Anomaly Inflow for M5-branes wrapping a Riemann Surface

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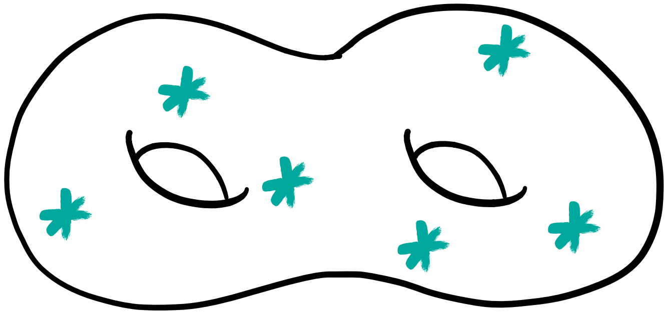
Geometrizing QFT

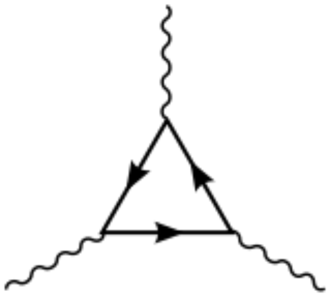
- Geometric Engineering of QFTs has become a powerful tool for exploring strongly coupled systems
- The space of 4D SCFTs can be studied and characterized by 6D SCFTs on punctured Riemann Surfaces — The class “S” program [Witten ’97; Gaiotto ’09; Gaiotto, Moore, Neitzke ’09]
- Boundary conditions at punctures lead to a large class of possible choices for flavor Symmetry in 4D
- Varying amount of supersymmetry can be preserved in 4D by choices of topological twist
- Program generalizable to systems with less supersymmetry and across different dimensions
- Pair-of-Pants decomposition of Riemann surfaces provide natural building blocks for 4D SCFTs
- In this talk, we will restrict to 6D (2,0) A_{N-1} SCFT — Worldvolume theory of a stack of N M5-branes in M-theory



Important Question: How does the geometric set-up encode the ’t Hooft Anomalies of SCFTs?

’t Hooft Anomalies

- ’t Hooft Anomalies: Gauge anomalies for global symmetries
- Exist for quantum systems in even dimensions
- For 4D QFTs, they can be obtained from the triangle diagram



$$\delta_\epsilon S_{eff}[A] = \int_{M_4} \epsilon^a \mathscr{A}_a = \int_{M_4} \epsilon^a D_\mu j^\mu_a$$

Background Gauge Field for Global symmetry

- Anomalies are one loop exact. They are preserved under RG flow
- Measures of degrees of freedom of quantum systems
- Anomalies provide strong constraints for the IR phases of quantum systems
- In superconformal field theories, the conformal anomaly coefficients (a, c, …) and flavor central charge are related to anomalies associated to the R-symmetry

Anomaly Polynomial

- Wess-Zumino consistency conditions imply that anomalies are naturally **geometric quantities**
- In even dimensions D,

$$\delta S_{\text{eff}} = \int_{M_D} I_D^{(1)}$$

order in gauge parameter

Descent procedure: $$\underbrace{I_{D+2}}_{} = dI^{(0)}_{D+1}, \qquad \delta I^{(0)}_{D+1} = dI^{(1)}_{D}$$

the anomaly polynomial,

a closed, gauge-invariant (*D*+2)-form in M_{D+2}

$f(\mathrm{Tr}F^m)$ for background gauge curvature form

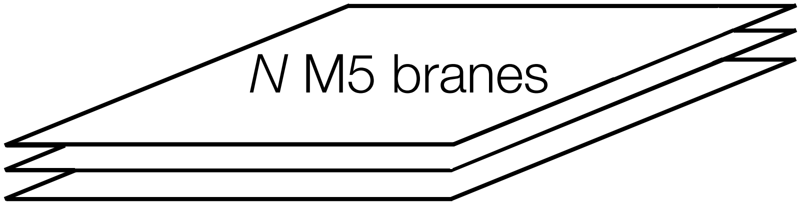
General structure: $$I_{D+2} = I^{\text{gauge}}_{D+2} + I^{\text{grav}}_{D+2} + I^{\text{mixed}}_{D+2}$$

$f(\mathrm{Tr}R^m)$ for curvature of spacetime tangent bundle

e.g. For D=4 $$I_6 = C_1 Tr(F^3) + C_2 Tr(F_1 F_2^2) + C_3 Tr(F) p_1(TM_4)$$

The C's are the anomaly coefficients

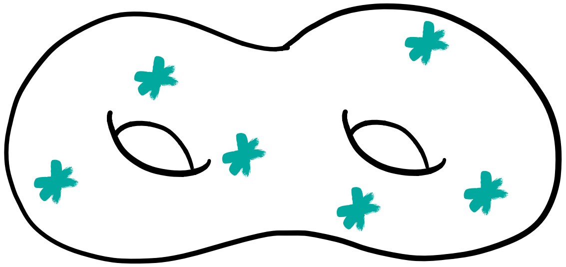
4D N=2 Class S Arena



Worldvolume theory is

6d (2,0) SCFTs of type A_{N-1}

Partial **topological twist** over $\Sigma_{g,n}$ (genus *g*, *n* punctures) to preserve susy



4d theory of "**Class *S***"

The 4d SCFTs are labeled by:

- Euler characteristic $\chi(\Sigma_{g,n}) = -2g + 2 - n$
- Local data at the punctures.

Regular punctures

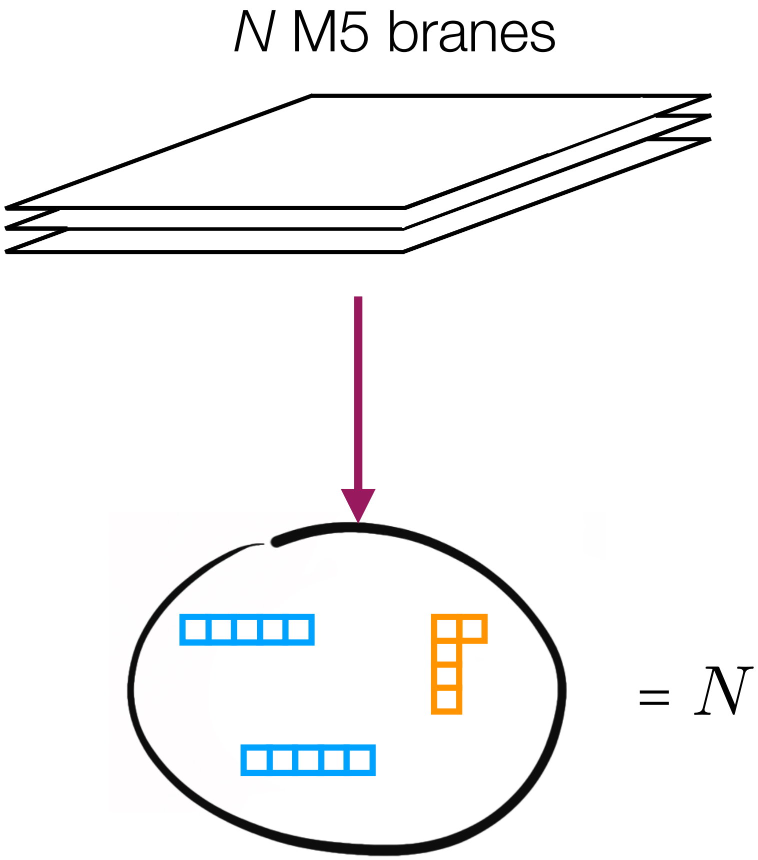
Boundary conditions for SCFT labeled by a partition of N, 1-1 with Young tableaux

$$N = \sum_k k n_k$$

Flavor symmetry at Puncture is fixed

$$G = S\left(\prod_k U(n_k)\right)$$

Example: Hypermultiplets



$SU(N)$ x $SU(N)$ x $U(1)$

$$p = -\chi(\Sigma_{g=0,n=3}) = 1$$

$= N^2$ free bifundamental hypermultiplets

Structure of anomalies with Regular Punctures

- Anomalies of Class S theories have been studied by using field theory methods — QFT dualities and anomaly matching in Higgs branch [Chacaltana, Distler, Tachikawa '12]

$$I_6^{\mathcal{S}} = -\chi(\Sigma_{g,n}) I_6^b + \sum_{\alpha=1}^{n} I_6(G_\alpha)$$

Universal contribution
Independent of puncture data

Contribution from each puncture
Independent of Riemann Surface data
Fixed by boundary data

Universal part fixed by reducing
6D anomaly of Riemann Surface

$$\int_{\Sigma_{g,n}} I_8[A_{N-1}] = -\chi(\Sigma_{g,n}) I_6^b$$

Goal: Provide a geometric derivation of anomaly polynomial by directly Considering the compactification of the M5-branes

Outline

- Anomaly Inflow for M5-branes
- Puncture Geometry

Anomaly Inflow for M5-Branes

Anomaly Inflow

- Gauge symmetries and diffeomorphisms can be broken classically when Gauge and Gravitational theories are taken over spaces with boundaries or when there are localized sources
- When gauge symmetries and diffeomorphisms are restricted on boundaries or on localized sources, they induce global symmetries
- The effective action of the localized degrees of freedoms at the boundaries or on the sources can be anomalous under the induced global symmetry
- Consistency of the sources and boundaries requires the quantum anomaly of the localized degrees of freedom to cancel the anomalous variation of the bulk action [Callan, Harvey '85]
- Dirichlet boundary conditions of bulk gauge fields are background fields for boundary global symmetry
- Anomaly inflow makes the higher dimensional nature of anomalies from descent natural

$$\delta S_{\text{eff}} = \int_{M_D} I_D^{(1)}$$

Descent procedure:

$$\underbrace{I_{D+2}}_{\text{the anomaly polynomial,}} = dI_{D+1}^{(0)}, \qquad \delta I_{D+1}^{(0)} = dI_D^{(1)}$$

the anomaly polynomial,

a closed, gauge-invariant (D+2)-form in M_{D+2}

M-theory with M5-brane sources

Consider the SUGRA action of 11D M-theory

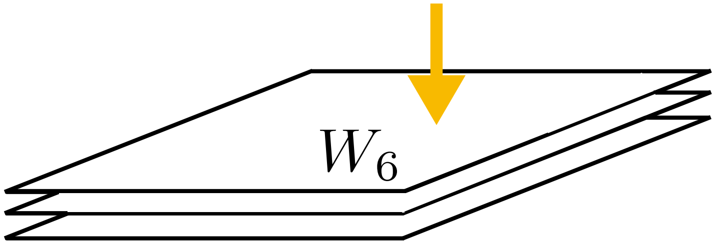
$$\frac{S_M}{2\pi} = \int_{M_{11}} \sqrt{g}\left[R - \frac{1}{2}|G_4|^2\right] - \frac{1}{6}C_3 \wedge G_4^2 - C_3 \wedge X_8$$

$$X_8 = \frac{1}{196}\left[p_1^2(TM_{11}) - 4p_2(TM_{11})\right]$$

$$p_1 \sim Tr(R^2), \qquad p_2 \sim Tr(R^4)$$

$$X_8 = dX_7^{(0)}, \qquad \delta_{diff}X_7^{(0)} = dX_6^{(1)}$$

In addition to diff, there is a gauge symmetry $C_3 \to C_3 + d\lambda_2$



$$M_{11} = W_6 \times \mathbb{R}^5 \qquad dG_4 = N\,\delta_{W_6}$$

$$\frac{\delta S_M}{2\pi} = -N\int_{M_{11}}\left[X_6^{(1)} + \lambda_2 \wedge G_4\right]\delta_{W_6}$$

In presence of M5-brane sources, the classical variation of the action is non-vanishing

$$TM_{11}\Big|_{W_6} = TW_6 \oplus NW_6$$

The normal bundle is the $SO(5)$ R-symmetry bundle
The gravitational and R-symmetry anomalies
must cancel the non-vanishing bulk variation

A more careful analysis is necessary to see the dependence of the bulk variation on the background fields

Variation with Background Fields

- Variation of action must be done with respect to a globally defined and closed object
- In presences of source, G_4 is singular and the action is ill defined
- Variation must account for background fields that live on the brane
- G_4 must be replaced with a suitable object that is gauge invariant, globally defined and non-singular [Witten '97; Freed, Harvey, Minasian, Moore '98; Harvey, Minasian, Moore '98]

$$dG_4 = N\delta^{(5)}(r)dr \wedge \underbrace{d\Omega_4}$$

angular form on the 5 transverse directions
$SO(5)$-bundle = $S^4 \times r$

$\downarrow$

Replace the RHS with characteristic classes which are smooth, well-defined on the full $SO(5)$-bundle.

$$dG_4 = \underbrace{d\rho(r)}_{\text{bump form}} \wedge \underbrace{E_4}_{\text{gauge-invariant, closed, globally defined angular form}}$$

$$dy^a \to Dy^a - A^{ab}_{SO(5)}y^b, \qquad y^a y^a = 1$$

$$D\Omega_4 = \frac{1}{4!}\epsilon_{a_1 \cdots a_5} Dy^{a_1} \cdots Dy^{a_4} y^{a_5}$$

$$E_4 = \frac{N}{V_4}\left[D\Omega_4 + \alpha_1 FDyy + \alpha_2 FFy\right]$$

The total number of branes $\displaystyle\int_{S^4} E_4 = N$

Satisfies Descent Relations

$$E_4 = dE_3^{(0)}, \qquad \delta E_3^{(0)} = dE_2^{(1)}$$

$$\boxed{G_4 = -\rho(r)E_4 + \cdots}$$

Variation with Background Fields — Answer

To express the answer after variation,
in the region near the branes, write

$$M_{11} = r \times M_{10}\,, \qquad S^4 \hookrightarrow M_{10} \to W_6$$

The variation of the M-theory action can be written as integral over a descent of a 12-form

$$\frac{\delta S_M}{2\pi} = \int_{M_{10}} \mathrm{I}_{10}^{(1)} \qquad \mathrm{I}_{12} = d\mathrm{I}_{11}^{(0)}\,, \qquad \delta \mathrm{I}_{11}^{(0)} = d\mathrm{I}_{10}^{(1)}$$

$$\boxed{\mathrm{I}_{12} = -\frac{1}{6}\left(E_4\right)^3 - E_4 \wedge X_8}$$

from $\delta(C_3 G_4^2)$ from $\delta\left(C_3 X_8\right)$

Inflow result for flat branes:

$$I_8^{inf} = \int_{S^4} \mathrm{I}_{12}$$

$$\boxed{I_8^{inf} + I_8[A_{N-1}] + I_8^{de} = 0}$$

Anomaly polynomial for 6D SCFT

Anomaly polynomial for a free 6D
(2,0) tensor multiplet
Center of mass degree of freedom

Reducing Anomaly Polynomial

Consider the case when the branes are wrapped on an even dimensional compact geometry

$$W_6 = \mathbb{R}^{1,5-k} \times X_k\,, \qquad M_{10} = \mathbb{R}^{1,5-k} \times M_{4+k}$$

$$S^4 \hookrightarrow M_{4+k} \to X_k$$

The S⁴ fibration is fixed by SUSY

Anomaly polynomial of field theory is obtain by Integrating over compact directions

$$I_{8-k} = \int_{M_{4+k}} I_{12}$$

1. Construct the form $\bar{E}_4$ on M_{4+k} — Flux that support M-theory background
2. Gauge $\bar{E}_4$ along the symmetries acting on M_{4+k} to obtain E_4
3. Integrate to obtain the lower dimensional anomaly polynomial

M5-branes on Riemann Surface

e.g. Consider the case when $X_2 = \Sigma_{g,n}$,

This configuration preserves
8 supercharges

$\Sigma_{g,n} \subset CY_2 = T^*\Sigma_{g,n}$

The 10D space near branes decomposes as

$$M_{10} = \underbrace{\mathbb{R}^{1,3} \times \Sigma_{g,n}}_{W_6} \times S^4 , \qquad M_{11} = r \times M_{10}$$

$$S^4 \hookrightarrow M_6 \to \Sigma_{g,n}$$

The space $M_6 = \Sigma_{g,n} \times S^4$ has boundaries and we cannot reduce the anomaly polynomial on it

Strategy for punctures:

Consider a closure of M_6 to $\widetilde{M}_6$ by gluing a space X_6^α at each puncture

$$\widetilde{M}_6 = M_6 \cup \bigcup_{\alpha=1}^{n} X_6^\alpha$$

We assume that X_6^α can be smoothly glued to M_6
The possible choices of punctures map to the
Possible choices for X_6^α

$$I_6^{inf} = \int_{\widetilde{M}_6} \mathrm{I}_{12} = \int_{M_6} \mathrm{I}_{12} + \sum_{\alpha=1}^{n} \int_{X_6^\alpha} \mathrm{I}_{12}$$

Contribution of puncture encoded in X_6^α and on E_4

Bulk Contribution to Anomaly

$$I_6^{inf} = \int_{\widetilde{M}_6} \mathrm{I}_{12} = \int_{M_6} \mathrm{I}_{12} + \sum_{\alpha=1}^{n} \int_{X_6^\alpha} \mathrm{I}_{12}$$

$$= I_6^{bulk} + \sum_{\alpha} I_6(G_\alpha)$$

Universal term fixed by 6D

$$I_6^{bulk} = \int_{M_6} \mathrm{I}_{12} = \int_{\Sigma_{g,n}} I_8^{inf} = -\chi(\Sigma_{g,n}) I_6$$

Integrating on the sphere yields
The anomaly for 6D theory

Integrate the 6D polynomial while
Implementing the twist

The game is to understand puncture geometry and how to fix the flux on it

Puncture Geometry and Puncture data

The non-puncture I

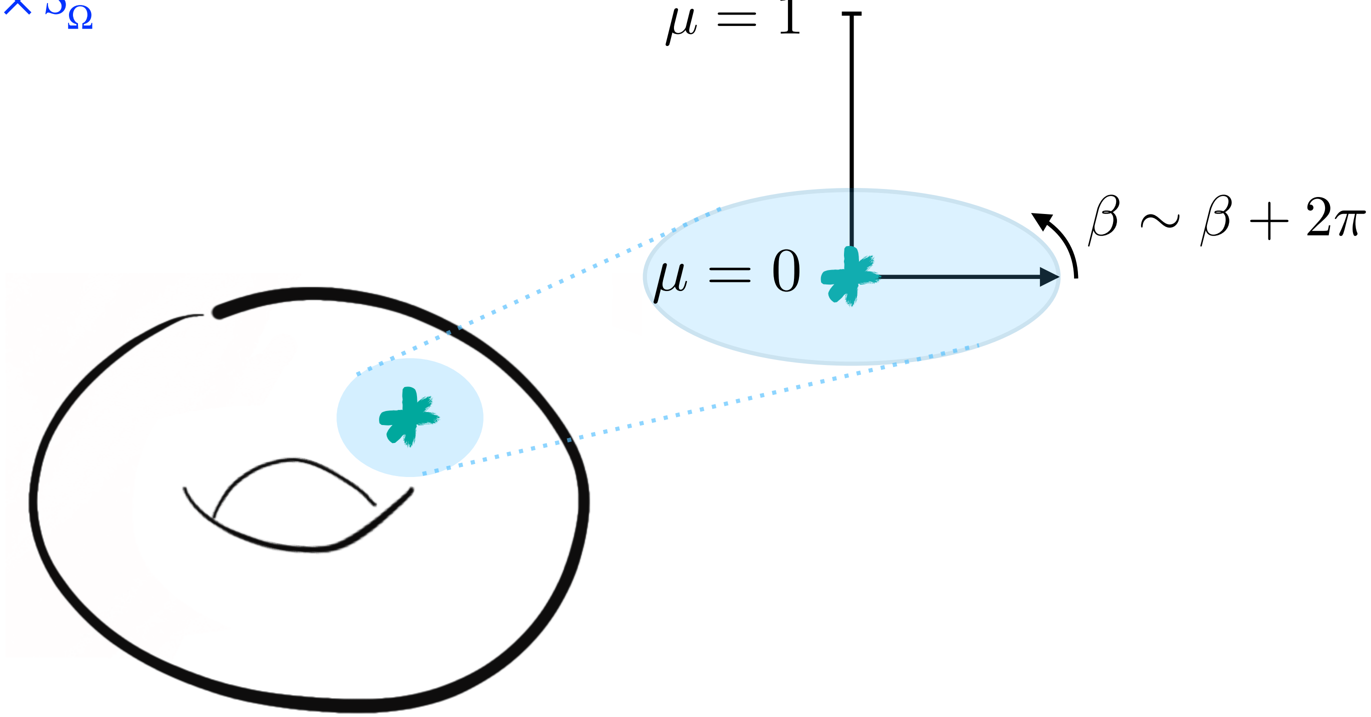
Consider a point on the Riemann Surface and a small disc D_α centered around it

Now, we have the product geometry the disc with the sphere fiber $X_6^\alpha = D_\alpha \times S^4$

The 4-sphere can be parametrized as $[\mu] \times S^1_\phi \times S^2_\Omega\,,\qquad \mu \in [0,1]\,,\qquad \begin{cases} S^2_\Omega \to \mu = 0 \\ S^1_\phi \to \mu = 1 \end{cases}$

The disc can be parametrized with (r_Σ, β)

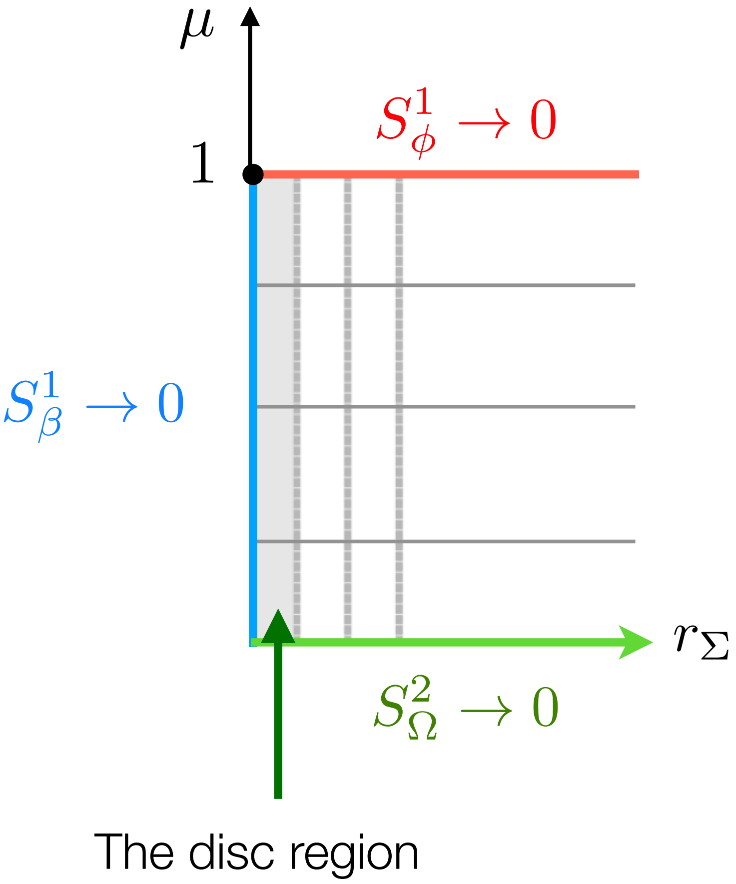
Also write $X_6^\alpha = X_4^\alpha \times S^2_\Omega$



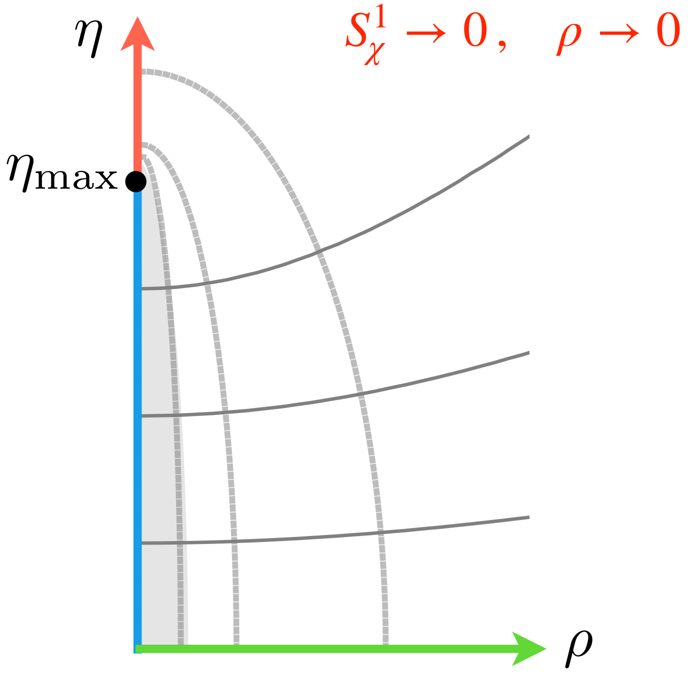
The non-puncture II

Perform a change of coordinates, trick motivated by holography [Gaiotto, Maldacena '09, IB '15]

$(r_\Sigma, \mu, \phi, \beta)$



$(\rho, \eta, \chi, \beta) \qquad \chi = \phi + \beta$



$S^1_\beta \hookrightarrow X^\alpha_4 \to \mathbb{R}^3\,, \quad D\beta = d\beta - Ld\chi$

$$\rho \to 0, \quad L = \begin{cases} 1 & \eta < \eta_{max} \\ 0 & \eta > \eta_{max} \end{cases}$$

The connection has monopole source with unit charge 1

Monopole Geometry as Puncture geometry

More interesting geometry obtained by adding more monopole sources

$D\beta - Ld\chi$

Add p monopoles at locations $\eta = \eta_a, \quad \eta_p = \eta_{max}$

In the region near $\rho = 0, \quad L(\rho, \eta)$ Piece-wise constant

Monopole charge given by the flux

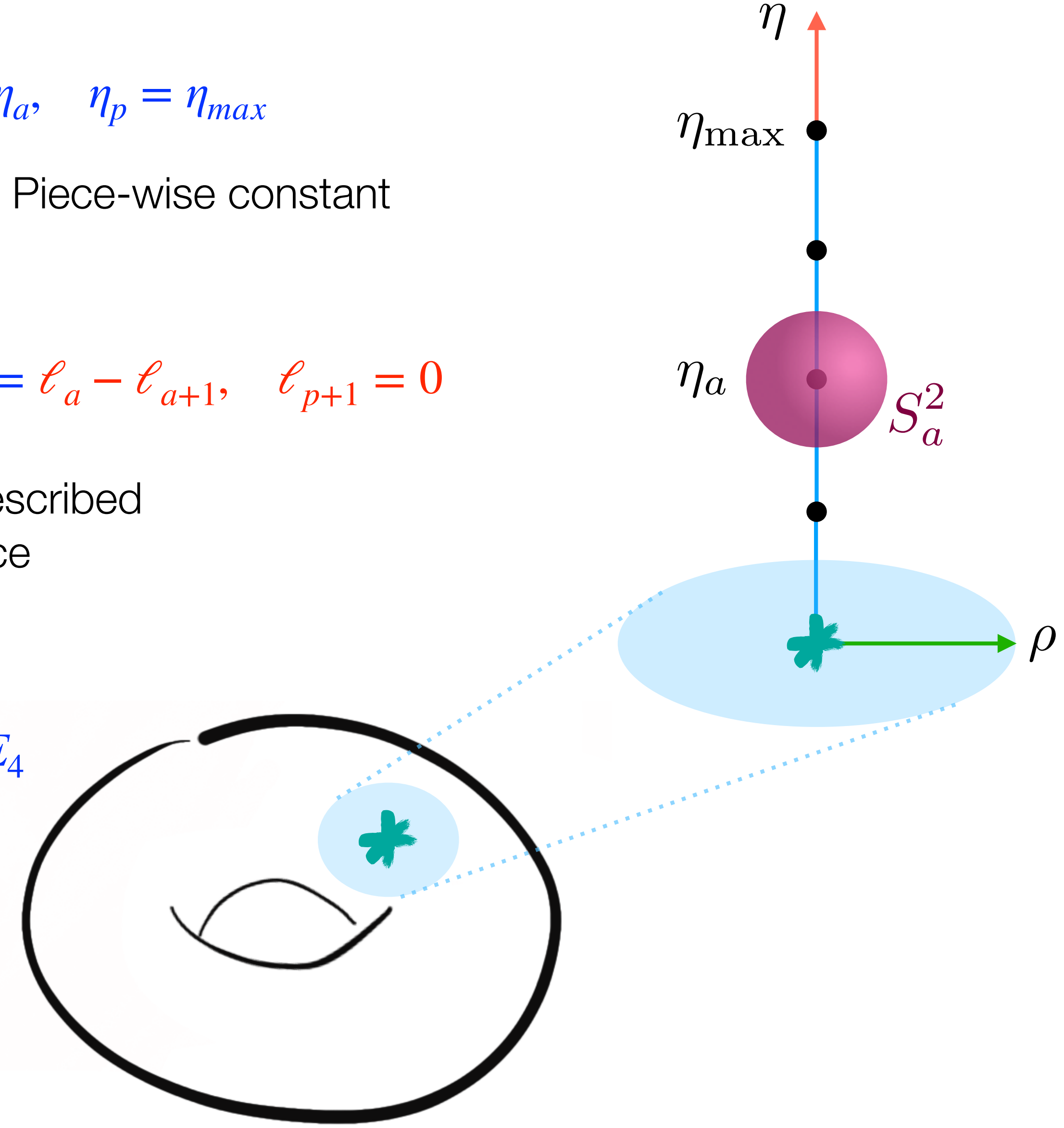
$$k_a = \int_{S_a^2} \frac{d(D\beta)}{2\pi} = -\Delta_{\eta=\eta_a} L(\rho=0) = \ell_a - \ell_{a+1}, \quad \ell_{p+1} = 0$$

The region near a monopole can be described by a single center Taub-Nut space

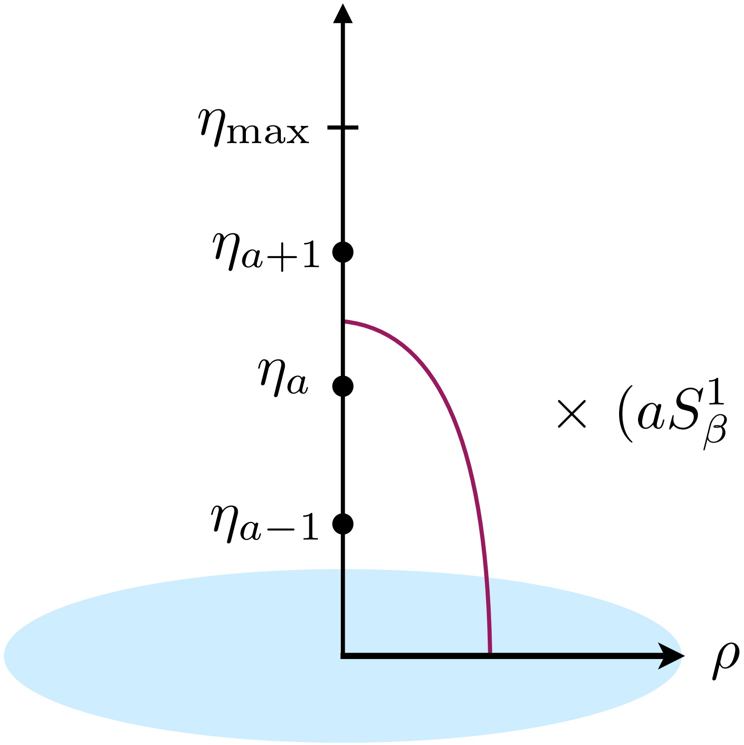
The 8-form X_8 picks up a contribution from each Taub-Nut space

Next we consider the possible flux for E_4

It is constrained by flux quantization and regularity



Cup Flux

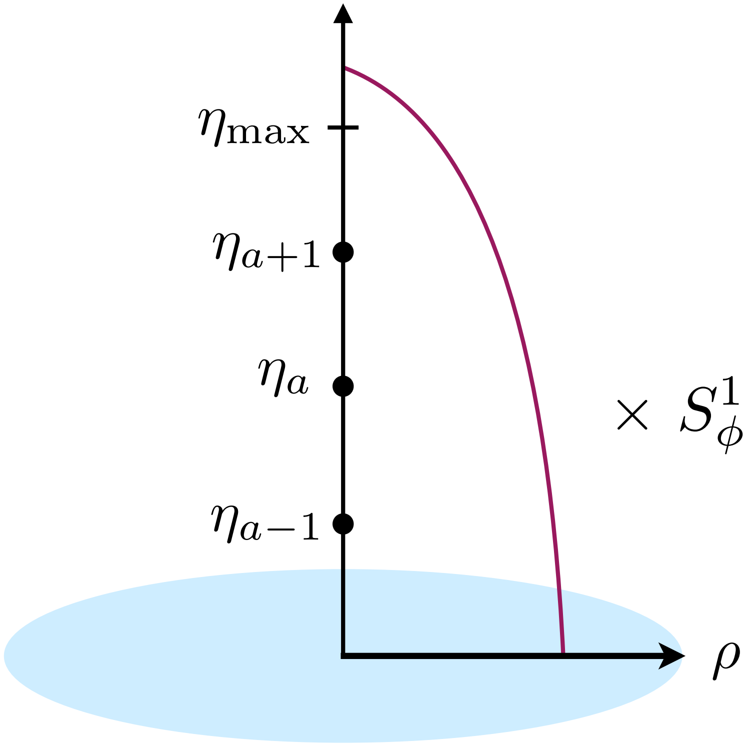


combination that vanishes on axis

$$\times\ (aS^1_\beta + bS^1_\chi)\ \times\ S^2_\Omega \simeq C^a_4$$

$$\int_{C^a_4} E_4 = y_a \in \mathbb{Z}$$

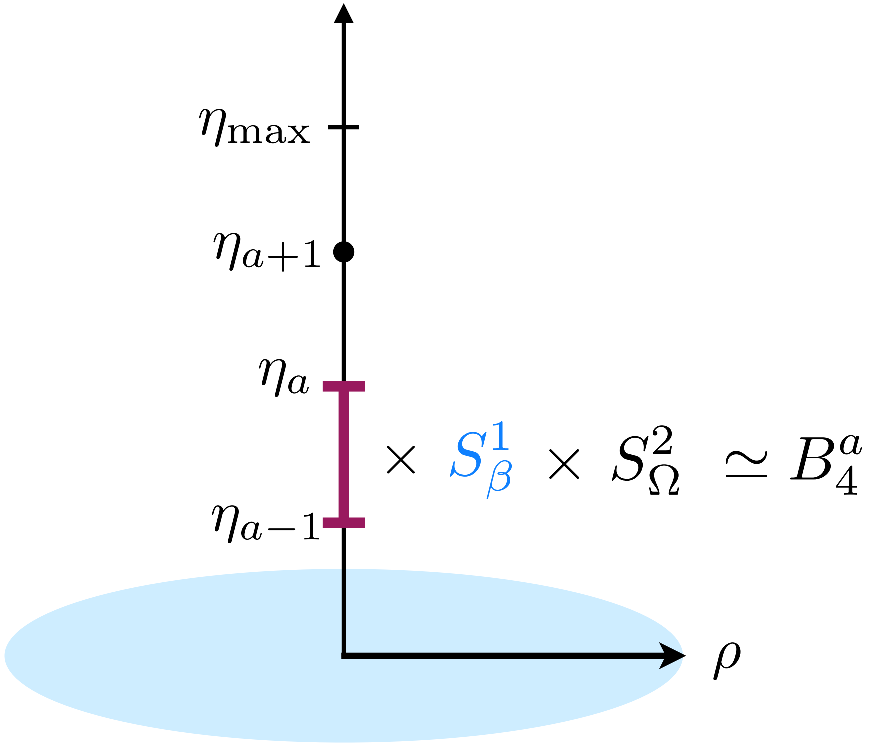
Total flux



$$\times\ S^1_\phi\ \times S^2_\Omega\ \simeq\ S^4$$

$$N = \int_{S^4} E_4 = y_p \in \mathbb{Z}$$

Bubble flux

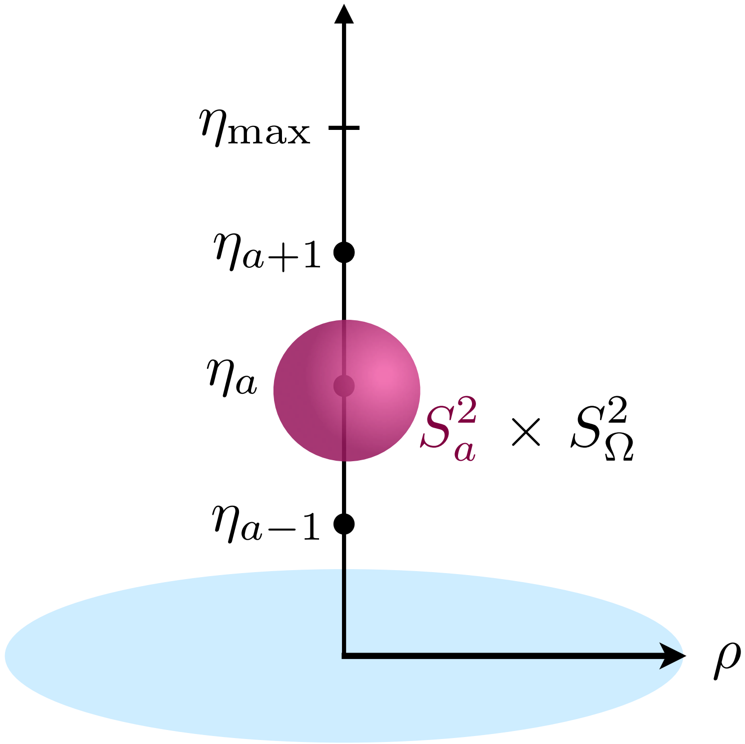


$$\int_{B_4^a} E_4 = w_a - w_{a-1} \in \mathbb{Z}_+$$

$$0 = w_0 < w_1 \cdots < w_p, \quad w_a \in \mathbb{Z}_+$$

Positivity of flux fixed by orientation of cycle

Regularity and Partition of N



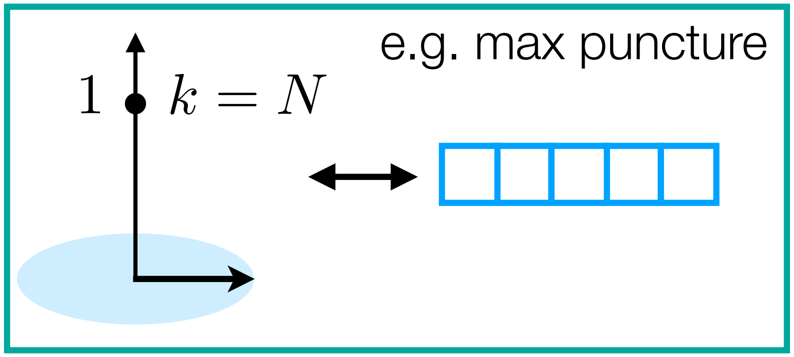
Regularity of flux near monopole implies

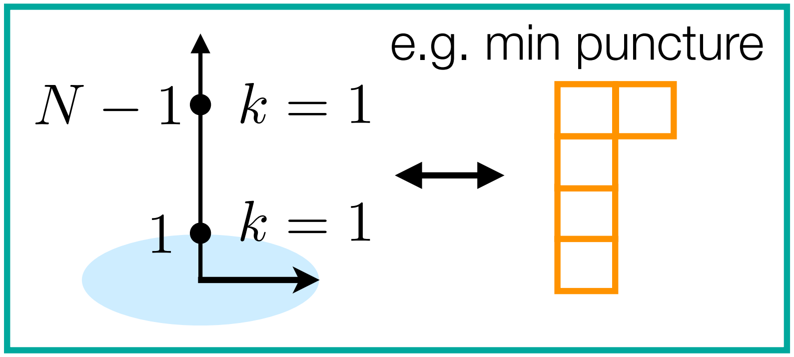
$$\int_{S_a^2\times S_\Omega^2} E_4 = 0$$

$$= y_a - y_{a-1} - w_a k_a$$

$$y_a = \sum_{b}^{a} w_a k_a, \qquad N = \sum_{b}^{p} w_a k_a$$

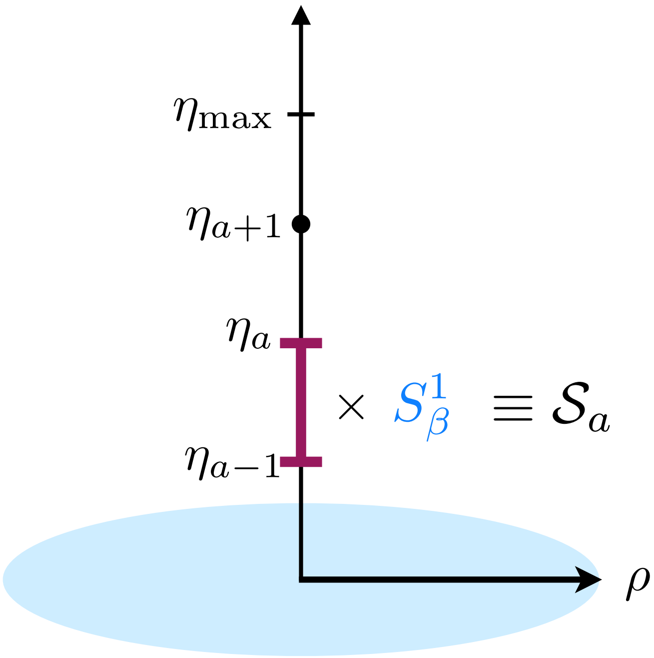
This partition of *N* defines a Young diagram.

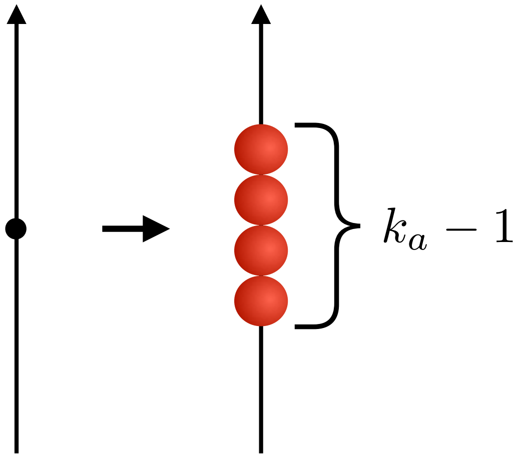




Flavor Symmetry

Flavor symmetry comes from two cycles in puncture geometry





Each two cycle has associated harmonic form, ω_a

$$E_4 \supset \frac{1}{2\pi}\sum_{a=2}^{p} \underset{\uparrow}{F_a} \wedge \omega_a$$

Background field for a $U(1)$ symmetry

Resolution of the monopole singularity leads to $(k_a - 1)\ \mathbb{P}^1$ with harmonic forms $\hat{\omega}_{a,I}$

$$E_4 \supset \frac{1}{2\pi}\sum_{a=1}^{p}\sum_{I=1}^{k_a-1} \underset{\uparrow}{\hat{F}^I_a} \wedge \hat{\omega}_{a,I}$$

Background fields for Cartan elements of $SU(k_a)$ symmetry

Total flavor symmetry at puncture: $G = S\left(\prod U(k_a)\right)$

Anomaly answer

The final answer can be compared to field theory analysis

$$I_6^{inf} + I_6^{CFT} + I_6^{decoupled} = 0$$

free tensor multiplet on smooth Riemann Surface

Flavor central charge: $k^{inf}_{SU(k_a)} + k^{CFT}_{SU(k_a)} = 0$

- We provide a geometric derivation of the anomaly polynomial for N=2 class S theories with regular punctures
- We are able to match with previous field theory analysis
- Anomaly data completely encoded in topology of M-Theory background
- We describe a class of geometries that capture data associated to regular punctures
- The description of the puncture is encoded in consistency of various flux on the puncture geometry!

Anomaly answer

With all the data the class E_4 can be constructed. Plug into I_{12} and integrate

$$I_6^{\text{inf}} = I_6^{\text{inf}}(\Sigma_{g,n}) + \sum_\alpha I_6^{\text{inf}}(P_\alpha)$$

$$I_6(\Sigma_{g,n})^{\text{inf}} = \frac{N\chi}{2}\left[\frac{(c_1^r)^3}{3} - \frac{c_1^r p_1(TM_4)}{12}\right] - \frac{(4N^3-N)\chi}{6} c_1^r c_2^R$$

$$I_6(P_\alpha)^{\text{inf}} = \frac{1}{2}\sum_{a=1}^{p} N_a k_a \left[\frac{(c_1^r)^3}{3} - \frac{c_1^r p_1(TM_4)}{12}\right]$$

$$- \sum_{a=1}^{p}\left[\frac{2}{3}\ell_a^2(w_a^3 - w_{a-1}^3) - \frac{1}{6}N_a k_a + \ell_a(N_a - w_a\ell_a)(w_a^2 - w_{a-1}^2)\right] c_1^r c_2^R$$

$$- \sum_{a=1}^{p} 2N_a c_1^r \ \mathrm{ch}_2(SU(k_a))$$

$$N_a = \sum_{b=1}^{a} \ell_b(w_b - w_{b-1})$$

$p =$ # of monopoles

Outlook

- The 12-form polynomial of M-theory can be used in holographic systems to compute anomalies
- What are geometries that capture the anomalies for systems with Irregular punctures?
- How are puncture geometries related to descriptions in terms of Hitchin's equations?
- What are the geometries that capture punctures for N=1 class S or class S_k?
- Do they provide a new method for obtaining puncture data?
- What is the associated polynomial for general (1,0) theories?

Thank you!