Integrable Deformations and Generalised Dualities

Holography, Generalised Geometry and Duality MITP

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Driezen, Sevrin, DT [1806.10712,1902.04142]

Demulder, Hassler, DT [1810.11446]

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Motivation

Dualities.....

....a catalyst for theoretical progress in diverse areas: statistical physics; QFT theory; condensed matter and of course String Theory.

- Target space T-duality – intrinsically stringy $\Rightarrow$ new geometric ideas e.g. generalised geometry or DFT
- More generally U-dualities $\Rightarrow$ M-theory?
- Gauge-gravity dualities or holography!

What other dualities?

What are their uses?

Motivation

A hierarchy of T-dualities

Bianchi–Conservation democracy ?

1. Abelian isometries $\Rightarrow$ Abelian T-duality

$$K = \partial_\theta \ , \qquad [K, K] = 0 \ , \qquad d \star J = 0$$

2. Non-Abelian isometries $\Rightarrow$ Non-Abelian T-duality Quevedo,De La Ossa

$$K_a = k_a^\mu \partial_\mu \ , \qquad [K_a, K_b] = f_{ab}{}^c K_c \ , \qquad d \star J_a = 0$$

3. Non-Abelian Non-isometries $\Rightarrow$ Poisson-Lie T-duality Klimick,Severa

$$K_a = k_a^\mu \partial_\mu \ , \qquad [K_a, K_b] = f_{ab}{}^c K_c \ , \qquad d \star J_a = \tilde{f}^{bc}{}_a J_b \wedge J_c$$

Motivation

Reasons to be skeptical ...apologia

- Quantum g_s and α' status unclear ... **Holography large N**
- Baroque or ugly geometries ... **wrong variables**

Reasons to care

- Non-Abelian T-duality **holographic backgrounds** for exotic quiver QFTs
- η- and λ- **integrable deformations** of AdS_5 superstring
- Close connection to **gauged supergravity**
- Examples of **generalised parallelisable** geometries
- A manifold structure for DFT

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Non-linear sigma model and principal chiral model

Strings in curved target space $\mathcal{M}$, $E_{ij} = G_{ij} + B_{ij}$:

$$S = \int \partial_+ X^i (G_{ij}(X) + B_{ij}(X)) \partial_- X^j$$

Suppose an isometry group G of vector field K_a then Noether currents

$$J_{\pm a} = K_a{}^i \left(G_{ij} \pm B_{ij}\right) \partial_\pm X^j$$

Useful example $\mathcal{M} = G$, a group manifold, and the PCM

$$S = \int \langle g^{-1}\partial_+ g, g^{-1}\partial_- g\rangle = \int L_+^a \kappa_{ab} L_-^b \ , \quad g = g(X) : \Sigma \to G$$

Left-invariant one-forms $L = g^{-1} dg$

Recap: the Principal Chiral Model

- **Classically (and Quantum) Integrable**: Lax formulation of e.q.m.

$$\mathcal{L}(z) = \frac{1}{1-z^2} g^{-1} dg + \frac{z}{1-z^2} \star g^{-1} dg \ , \quad d\mathcal{L} - \mathcal{L} \wedge \mathcal{L} = 0 \ ,$$

  $z \in \mathbb{C}$ an auxiliary parameter;
- **∞ of conserved charges** encoded in z-expansion of monodromy

$$T(z) = Pexp \int d\sigma \mathcal{L}_\sigma \ , \quad \partial_\tau T(z) = 0$$

Non-Abelian T-dual: The Buscher Procedure

Gauging procedure to obtain the non-Abelian T-dual geometry

1. **Gauge** G_L in PCM $\partial g \to Dg = \partial g - Ag$
2. **Double** the degrees of freedom with Lagrange multipliers

$$L_v = v_a F^a_{+-} \qquad F_{+-} = [D_+, D_-]$$

3. **Gauge Fix** $g = 1$ and integrate by parts
4. **Integrate out** non-propagating gauge fields to get new sigma model

$$S_{T-dual} = \frac{1}{\pi} \int \partial_+ v^a (\kappa^2 \delta_{ab} + F_{ab}{}^c v_c)^{-1} \partial_- v^b$$

Classical equivalence (canonical transformation) to PCM

Non-Abelian T-dual: Example of S^3

Lag. multipliers in spherical coordinates

$$(v_1, v_2, v_3) \mapsto (r, \theta, \phi)$$

Extract T-dual geometry

$$\widehat{ds^2} = \frac{dr^2}{\kappa^2} + \frac{r^2\kappa^2}{r^2+\kappa^4}\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

$$\widehat{B} = \frac{r^3}{r^2+\kappa^4}\sin\theta d\theta \wedge d\phi$$

$$\widehat{\Phi} = \phi_0 - \frac{1}{2}\log(r^2+\kappa^4)$$

Extends to RR sector and type II supergravity [Sfetsos,Thompson]

λ-deformations: The Sfetsos Procedure

Rather similar to the Buscher procedure this recipe produces integrable λ deformations [Sfetsos 1312] as a regularisation of non-Abelian T-duality

1. **Double** the d.o.f.: $\kappa^2 S_{PCM}[\tilde{g}] + kS_{WZW}[g]$
2. **Gauge** G_L in PCM and G_{diag} in WZW
3. **Gauge Fix** $\tilde{g} = 1$
4. **Integrate out** non-propagating gauge fields

$$S_\lambda = kS_{WZW} + \frac{k\lambda}{2\pi}\int Tr(g^{-1}\partial_+ g\mathcal{O}_g\partial_- gg^{-1})$$

$$\mathcal{O}_g = (1 - \lambda \mathrm{ad}_g)^{-1} \qquad \lambda = \frac{k}{\kappa^2 + k}$$

Integrable model for all values of λ!

Interpolation between CFT and non-Abelian T-duals

Nice behaviour in limits of small and large deformations:

- $\lambda \to 0$: current bilinear perturbation

$$S_\lambda|_{\lambda\to 0} \approx kS_{WZW} + \frac{k}{\pi}\int \lambda J_+^a J_-^a + \mathcal{O}(\lambda^2)$$

- $\lambda \to 1$: non-Abelian T-dual of PCM

$$S_\lambda|_{\lambda\to 1} \approx \frac{1}{\pi}\int \partial_+ X^a(\delta_{ab} + f_{ab}{}^c X_c)^{-1}\partial_- X^b + \mathcal{O}(k^{-1})$$

In this limit the gauged WZW in the Sfetsos Procedure becomes a Lagrange multiplier term of the Buscher Procedure

D-branes in the λ-model

Boundaries break symmetries but b.c. that preserve integrability?

Technique: Conserved boundary Monodromy Cherednik 84, Sklyanin 88

Transport the Lax from $0 \to \pi$, and reflect $\pi \to 0$

$$
\begin{aligned}
T^b(z) &= T^\Omega(0,\pi,-z)\,T(\pi,0,z) \\
&= Pexp \int_0^\pi \Omega(\mathcal{L}_\sigma(-z)) \cdot Pexp \int_\pi^0 \mathcal{L}_\sigma(z)
\end{aligned}
$$

$\Omega \in aut\, \mathfrak{g}$ automorphism encodes reflection at boundary.
Conserved charges $Q^{(n)} = Tr(T^b(z))^n$ if

$$
\partial_\tau T^b(z) = [T^b(z), N(z)]
$$

D-branes in the λ-model

- Using explicit form of Lax we find integrable boundary conditions:

$$\mathcal{O}_{g^{-1}}[g^{-1}\partial_- g]|_{\partial\Sigma} = -\Omega \cdot \mathcal{O}_g[\partial_+ g g^{-1}]|_{\partial\Sigma}$$

- Interpret these as a mix of Dirichlet and Neumann b.c.

$$\partial_\tau X^D = 0 \ , \quad \widehat{G}_{ab}\partial_\sigma X^{bN} = \mathcal{F}_{ab}\partial_\tau X^{bN} = (\widehat{B}_{ab} + 2\pi\alpha' F_{ab})\partial_\tau X^{bN}$$

with gauge flux $F = dA$ on the brane.

- D-branes are twisted conjugacy classes – matching beautiful results in CFT

Alekseev Schomerus, Felder et al., Stanciu, Stanciu Figueroa-O'Farrill

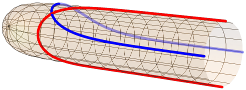
$$C_\omega(g) = \{hg\omega(h^{-1})|h \in G\} \ , \quad \omega(e^{tX}) \sim e^{t\Omega X} \ .$$

Asymmetric λ-model and D-branes

λ-deformations of G/H gauged WZW model. Integrable deformation of cigar black hole? Requires an additional twist to Sfetsos procedure by an outer automorphism in the gauging.

$$ds^2 = k\frac{1+\lambda^2}{1-\lambda^2}\frac{d\xi d\bar{\xi}}{1+|\xi|^2} + \frac{\lambda}{1-\lambda^2}\frac{d\xi^2 + d\bar{\xi}^2}{1+|\xi|^2}$$

- **D1** hairpins $\partial_\tau(\xi - \bar{\xi}) = 0$ and $\partial_\sigma(\xi - \bar{\xi}) = 0$



- **D0** living at the tip $\partial_\tau \xi = \partial_\tau \bar{\xi} = 0$, $\xi = \bar{\xi} = 0$
- **D2** with world volume gauge field

Open questions for application for integrable deformations of Sine-Liouville (FZZ conjecture) and matrix model dual?

Contents

Poisson-Lie Symmetry

Relax assumption of isometries but still keep T-duality? σ-model on group G with isometries broken in a special way [Klimcik Severa]

- Modified conservation law for currents:

$$d \star J_a = \tilde{F}^{\,bc}{}_a \, J_b \wedge J_c$$

- Constrains metric and B-field $E_{ij} = G_{ij} + B_{ij}$

$$L_{K_a} E_{ij} = \tilde{F}^{\,bc}{}_a K_b^m K_c^n E_{mi} E_{nj}$$

1. This condition can be solved!
2. When it is solved there is an equivalent dual σ-model on $\tilde{G}$

Drinfeld Double technology 1

Compatibility on $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ gives a cocycle condition, i.e. bi-Algebra structure

$$0 = F_{ab}{}^{c}\tilde{F}^{ec}{}_{a} - 2F_{d[a}{}^{c}\tilde{F}^{de}{}_{b]} + 2\tilde{F}^{dc}{}_{[a}F_{b]df}{}^{e} \, ,$$

Equivalent to Drinfeld double algebra $\mathfrak{d}$: $\mathbb{T}_A = (T_a, \tilde{T}^a)$

$$[\mathbb{T}_A, \mathbb{T}_B] = i\mathbb{F}_{AB}{}^{C}\mathbb{T}_C$$

$$[T_a, T_b] = iF_{ab}{}^{c}T_c \, , \quad [T_a, \tilde{T}^b] = i\tilde{F}^{bc}{}_{a}T_c - iF_{ac}{}^{b}\tilde{T}_c \, , \quad [\tilde{T}^a, \tilde{T}^b] = i\tilde{F}^{ab}{}_{c}\tilde{T}^c$$

Maximally isotropic subgroups $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$

$$\eta(T_a, T_b) = 0 \, , \quad \eta(T_a, \tilde{T}^b) = \delta_a^b \, , \quad \eta(\tilde{T}^a, \tilde{T}^b) = 0 \, ,$$

Drinfeld Double technology 2

Important combination of adjoint actions (the Poisson bi-vector)

$$\Pi = \Pi_g^{ab} T_a \otimes T_b : G \to \mathfrak{g} \wedge \mathfrak{g}$$

$$a_a{}^b = \eta(g T_a g^{-1}, \tilde{T}^b)\ , \quad b^{ab} = \eta(g \tilde{T}^a g^{-1}, \tilde{T}^b)\ , \quad \Pi_g = b_g a_{g^{-1}}$$

Nice behaviour under group multiplication

$$\Pi_{hg} = \Pi_g + (a_{g^{-1}} \otimes a_{g^{-1}})\Pi_h\ , \quad \Pi_e = 0$$

Essentially the integral of the $\tilde{F}^{ab}{}_c$:

$$d\Pi^{ab} = -L^c \tilde{F}^{ab}{}_c - 2L^c F_{cd}{}^{[a}\Pi^{b]d}$$

PLT Pairs of Sigma models

- PL T-duality equivalence between two σ-models

$$S[g] = \int g^{-1}\partial_+ g(E_0^{-1} + \Pi)^{-1} g^{-1}\partial_- g\,, \quad g \in G$$

$$\tilde{S}[\tilde{g}] = \int \tilde{g}^{-1}\partial_+ \tilde{g}(E_0 + \tilde{\Pi})^{-1} \tilde{g}^{-1}\partial_- \tilde{g}\,, \quad \tilde{g} \in \tilde{G}$$

- $E_0 = G_0 + B_0$ contains d^2 constant moduli (can promote to functions of spectators)
- The two models are related by a canonical transformation
- Normally very "ugly" target spaces, algebraic structure quite hidden

Examples

This set up subsumes both Abelian and non-Abelian T-duality and goes further

1. $\mathfrak{g} = u(1)^d$, $\tilde{\mathfrak{g}} = u(1)^d$, $\Pi = \tilde{\Pi} = 0 \Rightarrow$ **Abelian T-dual**
2. $\tilde{\mathfrak{g}} = u(1)^d$, $\Pi = 0$, $\tilde{\Pi}_{ab} = f_{ab}{}^c \tilde{X}_c \Rightarrow$ **non-Abelian T-dual**
3. $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ both non-Abelian $\Rightarrow$ **PL T-dual**
4. $\mathfrak{d} = \mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + (\mathfrak{a} + \mathfrak{n}) \Rightarrow$ **Integrable η-models**

Yang-Baxter and η Deformations

Integrable models [Klimcik '02] based on modified Yang-Baxter eq

$$[\mathcal{R}A, \mathcal{R}B] - \mathcal{R}([\mathcal{R}A, B] + [A, \mathcal{R}B]) = -c^2 [A, B] \ , \quad \forall A, B \in \mathfrak{g}$$

An integrable deformed PCM

$$S_\eta = \frac{1}{2\pi t} \int_\Sigma d^2\sigma \, Tr \left(\partial_+ g g^{-1}, \frac{1}{1 - \eta \mathcal{R}} \partial_- g g^{-1} \right)$$

- $c^2 = -1 \Rightarrow \eta$ Deformations
- $c = 0 \Rightarrow$ Includes e.g. TsT

PL-type with

$$\tilde{F}^{ab}{}_c = \mathcal{R}^{ae} F_{ec}{}^b - \mathcal{R}^{be} F_{ec}{}^a \ , \quad E_0 = \eta^{-1} - \mathcal{R}$$

Generalised Geometry for PL Geometries I

Curved Generalised Metric encodes physical $E_{ij} = G_{ij} + B_{ij}$ whereas **Flat Generalised Metric** encodes d^2 moduli $E_0 = G_0 + B_0$.

$$\mathcal{H}_{\hat{I}\hat{J}} = \begin{pmatrix} G_{ij} - BG^{-1}B & -BG \\ G^{-1}B & G^{-1} \end{pmatrix} \qquad \mathcal{H}_{AB} = \begin{pmatrix} (G_0)_{ab} - B_0 G_0^{-1} B_0 & -B_0 G_0 \\ G_0^{-1} B_0 & G_0^{-1} \end{pmatrix}$$

Similar $O(d,d)$ invariant pairing

$$\eta_{\hat{I}\hat{J}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \eta_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Generalised Frame Fields give a twisting matrix $\in O(d,d)$

$$\mathcal{H}_{\hat{I}\hat{J}} = \hat{E}_{\hat{I}}{}^{A} \mathcal{H}_{AB} \hat{E}_{\hat{J}}{}^{B}$$

PL T-duality as an $O(d,d)$ operation:

Un-twist $\hat{E}_{\hat{I}}^{A}$ $\oplus$ Invert $\mathcal{H}_{AB}$ $\oplus$ re-twist with $\hat{\hat{E}}_{\hat{I}}^{A}$

Generalised Geometry for PL Geometries 2

Explicit construction of **globally defined** frame fields as $\Gamma(TG + T^\star G)$

$$\hat{E}_A = \begin{cases} E^a = \Pi^{ab} V_b + L^a \\ E_a = V_a \end{cases}$$

Recall Lie derivative of $V = v + \nu$ on $W = w + \mu$:

$$\mathcal{L}_V W = [v, w] + \left(L_u v - i_v d\mu \right)$$

Frame algebra [Hassler]

$$\mathcal{L}_{\hat{E}_A} \hat{E}_B = \mathbb{F}_{AB}{}^C \hat{E}_C$$

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A doubled formalism for PL T-duality

- "Doubled Formalism": group element $\mathbb{g}(\mathbb{X})$ on $\mathbb{D}$ depends on $2d$ coordinates $\mathbb{X}^I$

$$\mathbb{L}(\mathbb{X}) = \mathbb{g}^{-1} d\mathbb{g}$$

- PL Dual Pairs follow from chiral-WZW [Klimcik & Severa; Sfetsos; Hull & Reid-Edwards; Driezen,Sevrin,DT]

$$\boxed{S_{\mathbb{D}} = \int_{\Sigma} -\mathbb{L}_\sigma^A \mathcal{H}_{AB} \mathbb{L}_\sigma^B + \mathbb{L}_\sigma^A \eta_{AB} \mathbb{L}_\tau^B + \int_{\mathcal{M}_3} \mathbb{F}_{AB}{}^D \eta_{DC} \mathbb{L}^A \wedge \mathbb{L}^B \wedge \mathbb{L}^C}$$

- RG β-function of $\mathcal{H}$ [Avramis, Derendinger, Prezas; Sfetsos-Siampos-DT]:

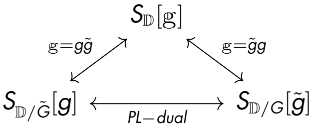
$$\frac{d\mathcal{H}_{AB}}{d\log\mu} = \mathcal{R}_{AB} = \frac{1}{8}(\mathcal{H}_{AC}\mathcal{H}_{BF} - \eta_{AC}\eta_{BF})(\mathcal{H}^{KD}\mathcal{H}^{HE} - \eta^{KD}\eta^{HE})\mathbb{F}_{KH}{}^C \mathbb{F}_{DE}{}^F$$

A doubled formalism for PL T-duality

Integrate out half the degrees of freedom reduces to conventional σ-models

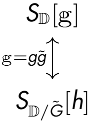
PL or η case

- $\mathfrak{d} = \tilde{\mathfrak{g}} + \mathfrak{g}$ two subalgebras: Drinfeld Double



λ case

- $\mathfrak{d} = \tilde{\mathfrak{g}} + \mathfrak{k}$ one subalgebra: Manin quasi-triple



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A doubled space time

Some questions:

- Can we connect directly the doubled worldsheet to supergravity?
- Can we make simple the underlying structures of these geometries?
- Can we extend PL symmetry to the Ramond-Ramond sector?

We can – utilising ideas of DFT on a group manifold [Blumenhagen,Hassler,Lust] and generalised geometry in SUGRA [Lee,Strickland-Constable,Waldram]

A more formal Courant Algebroid perspective see [Severa,Valach (1810.07763)]

DFT on $\mathbb{D}$

A dynamical space-time theory (DFT Hull, Zwiebach) for generalised metric $\mathcal{H}$ and scalar (density) d on a group manifold $\mathbb{D}$ [Blumenhagen,Hassler,Lust], see also Geissbuhler; Cederwall;

$$S_{\rm NS} = \int d^{2d}\mathbb{X} e^{-2d}\Big(\frac{1}{8}\mathcal{H}^{CD}\nabla_C\mathcal{H}_{AB}\nabla_D\mathcal{H}^{AB} - \frac{1}{2}\mathcal{H}^{AB}\nabla_B\mathcal{H}^{CD}\nabla_D\mathcal{H}_{AC} - 2\nabla_A d\nabla_B\mathcal{H}^{AB} + 4\mathcal{H}^{AB}\nabla_A d\nabla_B d + \frac{1}{6}F_{ACD}F_B{}^{CD}\mathcal{H}^{AB}\Big)$$

Group structure hides in derivatives:

$$\mathbb{L}_I(\mathbb{X}) = \mathrm{g}^{-1}\partial_I\mathrm{g}\ , \quad D_A = \mathbb{L}_A{}^I\partial_I\ , \quad [D_A, D_B] = \mathbb{F}_{AB}{}^C D_C$$

$$\nabla_A V^B = D_A V^B + \frac{1}{3}\mathbb{F}_{AC}{}^B V^C - wF_A V^B\ , \quad \mathbb{F}_A = D_A \log\det\mathbb{L}\ .$$

DFT on $\mathbb{D}$

Symmetry algebra of DFT requires a "section condition" constraint

- Conventional 2d diffeomorphisms

$$L_\xi V^A = \xi^B D_B V^A - w\xi^B F_B V^A + w D_B \xi^B V^A$$

- Generalised diffeomorphisms

$$\mathcal{L}_\xi V^A = \xi^B \nabla_B V^A - V^B \nabla_B \xi^A - \eta^{AB} \eta_{CD} V^C \nabla_B \xi^D + w D_B \xi^B V^A$$

- Section condition

$$\eta^{AB} D_A \bullet D_B \bullet = 0$$

Solve the section condition so fields that depend only on half the coordinates $\Rightarrow$ generalised geometry applied to SUGRA [Waldram et al]

Results

1. **At level of DFT on $\mathbb{D}$**
   - Equation of motion for $\mathcal{H}_{AB}$ in DFT on $\mathbb{D}$ match worldsheet $\beta^{\mathcal{H}}$
   - PL conditions extend to determine dilaton and RR fields

$$\mathbb{F}_{ABC}\Gamma^{ABC}G = 0\ , \quad G = -\mathcal{K}G\ ,$$

   G a MW $Spin(d,d)$ spinor

2. **On section, target space $\mathcal{M} = \mathbb{D}/\tilde{G}$**
   - Recover we conventional DFT for $\widehat{\mathcal{H}}_{\hat{I}\hat{J}}$ with section condition solved
   - non-unimodular case recover the correct modified supergravity e.q.m
   - Explicit examples of η and λ models show that this recovers the solutions to (modified-)sugra inc. fluxes

Conclusions

Conclusions

- Rich interplay between integrable models and generalised notions of duality
- Generalised dualities have concrete holographic application
- Poisson Lie geometries provide an elegant generalised geometry realisation
- A doubled approach, at both the worldsheet and space time, can expose their hidden simplicity

Appendix

The Squashed PCM

- Deform PCM to σ-model on a squashed S^3 [Cherednik '81]:

$$S = \frac{1}{2\pi} \int_{\Sigma} d^2\sigma \, Tr\left(g^{-1}\partial_+ g g^{-1}\partial_- g\right) + C J_+^3 J_-^3$$

$$J_\pm^3 = Tr(g^{-1}\partial_\pm g T^3)$$

- Integrable but $SU(2)_L \times SU(2)_R \to SU(2)_L \times U(1)_R$
- Non-local charges recover semi-classical version of (affine extension of) $\mathcal{U}_q(\mathfrak{sl}_2)$ [Kawaguchi, Matsumoto, Yoshida '11, '12]

$$\{Q_R^+, Q_R^-\}_{P.B.} = \frac{q^{Q_R^3} - q^{-Q_R^3}}{q - q^{-1}} \,, \quad q = \exp\left(\frac{\sqrt{C}}{1+C}\right)$$

Yang-Baxter and η Deformations

Integrable models [Klimcik '02] based on **modified** Yang-Baxter eq

$\mathcal{R}$-matrix: Solution of classical (modified) YB equation:

$$[\mathcal{R}A, \mathcal{R}B] - \mathcal{R}([\mathcal{R}A, B] + [A, \mathcal{R}B]) = -c^2 [A, B] \ , \quad \forall A, B \in \mathfrak{g}$$

An integrable deformed PCM

$$S_\eta = \frac{1}{2\pi t} \int_\Sigma d^2\sigma Tr \left(g^{-1}\partial_+ g, \frac{1}{1 - \eta\mathcal{R}} g^{-1}\partial_- g \right)$$

- Broken G_R recovered in a hidden quantum group symmetry $q = e^{\eta t}$
- $c^2 = -1 \Rightarrow \eta$ Deformations
- $c = 0 \Rightarrow$ Includes e.g. TsT

η Deformations and Supergravity

- Cosets and super-cosets e.g. $AdS_5 \times S^5$ superstring [Delduc, Magro, Vicedo 1309]
- κ-symmetric, solves **modified** SUGRA [Orlando et al 1607, Arutyunov et al. 1511]
- Weyl invariant (solve SUGRA) if unimodular [Borsato and Wulff 1608]

$$\mathcal{R}^B{}_A F^A{}_{BC} = 0$$

- Relation of modified to DFT established [Sakamoto et al; Baguet et al]

Update **Hoare, Seibold** genuine SUGRA solution for η-deformed $AdS_5 \times S^5$ (different choice of $\mathcal{R}$)

η and Poisson-Lie

- Modified conservation law for currents of broken G_R in η-model:

$$d \star J_a = \tilde{F}^{bc}{}_a J_b \wedge J_c$$

- $\tilde{f}^{bc}{}_a$ structure constants for $\mathfrak{g}_{\mathcal{R}}$

$$[A, B]_{\mathcal{R}} = [\mathcal{R}A, B] + [A, \mathcal{R}B]$$

- Mathematically $\mathfrak{g} \oplus \mathfrak{g}_{\mathcal{R}} \simeq \mathfrak{g}^{\mathbb{C}}$ defines a Drinfel'd Double
- $\star\mathcal{J}$ pure gauge in a dual algebra (Field Equations $\Leftrightarrow$ Bianchi identity)
- So although not isometric just the right structure for PL T-duality [Klimcik Severa]

For η deformation $E_0 = \eta^{-1} - \mathcal{R}$.

η, λ and Poisson-Lie

η and λ connected by generalised Poisson Lie T-duality

[Vicedo 1504; Hoare & Tseytlin 1504; Siampos Sfetsos DT 1506; Klimcik 1508]

- PL dualise η model + Analytic continue certain Euler angles and deformation parameters

$$\eta \to i\frac{1-\lambda}{1+\lambda} , \quad t \to \frac{\pi(1+\lambda)}{k(1-\lambda)}$$

- Acting on the parameter q we have

$$q = e^{\eta t} \leftrightarrow q = e^{\frac{i\pi}{k}}$$

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Recap: WZW model

Consider PCM + WZ term :

$$S_{WZW} = \frac{\kappa^2}{4\pi}\int_\Sigma d^2\sigma\, \langle g^{-1}\partial_+ g, g^{-1}\partial_- g\rangle + k\frac{1}{12\pi}\int_{\mathcal{M}_3}\langle g^{-1}dg, [g^{-1}dg, g^{-1}dg]\rangle$$

- Here $\partial\mathcal{M}_3 = \Sigma_2$ and $k \in \mathbb{Z}$ (say) $\Rightarrow$ path integral independent of $\mathcal{M}_3$.
- IR fixed point $\kappa^2 = k \Rightarrow$ we have a CFT
- $G_L \times G_R$ current algebra:

$$\partial_+(g^{-1}\partial_- g) = 0 \quad \partial_-(\partial_+ g g^{-1}) = 0$$

- Gauging 'anomaly' free sub-groups $\Rightarrow$ gauged-WZW $\Rightarrow$ coset CFTs

λ- Lax

Integrable model for all values of λ!

Gauge field e.q.m.:

$$A_+ = \lambda \mathcal{O}_g \partial_+ g g^{-1} \ , \quad A_- = -\lambda \mathcal{O}_{g^{-1}} \partial g^{-1} \partial_+ g \ , \quad \mathcal{O}_g = (1 - \lambda \mathrm{ad}_g)^{-1}$$

Lax ($z \in \mathbb{C}$)

$$\mathcal{L}_\pm(z) = \frac{2}{1+\lambda} \frac{A_\pm}{1 \mp z} \ , \quad d\mathcal{L} + \mathcal{L} \wedge \mathcal{L} = 0$$

$\mathcal{E}$-model for η, λ

η case

$$\mathfrak{d} = \mathfrak{g} + \mathfrak{g}_{\mathcal{R}} = \mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + \mathfrak{a} + \mathfrak{n}$$

$$<< Z_1, Z_2 >>= Im < Z_1, Z_2 >$$

$$\mathcal{E} : Z \to \frac{i}{2}(\eta - \eta^{-1})Z - \frac{i}{2}(\eta + \eta^{-1})Z^{\dagger}$$

$$\mathcal{H} = \begin{pmatrix} \eta\kappa & -\eta\kappa R \\ \eta R\kappa & \kappa\eta^{-1} - \eta R\kappa R \end{pmatrix}$$

λ case

$$\mathfrak{d} = \mathfrak{g} + \mathfrak{g}$$

$$\tilde{\mathfrak{h}} = \mathfrak{g}_{diag}$$

$$<< \{x_1, y_1\}, \{x_2, y_2\} >> \\ =< x_1, x_2 > - < y_1, y_2 >$$

$$\mathcal{H} = \begin{pmatrix} \frac{1-\lambda}{1+\lambda} & 0 \\ 0 & \frac{1+\lambda}{1-\lambda} \end{pmatrix}$$

- Generalised λ models, symmetries, S-matrix and quantisation

Appadu, Hollowood, Price, DT [1706.05322,1802.06016]

Generalised λ & YB-λ Theories

- Sfetsos Procedure can be generalised by replacing PCM:

$$kS_{WZW}[g] + S[\tilde{g}] = \int Tr\left(\tilde{g}^{-1}\partial_+ g \Theta \tilde{g}^{-1}\partial_- g\right)$$

- λ now a matrix Λ:

$$S_\lambda = kS_{WZW} + \frac{k}{2\pi}\int Tr(g^{-1}\partial_+ g \frac{1}{\Lambda^{-1} + Ad_g}\partial_- g g^{-1})$$

$$\Lambda = 1 + k^{-1}\Theta$$

- **Idea**: if Θ defined integrable PCM, Λ can define an integrable theory

Generalised λ & YB-λ Theories for $SU(2)$

λ-XXZ Model

$$\Theta = \text{diag}(\xi^{-1}, \xi^{-1}, \lambda^{-1})$$

Trigonometric Lax

$$\mathcal{L}_\sigma = f_+[z]^a \mathcal{J}_+^a T^a - f_-[z]^a \mathcal{J}_-^a T^a$$

RG invariant

$$\gamma'^2 = \frac{k^2}{4} \frac{(1-\xi^2)(1-\lambda)^2}{\lambda^2 - \xi^2}$$

λ-YB Model

$$\Theta = I + \frac{1}{kt}(1 - \eta \mathcal{R})^{-1}$$

Rational Lax

$$\mathcal{L}_\sigma = (c_+ + d\mathcal{R})\mathcal{J}_+ + (c_- + d\mathcal{R})\mathcal{J}_-$$

RG invariant

$$\Sigma = \frac{2\pi\eta\lambda}{k(1-\lambda)}$$

"Non ultra-local" i.e. central term in current algebra

$$\{\mathcal{J}_\pm^a(x), \mathcal{J}_\pm^b(y)\} = f_{ab}{}^c \mathcal{J}_\pm^c(x)\delta_{xy} \pm \frac{k}{2\pi}\delta^{ab}\delta'_{xy}$$

Classical Symmetries

- Expand monodromy to find symmetries but need to determine expansion points!

$$T(z) = P\exp\left(-\int \mathcal{L}_\sigma(z)\right)$$

- Determine Maillet r/s algebra

$$\{\mathcal{L}_\sigma^{1}, \mathcal{L}_\sigma^{2}\} = [r(z_1, z_2), \mathcal{L}_\sigma^{1} + \mathcal{L}_\sigma^{2}]\delta_{12} + [s(z_1, z_2), \mathcal{L}_\sigma^{1} - \mathcal{L}_\sigma^{2}]\delta_{12} - 2s(z_1, z_2)\delta'_{12}$$

- Locate special points $z_\star$ where $\lim_{\epsilon\to 0} r(z_\star, z_\star + \epsilon) = \mathit{finite}$

Charges and Symmetries

- Special points associated to Quantum Group Symmetries
- e.g. For $\lambda - YB$ model at $c(z_\star) = i\, d(z_\star)$ we find

$$Q^3 \sim \int \mathcal{J}_0^3 \,, \quad Q^\pm \sim \int (\mathcal{J}_0^1 \pm i\mathcal{J}_0^2) \exp\left[-i\Sigma \int_{-\infty}^{\pm x} \mathcal{J}_0^3(\pm y)dy\right]$$

$$q = \exp\left(\frac{2\pi\eta\lambda}{k(1-\lambda)}\right) = e^\Sigma \quad \textbf{Homogenous Gradation}$$

- For $\lambda - XXZ$ model similar with $q = \exp[\pi\sqrt{\gamma'^2}]$ **Principal Gradation**
- QG parameters are RG invariant
- Second quantum group point given by KM currents with

$$q'_{cl} = \exp\left(\frac{i\pi}{k}\right)$$

Exact S-Matrix

Based on symmetries, limits and RG behaviour, we find conjectured form for S-matrices using known blocks

- λ-XXZ Model in UV Safe Domain $\gamma'^2 < 0$ Bernard LeClair

$$\mathcal{S}_{\lambda-XXZ} = S_{SG}(\theta, \gamma') \otimes S^{(k)}_{RSOS}(\theta)$$

- λ-XXZ Model Other Domain (periodic in rapidity)

$$\mathcal{S}_{\lambda-XXZ} = S_{p}(\theta, \Sigma) \otimes S^{(k)}_{RSOS}(\theta)$$

- λ-YB Model (periodic in rapidity, parity broken)

$$\mathcal{S}_{\lambda-XXZ} = S_{h}(\theta, \Sigma) \otimes S^{(k)}_{RSOS}(\theta)$$

'Proving' S-matrix I

- Non-ultra-local *i.e.* δ' makes conventional techniques (QISM) inapplicable
- Alleviation Faddeev-Reshetikhin takes a limit, modifies UV but same IR properties

$$k \to 0 \,, \quad \frac{k}{\xi} \,, \frac{k}{\lambda} \; \textit{fixed}$$

- In this limit the Lax connection becomes ultra-local ($s(z,w) \to 0$) and can be regularised, and quantised, on a lattice
- Obtain a lattice theory, XXZ anisotropic spin chain.

$$H_{\frac{1}{2}} = \sum_{n=1}^{N} \left(\sigma_n^1 \sigma_{n+1}^1 + \sigma_n^2 \sigma_{n+1}^2 + \cos\gamma \sigma_n^3 \sigma_{n+1}^3 \right)$$

- Actually need a spin $S = \frac{k}{2}$ chain and identify

$$\gamma = \frac{\pi}{\gamma'} - k$$

'Proving' S-matrix II

- Ground state using TBA Kirillov-Reshetikhin find Dirac Sea dominated by k-Bethe strings whose density $\rho(z)$ obeys integral equation

$$\rho(z) + \rho_h(z) + \frac{1}{\pi}\int K(z-y)\rho(y)dy = \epsilon(z)$$

- Holes with density ρ_h are excitations above the ground state
- **Amazing fact**, these excitations scatter relativistically with a kernel

$$\tilde{K}(z) = \frac{d}{dz}LogS(z) = \int_0^\infty \cos(z\omega)\left(\coth(k\omega) + \coth(\gamma'\omega)\right)\tanh \pi\omega$$

- This corresponds exactly to the S-matrix of the λ-XXZ Model

Appendix: S-matrix Technology

Rapidity

$$E = m \cosh\theta \ , \quad P = m \sinh\theta$$

Axioms:

1. *Factorization* 2-body factorisation, no particle production
2. *Analyticity*. Only poles along the imaginary axis $0 < Im\theta < \pi$ associated to stable bound states.
3. *Hermitian analyticity*

$$S_{ij}^{kl}(\theta^*)^* = S_{kl}^{ij}(-\theta) \ .$$

4. *Unitarity*

$$\sum_{kl} S_{ij}^{kl}(\theta) S_{mn}^{kl}(\theta)^* = \delta_{im}\delta_{jn} \ , \qquad \theta \in \mathbb{R} \ .$$

5. *Crossing*

$$S_{ij}^{kl}(\theta) = \mathcal{C}_{kk'} S_{k'i}^{lj'}(i\pi - \theta) \mathcal{C}_{j'j}^{-1} = S_{\bar{k}i}^{l\bar{j}}(i\pi - \theta) \ ,$$

where $\mathcal{C}$ is the charge conjugation matrix.

Appendix: Gradation I

$$[H_i, E_j] = a_{ij}E_j, \quad [H_i, F_j] = -a_{ij}F_j \ , \quad [E_i, F_j] = \delta_{ij}H_j$$

Generalised Cartan matrix a_{ij} has off diagonal elements equal -2.
$K = H_0 + H_1$ is central. $K = 0$, i.e. centreless representations $\widehat{\mathfrak{su}(2)}$ becomes the loop algebra. Reps are the tensor of an $\mathfrak{su}(2)$ rep and functions of a variable z. Gradation is the relative action in $\mathfrak{su}(2)$ space and z-space.

homogenous gradation

$$E_1 = T^+, \quad F_1 = T^-, \quad E_0 = z^2 T^-, \quad F_0 = z^{-2} T^+, \quad H_1 = -H_0 = T^3$$

. *principal* gradation

$$E_1 = zT^+, \quad F_1 = z^{-1} T^-, \quad E_0 = zT^-, \quad F_0 = z^{-1} T^+, \quad H_1 = -H_0 = T^3$$

Appendix: Homogenous Gradation

	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\uparrow$	$+2$	$\mathfrak{Q}_2^+$	$\mathfrak{Q}_2^3$	$\mathfrak{Q}_2^-$
$z_* = +i\eta$	$+1$	$\tilde{\mathfrak{Q}}^+$	$\mathfrak{Q}_1^3$	$\mathfrak{Q}_1^-$
	0	$\mathfrak{Q}^+$	$\mathfrak{Q}^3 = -\tilde{\mathfrak{Q}}^3$	$\mathfrak{Q}^-$
$z_* = -i\eta$	-1	$\mathfrak{Q}^+_{-1}$	$\mathfrak{Q}^3_{-1}$	$\tilde{\mathfrak{Q}}^-$
$\downarrow$	-2	$\mathfrak{Q}^+_{-2}$	$\mathfrak{Q}^3_{-2}$	$\mathfrak{Q}^-_{-2}$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure: The charges and their grades for the expansion of the monodromy around the pair of special points $z = \pm i\eta$. The blue/red and positive/negative graded charges are associated to $\pm i\eta$, respectively. The red and blue charges generate the affine quantum group in homogenous gradation and all the other charges are obtained by repeated Poisson brackets of these charges.

Appendix: Principal Gradation

$\widehat{\mathfrak{su}(2)}_p$.

	$\vdots$	$\vdots$		$\vdots$
$\uparrow$	$+2$		$\mathfrak{Q}^3_2$	
$z_* = \infty$	$+1$	$\mathfrak{Q}^+$		$\tilde{\mathfrak{Q}}^-$
	0		$\mathfrak{Q}^3 = -\tilde{\mathfrak{Q}}^3$	
$z_* = -\infty$	-1	$\tilde{\mathfrak{Q}}^+$		$\mathfrak{Q}^-$
$\downarrow$	-2		$\mathfrak{Q}^3_{-2}$	
	$\vdots$	$\vdots$		$\vdots$

Figure: The charges and their grades for the expansion of the monodromy around the pair of special points $z = \pm\infty$ (or $0, \infty$ with a multiplicative spectral parameter). The blue/red and positive/negative graded charges are associated to $\pm\infty$, respectively. The red and blue charges generate the affine quantum group in principal gradation and all the other charges are obtained by repeated Poisson brackets of these charges.

RG in YB-λ model

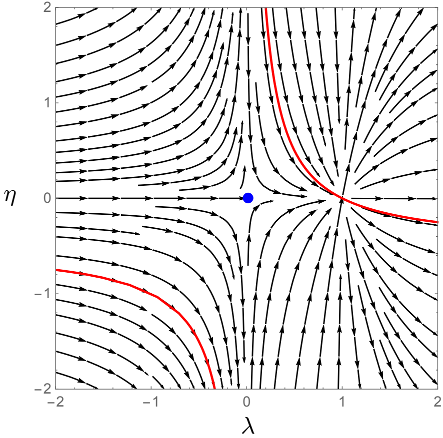

Figure: The RG flow of the YB lambda model (flows towards the IR). The WZW fixed point is the blue dot in the middle. The red curved is an example of a cyclic trajectory which has a jump from $\eta = +\infty$ to $-\infty$ at $\lambda = 0$ and a jump from $\lambda = -\infty$ to $\lambda = +\infty$.

RG in η-λ model

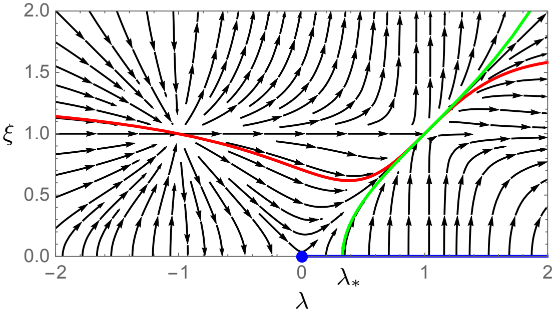

Figure: The RG flow (to the IR) of the XXZ lambda model. The WZW fixed point is identified by the blue blob. The blue line is a line of UV fixed points. The green curve is a UV safe trajectory that has $\gamma' \in \mathbb{R}$. The red curve is a cyclic RG trajectory with $\gamma' = i\sigma$, $\sigma \in \mathbb{R}$. The trajectory has a jump in the coupling λ from $-\infty$ to ∞, but is continuous in $1/\lambda$.

Contents

- Quantum aspects and resurgence of the η model

`Demulder, Dorigoni, DT [1604.07851]`

D-branes in the λ-model

- DBI action

$$S_{DBI} = \int e^{-\Phi}\sqrt{\widehat{G}+\mathcal{F}}$$

- λ enters spectrum of D-branes. E.g. $SU(2)$, δ a scalar fluctuation and g a gauge flucuation

$$\frac{d^2}{dt^2}\begin{pmatrix}\delta\\ g\end{pmatrix} = -\frac{1}{k\alpha'}\frac{1+\lambda^2}{1-\lambda^2}\begin{pmatrix} 2+\frac{(1+\lambda)^2}{1+\lambda^2}\Box & 2 \\ 2\,\Box & \frac{(1+\lambda)^2}{1+\lambda^2}\Box \end{pmatrix}\begin{pmatrix}\delta\\ g\end{pmatrix},$$

- Note δ not a moduli, D-branes are stabilised
- Flux quantisation $\Rightarrow$ D-branes stabilised to conjugacy classes of integrable highest weights Bachas, Petropolous; Stanciu Figueroa-O'Farrill
- e.g. $SU(2)_k$: 2 D0's and $k-1$ D2's wrapping S^2 whose size is a function of λ

λ Commentary

- λ deformations solve SUGRA with appropriate RR fields [Sfetsos DT, Borsato Wulff]
- Quantum group symmetry expected with $q = e^{\frac{i\pi}{k}}$ [Hollowood et al]
- Can be quantised on a light cone lattice as spin-k Heisenberg XXX spin-chain [Hollowood,Price,Appadu (+DT)]
- Also applied to cosets [Sfetsos], supercosets [Hollowood et al]
- One-loop marginal deformation in case of $PSU(2,2|4)$! [Appadu, Hollowood]