Mass-deformations of the ABJM theory: the holographic free energy

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What do we do?

Precision test of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence between

a mass deformed ABJM theory & a consistent trunction of M theory

- This test is made possible by:

  Exact field theory results obtained via supersymmetric localization

  [Jafferis '10] [Jafferis-Klebanov-Pufu-Safdi '11]

- It is inspired by and largely follows a similar calculation in

  - ABJM [Freedman-Pufu, 1302.7310]

  see, also

  - $\mathcal{N} = 2^*$ sYM [Bobev-Elvang-Freedman-Pufu, 1311.1508]
  - $\mathcal{N} = 1^*$ sYM [Bobev-Elvang-Kol-Olson-Pufu, 1605.00656]

  as well as other (nonconformal) holographic tests of localization

  [Martelli-Passias-Sparks '11], ...

The Plan

I. Field theory

(i) Deformations of ABJM

(ii) Free energies on S^3

II. (Super)Gravity

(i) $\mathcal{N} = 2$ truncation of $\mathcal{N} = 8$ $D = 4$ gauged supergravity (M theory)

(ii) BPS equation on S^3

(iii) Supersymmetric solutions

(iv) The holographic free energy on S^3

III. Comments and Conclusions *

* It works

$$\text{ABJM} \quad \equiv \quad \mathrm{U}(N)_k \times \mathrm{U}(N)_{-k} \text{ CS theory with } k=1$$

[ABJM '08]

- $\mathcal{N}=8$ SCFT with $\mathrm{SO}(8)_\mathrm{R}$ symmetry.
- Dual to M-theory on $\mathrm{AdS}_4 \times S^7$ that corresponds to the $\mathrm{SO}(8)$ vacuum of $\mathcal{N}=8$ $D=4$ SUGRA.
- As a $\mathcal{N}=2$ SCFT, it has 2 vector s-multiplets and 4 chiral s-multiplets, $A_1, A_2 \in (\boldsymbol{N}, \overline{\boldsymbol{N}})$ & $B_1, B_2 \in (\overline{\boldsymbol{N}}, \boldsymbol{N})$ with the superpotential

$$W_{\mathrm{ABJM}} = \frac{1}{4}\mathrm{Tr}\left(\epsilon^{ab}\epsilon^{cd} A_a B_c A_b B_d\right)$$

$\mathrm{SO}(8)_\mathrm{R} \to \mathrm{U}(1)_\mathrm{R} \times \mathrm{SU}(4)_\mathrm{F}$ and the s-conformal R-charges are

$$\Delta_{A_1} = \Delta_{A_2} = \Delta_{B_1} = \Delta_{B_2} = \frac{1}{2}$$

- Mixing between $\mathrm{U}(1)_\mathrm{R}$ with $\mathrm{U}(1)^3_\mathrm{F}$ provides for a more general $\mathrm{U}(1)'_\mathrm{R}$,

$$\Delta_{A_1} + \Delta_{A_2} + \Delta_{B_1} + \Delta_{B_2} = 2$$

Deformations of ABJM

Coupling to S^3 and real masses

[Jafferis '10] [Jafferis-Klebanov-Pufu-Safdi '11]

Different $\mathrm{U}(1)_{\mathrm{R}}$'s yield inequivalent $\mathcal{N}=2$ SUSY theories on S^3

$$\{Q, Q^\dagger\} = \sigma_i J_i + iq\sigma + \frac{1}{a} R\,, \qquad a \equiv R_{S^3}$$

with

$$L = L_{\mathrm{SCFT}} + \frac{1}{a^2}[\delta_1 - 2\delta_2\delta_3]\mathcal{O}_B^1 + \ldots + \frac{1}{a}\delta_1\mathcal{O}_F^2 + \ldots$$

where $\mathcal{O}_B^i$ are boson bilinears and $\mathcal{O}_F^i$ are fermion bilinears and

$$\Delta_{A_1} = \frac{1}{2} + \delta_1 + \delta_2 + \delta_3\,, \qquad \Delta_{A_2} = \frac{1}{2} + \delta_1 - \delta_2 - \delta_3\,, \qquad \text{etc.}$$

In the large N limit, the free energy can be computed via localization

$$F_{S^3}^{\mathrm{ABJM}} = \frac{4\sqrt{2}\pi}{3} N^{3/2} \sqrt{\Delta_{A_1}\Delta_{A_2}\Delta_{B_1}\Delta_{B_2}}$$

- The SCFT is recovered by F-maximization, $\delta_1 = \delta_2 = \delta_3 = 0$.
- The δ_i can be viewed as (i) deformed R-charges, (ii) real masses, (iii) constant scalar fields of the background vector mutiplets.
- In the holographic dual δ_i correspond to b-ry values of complex scalars in the STU-model inside $\mathcal{N}=8$ $D=4$ SUGRA. [FP '13]

Deformations of ABJM

Complex masses

Explicit mass deformation of the superpotential

$$W_{\text{ABJM}} \quad \longrightarrow \quad W_{\text{ABJM}} + m \, \text{Tr}\left[\left(\widetilde{T} A_1 \right)^2 \right]$$

The IR limit of the RG flow is a $\mathcal{N} = 2$ SCFT, which we call mABJM.

[Benna-Klebanov-Klose-Smedback '08] [Klebanov-Klose-Murugan '08]

- It is dual to the $\text{AdS}_4 \times_w M_7$ solution corresponding to the $\text{SU}(3) \times \text{U}(1)$ critical point of $\mathcal{N} = 8$ $D = 4$ SUGRA. [Warner '83] [CPW '01]
- The A_1 s-multiplet is integrated out, which yields a sextic superpotential of the $\mathcal{N} = 2$ SCFT with a new $\text{U}(1)_{\text{R}}$.
- Turn on arbitrary R-charges (real masses) on S^3 together with the mass deformation in the superpotential,

$$F_{S^3}^{\text{mABJM}} = \frac{4\sqrt{2}\pi}{3} N^{3/2} \sqrt{\Delta_{A_2} \Delta_{B_1} \Delta_{B_2}}$$

[Jafferis-Klebanov-Pufu-Safdi '11]

The SCFT free energy is recovered by F-maximization

$$\Delta_{A_2} + \Delta_{B_1} + \Delta_{B_2} = 1 \qquad \Longrightarrow \qquad \Delta_{A_2} = \Delta_{B_1} = \Delta_{B_2} = \frac{1}{3}$$

giving the correct result consistent with holography

$$F_{S^3}^{\text{mABJM}} = \frac{4\sqrt{2}\pi}{9\sqrt{3}} N^{3/2} = \frac{4}{3\sqrt{3}} F_{S^3}^{\text{ABJM}} \,, \qquad \frac{L_{\text{NW}}^2}{L_{\text{SO}(8)}^2} = \frac{\mathcal{P}_{\text{SO}(8)}}{\mathcal{P}_{\text{NW}}} = \frac{4}{3\sqrt{3}}$$

ABJM and mABJM

The holographic problem

$$F_{S^3}^{\mathrm{ABJM}} = \frac{4\sqrt{2}\pi}{3} N^{3/2} \sqrt{\Delta_{A_1}\Delta_{A_2}\Delta_{B_1}\Delta_{B_2}}\,, \qquad \Delta_{A_1} + \Delta_{A_2} + \Delta_{B_1} + \Delta_{B_2} = 2$$

$$\Downarrow \qquad \Delta_{A_1} \to 1\,, \qquad \delta_1 + \delta_2 + \delta_3 = \frac{1}{2}$$

$$F_{S^3}^{\mathrm{mABJM}} = \frac{4\sqrt{2}\pi}{3} N^{3/2} \sqrt{\Delta_{A_2}\Delta_{B_1}\Delta_{B_2}}\,, \qquad \Delta_{A_2} + \Delta_{B_1} + \Delta_{B_2} = 1$$

- No explicit dependence on the superpotential mass, m.

Our task

- Derive $F_{S^3}^{\mathrm{mABJM}}(\delta_1, \delta_2, \delta_3)$ by computing the on-shell action on suitable solutions of M theory.
- Understand how the constraint on the δ_i's arises.

Important hints

- It is sufficient to work within $\mathcal{N} = 8$ $D = 4$ sugra.
  - ABJM is captured by the STU model; $\mathrm{U}(1)^4$ truncation. [FP '13]
  - mABJM has an additional pseudoscalar; $\mathrm{SU}(3) \times \mathrm{U}(1)$ truncation. [W '83]

The holographic set-up

- A consistent truncation $\mathcal{N}=8$ $D=4$ SUGRA [BMP '18]

$$
\begin{array}{llcccc}
\mathrm{U}(1)^2: & \mathcal{N}=2 \text{ sugra} & + & 3 \text{ vector multiplets} & + & 1 \text{ hypermultiplet} \\
 & g_{\mu\nu},\ A^0_\mu & & A^i_\mu\,,\ z_i \in \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} & & \zeta_1,\zeta_2 \in \frac{\mathrm{SU}(3)}{\mathrm{SU}(2)\times\mathrm{U}(1)} \\
\mathrm{U}(1)^3: & g_{\mu\nu} & & z_i & & \zeta_1 = 0\,,\ z\equiv\zeta_2 \\
 & & & \delta_i & & m
\end{array}
$$

$$
\mathrm{U}(1)^2 \ \equiv\ \text{Cartan of SU}(3) \ \subset\ \mathrm{SU}(4)_{\mathrm{F}} \qquad \mathrm{U}(1)\ \equiv\ \mathrm{U}(1)^{\mathrm{mABJM}}_{\mathrm{R}}
$$

- Continue SUSY variations to the Euclidean domain,

$$
(z_i,\bar{z}_i) \ \to\ (z_i,\tilde{z}_i)\,, \qquad (z,\bar{z}) \ \to\ (z,\tilde{z})
$$

- S^3-sliced (Euclidean) metric Ansatz

$$
ds^2 = ds^2 = L^2 e^{2A(r)}\, ds^2_{S^3} + e^{2B(r)}\, dr^2
$$

Radial gauges: $\quad$ (CF) $\quad e^{2B(r)} = \frac{L^2}{r^2} e^{2A(r)}, \qquad$ (FG) $\quad e^{2B(\rho)} = L^2$

- Obtain BPS equations for $A(r)$, $z_i(r)$, $\tilde{z}_i(r)$, $z(r)$ and $\tilde{z}(r)$.

The BPS equations

The BPS equations can be recast into flow equations

$$z_j{}' = -\frac{e^B}{2L}\,(1 - z_j\,\tilde z_j)\,\frac{\mathfrak{G}^{1/2}}{\widetilde{\mathfrak{G}}^{1/2}}\,\widetilde{\mathfrak{F}}_j\,, \qquad \tilde z_j{}' = -\frac{e^B}{2L}\,(1 - z_j\,\tilde z_j)\,\frac{\widetilde{\mathfrak{G}}^{1/2}}{\mathfrak{G}^{1/2}}\,\mathfrak{F}_j$$

$$\frac{z\,'}{z} = \frac{e^B}{L}\,\mathfrak{G}^{1/2}\,\widetilde{\mathfrak{G}}^{1/2}\,, \qquad \frac{\tilde z\,'}{\tilde z} = \frac{e^B}{L}\,\mathfrak{G}^{1/2}\,\widetilde{\mathfrak{G}}^{1/2}$$

$$A\,' = \frac{e^B}{L}\left[\,\pm\,e^{-A} - \frac{1}{2}\,\frac{\widetilde{\mathfrak{G}}^{1/2}}{\mathfrak{G}^{1/2}}\,\mathfrak{W}\right] = \frac{e^B}{L}\left[\,\mp\,e^{-A} - \frac{1}{2}\,\frac{\mathfrak{G}^{1/2}}{\widetilde{\mathfrak{G}}^{1/2}}\,\widetilde{\mathfrak{W}}\right]$$

where

$$\mathfrak{W} = e^{K_V/2}\,\frac{1}{1 - z\,\tilde z}\,[\,2\,(z_1 z_2 z_3 - 1) + z\,\tilde z\,(1 - z_1)(1 - z_2)(1 - z_3)\,]$$

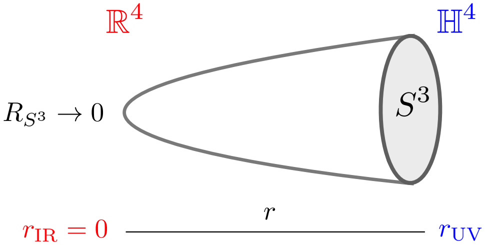
$$\mathfrak{F}_i = e^{K_V/2}\,\frac{1}{1 - z\,\tilde z}\left[\,2\left(\tilde z_i - \frac{z_1 z_2 z_3}{z_i}\right) + z\,\tilde z\,\frac{1 - \tilde z_i}{1 - z_i}(1 - z_1)(1 - z_2)(1 - z_3)\right]$$

$$\mathfrak{G} = e^{K_V/2}\left[\,2(z_1 z_2 z_3 - 1) + (1 - z_1)(1 - z_2)(1 - z_3)\,\right]$$

$$e^{K_V/2} = \prod_{i=1}^{3}\frac{1}{(1 - z_i\,\tilde z_i)^{1/2}}$$

Solutions

Euclidean AdS_4



- $\mathbb{H}^4$ solutions

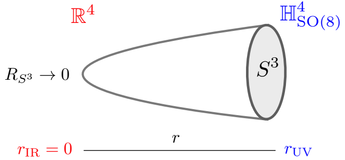
$$\mathrm{SO}(8): \quad z_i = \tilde{z}_i = 0\,, \qquad z = \tilde{z} = 0\,, \qquad \mathcal{P}_* = -6$$

$$\mathrm{W}: \quad z_i = \tilde{z}_i = \sqrt{3} - 2\,, \qquad z\,\tilde{z} = \frac{1}{3}\,, \qquad \mathcal{P}_* = -\frac{9\sqrt{3}}{2}$$

$$ds^2 = \frac{4L^2}{(1 - r^2/r_{\mathrm{UV}}^2)^2}\left(dr^2 + r^2\, ds_{S^3}^2\right), \qquad r_{\mathrm{UV}}^2 = -\frac{6}{\mathcal{P}_*}$$

Solutions

Asymptotics



- **UV asymptotics**

$$\text{FG gauge:} \qquad \frac{r}{r_{\mathrm{UV}}} = 1 - 2\,e^{-\rho} + 2\,e^{-2\rho} + \ldots .$$

$$z_i(\rho) = a_i\, e^{-\rho} + b_i\, e^{-2\rho} + \ldots\,, \qquad \tilde{z}_i(\rho) = \tilde{a}_i\, e^{-\rho} + \tilde{b}_i\, e^{-2\rho} + \ldots\,,$$

$$z(\rho) = a\, e^{-\rho} + b e^{-2\rho} + \ldots\,, \qquad \tilde{z}(\rho) = \tilde{a}\, e^{-\rho} + \tilde{b}\, e^{-2\rho} + \ldots\,,$$

The BPS equations determine b's in terms of a's (susy), which must satisfy

$$a_1 + a_2 + a_3 - \tilde{a}_1 - \tilde{a}_2 - \tilde{a}_3 = 4$$

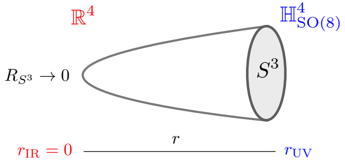
The map onto the field theory gives the identification

$$\Delta_i = \frac{1}{4}(a_i - \tilde{a}_i)\,, \qquad \Delta_1 + \Delta_2 + \Delta_3 = 1$$

This leaves 7 independent UV-parameters.

Solutions

Asymptotics



- **IR asymptotics**

  The S^3 smoothly shrinking to a point at $r = 0$, the metric becomes flat

$$A(r) = A_0 + \log r + \mathcal{O}(r^2)$$

  This imposes the following constraints on the scalar fields

$$z_i(0) = c_i\,, \qquad \tilde{z}_i(0) = \tilde{c}_i\,, \qquad z(0) = c\,, \qquad \tilde{z}(0) = \tilde{c}$$

  given by

$$c_i = \frac{2\,\tilde{c}_j\,\tilde{c}_k - x_0(1-\tilde{c}_j)(1-\tilde{c}_k)}{2 - x_0(1-\tilde{c}_j)(1-\tilde{c}_k)}\,, \qquad x_0 = c\,\tilde{c}\,, \qquad (ijk)\text{-cyclic}$$

  and a cubic constraint due to the hyperscalar

$$2(\tilde{c}_1\,\tilde{c}_2\,\tilde{c}_3 - 1) + (1-\tilde{c}_1)(1-\tilde{c}_2)(1-\tilde{c}_3) = 0$$

Exercise: Find the map IR $\to$ UV

$$(\tilde{c}_i; x_0) \qquad \longrightarrow \qquad (a_i, \tilde{a}_i; a, \tilde{a})$$

Solutions

IR to UV

For the vanishing hyperscalar, one can solve the BPS equations analytically

[FP '13]

In the IR

$$c_i = \frac{\tilde{c}_1 \tilde{c}_2 \tilde{c}_3}{\tilde{c}_i}, \qquad c = \tilde{c} = 0$$

Along the flow, e.g.,

$$e^{2A} = 4r^2 \frac{(1 - \tilde{c}_1 \tilde{c}_2 \tilde{c}_3 \, r^4/r_{\mathrm{UV}}^4)}{(1 - r^2/r_{\mathrm{UV}}^2)^2 (1 - \tilde{c}_1 \tilde{c}_2 \tilde{c}_3 \, r^2/r_{\mathrm{UV}}^2)^2}$$

In the UV

$$a_i = \frac{4\, c_i}{1 - \tilde{c}_1 \tilde{c}_2 \tilde{c}_3}, \qquad \tilde{a}_i = \frac{4\, \tilde{c}_i}{1 - \tilde{c}_1 \tilde{c}_2 \tilde{c}_3}$$

In particular

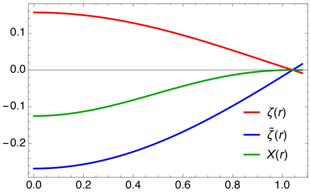
$$\Delta_1 = \frac{(1+\tilde{c}_2)(1+\tilde{c}_3)}{(1-\tilde{c}_2)(1-\tilde{c}_3)}, \qquad \Delta_2 = \frac{(1+\tilde{c}_3)(1+\tilde{c}_1)}{(1-\tilde{c}_3)(1-\tilde{c}_1)}, \qquad \Delta_3 = \frac{(1+\tilde{c}_1)(1+\tilde{c}_2)}{(1-\tilde{c}_1)(1-\tilde{c}_2)}$$

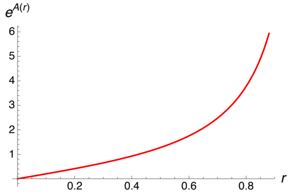
in terms of which the cubic constraint becomes simply

$$\Delta_1 + \Delta_2 + \Delta_3 = 1$$

Solutions

Symmetric sector





$$z_i(r) = \zeta(r)\,, \qquad \tilde{z}_i(i) = \tilde{\zeta}(i)\,, \qquad z(r)\tilde{z}(r) = X(r)\,; \qquad X(0) = x_0$$

- As expected, since $\Delta_1 = \Delta_2 = \Delta_3 = \frac{1}{3}$,

$$a_i = \frac{2}{3} - \frac{2}{3\sqrt{3}} + f(x_0)\,, \qquad \tilde{a}_i = -\frac{2}{3} - \frac{2}{3\sqrt{3}} + f(x_0)\,, \qquad f(0) = 0$$

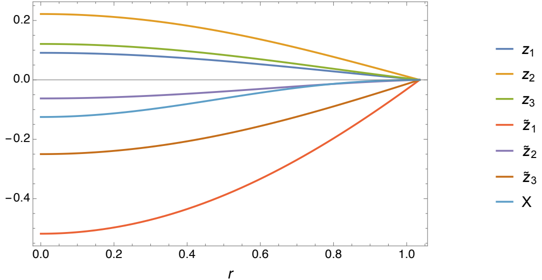
- An analytic solution

$$\zeta = -\tilde{\zeta}, \qquad z\tilde{z} = -\frac{4\zeta^2}{(1-\zeta^2)^2}\,, \qquad x_0 = -\frac{1}{3}\,, \qquad r_{\mathrm{UV}}^2 = \frac{32}{27}$$

$$\zeta = \mathcal{R} - 2\sqrt{1+\mathcal{R}^2}\cos\alpha\,, \quad \alpha = \frac{1}{3}\left[\mathrm{Arg}(\mathcal{R}+i) + \pi\right]\,, \quad \mathcal{R} = \frac{r_{\mathrm{UV}}^2 - r^2}{r_{\mathrm{UV}}^2 + r^2}$$

Solutions

General numerics



- For $|x_0| < 1/3$, the formal series solution is convergent for $0 \leq r \leq r_{\rm UV}$ and thus can be determined with arbitrary precision.

THEOREM.

1. The UV parameters, Δ_i, do *not* depend on x_0 and thus are given by their $x_0 = 0$ values.
2. The parameters a_i and $\tilde{a}_i$ are shifted from their $x_0 = 0$ values by the *same* function $f(\tilde{c}_i, x_0)$.

PROOF.

(i) Extensive numerical checks. [BMPR '18]

(ii) Perturbative expansion around the analytic solution. [Kim-Kim '19]

Towards the free energy

Making it finite

3 SUBTLE POINTS

1. Holographic renormalization [Skenderis et al '02]

   The on-shell action must be regularized consistent with SUSY.

$$S_{\text{reg}} = S_{\text{bulk}} + S_{\text{GH}} + S_{\mathcal{R}} + S_{\text{SUSY}}$$

where

$$S_{\text{SUSY}} = \int_{S^3} d\Omega_3 \left[L^2 e^{3A} (\mathfrak{W}\widetilde{\mathfrak{W}})^{1/2} \right]_{r=r_0}$$

[FP '13] [Freedman-Pufu-KP-Warner '16]

$$S_{\text{on-shell}} \equiv \lim_{r_0 \to r_{\text{UV}}} S_{\text{reg}}$$

Using the BPS equations, the on-shell action can be simplified

$$S_{\text{on-shell}} = -2\,\text{vol}_{S^3} L^2 \left[\int_0^{r_{\text{UV}}} dr \left(\frac{e^{2A}}{r} - \frac{r_{\text{UV}}}{(r - r_{\text{UV}})^2} \right) - \frac{1}{2} \right]$$

- The result depends on the IR value, x_0, of the hyperscalar.

Towards the free energy

Alternate quantization

2. It's the Legendre transform s...d, use the alternate quantization.

[Breitenlohner-Freedman '82] [Klebanov-Witten '99]

Scalars: $z_i + \tilde{z}_i$

Pseudoscalars: $z_i - \tilde{z}_i$, z and $\tilde{z}$

Scalars require alternate quantization,

$$\phi \qquad \longrightarrow \qquad \pi \equiv -\lim_{\rho_0\to\infty} \frac{\delta S_{\text{reg}}[\phi]}{\delta\phi} = -\lim_{\rho_0\to\infty} e^{-\rho_0}\Pi_\phi(\rho_0)$$

In particular,

$$\mathfrak{a}_i = \frac{L^2}{8}\left(\tilde{b}_i + \frac{a_1 a_2 a_3}{a_i} - \frac{a\,\tilde{a}}{2}\right) \overset{\text{BPS}}{=} -\frac{L^2}{4}\tilde{a}_i$$

$$\tilde{\mathfrak{a}}_i = \frac{L^2}{4} a_i\,, \qquad \mathfrak{a} = \tilde{\mathfrak{a}} = 0$$

The holographic dual of the free energy is the Legendre transform of the on-shell action! [FP '13]

$$J_{\text{on-shell}}(a_i, \tilde{a}_i, a, \tilde{a}) = S_{\text{on-shell}} + \frac{1}{2}\int_{S^3} d\Omega_3 \sum_{i=1}^{3} (a_i + \tilde{a}_i)(\mathfrak{a}_i + \tilde{\mathfrak{a}}_i)$$

3. A holographic Ward identity

Consider a *variation* of $J_{\text{on-shell}}(a_i, \tilde{a}_i, a, \tilde{a})$ w/t UV-parameters,

$$\frac{dS_{\text{reg}}}{d\mu} = -\int_{S^3} d\Omega_3 \left[\sum_{i=1}^{3} \left(\mathfrak{a}_i \frac{\partial a_i}{\partial \mu} + \tilde{\mathfrak{a}}_i \frac{\partial \tilde{a}_i}{\partial \mu} \right) + \mathfrak{a} \frac{\partial a}{\partial \mu} + \tilde{\mathfrak{a}} \frac{\partial \tilde{a}}{\partial \mu} \right]$$

[Bobev et al '16]

from which it follows that

$$\frac{dJ_{\text{on-shell}}}{d\mu} = \text{vol}_{S^3} \frac{L^2}{4} \sum_{i=1}^{3} (a_i + \tilde{a}_i) \frac{\partial}{\partial \mu} (a_i - \tilde{a}_i)$$

But

$$a_i - \tilde{a}_i = 4\,\Delta_i$$

By the THEOREM, Δ_i do *not* depend on the hyperscalars, in particular on x_0. Thus

$$\frac{dJ_{\text{on-shell}}}{dx_0} = 0$$

But, for $x_0 = 0$, we have an analytic solution!

The holographic free energy

Final result

Use [FP '13]

$$S_{\text{on-shell}}(\tilde{c}_i, x_0 = 0) = 2\,\text{vol}_{S^3} L^2\, \frac{1 + \tilde{c}_1 \tilde{c}_2 \tilde{c}_3}{1 - \tilde{c}_1 \tilde{c}_2 \tilde{c}_3}$$

Then

$$J_{\text{on-shell}} = 2\,\text{vol}_{S^3} L^2\, \frac{(1 - \tilde{c}_1^2)(1 - \tilde{c}_2^2)(1 - \tilde{c}_3^2)}{(1 - \tilde{c}_1 \tilde{c}_2 \tilde{c}_3)^2}$$

Identifying

$$\Delta_{A_2} \equiv \Delta_1 = \frac{(1 + \tilde{c}_2)(1 + \tilde{c}_3)}{(1 - \tilde{c}_2)(1 - \tilde{c}_3)} \qquad \Delta_{B_1} \equiv \Delta_2\,, \qquad \Delta_{B_2} \equiv \Delta_3$$

and using the standard AdS/CFT relation

$$\text{vol}_{S^3} L^2 = \frac{\pi}{3\sqrt{2}} N^{3/2}$$

we get

$$J_{\text{on-shell}} = \frac{4\sqrt{2}\pi}{3} N^{3/2} \sqrt{\Delta_1 \Delta_2 \Delta_3} = F_{S^3}^{\text{mABJM}}$$

Comments

1. UV vs IR

* The mass m is "discrete," the free energy depends on either $m = 0$ or $m \neq 0$.
* Similarly, the hyperscalar z is "discrete," and amounts to imposing an algebraic constraint on the vector scalars, z_i, in the IR.
* The same holds for the topologically twisted index in mABJM vs AdS black hole entropy. [BMP '18]
* More generally, similar mechanism is present for general AdS black holes in $\mathcal{N} = 2$ $D = 4$ SUGRA with hypermultiplets.
  [Halmagyi-Petrini-Zaffaroni '13] [Halmagyi et al '13-'15]
* However, there seems to be no simple way to predict how the constraints in the IR (or at the horizon) translate into relations in the UV without solving the BPS equations. This is a striking difference with the FT side.

Comments

2. Complex vs real R-charges

   * The map

   $$\tilde{c}_i \qquad \longrightarrow \qquad \Delta_i = \frac{(1+\tilde{c}_j)(1+\tilde{c}_k)}{(1-\tilde{c}_j)(1-\tilde{c}_k)} \qquad \& \quad \text{cyclic}$$

   provides linearization of the cubic constraint, $\Delta_1 + \Delta_2 + \Delta_3 = 1$.

   * We assumed $|\tilde{c}_i| < 1$ (SUGRA) and $\tilde{c}_i \in \mathbb{R}$. Then $0 < \Delta_i < 1$.
   * For complex $\tilde{c}_i$ satisfying in additon $\tilde{c}_1 \tilde{c}_2 \tilde{c}_3 < 1$ so that the metric is real, $0 < \mathrm{Re}\,\Delta_i < 1$, but the map is not onto.
   * Are there additional saddle points for complex Δ_i?

3. $1/N$ corrections?

Thank you