Anomalous dimensions from geometry and the effective action

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Based mainly on:

- work to appear: with G. Georgiou, P. Panopoulos & E. Sagkrioti, 1905.xxxxx [hepth].

General motivation

- The quantum behavior of interacting field theories is encoded on how the couplings change with energy and on how the operators acquire anomalous dimensions.
- Studied in the framework of RG (since [Wilson 71]). Results into 1st order non-linear differential systems, i.e. β-function eqs.
- Traditionally computed perturbatively. For example

$$\text{QCD}: \qquad \mu \frac{dg}{d\mu} = -\frac{7g^3}{16\pi^2} + \cdots$$

A rare occasion and very important to be able to compute them exactly.

Knowing the β-functions and anomalous dimensions exactly:

- We may discover new fixed point theories towards the IR.
- We may investigate the fate of degrees of freedom in the flow from the UV to the IR, i.e. Zamolodchikov's C-theorem.
- We will learn more on the structure of QFTs.

I will concentrate on 2-dim Conformal Field Theories as a basis:

- Systematic construction of new (integrable) deformations of (interacting) CFT's having explicit Lagrangian descriptions.
- Smooth RG flows (UV to IR) between CFTs.
- β-function and anomalous dimensions of general operators, essentially without loop computations.

Not to cover here at all, but serving as extra motivation:

- Embedding to type-II supergravity: [KS-Thompson 14, Demulder-KS-Thompson15,Borsato-Tseytlin-Wulff16,Chervonyi-Lunin16].
- Conceivable usage in an AdS/CFT context.

Outline

- The theories of interest
- Effective actions: Self-interacting theories.
- Exact β-functions and anomalous dimensions.

  Perturbative info + symmetry + analyticity $\Longrightarrow$ exact info

- A more powerful and easier to apply method.

  Effective action + geometry of couplings $\Longrightarrow$ exact info

- The anomalous dims of $d^{(m)}_{a_1 \dots a_m} J_+^{a_1} \dots J_+^{a_m}$ and $d_{abc} J_+^a J_+^b J_-^c$.
- Mutually interacting theories.
- Concluding remarks.

The theories of interest

Seeds in old works: The non-Abelian Thirring model

(1+1)-dim fermions ψ^a in the $\square$ of $SU(N)$ [Dashen-Frishman 73 & 75].

- A non-Abelian generalization of the Thirring model [Thirring 58]

$$\mathcal{L} = \overbrace{\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi}^{\text{free action}} - \frac{g_v}{2} \overbrace{J^a_{\mu}J^{a\mu}}^{\text{interaction}}$$

Fermion bilinear: $J^a_{\mu} = \bar{\psi}t^a\gamma_{\mu}\psi$, $a = 1, 2, \ldots, N^2 - 1$.

- Scale invariant for $g_v = 0$ and $g_v = \frac{4\pi}{N+1}$.
  The scaling dimension Δ of the fermions is

$$g_v = 0 : \qquad \Delta = \frac{1}{2}$$

$$g_v = \frac{4\pi}{N+1} : \qquad \Delta = \frac{1}{2} + \frac{N-1}{N}$$

- A non-trivial conformal point. Fermions get new dimension.
  Admits a large N limit, by keeping $gN = \text{fixed}$.

Bosonic version leads to generalization

Generalization via the steps:

- Replace the free fermion theory by a 2-dim CFT with action S_k having a current algebra for a group G (replaces $SU(N)$)
- Keep the interaction bilinear in currents the same

$$S_{k,\lambda} = S_k + \frac{k}{\pi}\int d^2\sigma\, \lambda_{ab} J_+^a J_-^b \ , \quad a,b = 1,2,\dots,\dim G \ .$$

- Near the CFT point

$$\partial_{\mp} J_{\pm}^a = \mathcal{O}(\lambda) \ .$$

- For $\lambda_{ab} = 0$ the $J_\pm$'s obey the OPE's

$$J^a(z)J^b(w) = i f_{abc}\frac{J^c}{z-w} + k\frac{\delta^{ab}}{(z-w)^2} \quad \text{(Kac–Moody algebra)} \ .$$

Infinite dimensional extension of the Lie-algebra.

Effective actions: Self-interacting theories [KS 14]

For a group G and group elements $g, \tilde{g} \in G$

$$S(g, \tilde{g}) = S_k(g) + S_{\mathrm{PCM},\kappa^2}(\tilde{g}) \ .$$

- The WZW action $S_k(g)$ is a CFT. It has a $G_{\mathrm{L,cur}} \times G_{\mathrm{R,cur}}$ current symmetry at level $k \in \mathbb{Z}^+$ generated by

$$J_+^a = -i\mathrm{Tr}(t^a \partial_+ g g^{-1}) \ , \quad J_-^a = -i\mathrm{Tr}(t^a g^{-1} \partial_- g) \ .$$

- The Principal Chiral model action (PCM) is

$$S_{\mathrm{PCM},\kappa^2}(\tilde{g}) = -\frac{\kappa^2}{\pi} \int \mathrm{Tr}(\tilde{g}^{-1} \partial_+ \tilde{g} \tilde{g}^{-1} \partial_- \tilde{g}) \ .$$

It is integrable with global $G_L \times G_R$ symmetry.

Introduction of interactions via gauge fields

Gauge the global symmetry group acting as

$$g \to \Lambda^{-1} g \Lambda \ , \qquad \tilde{g} \to \Lambda^{-1} \tilde{g} \ , \qquad \Lambda \in G$$

and we consider the action

$$S_{k,\kappa^2}(g,\tilde{g},A_\pm) = S_k(g,A_\pm) + S_{\rm PCM}(\tilde{g},A_\pm) \ .$$

- The gauged WZW action is

$$S_k(g,A_\pm) = S_k(g) + \frac{k}{\pi}\int \mathrm{Tr}\left(A_-\partial_+ g g^{-1} - A_+ g^{-1}\partial_- g + A_- g A_+ g^{-1} - A_- A_+\right) \ .$$

- The gauged PCM action is

$$S_{\rm PCM}(\tilde{g},A_\pm) = -\frac{\kappa^2}{\pi}\int \mathrm{Tr}(\tilde{g}^{-1}\widetilde{D}_+\tilde{g}\tilde{g}^{-1}\widetilde{D}_-\tilde{g}) \ ,$$

with the covariant derivatives being $D_\pm \tilde{g} = \partial_\pm \tilde{g} - A_\pm \tilde{g}$.

After gauge fixing $\tilde{g} = \mathbb{1}$ and integrating out the gauge fields

$$\boxed{S_{k,\lambda}(g) = S_k(g) + \frac{k}{\pi}\int J_+^a(\lambda^{-1}\mathbb{I} - D^T)^{-1}_{ab} J_-^b} ,$$

the λ-deformed σ-model action, where

$$\lambda = \frac{k}{k+\kappa^2} , \qquad 0 < \lambda < 1 ,$$

is the deformation parameter and

$$J_+^a = -i\mathrm{Tr}(t^a \partial_+ g g^{-1}) , \quad J_-^a = -i\mathrm{Tr}(t^a g^{-1}\partial_- g) ,$$
$$D_{ab} = \mathrm{Tr}(t_a g t_b g^{-1}) .$$

- Since D is orthogonal this action is non-singular.
- Singularities appear only when $\lambda \to \pm 1$. Letting also $k \to \infty$

$$(\lambda+1)k = \mathrm{finite} , \qquad (\lambda+1)^3 k = \mathrm{finite} ,$$

Non-Abelian & Pseudo-dual lims [KS 13, Georgiou-KS-Siampos 16].

Some properties

- Generalization to $\lambda\delta_{ab} \to \lambda_{ab}$ straightforward.
- For $\lambda \ll 1$

$$S_{k,\lambda}(g) = S_k(g) + \frac{k}{\pi}\lambda \int d^2\sigma\ J_+^a J_-^a + \dots\ .$$

- The full action $S_{k,\lambda}(g)$ has a duality-type symmetry [Itsios-KS-Siampos 14]

$$\boxed{k \to -k\ , \qquad \lambda \to \lambda^{-1}}\ , \qquad g \to g^{-1}\ .$$

It should be reflected as a symmetry in physical quantities.

- Integrable for special forms of λ_{ab}, eg. [KS 13, Hollowood-Miramontes-Schmidtt 14]

$$\lambda_{ab} = \lambda\delta_{ab}\ .$$

Not integrable at any finite order in λ.
For more integrable cases [KS-Siampos-Thompson 15].

Exact β-function and anomalous dims [Georgiou-Siampos-KS 15 & 16]

CFT and symmetry approach

- We want to compute the 2-point functions

$$\langle J^a(x_1)J^b(x_2)\rangle_\lambda = \langle J^a(x_1)J^b(x_2)e^{-\frac{\lambda}{\pi}\int J^a\bar{J}^a}\rangle \ ,$$
$$\langle J^a(x_1)\bar{J}^b(x_2)\rangle_\lambda = \langle J^a(x_1)\bar{J}^b(x_2)e^{-\frac{\lambda}{\pi}\int J^a\bar{J}^a}\rangle \ ,$$

perturbatively in λ by expanding the exponential.

- The basic correlators are

$$\langle J^a(x_1)J^b(x_2)\rangle = \frac{\delta_{ab}}{x_{12}^2}\ , \quad \langle J^a(x_1)J^b(x_2)J^c(x_3)\rangle = \frac{1}{\sqrt{k}}\frac{f_{abc}}{x_{12}x_{13}x_{23}}$$

and similarly for the $\bar{J}^a$'s. Mixed $J\bar{J}$ correlators vanish.

- For higher correlators use Ward dentities

The perturbative β-function and anomalous dimensions

With effort using "conventional" perturbative techniques:

- The β-function is

$$\beta = \frac{1}{2}\mu\frac{d\lambda}{d\mu} = -\frac{c_G}{2k}\left(\lambda^2 - 2\lambda^3\right) + \mathcal{O}(\lambda^4) \ ,$$

  where c_G is the quadratic Casimir in the adjoint.

- The anomalous dimension of the currents is

$$\gamma^{(J)} = \frac{c_G}{k}\left(\lambda^2 - 2\lambda^3\right) + \mathcal{O}(\lambda^4) \ .$$

Task: Extend these exactly in λ and leading in $1/k$.

Analyticity: λ-dependence of physical quantities

- Recall, that the action is singular at $\lambda = \pm 1$.
- The β-function & anomalous dims may have poles at $\lambda = \pm 1$.
- The β-function & anomalous dims should be invariant under

$$k \to -k \ , \qquad \lambda \to \frac{1}{\lambda} \ , \qquad (\text{for } k \gg 1) \ .$$

- Perturbative information to $\mathcal{O}(\lambda^2)$ and the above symmetry are enough to determine the β-function and the anomalous dimensions exactly in λ and to leading order in k.

The exact β-function and anomalous dimensions

The exact β-function and anomalous dimensions are of the form

$$\beta_\lambda = -\frac{c_G}{2k}\frac{f(\lambda)}{(1+\lambda)^2}\ , \qquad \gamma^{(J)} = \frac{c_G}{k}\frac{g(\lambda)}{(1-\lambda)(1+\lambda)^3}\ ,$$

where $f(\lambda)$ and $g(\lambda)$ are analytic in λ.

- They have a well defined non-Abelian and pseudodual limits as $\lambda \to \pm 1$ and $k \to \infty$, in a correlated way.
- Due to the symmetry $(k, \lambda) \mapsto (-k, \lambda^{-1})$ we have that

$$\lambda^4 f(1/\lambda) = f(\lambda)\ , \qquad \lambda^4 g(1/\lambda) = g(\lambda)\ .$$

- $f(\lambda)$ and $g(\lambda)$ are polynomials of degree four. It is fixed by the above symmetry and the two-loop perturbative result.

- The final result for the beta-function is

$$\beta^{\lambda} = -\frac{c_G}{2k}\frac{\lambda^2}{(1+\lambda)^2} \leqslant 0 \ .$$

- For the anomalous dimension

$$\gamma^{(J)} = \frac{c_G}{k}\frac{\lambda^2}{(1-\lambda)(1+\lambda)^3} \geqslant 0 \ .$$

- Agreement with perturbation theory to $\mathcal{O}(\lambda^3)$ and $\mathcal{O}(\lambda^4)$.

- Similarly for current 3-point functions and correlators of primary fields [Georgiou-KS-Siampos 16].

- Extended for the β-function for general λ_{ab} [Itsions-KS-Siampos 14, KS-Siampos 14, KS-Siampos-Sagkrioti 18] using the gravitational approach of [Ecker-Honerkamp 71, Friedan 80]

$$\frac{1}{2}\mu\frac{dG_{\mu\nu}}{d\mu} = R_{\mu\nu} + \cdots \ .$$

Questions and limitations

- What about the anomalous dimensions of general composite operators of the type

$$\boxed{\mathcal{O}^{(m,n)} = S_{a_1\ldots a_m;b_1\ldots b_n} J_+^{a_1}\ldots J_+^{a_m} J_-^{b_1}\ldots J_-^{b_n}}\ ,$$

  where the S-tensor belongs to some irrep of G.

- How are the operators dressed/modified in the process of the deformation?

- Clearly the previous approach is limited.

- How can we use the effective action to perform such computations?

Anomalous dims from the effective action

[Georgiou-Panopoulos-Sagkrioti-KS, to appear]

Recall the action we started with

$$S_{k,\kappa^2}(g,\tilde{g},A_\pm) = S_k(g,A_\pm) + S_{\rm PCM}(\tilde{g},A_\pm)\ ,$$

to which we add the term

$$\frac{ks}{\pi}\int S_{a_1\dots a_m;b_1\dots b_n}(\tilde{g}^{-1}D_+\tilde{g})^{a_1}\dots(\tilde{g}^{-1}D_+\tilde{g})^{a_m}(\tilde{g}^{-1}D_-\tilde{g})^{b_1}\dots(\tilde{g}^{-1}D_-\tilde{g})^{b_n}\ ,$$

where s is a coupling.

- This action is still gauge invariant, so we gauge fix as $\tilde{g} = \mathbb{1}$.
- At the end set $s = 0$; the λ-deformed action is recovered.
- The anomalous dimension of $\mathcal{O}^{(m,n)}$ will be extracted.
- The eqs. of motion are non-linear. Need the $\mathcal{O}(s)$-terms.

The dressed operators

Integrating out the gauge fields we obtain the action

$$S = S_{k,\lambda}(g) - \frac{ks}{\pi} \int d^2\sigma \; \mathcal{O}_\lambda^{(m,n)} + \mathcal{O}(s^2) \ ,$$

where

$$\boxed{\mathcal{O}_\lambda^{(m,n)} = S_{a_1 \dots a_m; b_1 \dots b_n} A_+^{a_1} \dots A_+^{a_m} A_-^{b_1} \dots A_-^{b_n} \ .}$$

with the classical values for the gauged fields

$$A_+ = i\left(\lambda^{-1}\mathbb{1} - D\right)^{-1} J_+ \ , \quad A_- = -i\left(\lambda^{-1}\mathbb{1} - D^T\right)^{-1} J_- \ .$$

- Obviously, as $\lambda \to 0$ we have that $A_\pm \sim J_\pm$. Therefore

$$\mathcal{O}_\lambda^{(m,n)} \sim S_{a_1 \dots a_m; b_1 \dots b_n} J_+^{a_1} \dots J_+^{a_m} J_+^{b_1} \dots J_-^{b_n} \ , \quad \text{as } \lambda \to 0 \ .$$

- In the λ-deformed theory the operators are modified .

The RG flow eqs: Essentials of the method

The eqs. of motion can be cast solely in terms of the gauge fields

$$\partial_+ A_- - \lambda \partial_- A_+ = [A_+, A_-] - m\, s\lambda\, D_+ \mathcal{O}_\lambda^{(m',n)} \ ,$$
$$\partial_- A_+ - \lambda \partial_+ A_- = -[A_+, A_-] - n\, s\lambda\, D_- \mathcal{O}_\lambda^{(m,n')} \ .$$

The prime implies a free index and $D_\pm$ are covariant derivatives.

Background field method (in this context by [Appadu-Hollowood 15])

- Choose a particular group element and construct a classical solution $A_\pm^{(0)}$ exact in λ and leading in s.
  Consider fluctuations around this $\delta A_\pm$.

- Cast them in the form

$$\hat{D} \begin{pmatrix} \delta A_- \\ \delta A_+ \end{pmatrix} = 0 \ , \quad \hat{D} \text{ is 1st order} \ .$$

To compute the contribution to the classical action

- Pass to the Euclidean regime and to momentum space.
- Split $\hat{D} = \hat{C} + \hat{F}$, where in $\hat{C}$ all momentum dependence.
- Integrating out the fluctuations, gives the effective Lagrangian

$$-\mathcal{L}_{\text{eff}} = \mathcal{L}^{(0)} + \int^{\mu} \frac{d^2 p}{(2\pi)^2} \ln(\det \hat{D})^{-1/2} \ .$$

- Logarithmically divergent with respect to the UV mass scale μ. We perform the large momentum expansion and keep terms proportional to $1/|p|^2$.
- Since $\hat{C}$ grows with $|p|$ we use the fact that

$$\ln \det \hat{D} = \ln \det \hat{C} + \mathrm{Tr}(\hat{C}^{-1}\hat{F}) - \frac{1}{2} \underbrace{\mathrm{Tr}(\hat{C}^{-1}\hat{F})^2}_{\text{relevant}-\text{term}} + \cdots \ .$$

The β-functions are obtained by demanding that

$$\frac{1}{2}\mu \frac{d\mathcal{L}_{\text{eff}}}{d\mu} = 0 \ .$$

- That gives

$$\beta^{\lambda} = -\frac{c_G}{2k}\frac{\lambda^2}{(1+\lambda)^2} \ .$$

- In general there is operators mixing under the RG flow.
- We will restrict to classes of operators where mixing doesn't occur. This will provide β^s.

The geometry of coupling space

For small values of λ and s

$$S = S_k(g) + \frac{k}{\pi}\int d^2\sigma\left(\lambda J_+^a J_-^a + \tilde{\lambda}\mathcal{O}^{(m,n)}\right) + \cdots \ ,$$

where $\tilde{\lambda} \sim s\lambda^{m+n}$.

- Consider a general perturbation with operators

$$\lambda^i \mathcal{O}_i \ , \quad \text{beta functions } \beta^i \ .$$

- There is metric in the space of couplings $G_{ij}^{(0)}$ defined via

$$G_{ij}^{(0)}(\lambda) = x^4 \langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle \ .$$

- Assuming renormalizability and using the Callan-Symanzik eq. the anomalous dimension matrix is [Kutasov 89]

$$\gamma_i{}^j = \partial_i \beta^j + G^{(0)jm}\left(G_{in}^{(0)}\partial_m \beta^n + \beta^n \partial_n G_{im}^{(0)}\right) \ .$$

Consider the two couplings: $\mathcal{O}_\lambda = J_+^a J_-^a$ and $\mathcal{O}_{\tilde{\lambda}} = \mathcal{O}^{(m,n)}$.

- $\mathcal{O}_{\tilde{\lambda}} = \mathcal{O}^{(m,n)}$ breaks Lorentz and/or scale invariance.
- Operator mixing and an anomalous dimension matrix $\gamma_i{}^j$.
- When $\tilde{\lambda} \to 0$, no mixing and only ${\gamma_\lambda}^\lambda, {\gamma_{\tilde{\lambda}}}^{\tilde{\lambda}} \neq 0$.
- Near $\tilde{\lambda} = 0$ the metric becomes diagonal

$$G_{ij}^{(0)}(\lambda, \tilde{\lambda}) = \delta_{ij} g_{ii}^{(0)}(\lambda) + \mathcal{O}(\tilde{\lambda}) \ .$$

This is consequence of the decoupling at $\tilde{\lambda} = 0$.

In the limit $\tilde{\lambda} \to 0$, the anomalous dimensions are:

$$\boxed{\gamma_{\mathcal{O}_\lambda} = {\gamma_\lambda}^\lambda = 2\partial_\lambda \beta^\lambda + \beta^\lambda \partial_\lambda \ln g^{(0)}_{\lambda\lambda}}$$

and that

$$\boxed{\gamma_{\mathcal{O}_{\tilde{\lambda}}} = {\gamma_{\tilde{\lambda}}}^{\tilde{\lambda}} = 2\partial_{\tilde{\lambda}} \beta^{\tilde{\lambda}} + \beta^\lambda \partial_\lambda \ln g^{(0)}_{\tilde{\lambda}\tilde{\lambda}}} \ .$$

- Hence, when $\tilde{\lambda} = 0$, we have the λ-deformed theory and as a bonus the anomalous dimension of $\mathcal{O}_{\tilde{\lambda}}$.
- In our case the metric has

$$g^{(0)}_{\lambda\lambda} = \frac{\dim G}{(1-\lambda^2)^2} \ , \qquad g^{(0)}_{\tilde{\lambda}\tilde{\lambda}} \sim \frac{1}{(1-\lambda^2)^{m+n}} \ .$$

  Computable with free field contractions (strict $k \to \infty$ limit).
- We first extract γ for $\mathcal{O}_\lambda = J^a_+ J^a_-$ [Georgiou, KS, Siampos 15]

$$\gamma_{J_+J_-} = -\frac{2c_G}{k} \frac{\lambda(1-\lambda(1-\lambda))}{(1-\lambda)(1+\lambda)^3} = -2\frac{c_G}{k}\lambda + \dots \ .$$

Example I: Arbitrary number of same chirality operators

We are interested in the anomalous dimension of

$$\boxed{\mathcal{O}^{(m,0)} = d^{(m)}_{a_1\dots a_m} J_+^{a_1} \dots J_+^{a_m}} \ ,$$

where $d^{(m)}_{a_1\dots a_m}$ is the completely symmetric rank-m tensor of $SU(N)$.

- $d^{(m)}_{a_1 a_2 \dots a_m} t^{a_1} t^{a_2} \dots t^{a_m}$ is a Casimir operator. We derive

  $$d^{(m)}_{ab(a_1\dots a_{m-2}} f_{c)bd} f_{dae} = -\frac{c_G}{m-1} d^{(m)}_{cea_1\dots a_{m-2}} \ , \quad m = 2, 3, \dots$$

  and we may show there is no operator mixing under RG flow.
- Apply the formalism and define $\tilde{\lambda} = s\lambda^m$ (effective coupling)

$$-\mathcal{L}_{\text{eff}} = -\frac{k}{2\pi}\left(\frac{1+\lambda}{1-\lambda} J_+ J_- + \frac{2\tilde{\lambda}}{(1-\lambda)^m}\mathcal{O}^{(m,0)}\right)\Big)$$

$$-\frac{c_G}{2\pi}\ln\mu^2 \frac{\lambda^2}{(1-\lambda^2)^2}\left(J_+ J_- + \frac{m\tilde{\lambda}\ \mathcal{O}^{(m,0)}}{(1-\lambda)^{m-1}(1+\lambda)}\right) + \mathcal{O}(\tilde{\lambda}^2) \ .$$

- Then, demanding $\partial_{\ln \mu^2} \mathcal{L}_{\rm eff} = 0$ we obtain

$$\beta^{\tilde{\lambda}} = \frac{c_G}{2k} \frac{m\tilde{\lambda}\lambda^3}{(1-\lambda)(1+\lambda)^3} + \mathcal{O}(\tilde{\lambda}^2) \ .$$

- Using that we obtain

$$\boxed{\gamma_{\mathcal{O}_\lambda^{(m,0)}} = 0 \ , \quad m = 2, 3, \dots} \ .$$

  This is quite general and robust.

- For $m = 1$: The anomalous dimension of a single current

$$\gamma_{\mathcal{O}_\lambda^{(1,0)}} = \gamma_{J_+} = \frac{c_G}{k} \frac{\lambda^2}{(1-\lambda)(1+\lambda)^3} \ ,$$

  as before.

Conserved currents

- Due to the classical equations of motion

$$\partial_\mp A_\pm = \mp \frac{1}{1+\lambda}[A_+, A_-] \ ,$$

as well as of $f_{ab(a_1} d^{(m)}_{a_2 a_3 \dots a_m)b} = 0$, chiral conservation laws

$$\boxed{\partial_- \mathcal{O}_\lambda^{(m,0)} = 0 \ , \quad m = 2, 3, \dots} \ .$$

The first one $\mathcal{O}_\lambda^{(2,0)} \sim T_{++}$.

- Since $\gamma_{\mathcal{O}^{(m,0)}} = 0$ this law holds up to $\mathcal{O}(1/k^2)$ thanks to

$$\langle \mathcal{O}^{(m,0)}(x_1) \mathcal{O}^{(n,0)}(x_2) \rangle \sim \frac{\delta_{mn}}{x_{12}^{2m}} + \mathcal{O}(1/k^2) \ ,$$

exactly in λ and to no operator mixing.

Example II: Mixed chirality operators

We are interested in the anomalous dimension of

$$\boxed{\mathcal{O}^{(2,1)} = d_{abc} J_+^a J_+^b J_-^c} \ ,$$

where d_{abc} is the symmetric tensor of $SU(N)$ of rank 3.

Following a similar procedure:

- We find that

$$\gamma_{\mathcal{O}_\lambda^{(2,1)}} = -\frac{2c_G}{k} \frac{\lambda(1-\lambda(1-\lambda))}{(1-\lambda)(1+\lambda)^3} \ .$$

- This is the same as that for

$$\gamma_{\mathcal{O}_\lambda^{(2,1)}} = \gamma_{J_+J_-} = \gamma_{\mathcal{O}_\lambda^{(1,1)}} \ ,$$

quite unexpected.

Perturbative checks

Important to check the above exact against perturbative results.

This is a laborious task ...

- We have computed

$$\gamma_{\mathcal{O}^{(m,0)}} = \mathcal{O}(\lambda^3) \ , \quad m = 3, 4, \ldots \ ,$$
$$\gamma_{\mathcal{O}^{(2,1)}} = -\frac{2c_G}{k}\lambda(1-3\lambda) + \mathcal{O}(\lambda^3) \ .$$

- In particular, for $\mathcal{O}^{(2,0)}$

$$\gamma_{\mathcal{O}^{(2,0)}} = \mathcal{O}(\lambda^4) \ .$$

In full agreement with the exact results.

The above result is robust and has a simple explanation.

- The anomalous dimension of any operator has the form

$$\gamma_{\mathcal{O}} = \frac{c_G}{k} \frac{\lambda^n f(\lambda)}{(1-\lambda)(1+\lambda)^3} , \qquad n = 0, 1, \dots .$$

  n is determined by the leading order perturbative result.

- The overall function $f(\lambda)$ is analytic in λ.
- Invariance under the symmetry $k \to -k$ and $\lambda \to \lambda^{-1}$ implies

$$\lambda^{2(2-n)} f(1/\lambda) = f(\lambda) .$$

- For $n = 0, 1, 2$, $f(\lambda)$ is a 4th, 2nd and 0th order polynomial.
- For $n \geqslant 3$ not possible, unless $f(\lambda) = 0$ leading to $\gamma_{\mathcal{O}} = 0$.
- Hence, vanishing at $\mathcal{O}(\lambda^2)$ implies $\gamma_{\mathcal{O}} = 0$ altogether.

Large N-limit

Consider groups admitting a large rank limit, i.e. $SU(N)$.
In that case $c_G \gg 1$.

- To make sense of the β-function, let $\lambda \to 0$ and keep $\zeta = c_G \lambda$ finite.
- Then
$$\frac{1}{2}\mu \frac{d\zeta}{d\mu} = -\frac{\zeta^2}{2k} \; .$$
- The anomalous dimensions of $\mathcal{O}(\lambda^2)$ vanish in that limit.
Those behaving as $\mathcal{O}(\lambda)$ remain finite.

Unequal level mutually interacting models [Georgiou-KS 17]

In the current bilinear the currents have different levels

$$S_{k,\lambda}(g_1, g_2) = S_{k_1}(g_1) + S_{k_2}(g_2) + \frac{\sqrt{k_1 k_2}}{\pi}\lambda \int d^2\sigma \; J^a_{1+} J^a_{2-} \ .$$

- That changes the β-function to

$$\beta^\lambda = -\frac{c_G}{2\sqrt{k_1 k_2}} \frac{\lambda^2(\lambda - \lambda_0)(\lambda - \lambda_0^{-1})}{(1-\lambda^2)^2} \ ,$$

- A new fixed point in the IR at $\lambda = \lambda_0 = \sqrt{\frac{k_1}{k_2}} < 1$. Then

$$S_{k,\lambda_0}(g_1, g_2) = S_{k_1}(g_2 g_1) + S_{k_2 - k_1}(g_2) \ .$$

The theory in the IR is $G_{k_1} \times G_{k_2 - k_1}$.

- The *C*-function has been computed exactly and obeys Zamolodchikov's criteria:
  - a) monotonically decreasing towards the IR
  - b) equals the central charges at the fixed points.

[Georgiou-Panopoulos-Sagrioti-KS 18] (1st example in literature)

- For composite operators in the case of unequals levels

$$\gamma_{\mathcal{O}_\lambda^{(m,0)}} = 0 \, ,$$

$$\gamma_{\mathcal{O}_\lambda^{(2,1)}} = \gamma_{J_{1+}J_{2-}} = \gamma_{\mathcal{O}_\lambda^{(1,1)}} \, ,$$

similarly to before.

Concluding remarks

Models of interacting current algebra theories.

- Focused on self- $J_+ J_-$ and mutually-interactions $J_{1+} J_{2-}$.
- There is generalization with many theories self- and mutually interacting, i.e. $J_{i+} J_{j-}$ [Georgiou-KS, 18]. New IR fixed points.
- IR theories correspond typically to different CFTs on the left and right sectors [Georgiou-KS-Pappas, to appear].
- Extension of the construction to exact coset CFTs [KS 13, Hollowood-Miramontes-Schmidtt 14,KS-Siampos 17]

Computed exactly the beta-function and anomalous dimensions.

- Based on leading order perturbative results and symmetries.
- Based on the effective action and geometry in coupling space.

Many Future directions...., but the most relevant for this talk are:

- Explain the high degeneracy of the anomalous dim spectrum.
- Adopt? extend ? this work for η-deformations [Klimcik 02 & 08], [Delduc-Magro-Vicedo 13] and [Delduc-Lacroix-Magro-Vicedo 18]. A relation between λ- and η-defs via Poisson Lie-T-duality.