T-duality & Integrable Models

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Outline

i) Non-Abelian T-duality w. r. t. non-semisimple groups

ii) Supergravity possesses innate knowledge of r-matrix solutions to the Classical Yang-Baxter Equation

NA T-duality

For simplicity, consider class of Bianchi cosmologies

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a_1^2\sigma_1 + a_2^2\sigma_2^2 + a_3^2\sigma_3^2, \quad a_i = a_i(t)$$

$$\mathrm{d}\sigma_i = \frac{1}{2} f^i_{jk}\sigma_j \wedge \sigma_k$$

Matrix inversion

de la Ossa, Quevedo (1992)

$$g'_{ij} + B'_{ij} = (\gamma_{ij} + \beta_{ij} + f^k_{ij}\lambda_k)^{-1}$$

$$\gamma_{ij} = \mathrm{diag}(a_1^2, a_2^2, a_3^2),$$
$$\beta_{ij} = 0$$

NA T-duality

Dilaton shift

de la Ossa, Quevedo (1992)

$$\Phi \to \Phi + \frac{1}{2} \ln \det N, \qquad N = (\gamma + \beta + f \cdot \lambda)^{-1}$$

Funny transformation, e. g. Bianchi IX

$$N = \begin{pmatrix} 1 & \lambda_3 & -\lambda_2 \\ -\lambda_3 & 1 & \lambda_1 \\ \lambda_2 & -\lambda_1 & 1 \end{pmatrix}$$

Problem

A Problem with Non-Abelian Duality

M. Gasperini R. Ricci G. Veneziano

(Submitted on 24 Aug 1993 (v1), last revised 7 Sep 1993 (this version, v2))

We investigate duality transformations in a class of backgrounds with non-Abelian isometries, i.e. Bianchi-type (homogeneous) cosmologies in arbitrary dimensions. Simple duality transformations for the metric and the antisymmetric tensor field, generalizing those known from the Abelian isometry (Bianchi I) case, are obtained using either a Lagrangian or a Hamiltonian approach. Applying these prescriptions to a specific conformally invariant $\s$ -model, we show that no dilaton transformation leads to a new conformal background. Some possible ways out of the problem are suggested.

Same procedure fails to work for non-semisimple groups

Bianchi V $$f_{12}^2 = f_{13}^3 = 1$$

Anomaly

Non-semisimple groups: there exists gauge-gravity anomaly.

Alvarez, Alvarez-Gaumé, Lozano (1994)

NATD sigma-model

$$S = \frac{1}{2\pi}\int \mathrm{d}^2 z \Big(F_{ij}\partial x^i \bar{\partial} x^j + (2\Phi + \ln\det N)\partial\bar{\partial}\sigma$$

$$+ (\partial\lambda_a - \partial x^i F^L_{ia} + \mathrm{tr} T_a \partial\sigma) N^{ab} (\bar{\partial}\lambda_b + F^R_{bj}\bar{\partial} x^j - \mathrm{tr} T_b \bar{\partial}\sigma) \Big),$$

$$N = (E(x) + \lambda_c f^c)^{-1}, \quad (f^c)_{ab} = f^c_{ab}$$

Elitzur, Giveon, Rabinovici, Schwimmer, Veneziano (1994)

Resolution

Anomaly contribution for Bianchi V

$$S = S_0 + S_1$$
$$\pi\delta_\sigma S_0 = \frac{1}{2}[\beta_{G_{ij}} + \beta_{B_{ij}}]\partial x^i \bar{\partial} x^j + \frac{1}{2}\beta_\Phi \partial\bar{\partial}\sigma \quad \text{(one-loop)}$$
$$\pi\delta_\sigma S_1 = -\frac{1}{2}[\beta_{G_{ij}} + \beta_{B_{ij}}]\partial x^i \bar{\partial} x^j$$

Modification of $\boldsymbol{\beta}$-functions

Elitzur, Giveon, Rabinovici, Schwimmer, Veneziano (1994)

Argument can be easily extended to generic (g, B)

Hong, Kim, ÓC (2018)

Yang-Baxter σ-model

Integrable deformations of $AdS_5 \times S^5$ based on the Yang-Baxter σ-model.

Klimčík (2002); Delduc, Magro, Vicedo (2013); Kawaguchi, Matsumoto, Yoshida (2014)

Deformations that are not supergravity solutions.

Birth of “Generalized Supergravity”.

Connection to NA T-duality rediscovered.

Hoare, Tseytlin (2016); Borsato, Wulff (2016 - 2018); Hoare, Thompson (2017)

Hassler (2017); Demulder, Hassler, Thompson (2018)

Generalized Supergravity

Arutyunov, Hoare, Frolov, Roiban, Tseytlin (2015); Tseytlin, Wulff (2016)

$$R_{MN} - \frac{1}{4}H_{MKL}H_N{}^{KL} - T_{MN} + \nabla_M X_N + \nabla_N X_M = 0$$

$$\frac{1}{2}\nabla^K H_{KMN} + \frac{1}{2}\mathcal{F}^K\mathcal{F}_{KMN} + \frac{1}{12}\mathcal{F}_{MNKLP}\mathcal{F}^{KLP} = X^K H_{KMN} + \nabla_M X_N - \nabla_N X_M$$

$$R - \frac{1}{12}H^2 + 4\nabla_M X^M - 4X_M X^M = 0$$

$$T_{MN} \equiv \frac{1}{2}\mathcal{F}_M\mathcal{F}_N + \frac{1}{4}\mathcal{F}_{MKL}\mathcal{F}_N{}^{KL} + \frac{1}{96}\mathcal{F}_{MPQRS}\mathcal{F}_N{}^{PQRS} - \frac{1}{4}G_{MN}(\mathcal{F}_K\mathcal{F}^K + \frac{1}{6}\mathcal{F}_{PQR}\mathcal{F}^{PQR})$$

$$X = \mathrm{d}\Phi + I + i_I B, \quad \mathcal{F} = e^\Phi F$$

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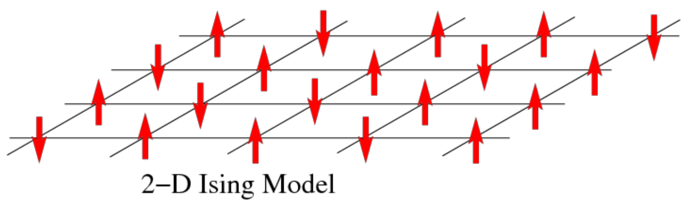
ii) Supergravity possesses innate knowledge of r-matrix solutions to the Classical Yang-Baxter Equation

Yang-Baxter

QYBE is a hallmark of exact solvability (integrability)

- Statistical Physics
- PDEs - KdV equation
- Knot theory
- Chern-Simons theory
- Supergravity (CYBE)



$$\frac{\partial u}{\partial t} + au\frac{\partial u}{\partial x} + b\frac{\partial^3 u}{\partial x^3} = 0$$

Classical Yang-Baxter

"Classical limit" of the QYB: simpler equation

$$[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) = -c^2[X, Y]$$

$$X, Y \in \mathfrak{g}, \quad c \in \mathbb{C}$$

$$R(X) = r^{ij}\, b_i\, \mathrm{Tr}[b_j X] \qquad\qquad r = \frac{1}{2} r^{ij} b_i \wedge b_j$$

$$\mathrm{Tr}[b_i X]\mathrm{Tr}[b_j Y] b_k \left(r^{l_1 i} r^{l_2 j} f_{l_1 l_2}{}^{k} + r^{l_1 j} r^{l_2 k} f_{l_1 l_2}{}^{i} + r^{l_1 k} r^{l_2 i} f_{l_1 l_2}{}^{j} \right) = -c^2[X, Y]$$

Statement

Given any solution to supergravity with an isometry group, there exists a deformation where the equations of motion reduce to the (homogeneous) Classical Yang-Baxter Equation.

$$f_{l_1 l_2}{}^{i} r^{l_1 j} r^{l_2 k} + f_{l_1 l_2}{}^{j} r^{l_1 k} r^{l_2 i} + f_{l_1 l_2}{}^{k} r^{l_1 i} r^{l_2 j} = 0$$

Yang-Baxter σ-model

In the target space, recover open-closed string map from noncommutativity in string theory.

$$(G^{-1} + \Theta) = (g + B)^{-1}$$

Realized this from studying AdS/CFT picture. This itself is an O(d,d) transformation.

van Tongeren (2015/6),
Araujo, Bakhmatov, ÓC, Sakamoto, Sheikh-Jabbari, Yoshida (2017)

Modification related to the deformation: $I^{\mu} = \nabla^{G}_{\nu} \Theta^{\nu\mu}$

Generic recipe

Classical Yang-Baxter Equation from Supergravity

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We promote the open-closed string map, originally formulated by Seiberg & Witten, to a solution generating prescription in generalized supergravity. The approach hinges on a knowledge of an

1. Extract g, B from open-closed string map
2. Dilaton from known T-duality invariant (covers TsT)
3. Killing vector from divergence of “NC parameter”

Embed in DFT **Fernandez-Melgarejo, Sakamoto, Sakatani, Yoshida (2017)**

One Unknown

For simple examples, e. g. D=2, can solve for NC parameter.

Supergravity knows it is the r-matrix (AdS_2, S^2, etc.)

More generally, can assume it is bi-Killing and nice things happen.

$$\Theta^{\mu\nu} = r^{ij} K_i^\mu K_j^\nu \qquad r^{ij} = -r^{ji}$$

$$I^\mu = \frac{1}{2} r^{ij} f_{ij}{}^k K_k^\mu$$

$$\Theta^{[\alpha\rho} \nabla_\rho \Theta^{\beta\gamma]} = K_i^\alpha K_j^\beta K_k^\gamma f_{l_1 l_2}{}^{[i} r^{j l_1} r^{k] l_2} = 0$$

Perturbative Proof

Expand in NC parameter, plug into GS EOMs.

$$g_{\mu\nu} = G_{\mu\nu} + \Theta_{\mu}{}^{\alpha}\Theta_{\alpha\nu} + \mathcal{O}(\Theta^4),$$
$$B_{\mu\nu} = -\Theta_{\mu\nu} - \Theta_{\mu\alpha}\Theta^{\alpha\beta}\Theta_{\beta\nu} + \mathcal{O}(\Theta^5),$$
$$\phi = \Phi + \frac{1}{4}\Theta_{\rho\sigma}\Theta^{\rho\sigma} + \mathcal{O}(\Theta^4)$$

$$K_i^{\alpha}K_k^{\beta}\nabla_{\alpha}K_{\beta m}\left(f_{l_1l_2}{}^{m}r^{il_1}r^{kl_2} + f_{l_1l_2}{}^{k}r^{ml_1}r^{il_2} + f_{l_1l_2}{}^{i}r^{kl_1}r^{ml_2}\right) +$$
$$\left(\Theta^{\beta\gamma}\Theta^{\alpha\lambda} + \Theta^{\alpha\beta}\Theta^{\gamma\lambda} + \Theta^{\gamma\alpha}\Theta^{\beta\lambda}\right)R_{\beta\gamma\alpha\lambda} = 0.$$

Bakhmatov, ÓC, Yavartanoo, Sheikh-Jabbari (2018)

Derivation of map from NA T-duality **Borsato, Wulff (2018)**

Example

Method works for all geometries, e .g. Schwarzschild, etc.

$$\mathrm{d}s^2 = \frac{(-\mathrm{d}t^2 + \mathrm{d}z^2)}{z^2} + \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 + \mathrm{d}s^2(T^6),$$

$$F_5 = (1 + *_{10})\frac{1}{\sqrt{2}z^2}\mathrm{d}t \wedge \mathrm{d}z \wedge (\omega_r - \omega_i)$$

Focus on AdS_2 (here I can solve for deformation in general)

$$K_1 = -t\partial_t - z\partial_z, \quad K_2 = -\partial_t, \quad K_3 = -(t^2 + z^2)\partial_t - 2tz\partial_z$$

Example

$$\begin{aligned}\Theta &= \alpha K_1 \wedge K_2 + \beta K_2 \wedge K_3 + \gamma K_3 \wedge K_1 \\ &= \big(-\alpha z + \beta 2tz + \gamma z(-t^2 + z^2)\big)\, \partial_t \wedge \partial_z \equiv \zeta \partial_t \wedge \partial_z\end{aligned}$$

Deformed solution

$$\mathrm{d}s^2 = \frac{z^2}{(z^4 - \zeta^2)}(-\mathrm{d}t^2 + \mathrm{d}z^2), \quad B = \frac{\zeta}{(z^4 - \zeta^2)}\mathrm{d}t \wedge \mathrm{d}z,$$

$$\Phi = -\frac{1}{2}\log\left[\frac{(z^4 - \zeta^2)}{z^4}\right], \quad I = -\alpha T_1 - 2\beta T_2 + \gamma T_3,$$

$$F_5 = (1 + *_{10})\frac{z^2}{\sqrt{2}(z^4 - \zeta^2)}\mathrm{d}t \wedge \mathrm{d}z \wedge (\omega_r - \omega_i), \quad F_3 = -\frac{\zeta}{\sqrt{2}z^2}(\omega_r - \omega_i)$$

Example

Einstein equation

$$R_{tt} + 2\nabla_t X_t = \frac{z^2(1 - 4\beta^2 + 4\alpha\gamma)(z^4 + \zeta^2)}{(z^4 - \zeta^2)^2}$$

$$T_{tt} = \frac{z^2(z^4 + \zeta^2)}{(z^4 - \zeta^2)^2} \qquad -\beta^2 + \alpha\gamma = 0$$

Two ways to solve EOMs:

i) impose homogeneous CYBE

ii) absorb constant in dilaton (modified CYBE)

Example

Einstein equation

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Two ways to solve EOMs:

i) impose homogeneous CYBE

ii) absorb constant in dilaton (modified CYBE)

$T\bar{T}$

Deformations of 2D CFTs by irrelevant operator

Zamalodchikov, Smirnov (2016)

holographically can be described by marginal deformations of WZW models

Giveon, Itzhaki, Kutasov (2017)

necessary condition on marginal deformation (perturbatively)

$$\delta\mathcal{L} = g\sum_{ij} c^{ij} J_i(z)\bar{J}_j(\bar{z})$$

Chaudhuri, Schwartz (1988)

$$\sum_{i,j} c^{il}c^{jm}f_{ij}{}^{k} = \sum_{i,j} c^{li}c^{mj}\bar{f}_{ij}{}^{k} = 0, \quad \text{for all } l,m,k$$

$T\overline{T}$

For Abelian chiral currents conjectured that the finite transformation was an O(d,d) transformation **Hassan, Sen (1992)**

Conjecture later proved. **Kiritsis (1993); Henningson, Nappi (1993)**

The input from YB deformations is that the Abelian condition can be relaxed.

For unimodular YB deformations with chiral split, the CYBE condition is the CS condition.

Conclusions

Generalized Supergravity can be traced to NA T-duality.

Through this open-closed map - an O(d,d) transformation - Supergravity equations of motion reduce to the CYBE.

True for any supergravity solution with an isometry group.