the condition

$$\mathrm{WF}(f_n) \cap (M^n, \overline{V_+}^n \cup \overline{V_-}^n) = \emptyset \ .$$

This includes in particular the local functionals where the support of f_n is restricted to the thin diagonal

$$\mathrm{Diagonal}(M^n) = \{(x_1, \dots, x_n) \in M^n | x_1 = \cdots = x_n\} \ .$$

One can now enlarge the algebra of regular functionals by normal ordered microcausal functionals where the product is defined by (1). This algebra does not depend on the choice of H; a change of H amounts only to a relabeling of its elements in terms of normal ordered microcausal functionals,

$$:F:_{H'} = \sum_{n=0}^{\infty} \frac{1}{2^n n!} :\langle F^{(2n)}, (H' - H)^{\otimes n} \rangle :_H$$

where we exploited the fact that the difference of two Hadamard functions is smooth.

3. Interactions

Interactions are induced by normal ordered local functionals. In order to avoid infrared problems we restrict ourselves to local functionals V which have compact spacetime support

$$\mathrm{supp}(V) = \overline{\bigcup_{\phi} \mathrm{supp} V^{(1)}[\phi]} \ .$$

The easiest way of constructing the algebra of interacting observables uses the time ordered product. On regular observables it is defined by

$$F \cdot_T G = \sum_{n=0}^{\infty} \frac{i^n}{n!} \langle F^{(n)}, \Delta_D^{\otimes n} G^{(n)} \rangle$$

with the *Dirac propagator*

$$\Delta_D = \frac{1}{2}(\Delta_R + \Delta_A)$$

For normal ordered regular functionals one obtains

$$:F:_H \cdot_T :G:_H = :F \cdot_{T,H} G:_H$$

with

$$F \cdot_{T,H} G = \sum_{n=0}^{\infty} \frac{1}{n!} \langle F^{(n)}, (H + i\Delta_D)^{\otimes n} G^{(n)} \rangle$$

Here $H + i\Delta_D \doteq \Delta_F^H$ is the H-dependent generalization of the Feynman propagator for generic spacetimes. For arguments x, x' with x' outside of the future of x, it coincides with $H + \frac{i}{2}\Delta$, and for x' outside of the past of x with its complex conjugate. As a Green's function, it has at coinciding points the same singular directions as the δ-function. Therefore its wave front set is

$$\mathrm{WF}(\Delta_F^H) = \{(x, x; k, -k) | x \in M, k \neq 0\} \cup \{(x, x'; k, k') \in \mathrm{WF}(\Delta) | k \in \overline{V_\pm} \text{ if } x' \text{ is in the future (past) of } x\} \ .$$

By the criterion for multiplicability of distributions we see that the time ordered products of normal ordered local functionals are well defined if their supports are disjoint.

The extension of the time ordered product to arbitrary normal ordered local functionals corresponds to renormalization. Such an extension (a *renormalized time ordered product*) is always possible but in general not unique. The possible extensions are related by the renormalization group in the original sense of Stückelberg and Petersen [18].

Once the time ordered product is given, one defines the interacting observables by Bogoliubov's formula

$$F_V = S(:V:_H)^{-1} \star (S(:V:_H) \cdot_T F)$$

with the time ordered exponential (*formal S-matrix*)

$$S(G) = \sum_{n=0}^{\infty} \frac{1}{n!} G^{\cdot_T n}$$

Note that the interacting theory depends on the choice of H and on the choice of the renormalized time ordered product.

4. Adiabatic limit and general covariance

On the side of the algebra of local observables it is straightforward to eliminate the restriction to compactly supported interactions. Namely, due to the causal properties of the propagators the algebraic relations for the interacting observables depend only on the interaction within a causally closed region where the observables are localized. This can be used to define the algebra of all local observables for an arbitrary local interaction (*algebraic adiabatic limit*) [19, 14].

It is more difficult to satisfy the requirement of general covariance. In Minkowski space the possible choices of the time ordered product are restricted by the condition of Poincaré invariance. To find a corresponding condition in the general case where the symmetry group generically is trivial one has to relate theories on different spacetimes. The leading principle (*local covariance*) is that for structure preserving embeddings of spacetimes the correponding algebras should also be properly embedded. Hence consider a diffeomorphic embedding χ of a spacetime N into a spacetime M. χ is called admissible if it is isometric and preserves the causal relations. In particular, any causal curve connecting points in the image should be the image of a causal curve in N. Let $\mathcal{A}(N), \mathcal{A}(M)$ denote the corresponding algebras of local observables. Then there should exist an injective homomorphism $\alpha_\chi : \mathcal{A}(N) \to \mathcal{A}(M)$ such that for admissible embeddings $\chi : N \to M$ and $\psi : M \to L$ holds

$$\alpha_\psi \alpha_\chi = \alpha_{\psi\circ\chi} \ .$$

In more mathematical terms, the association of algebras to spacetimes should be a functor between the corresponding categories, endowed with the appropriate morphisms [20].

One can now characterize locally covariant interactions and locally covariant time ordered products in terms of natural transformations between the functor $\mathcal{A}$ of observable algebras and other functors on the category of spacetimes.

An important example is the concept of a locally covariant scalar field. It is an association of fields A_M for each spacetime M such that for any admissible embedding $\chi : N \to M$ the condition of local covariance

$$\alpha_\chi(A_N(x)) = A_M(\chi(x))$$

does hold. Note that this condition specializes on Minkowski space to the usual concept of a scalar field with the appropriate transformation under Poincaré transformations. The normal ordering prescription $: \bullet :_H$ in general does not produce locally covariant fields since there does not exist a choice of Hadamard functions H_M for every M such that $\chi^* H_M = H_N$. Instead one replaces in the normal ordering prescription for local functionals H by the first terms in the asymptotic expansion. Since this approximation of H is, in general, no longer a bisolution, one can find anomalies as e.g. the conformal anomaly of the energy momentum tensor [21]

For the discussion of a locally covariant time ordered product we use the renormalization group. We want to have on each spacetime M a renormalized time ordered $\cdot_{T_M}$ such that for an admissible embedding $\chi : N \to M$ and local functionals $F_1, \ldots, F_n$ the equation

$$\alpha_\chi(F_1 \cdot_{T_N} \cdots \cdot_{T_N} F_n) = \alpha_\chi(F_1) \cdot_{T_M} \cdots \cdot_{T_M} \alpha_\chi(F_n)$$

holds. If we performed the extension of the time ordered product on every spacetime independently we would obtain the equation above only after a finite renormalization. In order to analyze this structure it is convenient to pass to the generating functionals of the time ordered products (the *formal S-matrices*). According to the Main Theorem of renormalization (Stora-Popineau [22]) two possible time ordered products are related by the following relation between their respective formal S-matrices

$$\hat{S} = S \circ Z$$

where Z is an bijective map on the space of local functionals (with some further restrictions). Let now for each spacetime M a formal S-matrix S_M be given. Then, for any embedding $\chi : N \to M$, there exists a renormalization group element $Z(\chi)$ such that

$$\alpha_\chi \circ S_N = S_M \circ \alpha_\chi \circ Z(\chi) \ .$$

The association $\chi \to Z(\chi)$ satisfies the cocycle condition

$$\alpha_{\psi\circ\chi} \circ Z(\psi \circ \chi) = \alpha_\psi \circ Z(\psi) \circ \alpha_\chi \circ Z(\chi) \ .$$

The cocycle is trivial if there exist renormalization group transformations Z_N such that

$$\alpha_\chi \circ Z(\chi) = Z_M^{-1} \circ \alpha_\chi \circ Z_N \ .$$

In this case we define a covariant time ordered product in terms of generating functions by $\hat{S}_M = S \circ Z_M$.

A solution of this cohomological problem was found by Hollands and Wald [15, 16].

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