Positivity Bounds in EFTs (of Gravity).

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Positivity

$$\langle \Omega | \hat{\phi}(f)^2 | \Omega \rangle \geq 0$$

aka unitarity.

$$\hat{\phi}(f) = \int d^4x \, \phi(x) f(x).$$

'Unitarity' + assumptions of 'analyticity' ≡ causality

'polynomial boundedness' ≡ locality

⇒ constraints on signs and magnitudes of co-efficients in a LEEFT.

All 'cosmological' theories of inflation + dark energy are non-renormalizable.

eg Higgs inflation $\xi R |H|^2$

cutoff $\Lambda^2 \sim \frac{M_{pl}^2}{\xi}$

DBI / P-inflation / pseudo-Goldstone ...

$$\mathcal{L} = -\Lambda^4 p(X) \qquad X \sim \frac{(\partial\phi)^2}{\Lambda^4}$$

Wilsonian Effective action

TOP DOWN

$$e^{\frac{i}{\hbar} S_W[\text{Light}]} = \int D\,\text{Heavy}\; e^{\frac{i}{\hbar} S_{UV}[\text{Light, Heavy}]}$$

BOTTOM UP

$$S_W[\text{Light}] = \Lambda^4 F\left(\frac{\text{Boson}}{\Lambda}, \frac{\text{Fermion}}{\Lambda^{3/2}}, \frac{\partial_\mu}{\Lambda}\right)$$

'cutoff' or 'strong coupling scale'.

Weak or Strong

We introduce the notion of a weakly coupled EFT

Weakly coupled = New 'classical' tree level physics enters at scales below scale at which loops ~ trees.

$$S_W = \frac{1}{g_*^2} \int d^Dx \, \Lambda^D F_0 \left[\frac{\text{Boson}}{\Lambda^{\left(\frac{D-2}{2}\right)}}, \frac{\text{Fermion}}{\Lambda^{\left(\frac{D-1}{2}\right)}}, \frac{\partial}{\Lambda} \right]$$

$$+ \; g_*^0 \int d^Dx \, \Lambda^D F_1 [\, \dots \,] \qquad \text{(one-loop)}$$

$$+ \; g_*^2 \int d^dx \, \Lambda^D F_2 [\, \dots \dots \,] \qquad \text{(two-loop).}$$

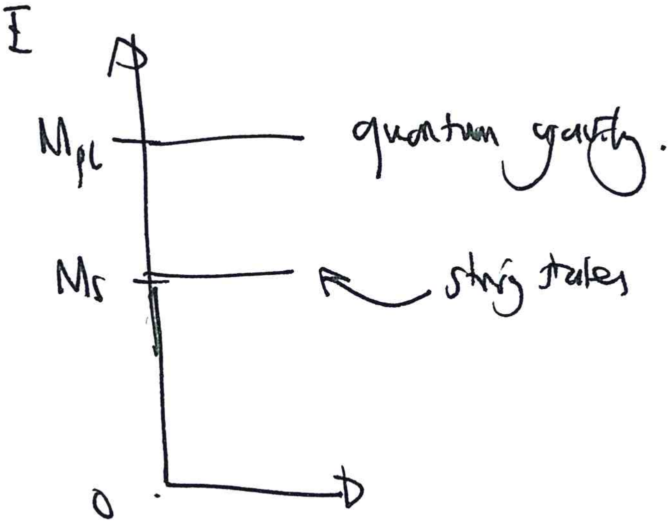
$$+ \; g_*^{2(n-1)} \; \dots \qquad \text{(n-loops)}$$

Example 1 - String theory.

$$\Lambda = M_s \qquad M_s^2 \sim \frac{1}{\alpha'} \qquad g_* = g_s$$

$$S \sim \int d^{10}x \, \frac{M_s^8}{g_s^2} \left(R + \ldots \, \alpha'^{p} R^{p+1} \ldots \right)$$

$$M_{pl}^8 = \frac{M_s^8}{g_s^2}$$



Example 2 - Kaluza-Klein theory

$$\int d^{4+N}x \; M_{4+N}^{2+N} R \sim \int \frac{M_{4+N}^{2+N}}{M_{KK}^{N}} R$$

$$M_{KK} \sim \frac{1}{L} \qquad \frac{1}{g_*^2} = \left(\frac{M_{4+N}}{M_{KK}}\right)^{2+N}$$

size of extra dimension

Strong coupling $\quad g_* \sim 1 \quad$ loops + trees are of same order at cutoff

eg $\quad S = \int d^4x \, \frac{M_{pl}^2}{2}\left[R + \alpha \frac{R^2}{M_{pl}^2} + \beta \frac{R_{\mu\nu}^2}{M_{pl}^2} + \dots \right]$

'traditional quantum gravity'.

Bottom up perspective – co-efficients of local operators in LEEFT cannot be determined and are just 'matched' against observations.

CLAIM – this is not true entirely, positivity bounds impose conditions on 'signs' and 'magnitudes' of Wilson coefficients that imply not all LEEFTs admit UV completion.

Kallen–Lehmann spectral rep

Consider scalar operator $\hat{O}(x)$ in Minkowski.

Locality tells us $[\hat{O}(x), \hat{O}(y)] = 0$, $(x-y)^2 > 0$

Unitary tells us that $\langle \psi | \hat{O}(f)^2 | \psi \rangle \geq 0$

$$\hat{O}(f) = \int d^4x \, f(x) \, \hat{O}(x).$$

Together with Poincaré invariance these imply

$$\langle 0 | T \hat{O}(x) \hat{O}(y) | 0 \rangle = \int \frac{d^d k}{(2\pi)^d} e^{ik.(x-y)} G_0(k)$$

$$G_0(k) = \frac{Z_p}{k^2 + m^2 - i\varepsilon} + S_{N-1}(k^2)$$

$$+ (-k^2)^N \int_{4m^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^N (k^2 + \mu - i\varepsilon)}$$

$\rho(\mu) \geq 0$ is the KL spectral density.

$$S_{N-1}(k^2) = \sum_{r=0}^{N-1} \frac{c_r}{r!} (-k^2)^r$$

i.e. $$\int \frac{d^dk}{(2\pi)^d} S_{N-1}(k^2) e^{ik.(x-y)} = \sum_{r=0}^{N-1} \frac{c_r}{r!} \Box^r \delta^d(x-y)$$

N = No. of subtractions

ie assumption is $\qquad \rho(\mu) < a\mu^N \qquad$ (Polynomial boundedness)

<u>Example - Conformal field theory</u>

$$\langle 0|\hat{O}(x)\hat{O}(y)|0\rangle = \frac{C_0}{((x-y)^2)^{\Delta}}$$ Δ conformal dimension.

$$= \int \frac{d^dk}{(2\pi)^d} e^{ik.(x-y)} c\int_0^\infty d\mu \frac{\mu^{-\frac{d}{2}+\Delta}}{(k^2+\mu-i\varepsilon)}$$

$$\Rightarrow \int \frac{d^dk}{(2\pi)^d} e^{ik.(x-y)} c(-k^2)^N \int_0^\infty d\mu \frac{\mu^{-\frac{d}{2}+\Delta-N}}{(k^2+\mu-i\varepsilon)}$$

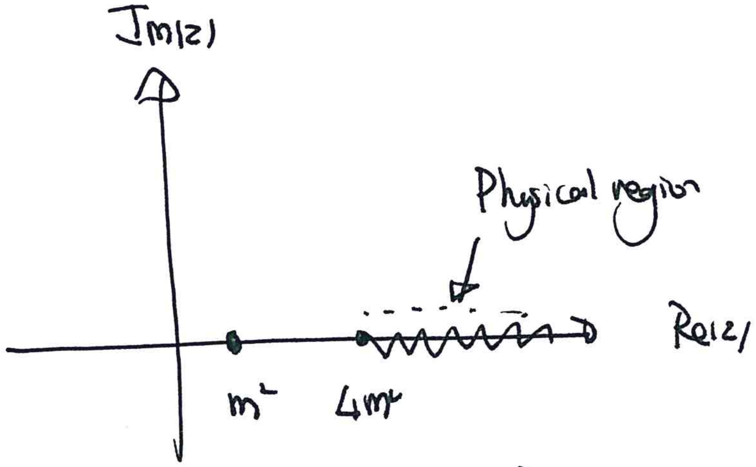
So every time $\Delta = \frac{d}{2} + k$ for $k = 0, 1, \ldots \infty$

we have to increase no of subtractions.

$\therefore$ For any field theory with a UV fixed point

$N_0 =$ Integer k just below $\left(\Delta - \frac{d}{2}\right)$.

Analytic structure



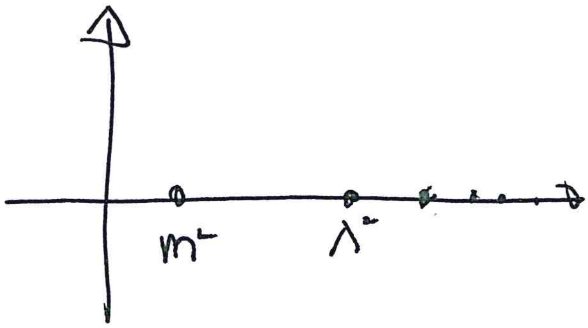
$$G_0(z) = \frac{Z_p}{m^2 - z} + S_{N-1}(z) + z^N \int_{4m^2}^{\infty} d\mu \, \frac{\rho(\mu)}{\mu^N (\mu - z)}$$

pole subtracted

$$\boxed{\frac{1}{M!} \frac{d^M G_0'}{dz^M}\bigg|_{z=0} = \int_{4m^2}^{\infty} d\mu \, \frac{\rho(\mu)}{\mu^{M+1}} > 0}$$

Assume weak coupling

Branch cut $\Rightarrow$ sum of poles



$$\rho(\mu) = \sum_i g_i^2 \, \delta(\mu - m_i^2)$$

$$\Lambda^2 = \mathrm{Min}(m_i^2)$$

What does this tell us about EFT ?

$$S = \int d^dx \; \hat{O}(x) \left[\Box + a_1 \frac{\Box^2}{\Lambda^2} + a_2 \frac{\Box^3}{\Lambda^4} + \dots \right] \hat{O}(x)$$

Tree level Feynman propagator is

$$G_0 = - \frac{1}{z + a_1 \frac{z^2}{\Lambda^2} + a_2 \frac{z^3}{\Lambda^4} + \dots}$$

$$G_0' = \frac{a_1}{\Lambda^2} + \frac{(a_2 - a_1^2) z}{\Lambda^4} + O(z^2) \qquad \text{(assuming no subtractions)}$$

Positivity bounds $\quad a_1 > 0 \qquad a_2 > a_1^2$

More precisely $\quad 0 < a_2 - a_1^2 < a_1 \quad$ since $\displaystyle \int_{\Lambda^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^2} < \frac{1}{\Lambda^2} \int_{\Lambda^2}^{\infty} \frac{d\mu \, \rho}{\mu}$

More generally ...

<u>UV theory</u>

$$G^{\prime}_{0\,UV}(z) = S_{N_0-1}(z) + z^{N_0}\int_{4m^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^{N_0}(\mu - z)}$$

<u>LEEFT</u>.

$$G_0^{LEEFT}(z) \sim \frac{1}{z}\left(1 + \sum c_{nm}\frac{z^n}{\Lambda^{2n}}(\ln(z))^m \cdots\right)$$

$$G_0^{LEEFT}(z) = S_{N_{EFT}}(z) + z^{N_{EFF}}\int_{4m^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^{N_{EFT}}(\mu - z)}$$

In general $N_{EFT} \gg N_0$

For $N_{EFT} > r \geq N_0$

$$\boxed{c_r = \int_{4m^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^r(\mu)} > \int_{4m^2}^{(\epsilon\Lambda)} d\mu \frac{\rho(\mu)}{\mu^{r+1}}}$$

What about gravity?

eg extra dimensional models (Weakly coupled).

$\Downarrow$

4d gravity + ∞ tower of spin 2, 0 modes

$$S \approx \frac{M_p^2}{2}\int d^4x\sqrt{g_E}\Big[R_E - \frac{1}{2}\sum_i(\nabla X_i)^2 - \frac{1}{2}m_{2i}^2(X_i^2 - X_i^2)$$

$$- \frac{1}{2}\sum_j (\nabla\phi_j)^2 - \frac{1}{2}m_j^2\phi_j^2$$

$$+ \mathcal{L}_M[g_J, X_M]$$

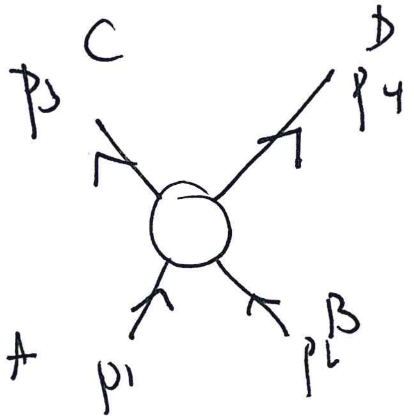
$$g_J \approx g^E_{E\mu\nu} + \sum_i \frac{\alpha_i}{\Lambda}X_{\mu\nu} + \sum_j \frac{\beta_j}{\Lambda} g^E_{\mu\nu} X_j$$

$$\Rightarrow \quad S = \int d^4x\sqrt{g_J}\,\frac{M_p^2}{2}\Big[R_J + \frac{a}{\Lambda^2}\overbrace{(R^J_{\mu\nu}R_J^{\mu\nu} - \tfrac{1}{3}R_J^2)}^{\text{spin 2}}$$

$$+ \frac{b}{\Lambda^2}\underbrace{R^2}_{\text{scalar part.}}$$

1960's S-matrix assumptions

1. Unitarity : $\hat{S}^{+}\hat{S} = \hat{I}$
2. Locality : Scattering amplitude polynomial (exponentially bounded) $|A(k)| < \alpha e^{\beta|k|}$
3. Causality : Analytic function of Mandelstam variables (modulo poles + cuts).
4. Poincaré Invariance
5. Crossing Symmetry : Follows from above
6. Mass-Gap :



S-channel

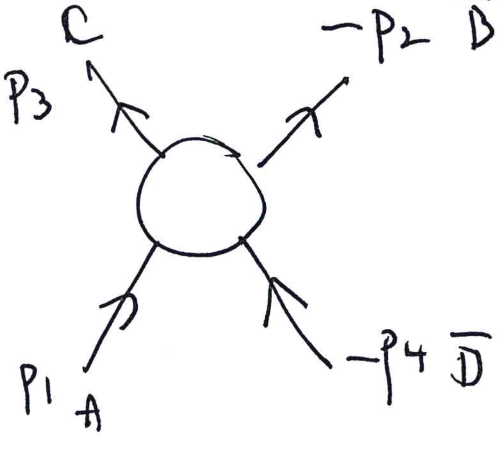
$A + B \to C + D$

$s + t + u = 4m^2$

$s = (p_1 + p_2)^2$

$t = (p_1 - p_3)^2$

$u = (p_1 - p_4)^2$



u-channel

$A + \bar{D} \to C + \bar{B}$

Jin-Martin (generalization of Froissart).

$$-\#m^2 < t < 4m^2 \qquad |A(s,t)| < s^2$$

(usual Froissart $|A(s,0)| < s \ln s$).

$\boxed{0 \leq t < 4m^2}$

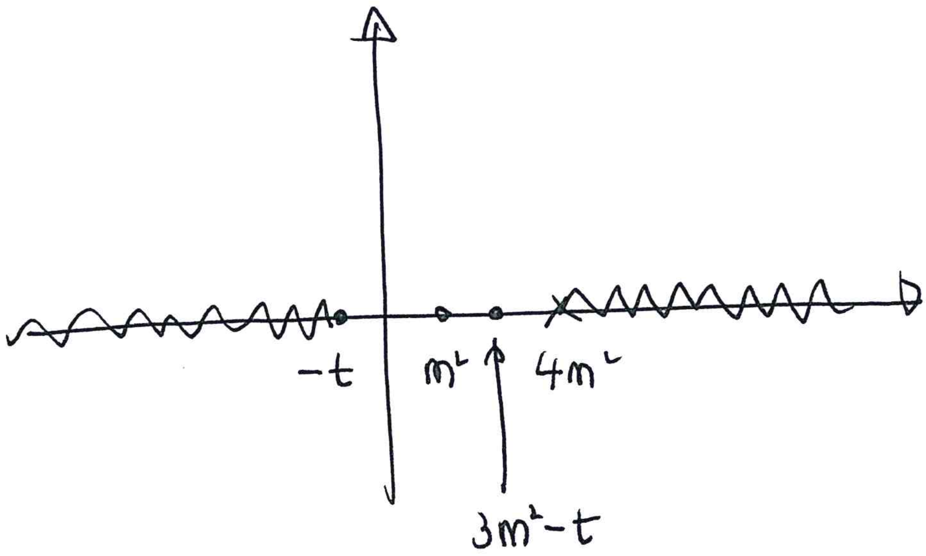
$$\boxed{\begin{aligned} A_{AB\to CD}(s,t) = \; & a(t) + b(t)\,s \\ & + \frac{s^2}{\pi}\int_{4m^2}^{\infty} d\mu \, \frac{\operatorname{Im} A_{AB\to CD}(\mu,t)}{\mu^2(\mu - s)} \\ & + \frac{u^2}{\pi}\int_{4m^2}^{\infty} d\mu \, \frac{\operatorname{Im} A_{A\bar{D}\to C\bar{B}}(\mu,t)}{\mu^2(\mu - u)} \end{aligned}}$$

s-channel

u-channel.

$$\boxed{\left.\frac{\partial^n}{\partial t^n} \operatorname{Im} A_{AB\to CD}(\mu,t)\right|_{t=0} > 0 \quad \forall n.}$$

AND same for u-channel. $\left(\frac{\partial^n}{\partial t^n}\left(P_L(\cos\theta)\right) > 0 \right)$



Implications . . . .

$$\frac{\partial^{2+N}}{\partial s^{2+N}} A'_{AB\to CD}(s,t) > 0 \qquad \underline{0 \le t < 4m^2}.$$

$M = 2+N$

$$\frac{1}{M!}\frac{d^M}{ds^M} A'_{AB\to CD}(s,t)\Big|_{s=2m^2} = \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{\operatorname{Im} A_s(\mu,t)}{(\mu-2m^2+t/2)^{M+1}}$$

$$+ \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{\operatorname{Im} A_u(\mu,t)}{(\mu-2m^2+t/2)^{M+1}} > 0.$$

eg $\quad t=0 \qquad S = -\frac{1}{2}(\partial\phi)^2 + \frac{c}{\Lambda^4}\left((\partial\phi)^2\right)^2 \ldots$

$$A \sim \frac{c}{\Lambda^4}(s^2+t^2+u^2 \cancel{-4m^4}) \Rightarrow \qquad c>0$$

Non-forward limit bounds

define $\quad f(s,t) = \frac{1}{2}\frac{\partial^2}{\partial s^2} A'(s,t)$

We derive: $\quad f(s,t) > 0$

$$\frac{\partial f(s,t)}{\partial t} + \frac{3}{2M^2} f(s,t) > 0$$

$$\frac{1}{2}\frac{\partial^2 f(s,t)}{\partial t^2} + \frac{3}{2M^2}\left(\frac{\partial f(s,t)}{\partial t} + \frac{3}{2M^2} f(s,t)\right) > 0$$

$\vdots$

where $M = 2m$ or Λ

(

Application to general spin

Problems $\cos\theta = -\frac{2t}{(s-4m^2)}$

Kinematic poles $s=4m^2$

$\sqrt{stu}$ branch cuts

For DF scalars $\sqrt{-su}$ branch cuts

Crossing complicated in helicity

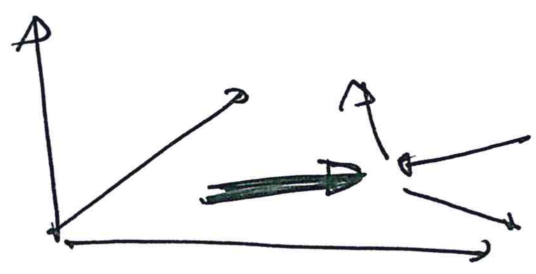
$$A^s_{\lambda_1\lambda_2\lambda_3\lambda_4}(s,t) = \text{Phase} \times$$

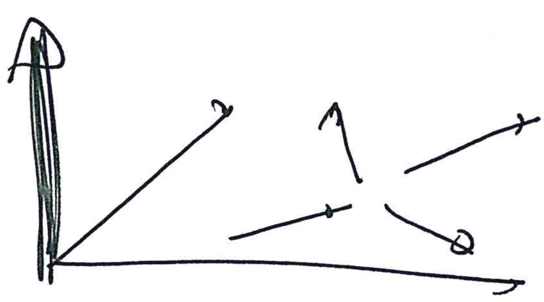
$$\sum_{\lambda_1'\lambda_2'\lambda_3'\lambda_4} d^{s_1}_{\lambda_1'\lambda_1}(\pi-\chi)\, d^{s_2}_{\lambda_2'\lambda_2}(\chi)\, d^{s_3}_{\lambda_3'\lambda_3}(\chi-\pi)\, d^{s_4}_{\lambda_4'\lambda_4}(-\chi)$$

$$\times \quad A^u_{\lambda_1'\lambda_4'\lambda_3'\lambda_2'}(u,t)$$

Wigner J matrices

$$\sin\chi = \frac{-2m\sqrt{t}}{\sqrt{(s-4m^2)(u-4m^2)}}$$





$$T_{\tau_1\tau_2\tau_3\tau_4} = \sum_{\lambda_1\lambda_2\lambda_3\lambda_4} u^{s_1}_{\lambda_1\tau_1} u^{s_2}_{\lambda_2\tau_2} u^{s_3 *}_{\tau_3\lambda_3} u^{s_4 *}_{\tau_4\lambda_4} A_{\lambda_1\lambda_2\lambda_3\lambda_4}$$

$$T^s_{\tau_1\tau_2\tau_3\tau_4}(s,t,u) = e^{-i\sum_i \tau_i \chi}\, T^u_{-\tau_1-\tau_4-\tau_2-\tau_3}(u,t,s)$$

Regularized transversity amplitudes

$$\boxed{T^{+}_{T_1T_2T_3T_4}(s,\theta) = \left(\sqrt{-su}\right)^{\xi}\left(s(s-4m^2)\right)^{S_1+S_2} \times \left(T_{T_1T_2T_3T_4}(s,\theta) + T_{T_1T_2T_3T_4}(s,-\theta)\right)}$$

def $N_s = 2 + \xi + 2(S_1+S_2)$

$$\boxed{f_{T_1T_2}(s,t) = \frac{1}{N_s!}\,\frac{d^{N_s}}{ds^{N_s}}\,T^{+\,\prime}_{T_1T_2T_1T_2}(s,t)}$$

$$s = v + 2m^2 - t/2$$

$$\boxed{f_{T_1T_2}(v,t) = \frac{1}{\pi}\int_{4m^2}^{\infty} d\mu\, \frac{\mathrm{Abs}_s\, T^{+}_{T_1T_2T_1T_2}(\mu,t)}{(\mu - 2m^2 + t/2 - v)^{N_s+1}} + \frac{1}{\pi}\int_{4m^2}^{\infty} d\mu\, \frac{\mathrm{Abs}_u\, T^{+}_{u\,T_1T_2T_1T_2}(\mu,t)}{(\mu - 2m^2 + t/2 + v)^{N_s+1}}}$$

Application 1 – Weak Gravity conjecture

Bellazzini et al.

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{pl}^2}{2} R - \frac{1}{4} F_{\mu\nu}^2 \right.$$

$$+ \frac{\alpha_1}{4M_{pl}^4} \left(F_{\mu\nu}^2\right)^2 + \frac{\alpha_2}{4M_{pl}^4} \left(\tilde{F}^{\mu\nu} F_{\mu\nu}\right)^2$$

$$\left. + \frac{\alpha_3}{2M_{pl}^2} F_{\mu\nu} F_{\alpha\beta} W^{\mu\nu\alpha\beta} \right]$$

Weyl

$$A^{++}_{4\gamma} = \underbrace{\frac{-s^2}{M_{pl}^2 t}}_{\text{if removed}} - \frac{s}{M_{pl}^2} + \frac{2s^2(2\alpha_1 - \alpha_3)}{M_{pl}^4}$$

$$\Rightarrow \quad (2\alpha_1 - \alpha_3) > 0$$

$$\left(\frac{\sqrt{2}\,|Q|}{M/M_{pl}}\right)_{ext} = 1 + \frac{4}{5}\,\frac{(4\pi)^2 M_{pl}^2\,(2\alpha_1-\alpha_3)}{M^2} > 1$$

Implies Mild form of weak gravity conjecture

(If no $Q>M$ since $M > \sum_i m_i$ $Q=\sum q_i$
kinematically forbidden).