The dual of generic $\mathcal{N} = 1$ marginal deformations

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Introduction

Type IIB gravity duals of $\mathcal{N} = 1$ SCFT

- Sasaki–Einstein (S^5, $T^{1,1}$, $Y_{p,q}$, ...) ($F_5 \neq 0$)
- Pilch–Warner, β-deformation, ... (all fluxes, but also isometries)

Are there more generic constructions?

Classic example: $\mathcal{N} = 1$ marginal deformations for $\mathcal{N} = 4$

Superpotential deformation [*Leigh & Strassler*]

$$\mathcal{W} = \epsilon_{ijk} \operatorname{tr} Z^i Z^j Z^k + f_{ijk} \operatorname{tr} Z^i Z^j Z^k$$

- f_{ijk} symmetric giving 10 complex marginal deformations
- but beta-function constrains moment map for SU(3) symmetry

$$f_{ikl}\bar{f}^{jkl} - \tfrac{1}{3}\delta_i^j f_{klm}\bar{f}^{klm} = 0$$

- exactly marginal deformation as symplectic quotient [*Kol; Green et al*]

$$\widetilde{\mathcal{M}} = \{f_{ijk}\} /\!\!/ \mathrm{SU}(3)$$

Can choose, including gauge coupling,

$$\Delta\mathcal{W} = \lambda_1 \operatorname{tr} Z^1 Z^2 Z^3 + \lambda_2 \operatorname{tr}\left[(Z^1)^3 + (Z^2)^3 + (Z^3)^3\right] + \Delta\tau \operatorname{tr} W_\alpha W^\alpha$$

- $\lambda_2 = 0$: "beta-deformation", $U(1)^3$ isometry, exact dual solution [*Lunin & Maldacena*]
- generic: no isometries, perturbative tour de force to 3rd order [*Aharony, Kol & Yankielowicz*]

can one find the generic dual geometry?

Idea: analogue of Calabi–Yau

Given explicit Kähler SU(3) structure (Ω, ω) with $\frac{1}{8}\mathrm{i}\Omega \wedge \bar{\Omega} = \frac{1}{6}\omega^3$

$$\mathrm{d}\Omega = 3\mathrm{i}\alpha \wedge \Omega, \qquad \mathrm{d}\omega = 0$$

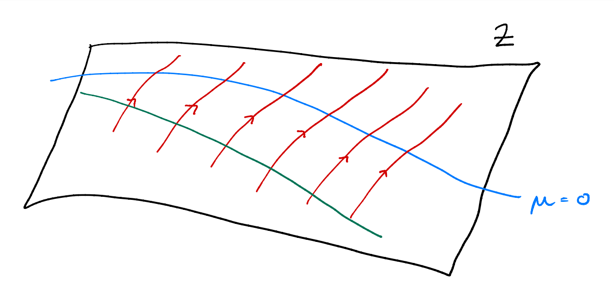
then vary Kähler form within cohomology class

$$\omega' = \omega + \mathrm{i}\partial\bar{\partial}h, \qquad \Omega' = \lambda\Omega$$

there exists Calabi–Yau solution (Ω_*, ω_*)

Symplectic quotient *[Donaldson, ...]*

Typical of supersymmetry conditions: first solve F-terms (holomorphic)



- Z is Kähler (infinite-dimensional) with group action G
- orbits for $G_{\mathbb{C}}$ intersect $\mu = 0$ (if "stable" – algebraic condition)

Kähler–Einstein, Sasaski–Einstein, Hermitian Yang-Mills, ...

Structures in $E_{6(6)}$ generalised geometry

Warped compactification, 8 supercharges, all fluxes

$$\mathrm{d}s^2 = \mathrm{e}^{2\Delta}\mathrm{d}s^2(\mathrm{AdS}_5) + \mathrm{d}s^2(M)$$

- "exceptional Sasaki–Einstein"
- "H structure" and "V structure" on M generalise notion of complex and symplectic structures (on cone)
- adapted to field theory Kähler and superpotential

[*Ashmore, Petrini & DW*]

$E_{6(6)} \times \mathbb{R}^+$ generalised geometry

Generalised tangent space

$$E \simeq TM \oplus 2T^*M \oplus \Lambda^3 T^*M \oplus 2\Lambda^5 T^*M$$

$$V^M = (v^m, \lambda^i_m, \rho_{mnp}, \sigma^i_{m_1 \dots m_5})$$

- transforms as **27** under $\mathrm{E}_{6(6)}$
- parametrises diffeomorphism and gauge symmetry

Generalised tensors

For example, adjoint **78** includes potentials

$$\mathrm{ad}\,\tilde{F} \simeq 3\mathbb{R} \oplus (TM \otimes T^*M) \oplus 2\Lambda^2 T^*M \oplus 2\Lambda^2 TM \oplus \Lambda^4 T^*M \oplus \Lambda^4 TM$$

$$A^M{}_N = (\dots, B^i_{mn}, \dots, C_{mnpq})$$

Can "twisting" of generalised tensors by gauge potentials

$$V = \mathrm{e}^{B^i + C}\,\tilde{V} \qquad A = \mathrm{e}^{B^i + C}\,\tilde{A}\,\mathrm{e}^{-B^i - C}$$

Generalised Lie derivative

"Generalised diffeomorphism, GDiff" symmetries

$$L_V = \text{diffeo} + \text{gauge transf}$$
$$= \mathcal{L}_v - (\mathrm{d}\lambda^i + \mathrm{d}\rho)\,\cdot$$

where forms act via adjoint (n.b. $L_V W \neq -L_W V$)

$E_{6(6)}$ cubic invariant

$$c(V,V,V) = \epsilon_{ij}(i_v\lambda^i)\sigma^j - \tfrac{1}{2} i_v\rho\wedge\rho - \tfrac{1}{2}\epsilon_{ij}\rho\wedge\lambda^i\wedge\lambda^j \in \Gamma(\Lambda^5 T^*M)$$

V structure

Generalised vector

$$K \in \Gamma(E) \qquad \text{such that } c(K, K, K) > 0$$

- stabilised by $\mathrm{F}_{4(4)} \subset \mathrm{E}_{6(6)}$
- L_K generates $\mathrm{U}(1)_R$ symmetry
- very special real geometry on space of structures (vector mulitplet)

H structure

Weighted adjoint tensors

$$J_\alpha(x) \in \Gamma(\mathrm{ad}\,\tilde{F} \otimes (\det T^*M)^{1/2})$$

forming highest root $\mathfrak{su}_2$ algebra, with $\kappa^2 \in \Gamma(\det T^*M)$

$$[J_\alpha, J_\beta] = 2\kappa\epsilon_{\alpha\beta\gamma}J_\gamma$$
$$\mathrm{tr}\, J_\alpha J_\beta = -\kappa^2\delta_{\alpha\beta}$$

- stabilised by $\mathrm{SU}^*(6)$
- hyper-Kähler geometry on space of structures (hypermultiplet)

Compatible structures

The H and V structures are compatible if

$$J_\alpha \cdot K = 0 \qquad\qquad c(K, K, K) = \kappa^2$$

(analogues of $\omega \wedge \Omega = 0$ and $\frac{1}{6}\omega^3 = \frac{1}{8}\mathrm{i}\Omega \wedge \bar{\Omega}$ on cone)

the compatible pair $\{J_\alpha, K\}$ define an USp(6) structure

J_α and K come from Killing spinor bilinears.

Example: Sasaki–Einstein $(\xi, \sigma, \omega, \Omega)$

$$\mathrm{d}s^2(M) = \sigma \otimes \sigma + \mathrm{d}s_4^2, \qquad F = 4\,\mathrm{vol}_5,$$
$$\mathrm{d}\sigma = 2\omega, \qquad \mathrm{d}\Omega = 3\mathrm{i}\sigma \wedge \Omega.$$

where $\sigma = \mathrm{d}\psi + a$ and $\xi = \partial/\partial\psi$ embeds as CR structure

$$J_+ = \tfrac{1}{2}\kappa u^i \left(\Omega - \mathrm{i}\Omega^\sharp\right), \qquad u^i = \tau_2^{-2}(\tau, 1)^i, \quad \kappa^2 = \mathrm{vol}_5$$

$$\mathrm{ad}\,\tilde{F} \simeq 3\mathbb{R} \oplus (TM \otimes T^*M) \oplus \Lambda^2 T^*M \oplus \Lambda^2 TM \oplus \Lambda^4 T^*M \oplus \Lambda^4 TM,$$

and

$$K = \xi - \sigma \wedge \omega \qquad \text{contact structure}$$

$$E \simeq TM \oplus 2T^*M \oplus \Lambda^3 T^*M \oplus 2\Lambda^5 T^*M,$$

Differential conditions

Define $J_+ = J_1 + \mathrm{i}J_2$ then

$$\mu_+(V) = 0, \qquad \mu_3(V) = \int_M c(K, K, V),$$

$$L_K J_+ = 3\mathrm{i}J_+, \qquad L_K K = 0.$$

What are μ_α??

Triplet of moment maps

Infinitesimally GDiff parametrised by $V \in \Gamma(E) \simeq \mathfrak{gdiff}$ and acts by

$$\delta J_\alpha = L_V J_\alpha$$

preserves HK structure on space of J_α giving triplet of moment maps

$$\mu_\alpha(V) = -\tfrac{1}{2}\epsilon_{\alpha\beta\gamma} \int_M \operatorname{tr} J_\beta (L_V J_\gamma)$$

Universal contact structure

In all cases [*c.f. Gabella, Gauntlett, Sparks, DW*]

$$K = \xi - \sigma \wedge \omega$$

with $i_\xi \sigma = 1$, $i_\xi \omega = 0$ and $\mathrm{d}\sigma = 2\omega$ and

$$\frac{1}{\text{central charge}} \propto \int_M c(K, K, K)$$

c.f. a-max ...

Exceptional Sasaki geometry

“Exceptional Sasaki” geometry

We would like to relax one condition (analogue of Kähler on cone)

$$\mu_+(V) = 0, \qquad \cancel{\mu_3(V) = \int_M c(K,K,V)},$$

$$L_K J_+ = 3\mathrm{i} J_+, \qquad L_K K = 0.$$

and $\kappa^2 \neq c(K,K,K)$.

(Note that if only fiveform flux, then Exceptional Sasaki = Sasaki)

Final condition: GDiff moment map

- ∞-dim. space of exceptional Sasaki structures (J_+, K) is Kähler
- condition is single moment map for GDiff
- complex $\mathrm{GDiff}_{\mathbb{C}}$ orbit $X = \kappa J_+$ (holomorphic object)

$$\delta X = L_V X \qquad V \in \Gamma(E_{\mathbb{C}}) \simeq \mathfrak{gdiff}_{\mathbb{C}}$$

- intersects moment map condition on susy background (X_*, K_*)

Physical interpretation

From supersymmetric multiplet structure

- fixing orbit, $[X] = \mathrm{GDiff}_{\mathbb{C}} \cdot X$ fixes superpotential $\mathcal{W}$
- the condition $L_K X = 3\mathrm{i}X$ fixes $\Delta = 3$
- motion in orbit is renormalisation flow of Kähler potential

Field theory implies version of "Calabi–Yau theorem" [*Kol,Green et al.*]

- if $\mathcal{W}$ is exactly marginal then orbit will intersect
- only obstruction is extra global symmetry from fixed point of GDiff [*Ashmore, Gabella, Graña, Petrini, DW*]

Aside: c.f. GMPT

Translating into pure spinors on the cone [*GMPT*, *Tomasiello*]

$$\mathrm{d}\Phi_- = 0,$$
$$\mathrm{d}^{\mathcal{J}_-}(\mathrm{e}^{-3A}\,\mathrm{Im}\,\Phi_+) = F_{\mathrm{RR}}$$
$$\mathrm{d}(\mathrm{e}^{-A}\,\mathrm{Re}\,\Phi_+) = 0 \qquad \text{moment map}$$

(Already hard to solve first two equations...)

Marginal deformations of Sasaki–Einstein duals

$[X]$ for Sasaki–Einstein

We have, up to $\mathrm{GDiff}_{\mathbb{C}}$,

$$X = \tfrac{1}{2}\,\mathrm{vol}_5\, u^i \left(\Omega - \mathrm{i}\Omega^\sharp\right) = \mathrm{e}^{\frac{1}{4}\mathrm{id}(\sigma\wedge\omega)} \cdot \left(\tfrac{1}{2} u^i \sigma \wedge \Omega\right) \sim \tfrac{1}{2} u^i \sigma \wedge \Omega$$

$$\in \Gamma(\,T^* M \oplus \Lambda^3 T^* M \oplus \dots\,)$$

Marginal deformation data

- mesonic operators $\Leftrightarrow$ holomorphic function f on cone
- marginal $\Leftrightarrow$ $\mathcal{L}_\xi f = 3\mathrm{i}f$
- for example on S^5: $f = \frac{1}{6} f_{ijk} z^i z^j z^k$

Solution for deformed background

We find new family of Exceptional Sasaki solutions

$$
\boxed{\begin{aligned} K &= \xi - \sigma \wedge \omega \\ X &= \mathrm{e}^{b^i(\tau, f)} \left(\mathrm{d}f + v^i(\tau, f)\, \sigma \wedge \Omega \right) \end{aligned}}
$$

with $b^i \in \Gamma(\Lambda^2 T^*_{\mathbb{C}} M)$ linear in $\mathrm{d}f$ and v^i quadratic

- complicated deformed metric g, axion-dilaton and fluxes
- for S^5 matches leading parts of Aharony et al. and $\mathrm{GDiff}_{\mathbb{C}}$ action gives $\{f_{ijk}\} /\!\!/ \mathrm{SU}(3)$
- can also check $f = z^1 z^2 z^3$, β-deformation is $\mathrm{GDiff}_{\mathbb{C}}$ of LM solution

What can we calculate?

$[X]$ fixes superpotential so should encode holomorphic information

e.g. space of mesonic operators $\mathcal{O} = \operatorname{tr} \phi \cdots \phi$

Normally, involved calculation in supergravity

- harmonic expansion of fields on background
- count chiral multipets of fixed R-charge $\Rightarrow$ Hilbert series

S^5, $T^{1,1}$ by isometries; generic Sasaki–Einstein [*Eager, Schmude, Tachikawa*]

Here counted by cohomology

Space of integrable structures $\mathcal{Z}$ so $\delta X \in T\mathcal{Z}$

$$E_{\mathbb{C}} \xrightarrow{L.X} T\mathcal{Z} \xrightarrow{\delta\mu_+} E^*_{\mathbb{C}}$$

cohomology counts operators

$$\text{space of mesonic ops} = \frac{\{\delta X : \delta\mu_+ = 0\}}{\{\delta X = L_V X\}}$$

If $\mathrm{d}f$ nowhere vanishing (generic) then can write

$$X = \mathrm{e}^{\tilde{b}^i(f,\tau)+\tilde{c}(f,\tau)}\,\mathrm{d}f$$

and cohomology reduces to

$$\boxed{\text{space of mesonic ops} = \frac{\{\mathrm{d}\alpha = 0\}}{\{\alpha = \mathrm{d}f \wedge \mathrm{d}\lambda\}}}$$

where $\alpha := \mathrm{d}f \wedge \delta b \in \Gamma(\Lambda^3 T^*_{\mathbb{C}} M)$

S^5 example

Generic deformation

$$f(z) = \lambda_1 z^1 z^2 z^3 + \lambda_2 \left[(z^1)^3 + (z^2)^3 + (z^3)^3\right]$$

Graded by R-charge $L_K \delta X = \mathrm{i}q\, \delta X$ cohmology gives Hilbert series

$$H(t) = 1 + 3t + 3t^2 + 2t^3 + 3t^4 + 3t^5 + 2t^6 + \cdots = \frac{(1+t)^3}{1-t^3}$$

agrees with field theory! (cyclic cohomology [*Van den Burgh*])

Summary

We have solved for the "holomorphic" structure

Questions/Extensions

- generic calculation of Hilbert series
- full solution?
- same formalism for M-theory AdS_5
- similar formalism for M-theory AdS_4 – again solve for deformations of $d = 7$ Sasaki–Einstein
- cohomology gives index ...
- generic a-max dual ...