# Discrete gauge theories of charge conjugation

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Based on `arXiv:1903.06662`, `arXiv:1804.01108`, w/G.Arias-Tamargo, A.Bourget & A.Pini

- A huge effort has been/is dedicated to Gauge Theories. It is probably fair to say that most studies are for connected gauge groups (at least comparatively).

- In fact…

VOLUME 62, NUMBER 11 PHYSICAL REVIEW LETTERS 13 MARCH 1989

**Discrete Gauge Symmetry in Continuum Theories**

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(Received 16 December 1988)

We point out that local symmetries can masquerade as discrete global symmetries to an observer equipped with only low-energy probes. The existence of the underlying local gauge invariance can, however, result in observable Aharonov-Bohm–type effects. Black holes can therefore carry discrete gauge charges—a form of nonclassical "hair." Neither black-hole evaporation, wormholes, nor anything else can violate discrete gauge symmetries. In supersymmetric unified theories such discrete symmetries can forbid proton-decay amplitudes that might otherwise be catastrophic.

PACS numbers: 11.30.−j, 13.30.Ce, 97.60.Lf

Although it is a common and fruitful practice to consider local gauge invariance under discrete groups in lattice theories, the implications of such invariance in the continuum have not been widely discussed. (They have been invoked in one class of solutions to the axion domain-wall problem.[1,2])

At first sight the notion of local discrete symmetry in the continuum appears rather silly. Indeed, the most important dynamical consequence of a continuous local symmetry is the existence of a new field, the gauge field. This field is introduced in order to formulate covariant derivatives. Covariant derivatives are, of course, necessary so that invariant interactions involving gradients may be formed; such interactions in turn are necessary in *cal symmetry is a statement that the variables used in a physical theory are redundant.* In language that may be more familiar, this redundancy is often stated as the fact that in a gauge theory, only gauge-invariant quantities are physically meaningful.

From this point of view, it is clear that no processes, not even such exotic ones as black-hole evaporation or wormhole tunneling, can violate a gauge symmetry. There are two striking theoretical consequences of this observation:

(i) It has been argued recently that wormhole tunneling induces all local interactions consistent with *continuous* gauge symmetries.[3] (The restriction to continuous local symmetries is not always made explicitly, but has

- Nevertheless, on a second thought, discrete gauge theories can find many interesting applications e.g.

    - Many Condensed Matter (and lattice) models include discrete gauge theories.

    - Discrete symmetries are often needed in BSM to avoid forbidden processes (such as proton decay). If only global, when coupled to gravity, BH's would wash out their effect. This suggests to gauge them (indeed, in pheno scenarios, discrete gauge theories appear ``often")

- The last point suggests that all global symmetries, including discrete ones, should be gauged at a fundamental level. Indeed, this story has been recently revived

  Banks & Seiberg'10

- Discrete gauge theories also appear associated to the global properties of the gauge group: for instance, in a O(2N+1) gauge theory, the gauge group is the direct product of SO(2N+1) and a discrete Z2

- In the very controlled set-up of (at least) N=2 4d theories, the "landscape" contains theories which include discrete components in the gauge group

These "corners of the N=2 landscape" actually hide very interesting treasures: upon gauging appropriately chosen discrete symmetries (inlcuding S-duality), one may find N=3 theories

Garcia-Etxebarria & Regalado'16

- Today we will explore the simplest such case: the gauging of charge conjugation symmetry

In the past, this has been partially considered for phenomenological aplications by Schwartz in 1982, leading to the famous Alice strings (in fact, it has found a Condensed Matter avatar by Leonhardt & Volovik'00 )

# Contents

- Introduction
- A primer in Principal Extensions
- 4d N=2 theories based on Principal Extensions
- Coulomb branches
- Higgs branches
- ~~Conclusions~~ Open questions

- A first approach to gauging charge conjugation is to take some theory with such symmetry and quotient by it…

- …however this is subtle See *e.g.* Argyres & Martone

- Charge conjugation is essentially complex conjugation. It mixes nontrivially with gauge transformations

$$(G_2 \circ C \circ G_1)\psi = G_2 C(e^{i\alpha_1}\psi) = G_2(e^{-i\alpha_1}\psi^\star) = e^{i(-\alpha_1+\alpha_2)}\psi^\star$$
$$(G_1 \circ C \circ G_2)\psi = G_1 C(e^{i\alpha_2}\psi) = G_1(e^{-i\alpha_2}\psi^\star) = e^{i(\alpha_1-\alpha_2)}\psi^\star$$

- So the combination of G and C cannot simply be the direct product G x C

- So one needs a fresh start. A natural approach is to consider a gauge group which, *ab initio* includes the gauging of charge conjugation

- If such thing exists (not obvious *a priori*), the standard technology can be directly imported

- Today we will argue for the existence of such gauge groups: in the math literature they are called Principal Extensions

- They lead to very surprising consequences

    - Non-freely generated Coulomb branches (contrary to standard lore, first example of such thing!!!)

    - An “exotic” pattern of global symmetries

- Note that, starting with these “new” gauge groups we may consider gauge theories in arbitrary dimensions…

- Today we will concentrate on the 4d N=2 case for definitness…

…but there is a whole new world to explore!!!

# A primer in Principal Extensions

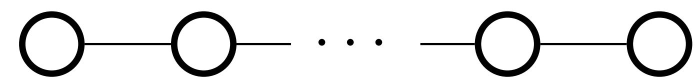
- Charge conjugation is essentially complex conjugation. It mixes nontrivially with gauge transformations

$$(G_2 \circ C \circ G_1)\psi = G_2 C(e^{i\alpha_1}\psi) = G_2(e^{-i\alpha_1}\psi^\star) = e^{i(-\alpha_1+\alpha_2)}\psi^\star$$
$$(G_1 \circ C \circ G_2)\psi = G_1 C(e^{i\alpha_2}\psi) = G_1(e^{-i\alpha_2}\psi^\star) = e^{i(\alpha_1-\alpha_2)}\psi^\star$$

- So the combination of G and C cannot simply be the direct product G x C

- Let us take instead a fresh start…Let's consider the group SU(N). Its Dynkin diagram is

$A_{N-1}$

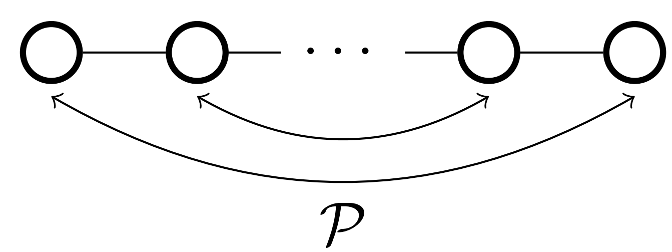


- As a graph, it has an automorphism group $\Gamma$ of order 2

$$\Gamma = \{1,\, \mathcal{P}\} \sim \mathbb{Z}_2$$

- In the graph

$A_{N-1}$




- Flipping the Dynkin diagram is like exchanging fundamental and antifundamental: we want an extension of SU(N) by the outer automorphism group of the Dynkin diagram and use that as gauge group (Principal Extension*)

*In an enlarged sense

- An extension is the exact sequence of groups

$$1 \to N \xrightarrow{\iota} \widetilde{G} \xrightarrow{q} G \to 1$$

- It turns out that if one can define a morphism as below the sequence is said to be split and the extension is a semidirect product

$$\sigma : G \to \widetilde{G} \quad / \quad q \circ \sigma = \mathrm{id}_G$$

- Let us work out an example: consider O(N).

$$M^T M = \mathbb{1}$$

$$A = R \qquad B = R\, I \qquad I = \left(\begin{array}{c|c} -1 & \\ \hline & \mathbb{1}_{N-1} \end{array}\right) \qquad g(R,\, \gamma) \to M\, \mathcal{I} \qquad \mathcal{I} = \{\mathbb{1},\, I\}$$

- Note that the product rule is that of a semidirect product

$$g(R_1,\, \gamma_1)\, g(R_2,\, \gamma_2) = g(R_1\, \gamma_1\, R_2\, \gamma_1,\, \gamma_1\, \gamma_2)$$

- We can also regard it as an extension of SO(N) by Z(2)

$$\begin{array}{rccc} q: & O(N) & \to & \Gamma \\ & g(R,\, \gamma) & \mapsto & \gamma \end{array} \qquad \begin{array}{rccc} \iota: & SO(N) & \to & O(N) \\ & R & \mapsto & g(R,\, e) \end{array} \qquad \begin{array}{rccc} r: & \Gamma & \to & 0 \\ & \gamma & \mapsto & 0 \end{array} \qquad \begin{array}{rccc} l: & 0 & \to & SO(N) \\ & 0 & \mapsto & \mathbb{1} \end{array}$$

$$\mathrm{Ker}\, r = I = \mathrm{Im}\, q \qquad \mathrm{Im}\, l = 1 = \mathrm{Ker}\, \iota \quad \longrightarrow \quad 1 \to SO(N) \xrightarrow{\iota} O(N) \xrightarrow{q} \mathbb{Z}_2 \to 1$$

- Note that the sequence is split!

$$\begin{array}{rccc} \sigma: & \Gamma & \to & O(N) \\ & \gamma & \mapsto & g(\mathbb{1},\, \gamma) \end{array} \qquad q \circ \sigma = e$$

- Actually, if N is odd it becomes a direct product. If N is even this is actually a Principal Extension!

- Likewise, one may do the same with SU(N) and extend it with the order 2 group of automorphisms of the Dynkin diagram

$$\begin{array}{cccc} & SU(N) & & \widetilde{SU(N)} \\ \iota: & N & \to & \widetilde{G} \\ & A & \mapsto & (A,\, 1) \end{array} \qquad\qquad \begin{array}{cccc} & \widetilde{SU(N)} & & \Gamma \\ q: & \widetilde{G} & \to & G \\ & (A,\, \alpha) & \mapsto & \alpha \end{array}$$

- With this one can construct the Principal Extension

$$1 \to \overset{SU(N)}{N} \xrightarrow{\iota} \overset{\widetilde{SU(N)}}{N \rtimes_S G} \xrightarrow{q} \overset{\Gamma}{G} \to 1$$

- The sequence is split and so again the Principal Extension will be a semidirect product

$$\begin{array}{cccc} \sigma: & G & \to & \widetilde{G} \\ & \alpha & \mapsto & (1,\, \alpha) \end{array}$$

- Let's try to find a matrix representation

$$G = \left\{ \mathbf{U} = \begin{pmatrix} M & 0 \\ 0 & M^\star \end{pmatrix} \middle| M \in \mathrm{SU}(N) \right\} \cong \mathrm{SU}(N)\,, \qquad \Gamma_A = \left\{ \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \begin{pmatrix} 0 & A \\ A^{-1} & 0 \end{pmatrix} \right\} \cong \mathbb{Z}_2\,,$$

- Demanding that these matrices consistently multiply according to the semidirect product rule gives

$$A\,A^\star = \begin{cases} \pm\mathbb{1} & \text{for } N \text{ even} \\ +\mathbb{1} & \text{for } N \text{ odd}\,. \end{cases}$$

- Thus there are two types of Principal Extension groups

$$\begin{cases} \widetilde{SU}(N)_I : & A = -A^T \\ \widetilde{SU}(N)_{II} : & A = A^T \quad (\text{even}\, N) \end{cases}$$

Arias-Tamargo, Bourget, Pini & D.R-G'19

- The semidirect product is then defined through the homomorphism in its product rule

- This homomorphism has to be an involution of G.

- Hence another way (actually more rigurous) to argue for our groups is by studying the classification of involutive automorphisms of G.

- This problem as solved by Cartan: our problem boils down to checking the classification of symmetric spaces!

| Cartan Class | $G$ | $K$ | $\dim K$ | Involution $\Theta$ | |
|---|---|---|---|---|---|
| $AI$ | $\mathrm{SL}(N,\mathbb{R})$ | $\mathrm{SO}(N)$ | $\frac{1}{2}N(N-1)$ | $g \mapsto (g^{-1})^T$ | $\Leftarrow \ \widetilde{SU}(N)_I$ |
| $AII$ ($N$ even) | $\mathrm{SL}(N/2,\mathbb{H})$ | $\mathrm{USp}(N)$ | $\frac{1}{2}N(N+1)$ | $g \mapsto -J_N(g^{-1})^T J_N$ | $\Leftarrow \ \widetilde{SU}(N)_{II}$ |
| $AIII, AIV$ | $\mathrm{SU}(p, N-p)$ | $S(\mathrm{U}(p)\times \mathrm{U}(N-p))$ | $p^2+(N-p)^2-1$ | $g \mapsto I_{p,N-p}\, g\, I_{p,N-p}$ | |

Arias-Tamargo, Bourget, Pini & D.R-G'19

- We have established the existence of our groups and exhausted their classification

- In particular, the matrix representation above provides a concrete realization. Recall

$$G = \left\{ \mathbf{U} = \begin{pmatrix} M & 0 \\ 0 & M^\star \end{pmatrix} \middle| M \in \mathrm{SU}(N) \right\} \cong \mathrm{SU}(N)\,, \qquad \Gamma_A = \left\{ \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \begin{pmatrix} 0 & A \\ A^{-1} & 0 \end{pmatrix} \right\} \cong \mathbb{Z}_2\,,$$

- With this we can explicitly construct a few particularly interesting representations for our purposes

- This acts on a 2N dim. space: the fundamental representation. Explicitly

$$\mathbf{Q} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \overline{\mathbf{Q}} = \mathbf{Q}^T \, \mathbf{\Gamma_0} \,, \qquad \text{with} \qquad \mathbf{\Gamma_0} = \begin{pmatrix} 0 & \mathbb{1} \\ -c\,\mathbb{1} & 0 \end{pmatrix} \; \vdots \; c = \begin{cases} -1 & \text{for type I} \\ +1 & \text{for type II} \end{cases}$$

- The transformation rule is

$$\mathbf{Q} \to \widetilde{\mathbf{U}}\,\mathbf{Q}\,, \qquad \overline{\mathbf{Q}} \to \overline{\mathbf{Q}}\,\widetilde{\mathbf{U}}^\dagger\,.$$

- The adjoint in turn is simply

$$\mathbf{\Phi} = \begin{pmatrix} \Phi & 0 \\ 0 & -\Phi^\star \end{pmatrix}\,.$$

- The transformation rule is

$$\mathbf{\Phi} \to \widetilde{\mathbf{U}}\,\mathbf{\Phi}\,\widetilde{\mathbf{U}}^\dagger\,.$$

- We may now constuct some interesting invariants

  - meson-like: $\mathbf{M}_{IJ} = (\overline{\mathbf{Q}}_I)_{\widetilde{\alpha}}\,(\mathbf{Q}_J)^{\widetilde{\alpha}} \equiv \overline{\mathbf{Q}}_I\,\mathbf{Q}_J\,.$ $\mathbf{M}_{IJ} = -c\,\mathbf{M}_{JI}$
  - superpotential-like: $\overline{\mathbf{Q}}_I\,\mathbf{\Phi}\,\mathbf{Q}_J\,.$ $\overline{\mathbf{Q}}_I\,\mathbf{\Phi}\,\mathbf{Q}_J = c\,\overline{\mathbf{Q}}_J\,\mathbf{\Phi}\,\mathbf{Q}_I$
  - Kahler-like: $\mathbf{Q}_I^\dagger\,\mathbf{Q}_J$

- We can also write an explicit Haar measure on the groups

$$\int_{\widetilde{\mathrm{SU}}(N)_{\mathrm{I,II}}} d\mu(X) f(X) = \frac{1}{2}\left(\int_{\mathrm{SU}(N)} d\mu^+(z) f(z) + \int_{\mathrm{SU}(N)\Theta_{\mathrm{I,II}}} d\mu^-_{\mathrm{I,II}}(z) f(\Theta_{\mathrm{I,II}}(z))\right),$$

(see Arias-Tamargo, Bourget, Pini & D.R-G. for details)

# $4\mathrm{d}\ \mathcal{N}=2$ based on $\widetilde{SU}(N)$

- One may imagine gauge theories based on Principal Extensions

- Since, at the end of the day, Principal Extensions are simply Lie groups, one can construct gauge theories following the textbook procedure

- This can be done in arbitrary dimensions. Today concentrate on N=2 in 4d as proof of concept

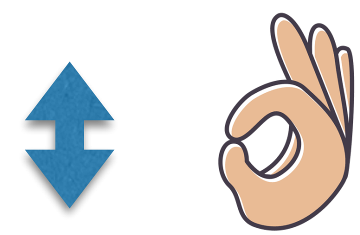
- To that matter we will use the fundamental and adjoint as above

- The W will come from the superpotential-like invariant. Note that

$$\overline{\mathbf{Q}}_I\,\mathbf{\Phi}\,\mathbf{Q}_J = c\,\overline{\mathbf{Q}}_J\,\mathbf{\Phi}\,\mathbf{Q}_I$$

$$\mathbf{M}_{IJ} = -c\,\mathbf{M}_{JI}$$

The meson is Symm (adjoint of Sp)/Antisymm (adjoint of SO) depending on the type or rep.



This quantity is Antisymm/Symm depending on the type of rep. We can form an invariant (the W!) contracting with the symplectic matrix/identity matrix

$$W = \overline{\mathbf{Q}}_J\,\mathbf{\Phi}\,\mathbf{Q}_I\,\mathbf{G}^{IJ} \longrightarrow$$

- $\widetilde{\mathrm{SU}}(N)_{\mathrm{I}}$: $\mathbf{G}$ is antisymmetric. The global symmetry is $\mathrm{Sp}(\frac{F}{2})$.
- $\widetilde{\mathrm{SU}}(N)_{\mathrm{II}}$: $\mathbf{G}$ is symmetric. The global symmetry is $\mathrm{SO}(F)$.

# Higgs/Coulomb branches

- One particularly powerful tool to study theories is to compute their index: information about the protected operators

$$I = \int d\eta_{\widetilde{SU(N)}}\,\mathrm{PE}[f]$$

"Single particle" contribution

- In particular, we have the integration formula over Principal Extensions, and so we can hope to extract protected useful information

# Higgs branches

- We are considering SQCD-like theories, with one vector multiplet and F matter fields (real/pseudo representations)

- In the SU theory the flavor symmetry would be U(F). What about the Principal Extension?

- We can use the index as a probe. This time we will compute the Hall-Littlewood limit of the index, a.k.a. Higgs branch Hilbert series Gadde, Rastelli, Razamat& Yan'13

$$HS_{(N,\,N_f)} = \int_{G^\circ} d\eta_{G^\circ}(X)\, \frac{\det\left(1 - t^2 \Phi_{\mathrm{Adj}}(X)\right)}{\det\left(1 - t\Phi_{\mathrm{F\bar{F}}}(X)\right)^{N_f}} \,,$$

- For instance

$$\mathrm{HS}^{\mathrm{I}}_{(4,8)}(t) = \frac{1}{(1-t^2)^{34}(1+t^2)^{17}}\Big(1 + 19t^2 + 621t^4 + 9672t^6 + 115781t^8 + 1012392t^{10} + 6929353t^{12} +$$
$$37647616t^{14} + 166763191t^{16} + 610159441t^{18} + 1871499527t^{20} + 4855440684t^{22} + 10751422823t^{24} +$$
$$20435224870t^{26} + 33521903017t^{28} + 47610887368t^{30} + 58717583354t^{32} + 62951199956t^{34} +$$
$$\ldots \; + \; \text{palindrome} \; + \ldots + t^{68}\Big) \; ,$$

$$\mathrm{HS}^{\mathrm{II}}_{(4,8)}(t) = \frac{1}{(1-t^2)^{34}(1+t^2)^{17}}\Big(1 + 11t^2 + 749t^4 + 8520t^6 + 123173t^8 + 975504t^{10} +$$
$$7079801t^{12} + 37130520t^{14} + 168290287t^{16} + 606231681t^{18} + 1880386783t^{20} + 4837617956t^{22} +$$
$$10783278743t^{24} + 20384258878t^{26} + 33595129641t^{28} + 47516178744t^{30} + 58828027690t^{32} +$$
$$62834962052t^{34} + \; \ldots \; + \; \text{palindrome} \; + \ldots + t^{68}\Big) \; ,$$

For more examples, refined & unrefined, check Bourget, Pini & D.R-G'18 and Arias-Tamargo, Bourget, Pini & D.R-G'19

- In particular, just as expected

$$\begin{cases} \widetilde{SU}(N)_I \, : \; Sp(\frac{F}{2}) \text{ global symmetry} \quad (\text{even } F) \\ \\ \widetilde{SU}(N)_{II} \, : \; SO(F) \text{ global symmetry} \quad (\text{even } N) \end{cases}$$

# Coulomb branches

- Let us now turn to the Coulomb branch
- We now explicitly need to be inside the conformal window (otherwise very complicated!)
- …so assume N, F are tuned so that we have a CFT.

- Use the index…one limit is sensitive only to the Coulomb branch!

$$f^{\frac{1}{2}H}=0 \qquad f^{V}=t$$

Gadde, Rastelli, Razamat& Yan'13

- The Coulomb branch index becomes

$$\mathcal{I}^{\text{Coulomb}}_{\widetilde{\text{SU}}(N)}(t)=\frac{1}{2}\left[\prod_{i=2}^{N}\frac{1}{1-t^i}+\prod_{i=2}^{N}\frac{1}{1-(-t)^i}\right].$$

- This can be re-written as

A.Bourget, A.Pini & D.R-G'18
Argyres & Martone'18
Bourton, Pini & Pomoni'18

$$\mathcal{I}^{\text{Coulomb}}_{\widetilde{\text{SU}}(N)}(t)=\frac{\sum\limits_{k_1<\cdots<k_r\ \text{odd}} t^{k_1+\cdots+k_r}}{\prod\limits_{i\ \text{even}}(1-t^i)\prod\limits_{i\ \text{odd}}(1-t^{2i})},$$

**Non-freely generated Coulomb branch in general!!!!**

- Being more explicit

| $N$ | PL of $\mathcal{I}^{\text{Coulomb}}_{\widetilde{\text{SU}}(N)}(t)$ |
|---|---|
| 2 | $t^2$ |
| 3 | $t^2 + t^6$ |
| 4 | $t^2 + t^4 + t^6$ |
| 5 | $t^2 + t^4 + t^6 + t^8 + t^{10} - t^{16}$ |
| 6 | $t^2 + t^4 + 2t^6 + t^8 + t^{10} - t^{16}$ |
| 7 | $t^2 + t^4 + 2t^6 + t^8 + 2t^{10} + t^{12} + t^{14} - t^{16} - t^{18} + \ldots$ (infinite) |

- Thus, from N=5 on we have a non-frely generated Coulomb branch. Note that N=4 is secretly O(6) (which should have a freely generated CB) and N=2 is trivial (and so should have a freely generated CB)

# Summary

- We have introduced a “new” family of gauge groups on which gauge theories can be based
- Actually one particular case is SO vs. O. Also the SU case made a modest appeareance
- The rules etc. to construct them are therefore just the usual ones. We could construct them in arbitrary dimensions. Today focused on 4d N=2.

- A powerful tool to explore the theories is the index (in particular because we have an integration formula). Using it we have seen that

    - The Coulomb brach is generically non-freely generated (first example of such a thing!)

    - The Higgs branch exhibits an “exotic” pattern of global symmetries

# Open questions

- This only touches upon the tip of the iceberg… there are loads of things to explore. For instance, in random order
  - String embedding??? Harvey & Royston'07
  - Global properties of the theories, spectrum of line operators…
  - Construction of quivers???
  - Versions in other dimensions (where perhaps other phenomena manifest)??
  - …

Thanks!