CONSISTENT TRUNCATIONS WITH DIFFERENT AMOUNT OF SUPERSYMMETRY

[D. CASSANI, G. JOSSE, M.P, C., D. WALDRAM, arXiv:1906...]

INTRODUCTION

- A major question in string theory/sugra is how to derive lower dimensional effective actions
  - effective low-dimensional theories in string compactifications
  - gravity solutions in the gauge/gravity duality

- Look for 10D solutions of the form

$$M_{10} = X_{10-d} \times M_d$$

maximally symmetric $\hookleftarrow$ $\hookrightarrow$ compact

  - expand the 10d field in harmonics on M_d
  - obtain a $10-d$ action with infinite tower of KK modes
  - truncate the theory to a finite set of fields in a consistent way
    - no dependence on the internal manifold in the eom and susy variations
    - recover a fully 10 -d interacting theory
    - all 10 - d solutions lift to solutions of the higher dimensional ones

- Consistent truncations are rare and non-trivial : often the truncation ansatz is ensured by geometric properties of the internal manifold
  - cohomologies on Calabi-Yau manifolds $\rightarrow$ massless modes
  - symmetries of group manifolds, cosets and G-structure manifolds $\rightarrow$ invariant modes

- Consistent reductions are not a mathematical curiosity
  - establish a map between sugra theories in different dimensions
  - insight on the higher dimensional origin of the lower dimensional gauge symmetries
  - powerful tool in AdS/CFT
  - embed into string theory AdS vacua, black holes, domain walls, and non-relativistic backgrounds

- Renewed interest for extended symmetries in string theory and M-theory
  - the $10-d$ effective action of type II and M-theory compactified on tori has large symmetry groups: U-duality groups
    - U-duality contains $O(d,d)$ (T-duality) as a subgroup
    - U-duality cannot be understood from symmetries of the conventional formulation of type II or M-theory
  - new formulations of type II and M-theory
    - DFT/EFT: Double and/or Exceptional Field Theories [hull, zwiebach 09; hohm, hull, zwiebach 10; hohm, samtleben 13; ...]
    - GCG/EGG: Generalised Complex and/or Exceptional Generalised Geometry [hitchin 02; gualtieri 04; hull 07; pacheco, waldram 08, ...]

    where

    - extended symmetries have a geometrical interpretation
    - role of the higher rank gauge fields

- GCG/EGG and EFT are natural frameworks to study consistent truncations
  - the symmetries in low dimensions come from isometries and gauge symmetries of the higher-dimensional theory
  - sphere reductions are a good example
    - consistency is not guaranteed by symmetry
    - only S^1, S^3, anf S^7 are parallelisable
    - understanding of such reduction requires explicit use of the U-duality symmetry
      - reformulation with manifest SU(8) symmetry [de Wit and Nicolai 87;...]
      - generalised Scherck-Schwarz reductions in EGG or EFT [lee, strickland-constable, waldram 14; hohm, samtleben 14, ...]

- In this talk I will focus on Generalised Geometry
  - summary of the main features of EGG for type IIB
  - generalised structures as a unified framework to study truncations with different amount of supersymmetry

EXCEPTIONAL GENERALISED GEOMETRY FOR IIB ON $X_5 \times M_5$

[hitchin 02; gualtieri 04; hull 07; pacheco, waldram 08, ...]

- Geometrise the gauge symmetries of RR and NS potentials by enlarging the tangent space

	Riemannian	G C G	E G G
tangent b.	TM	$TM \oplus T^*M$	$T \oplus T^* \oplus \Lambda^- \oplus \Lambda^5 T^* \oplus (T^* \otimes \Lambda^6 T^*)$
structure group	$SO(5)$	$O(5,5)$ T-duality	$E_{6(6)}$ U-duality

- charges under the various symmetries
- the transition functions involve RR and NS potentials as generalised diffeomorphims
- the structure group is the duality group on the internal manifold

Generalised tangent bundle

- Type IIB potentials (democratic formulation [bergshoeff et al. 01])

NS two-form	RR polyform	NS six-form
B	$C = \sum_{k=1/2}^{9/2} C_{2k-1}$	$\tilde{B}$
$H = \mathrm{d}B$	$F = \mathrm{d}C - H \wedge C$	$\tilde{H} = \mathrm{d}\tilde{B} - \frac{1}{2}[s(F) \wedge C]_{(7)}$

- with gauge transformations

$$\delta_V B = \mathcal{L}_v B - \mathrm{d}\lambda$$

$$\delta_V C = \mathcal{L}_v C - e^B \wedge \mathrm{d}\omega$$

$$\delta_V \tilde{B} = \mathcal{L}_v \tilde{B} - \mathrm{d}\tilde{\lambda} - \frac{1}{2}\left[e^B \wedge \mathrm{d}\omega \wedge s(C)\right]_6$$

- The generalised tangent bundle on M_5 is $E \in \mathbf{27}$ of $E_{6(6)}$
  - under $GL(5) \times SL(2) \subset E_{6(6)}$ it decomposes as

$$E \simeq TM \oplus T^*M \oplus \Lambda^5 T^*M \oplus (T^*M \oplus \Lambda^3 T^*M \oplus \Lambda^5 T^*M)$$

  - sections of E are generalised vectors

$$V \sim v + \lambda^\alpha + \rho + \sigma^\alpha$$

  - has a non-trivial fibration structure: the patching of generalised vectors on $U_{(\alpha)} \cap U_{(\beta)}$ implies the NS and RR gauge transformations

$$\left.\begin{aligned} \delta_V B|_\alpha &= \delta_V B|_\beta \\ \delta_V C|_\alpha &= \delta_V C|_\beta \end{aligned}\right\} \quad \Longleftrightarrow \quad V_{(\alpha)} = e^{\mathrm{d}\Omega_{(\alpha\beta)}}\, e^{-\mathrm{d}\Lambda_{(\alpha\beta)}} \cdot V_{(\beta)}$$

$E_{6(6)}$ adjoint action

- The $E_{6(6)}$ adjoint action is defined as

$$V' = R \cdot V \Leftrightarrow \begin{cases} v' &= lv + r \cdot v + \gamma \lrcorner \rho + \epsilon_{\alpha\beta} \beta^\alpha \lrcorner \lambda^\beta \,, \\ \lambda'^\alpha &= l\lambda^\alpha + r \cdot \lambda^\alpha + a^\alpha{}_\beta \lambda^\beta - \gamma \lrcorner \sigma^\alpha + v \lrcorner B^\alpha + \beta^\alpha \lrcorner \rho \,, \\ \rho' &= l\rho + r \cdot \rho + v \lrcorner C + \epsilon_{\alpha\beta} \beta^\alpha \lrcorner \sigma^\beta + \epsilon_{\alpha\beta} \lambda^\alpha \wedge B^\beta \,, \\ \sigma'^\alpha &= l\sigma^\alpha + r \cdot \sigma^\alpha + a^\alpha{}_\beta \sigma^\beta - C \wedge \lambda^\alpha + \rho \wedge B^\alpha \,. \end{cases}$$

- Also define twisted generalised vectors

$$V = e^{B^\alpha + C} \check{V} = v + \lambda^\alpha + \rho + \sigma^\alpha$$

and in components

$$\begin{aligned} v &= \check{v} \\ \lambda^\alpha &= \check{\lambda}^\alpha + \check{v} \lrcorner B^\alpha \\ \rho &= \check{\rho} + \check{v} \lrcorner C + \epsilon_{\alpha\beta} \check{\lambda}^\alpha \wedge B^\beta + \frac{1}{2} \epsilon_{\alpha\beta}\, \check{v} \lrcorner B^\alpha \wedge B^\beta \\ \sigma^\alpha &= \check{\sigma}^\alpha - C \wedge \check{\lambda}^\alpha + \check{\rho} \wedge B^\alpha - \frac{1}{2} \check{v} \lrcorner B^\alpha \wedge C + \frac{1}{2} \left(\check{v} \lrcorner C + \epsilon_{\alpha\beta} \check{\lambda}^\beta \wedge B^\gamma \right) \wedge B^\alpha \end{aligned}$$

Dorfman Derivative

- The ordinary Lie derivative generates diffeomorphisms

$$\mathcal{L}_v v'^m = v^n \partial_n v'^m - v'^n \partial_n v^m = v^n \partial_n v'^m - (\partial \times_{\rm ad} v)^m{}_n v'^n$$

GL(5) adjoint action $\hookleftarrow$

- The Dorfmann derivative generates generalised diffeos: diffeos plus gauge transf.

$$L_V V'^M = V^N \partial_N V'^M - (\partial \times_{\rm ad} V)^M{}_N V'^N$$

$\partial_N = (\partial_n, 0, 0, 0, 0) \in E^* \hookleftarrow$ $\hookrightarrow$ $E_{6(6)}$ adjoint action

- in components

$$L_V V' = \mathcal{L}_v v' + (\mathcal{L}_v \lambda'^\alpha - \iota_{v'} \mathrm{d}\lambda^\alpha) + (\mathcal{L}_v \rho' - \iota_{v'} \mathrm{d}\rho + \epsilon_{\alpha\beta} \mathrm{d}\lambda^\alpha \wedge \lambda'^\beta) + \mathcal{L}_v \sigma'^\alpha - \mathrm{d}\lambda^\alpha \wedge \rho' + \lambda'^\alpha \wedge \mathrm{d}\rho$$

Generalised Metric and Frame

- Define a generalised metric containing all the bosonic degrees of freedom on M

$$G \in \frac{E_{6(6)}}{Usp(8)}$$

  - acting on two generalised vectors G is

$$G(V, V') = v^m v'_m + h_{\alpha\beta}\lambda^\alpha \lrcorner \lambda'^\beta + \rho \lrcorner \rho' + h_{\alpha\beta}\sigma^\alpha \lrcorner \sigma'^\beta$$

$$\hookrightarrow h_{\alpha\beta} = e^\phi \begin{pmatrix} C_0^2 + e^{-2\phi} & -C_0 \\ -C_0 & 1 \end{pmatrix} \in SL(2)/SO(2)$$

- The inverse generalised metric is defined by its action on $Z = \in \bar{\mathbf{27}}$

$$G^{-1}(Z, Z') = \tilde{v} \lrcorner \tilde{v}' + h^{\alpha\beta}\tilde{\lambda}_\alpha \lrcorner \tilde{\lambda}'_\beta + \tilde{\rho} \lrcorner \tilde{\rho}' + h^{\alpha\beta}\tilde{\sigma}_\alpha \lrcorner \tilde{\sigma}'_\beta$$

where

$$Z = \tilde{v} + \tilde{\lambda}_\alpha + \tilde{\rho} + \tilde{\sigma}_\alpha \in E^* \sim T^*M \oplus \Lambda^5 TM \oplus (TM \oplus \Lambda^3 TM \oplus \Lambda^5 TM)$$

- We can define a generalised frame for the generalised tangent bundle E
  - take a frame and coframe on M_5

$$\{e^a\} \in TM \qquad \{e_a\} \in T^*M \qquad a = 1, \ldots, 5$$

  - define a generalised frame as

$$E_A = e^{\tilde{B}} e^{-B} e^{C} e^{\Delta} e^{\phi} \cdot \hat{E}_A$$
$$\hat{E}_A = \{\hat{e}_a\} \cup \{e^a\} \cup \{e^{a_1 \ldots a_5}\} \cup \{e^a\} \cup \{e^{a_1 a_2 a_3}\} \cup \{e^{a_1 \ldots a_5}\}$$

  - the generalised metric can be written as

$$G^{-1} = \delta^{AB} E_A . \otimes E_B$$

CONSISTENT TRUNCATIONS AND EGC

- In ordinary geometry, the field content and gaugings of a consistent truncations are determined by the G-structure on the internal manifold

  - given the structure group $K \subset GL(d)$ and the metric structure $SO(d)$

    - scalar manifold $H \in \frac{C_{GL(d)}(K)}{C_{SO(d)}(K)}$ $C_G(K) \to$ commutant of K in G
    - vector fields $A^a k_a$ $k_a \to$ globally defined vectors on TM
    - gauge group $[k_a, k_b] = f_{ab}^c k_c$ $f_{ab}^c \to$ K-singlets of the intrinsic torsion

  - Examples

	K	H	k_a
group manifold	$\mathbb{I} \subset GL(d)$	$H \in \frac{GL(d)}{SO(d)}$	$\hat{e}_a$ frame on TM
Sasaki Einstein	$SU(n) \subset GL(2n+1)$	$H \in \frac{\mathbb{C} \times \mathbb{R}^+}{U(1)}$	$\xi = \partial_\psi$ Reeb vector

- Extend the construction to generalised structure on E
  - maximally susy : generalised parallelisability and generalised Scherk-Schwarz
    - consistency of truncations on spheres [lee, strickland-constable, waldram 14]
    - sphere truncations in massive IIA [cassani, de felice, m.p. strickland-constable, waldram 16]
  - less susy reductions: generalised G-structures
    - revisit half-maiximal truncations on Sasaki-Einstein manifolds
    - compute truncations on β-deformed SE manifolds

MAXIMAL SUSY:
GENERALISED SCHERK-SCHWARZ REDUCTiON

- Ordinary reduction on a group manifold M_d
  - M_d admits globally defined left-invariant vector fields $\{\hat{e}_a\}$
  - $\{\hat{e}_a\}$ give a globally defined frame $\rightarrow M_d$ is parallelisable
  - the left-invariant vectors satisfy the algebra

$$\mathcal{L}_{\hat{e}_a}\hat{e}_b = f_{ab}{}^c\hat{e}_c \qquad\qquad f_{ab}{}^c \text{ constant}$$

  - $\{\hat{e}_a\}$ generate the (right) isometries of the metric

- Scherk-Schwarz truncation ansatz
  - define the twisted frame and the internal metric

$$\hat{e}'_a{}^m(x,z) = U_a{}^b(x)\,\hat{e}_b{}^m(z) \qquad U_a{}^b \in GL(d)$$

$$g^{mn}(x,z) = \mathcal{M}^{ab}(x)\,\hat{e}_a{}^m(z)\,\hat{e}_b{}^n(z) \qquad \mathcal{M}^{ab} = \delta^{cd}\,U_c{}^a U_d{}^b \in GL(d)/\,SO(d)$$

  - expand the 10d SUGRA fields

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathcal{M}_{ab}(x)(e^a - \mathcal{A}^a_\mu(x)\mathrm{d}x^\mu)(e^b - \mathcal{A}^b_\nu(x)\mathrm{d}x^\nu)$$

$$C_2(x,z) = \frac{1}{2}C_{\mu\nu}(x)\mathrm{d}x^{\mu\nu} + C_{\mu b}(x)\mathrm{d}x^\mu \wedge (e^a - \mathcal{A}^a_\nu(x)\mathrm{d}x^\nu)$$
$$+\frac{1}{2}C_{ab}(x)(e^a - \mathcal{A}^a_\mu(x)\mathrm{d}x^\mu) \wedge (e^b - \mathcal{A}^b_\nu(x)\mathrm{d}x^\nu) + c_2$$

$$\ldots$$

  where $g_{\mu\nu}$, $\mathcal{M}_{ab}$, $\mathcal{A}^a_\mu$ ..., are $D-d$- dimensional fields

- Features of the truncated theory
  - the reduction is consistent by symmetry
  - being parallelisable M_d has globally defined spinors $\Rightarrow$ maximal SUSY
  - the gauge group comes from isometries and RR and NS gauge transformations

Generalised Scherck-Schwarz reduction

- Extend to EGG the notion of parallelisability $\rightarrow$ Generalised Leibniz parallelisation
  - $\exists$ a globally defined frame $\{E_A\}$ for the $E_{d+1(d+1)} \times \mathbb{R}^+$ generalised tangent bundle on M_d.
  - the frame must satisfy the algebra

$$L_{E_A} E_B = X_{AB}{}^C E_C \qquad\qquad X_{AB}{}^C \text{ constant}$$

  - $X_{AB}{}^C$ generate the gauge algebra

$$[X_A, X_B] \; = \; -X_{AB}{}^C X_C$$

    and are related to the embedding tensor of the $D-d$ gauged supergravity

$$X_{AB}{}^C = \Theta_A{}^\alpha (t_\alpha)_B{}^C$$

embedding tensor $\hookleftarrow$ $\hookrightarrow$ U-duality generators

  - GLP implies the manifold is a coset $M_d \sim G/H$ and maximal SUSY

- Generalised Scherck-Schwarz ansatz
  - decompose all 10d fields according to $SO(1,9) \to SO(1,4) \times SO(5)$ and arrange them in $E_{6(6)} \times \mathbb{R}^+$ tensors
    - scalars $\to$ gen metric

$$\{g_{mn},\, B_{mn},\, C_0,\, C_{mn},\, C_{mnpq}\}$$

    - vectors $\to$ gen vector $\mathcal{A}_\mu{}^M \in E$

$$\mathcal{A}_\mu{}^M = \{h_\mu{}^m,\, B_{\mu n},\, C_{\mu n},\, C_{\mu npq}\}$$

    - two-forms $\to \mathcal{B}_{\mu\nu}{}^{MN} \in N' \subset (E \otimes E)_{\text{sym}}$

$$\mathcal{B}_{\mu\nu}{}^{MN} = \{B_{\mu\nu},\, C_{\mu\nu},\, C_{\mu\nu pq}\}$$

  - covariance under gen diffeos $\to$ field redefinition in the lower dimensional theory

$$\begin{aligned} B_\mu &= \overline{B}_\mu & B_{\mu\nu} &= \overline{B}_{\mu\nu} + \iota_{h_{[\mu}} B_{\nu]} \\ C_\mu &= e^{-B} \wedge \overline{C}_\mu & \tilde{B}_{\mu\nu} &= \overline{\tilde{B}}_{\mu\nu} - \tfrac{1}{2}\big[\,\overline{C}_{\mu\nu} \wedge s(C)\,\big]_4 + \iota_{h_{[\mu}} \tilde{B}_{\nu]} \\ \tilde{B}_\mu &= \overline{\tilde{B}}_\mu - \tfrac{1}{2}[\,\overline{C}_\mu \wedge s(C)\,]_5 & C_{\mu\nu} &= e^{-B} \wedge \overline{C}_{\mu\nu} + \iota_{h_{[\mu}} C_{\nu]} + B_{[\mu} \wedge C_{\nu]} \end{aligned}$$

- twist the generalised frame and metric

$$E'_A{}^M(x,z) = U_A{}^B(x) E_B{}^M(z) \qquad \mathcal{G}^{MN}(x,z) = \mathcal{M}^{AB}(x) E_A{}^M(z) E_B{}^N(z)$$

$$E_{6(6)} \hookleftarrow \qquad \hookrightarrow \frac{E_{6(6)}}{Usp(8)}$$

with $Usp(8)$ maximally compact subgroup of $E_{6(6)}$

- write the 10d fields as

$$\mathcal{A}_\mu{}^M(x,z) = \mathcal{A}_\mu{}^A(x) E_A{}^M(z)$$
$$\mathcal{B}_{\mu\nu}{}^{MN}(x,z) = \frac{1}{2}\mathcal{B}_{\mu\nu}{}^{AB}(x)(E_A \otimes_{N'} E_B)^{MN}(z)$$

a similar construction should work for higher rank forms

- Partial check of the consistency of the reduction $\rightarrow$ recover the gauge transformations of the lower dimensional gauged SUGRA

- Example: S^5 reduction in type IIB [lee, strickland-constable, waldram 14]
  - For any sphere backgrounds

$$S^d = \frac{SO(d+1)}{SO(d)} \quad \text{plus} \quad F_d = \mathrm{d}C_{d-1}$$

one can define the gen. tangent bundle with twisting given by C_{d-1}

$$E_{GL(d+1)} = T \oplus \Lambda^{d-2}T^*$$

  - $E_{GL(d+1)}$ is in $\frac{1}{2}d(d+1)$ dim. bivector representation of $GL(d+1,\mathbb{R})$ and

$$E_{GL(d+1)} = W \wedge W \qquad W \sim (\det T^*)^{1/2}(T + \Lambda^d T)$$

fundamental of $GL(d+1)$

  - $E_{GL(d+1)}$ always admits a globally defined frame

$$E_{ij} = E_i \wedge E_j \qquad (E_j \text{ frame on } W)$$

$$E_{ij} = v_{ij} + \tfrac{R^{d-2}}{(d-2)!}\epsilon_{ijk_1\dots k_{d-1}} y^{k_1}\mathrm{d}y^{k_2} \wedge \dots y^{k_{d-1}} + \iota_{v_{ij}} C$$

$SO(d+1)$ killing vectors $\hookleftarrow$ $\hookrightarrow$ $y^i \in \mathbb{R}^d : \delta_{ij}y^i y^j = 1$

  - all spheres are generalised parallelisable

- E_{ij} extend to a globally defined frame for the full $E_{6(6)}$ tangent bundle
  - decompose the $E_{6(6)}$ tangent bundle in $GL(6) \times SL(2)$ reprs

$$E = E^{(0)} \oplus E^{(\alpha)} \quad \Rightarrow \quad \begin{cases} E^{(0)} = TM \oplus \Lambda^3 T^* M \in (\mathbf{15}, \mathbf{1}) \\ E^{(\alpha)} = TM \oplus \Lambda^3 T^* M \in (\mathbf{6}, \mathbf{2}) \end{cases}$$

  - the globally defined frame is

$$E_A = E_{ij} + E^I_{\hat{\alpha}} \qquad \begin{cases} E_{ij} = v_{ij} + \frac{R^3}{3!} \epsilon_{ijk_1 \dots k_4} y^{k_1} \mathrm{d}y^{k_2} \wedge \dots \wedge \mathrm{d}y^{k_4} + \iota_{v_{ij}} C \\ E^I_{\hat{\alpha}} = f^\alpha_{\hat{\alpha}} (R \mathrm{d}y^i + y^i \mathrm{vol} + R \mathrm{d}y^i \wedge C) \end{cases}$$

- E_A satisfies the Leibniz algebra

$$\begin{aligned} &L_{E_{ij}} E_{kl} = R^{-1} (\delta_{ik} E_{jl} - \delta_{il} E_{jk} - \delta_{jk} E_{il} + \delta_{jl} E_{ik}) \qquad && L_{E^i_{\hat{\alpha}}} E_{jk} = 0 \\ &L_{E_{ij}} E^k_{\hat{\alpha}} = R^{-1} (\delta_{il} \delta^k_j E^l_{\hat{\alpha}} - \delta_{jl} \delta^k_i E^l_{\hat{\alpha}}) && L_{E^i_{\hat{\alpha}}} E^j_{\hat{\alpha}} = 0 \end{aligned}$$

which gives the $SO(6)$ embedding tensor

$$\left. \begin{aligned} X^{kk'}_{ii',jj'} &= R^{-1} (\delta_{ij} \delta^{kk'}_{i'j'} - \delta_{i'j} \delta^{kk'}_{ij'} - \delta_{ij'} \delta^{kk'}_{i'j} + \delta_{i'j'} \delta^{kk'}_{ij'} \\ X^{j\hat{\gamma}}_{ii',\hat{\beta}k} &= R^{-1} (\delta_{ik} \delta^j_{i'} - \delta_{ik} \delta^j_i) \end{aligned} \right\} \quad \in (\mathbf{21}, \mathbf{1}) \subset \mathbf{321}$$

HALF-MAXIMAL SUSY: generalised G-structures

- Compactifications on M_5: a half-maximal structure is an $SO(5) \subset E_{6(6)}$ structure
  - half-maximal susy corresponds to the breaking [see malek 17, for the EFT version]

$$USp(8) \supset USp(4)_R \times USp(4)_S \qquad \begin{cases} USp(4)_R \text{ R-symmetry} \\ USp(4)_S \text{ structure group} \end{cases}$$

  - the half-maximal structure is defined precisely by the singlets of $USp(4)_S$

$$E_{6(6)} \supset SO(1,1) \times SO(5,5) \supset SO(1,1) \times SO(5)_R \times SO(5)_S$$

$$\mathbf{27} \to \mathbf{10}_2 \oplus \mathbf{16}_{-1} \oplus \mathbf{1}_{-4} \to (\mathbf{5},\mathbf{1})_2 \oplus (\mathbf{1},\mathbf{5})_2 \oplus (\mathbf{4},\mathbf{4})_{-1} \oplus (\mathbf{1},\mathbf{1})_{-4}$$

- it is defined by six generalised vectors

$$K_0, K_a \in \Gamma(E) \qquad a = 1, \dots, 5$$

  such that

$$c(K_0, K_0, V) = 0, \qquad \forall V \in \Gamma(E)$$
$$c(K_0, K_a, K_b) = \delta_{ab}\,\mathrm{vol}_5$$

  where $c(V, V', V'')$ is the E_{66} cubic invariant,

$$c(V, V', V'') = -\tfrac{1}{2}\left(\iota_v \rho' \wedge \rho'' + \epsilon_{\alpha\beta}\, \rho \wedge \lambda'^{\alpha} \wedge \lambda''^{\beta} - 2\epsilon_{\alpha\beta}\, \iota_v \lambda'^{\alpha} \sigma''^{\beta}\right) + \text{symm. perm.}\,.$$

- we also need

$$\left.\begin{aligned} K_0^* &= -\tfrac{1}{5}\delta^{ab} c(K_a, K_b, \cdot\,) \\ K_a^* &= c(K_0, K_a, \cdot\,) \end{aligned}\right\} \in \Gamma(\det T^*M \otimes E^*) \qquad \begin{aligned} \langle K_0^*, K_0 \rangle &= 1 \\ \langle K_a^*, K_b \rangle &= \eta_{ab} \\ \langle K_0^*, K_a \rangle &= 0 \end{aligned}$$

- The generalised metric is computed using the $SO(5,5)$ structure defined by $\{K_0, K_0^*\}$

$$SO(5) \subset SO(5) \times SO(5) \subset SO(5,5) \subset E_{6(6)}$$

- the $SO(5,5)$ structure gives a decomposition of the generalised tangent bundle

$$V = V_0 + \tilde{V} + \Psi \quad \in \quad E = E_0 + E_{10} + E_{16}$$

$$\mathbf{27} = \mathbf{1} + \mathbf{10} + \mathbf{16}$$

- the generalised metric on E splits into metrics on E_0, E_{10} and E_{16}

$$\begin{aligned} G &= G_0 + G_{10} + G_{16} \\ &= \langle K_0^*, V\rangle^2 + \delta^{ab}\langle K_a^*, V\rangle\langle K_b^*, V\rangle + \frac{c(K_0, V, V)}{\text{vol}} - 4\sqrt{2}\langle V, K_1 \cdots K_5 \cdot V\rangle \end{aligned}$$

where

$$\langle Z, V\rangle = \hat{v}_m v^m + \hat{\lambda}^m_\alpha \lambda^\alpha_m + \tfrac{1}{3!}\,\hat{\rho}^{mnp}\rho_{mnp} + \tfrac{1}{5!}\,\hat{\sigma}^{mnpqr}_\alpha \sigma^\alpha_{mnpqr}$$

- G_0 : projection onto the singlet,
- G_{10} : an $SO(5)\times SO(5)\subset SO(5,5)$ structure splits E_{10} into positive- and negative-definite eigenspaces

$$E_{10}=C_+\oplus C_- \qquad \Leftrightarrow \qquad \begin{array}{l} \eta=G_+-G_- \\ G_{10}=G_++G_- \end{array}$$

with (K_a form a basis for C_-)

$$\eta(\tilde{V},\tilde{V})=\frac{c(K_0,\tilde{V},\tilde{V})}{\mathbf{vol}_5} \qquad G_-(V,V)=\delta^{ab}\langle K_a^*,V\rangle\langle K_b^*,V\rangle$$

- G_{16} : inner product between $SO(d,d)$ spinors

$$\begin{array}{c}\langle\Psi,\Gamma^{(+)}\Psi\rangle \\ \Updownarrow \\ \langle V,K_1\cdots K_5\cdot V\rangle\end{array} \qquad \Gamma^+=\Gamma_1^+\cdots\Gamma_5^+ \text{ chirality matrix}$$

- Generalised $SO(5-n)$ structures
  - the structure is further reduced to $SO(5-n) \subset SO(5)$ by globally-defined generalised vectors in the $\mathbf{27}$

$$(K_0, K_A) \quad A = 1, \dots, n+5 \qquad \begin{aligned} c(K_0, K_0, V) &= 0, \quad \forall V \in \Gamma(E), \\ c(K_0, K_A, K_B) &= \eta_{AB} \mathrm{vol}_5 \\ c(K_A, K_B, K_C) &= 0 \end{aligned}$$

with $\eta_{MN} = \mathrm{diag}(-1,-1,-1,-1,-1,+1,\dots,+1)$ flat $SO(5,n)$ metric.

  - dual generalised vectors

$$K_0^* = \frac{1}{5+n} \eta^{AB} c(K_A, K_B, \cdot)$$
$$K_A^* = c(K_0, K_A, \cdot)$$

with $\langle K_0^*, K_0 \rangle = 1$, $\langle K_A^*, K_B \rangle = \eta_{AB}$ and $\langle K_0^*, K_a \rangle = 0$

  - the generalised metric is compute as before

$$G = \langle K_0^*, V \rangle^2 + \delta^{ab} \langle K_a^*, V \rangle \langle K_b^*, V \rangle + \frac{c(K_0, V, V)}{\mathrm{vol}} - 4\sqrt{2} \langle V, K_1 \cdots K_5 \cdot V \rangle$$

- truncation ansatz for the scalars
  - the scalar manifold is given by the coset

$$H \in \frac{C_{E_{6(6)}}(SO(5-n))}{C_{SU(8)}(SO(5-n))} = O(1,1) \times \frac{SO(5,n)}{SO(5)\times SO(n)}$$

  - the generalised metric is built out of dressed generalised vectors

$$\tilde{K}_0 = \Sigma^2 K_0. \qquad \tilde{K}_a = \Sigma^{-1}\,\mathcal{V}_a{}^A K_A \qquad \tilde{K}_{\hat{a}} = \Sigma^{-1}\,\mathcal{V}_{\hat{a}}{}^A K_A$$
$$\tilde{K}_0^* = \Sigma^{-2} K_0^* \qquad \tilde{K}_a^* = \Sigma\,\mathcal{V}_a{}^A K_A^* \qquad \tilde{K}_{\hat{a}}^* = \Sigma\,\mathcal{V}_{\hat{a}}{}^A K_A^*$$

where $\Sigma \in O(1,1)$, $\{\mathcal{V}_A{}^a, \mathcal{V}_A{}^{\hat{a}}\} \in SO(5,n)$, $a = 1,\ldots,5, \hat{a} = 1,\ldots,n$

$$\eta_{AB} = \delta_{ab}\,\mathcal{V}_A{}^a\mathcal{V}_B{}^b - \delta_{\hat{a}\hat{b}}\,\mathcal{V}_A{}^{\hat{a}}\mathcal{V}_B{}^{\hat{b}} \quad M_{AB} = \delta_{ab}\,\mathcal{V}_A{}^a\mathcal{V}_B{}^b + \delta_{\hat{a}\hat{b}}\,\mathcal{V}_A{}^{\hat{a}}\mathcal{V}_B{}^{\hat{b}}$$

  - the generalised metric is

$$\begin{aligned} G &= G_0 + G_{10} + G_{16} \\ &= \Sigma^{-4}\,\langle V K_0^*\rangle^2 + \Sigma^2\left(2\;\delta^{ab}\mathcal{V}_a{}^C\mathcal{V}_b{}^D\,\langle V, K_A^*\rangle\langle V, K_B^*\rangle + \frac{c(K_0,V,V)}{\mathrm{vol}}\right) \\ &\quad -\tfrac{4\sqrt{2}}{5!}\,\Sigma^{-1}\,\epsilon^{abcde}\mathcal{V}_a{}^A\mathcal{V}_b{}^B\mathcal{V}_c{}^C\mathcal{V}_d{}^D\mathcal{V}_e{}^E\,\langle V, K_A\cdots K_E^*\cdot V\rangle \end{aligned}$$

- Example: Sasaki Einstein reduction in type IIB
  - truncation on squashed SE manifolds in 5 d [cassani, faedo, dall'agata 10]
    - the theory is $\mathcal{N}=4$ 5d sugra with two vector multiplets and with $Heis_3 \times U(1)$ gauging and scalars parameterise the coset

$$\mathcal{M}_{\rm scal} \;=\; SO(1,1) \times \frac{SO(5,2)}{SO(5)\times SO(2)}$$

  - SE geometry
    - $U(1)$ fibration over a KE base

$$\mathrm{d}s^2_{SE} = \mathrm{d}s^2_{KE} + \eta^2 = \sum_{i=2}^{5} (e^i)^2 + (e^1)^2 \qquad F_5 = \kappa \mathrm{vol}_5$$

    - 5d SE are $SU(2)$ structure manifolds

$$\begin{aligned} J_1 &= e^2\wedge e^5 - e^3\wedge e^4 \\ J_2 &= e^2\wedge e^4 + e^3\wedge e^5 \\ J_3 &= e^2\wedge e^3 - e^4\wedge e^5 \end{aligned} \qquad \Longleftrightarrow \qquad \begin{aligned} \Omega &= J_1 + iJ_2 \\ \omega &= J_3 \\ \eta &= -e^1 \end{aligned}$$

- the $SU(2)$ structure extend to a generalised $SU(2) \subset SO(5)$ structure
  - generalised vectors

$$K_0 = \xi \qquad K_{1,2,3} = \tfrac{1}{\sqrt{2}}\, \eta \wedge J_{1,2,3}$$

$$\begin{aligned} K_4 &= \tfrac{1}{\sqrt{2}}(n\eta - r\mathrm{vol}) \\ K_5 &= \tfrac{1}{\sqrt{2}}(-r\eta - n\mathrm{vol}) \\ K_6 &= \tfrac{1}{\sqrt{2}}(n\eta + r\mathrm{vol}) \\ K_7 &= \tfrac{1}{\sqrt{2}}(-r\eta + n\mathrm{vol}) \end{aligned}$$

  where $n = (1, 0)$ and $r = (0, 1)$.

  - under the commutant $SO(1, 1) \times SO(5, 2)$ of $SU(2)$ in $E_{6(6)}$

$$K_0 \in \mathbf{1}_{-1} \qquad K_A \in \mathbf{7}_{1/2}$$

- scalar ansatz
  - the scalar parameterise the coset

$$O(1,1) \times \frac{SO(5,2)}{SO(5) \times SO(2)}$$

  - generalised metric is built using the dressed vectors

$$\begin{pmatrix} \tilde{K}_a \\ \tilde{K}_{\underline{a}} \end{pmatrix} = e^{-(B^+ + B^- + C)} \cdot m \cdot r \cdot e^{-l} \cdot \begin{pmatrix} K_a \\ K_{\underline{a}} \end{pmatrix} = \Sigma^{-1} \begin{pmatrix} \mathcal{V}_a{}^B \\ \mathcal{V}_{\underline{a}}{}^B K_B \end{pmatrix}$$

with

$$B^\alpha = (n^\alpha b_i + r^\alpha c_i) J_i \,, \qquad C = -a J_3 \wedge J_3$$

$$m^\alpha{}_\beta = \begin{pmatrix} e^{\frac{\phi}{2}} & 0 \\ e^{\frac{\phi}{2}} C_0 & e^{-\frac{\phi}{2}} \end{pmatrix} , \quad r = \mathrm{diag}\left(e^V, e^U, e^U, e^U, e^U\right) \qquad l = \frac{\mathrm{tr}(r)}{3} = \frac{4U+V}{3}$$

  - a lengthy but straightforward computation reproduce the scalars of SE truncation

[cassani, dall'agata, faedo 10]

- gauge group $\rightarrow$ Dorfman derivative
  - the gauge generators

$$D_\mu = \nabla_\mu - gA_\mu^{M\alpha}\Theta_{\alpha M}{}^{NP} t_{NP} + gA_\mu^{M(\alpha}\epsilon^{\beta)\gamma}\xi_{\gamma M} t_{\alpha\beta}$$
$$D_\mu = \nabla_\mu - gA_\mu^{M\alpha}X_{M\alpha} + gA_\mu^{M\alpha}Y_{M\alpha}$$

are determined by embedding tensor of $\mathcal{N}=4$ sugra [schon, weidner 06]

$$(\xi_{\alpha M}, f_{\alpha MNP}) \iff \begin{cases} \Theta_{\alpha MNP} = f_{\alpha MNP} - \xi_{\alpha[N}\eta_{P]M} \\ \hat{f}_{\alpha MNP} = f_{\alpha MNP} - \xi_{\alpha[M}\eta_{P]N} - \frac{3}{2}\xi_{\alpha N}\eta_{MP} \end{cases}$$

  - Dorfman derivative

$$L_{K^t_A} K^t_B = F_{AB}{}^C K^t_C \qquad K^t_A = e^C K_A$$

with

$$F_{15}{}^6 = -3\,, \quad F_{23}{}^4 = -2\,, \quad F_{12}{}^8 = -\kappa\,, \quad F_{42}{}^8 = F_{43}{}^7 = 4$$
$$F_{16}{}^5 = 3\,, \quad F_{32}{}^4 = 2\,, \quad F_{13}{}^7 = -\kappa\,, \quad F_{24}{}^8 = F_{34}{}^7 = -4$$

  - the gauge generators $(X_A)_B{}^C = F_{AB}{}^C$ satisfy the algebra

$$(X_A)_B{}^c = F_{AB}{}^C \qquad \Longrightarrow \qquad Heis_3 \times U(1)$$

SUMMARY AND OUTLOOK

- Generalised Geometry and Exceptional field theories are unified frameworks to study consistent truncations
  - Generalised Scherk-Schwarz reduction for maximal supersymmetry
    - consistency of sphere reductions [lee, strickland-constable, waldram 14; hohm, samtleben 14, ...]
    - consistent truncations for massive IIA [ciceri, guarino, inverso 16; cassani, de felice, m.p. strickland constable, waldram 16]
  - Extension of the formalism to truncations with less SUSY
    - use of generalised structures [see also Malek 17 in EFT]
    - find new truncation with truly exceptional structure, e.g truncation on beta-deformed backgrounds
    - extend it to M-theory and different dimensions
    - interesting applications to AdS/CFT