Symmetries of Feynman Integrals: a new formulation for FI evaluation

Barak Kol
Hebrew Un., Jerusalem
Mainz ITP, 21 March 2019

based on work w. Philipp Burda, Subhajit Mazumdar, Amit Schiller and Ruth Shir in 8 papers since 2015, incl. 1 JHEP, 2 PRD.

Introduction

Motivation

At risk of stating the obvious:

Feynman diagrams are at the computational core of Quantum Field Theory.

Do we have a general theory?

Overview of main results

Definitions

A Feynman integral

$$I(\mu_1, \ldots, \mu_P, p_1^\mu, \ldots, p_X^\mu) = \int \frac{d^d l_1 \ldots d^d l_L}{\Pi_{i=1}^P (k_i^2 - \mu_i + i0)}$$

$$\mu := m^2$$

Parameter space: masses and kinematical invariants

$$X = \{\mu_i,\, p_r \cdot p_s\}$$

Comment: Numerators, spacetime dimension

IBP + DE operators

- $l_1, l_2, \ldots, l\_L$ - loop currents
- $p_1, p_2, \ldots, p_{n-1}$ - independent external currents

$$\partial_l \, l, \;\; \partial_l \, p, \;\; p \, \partial_p$$

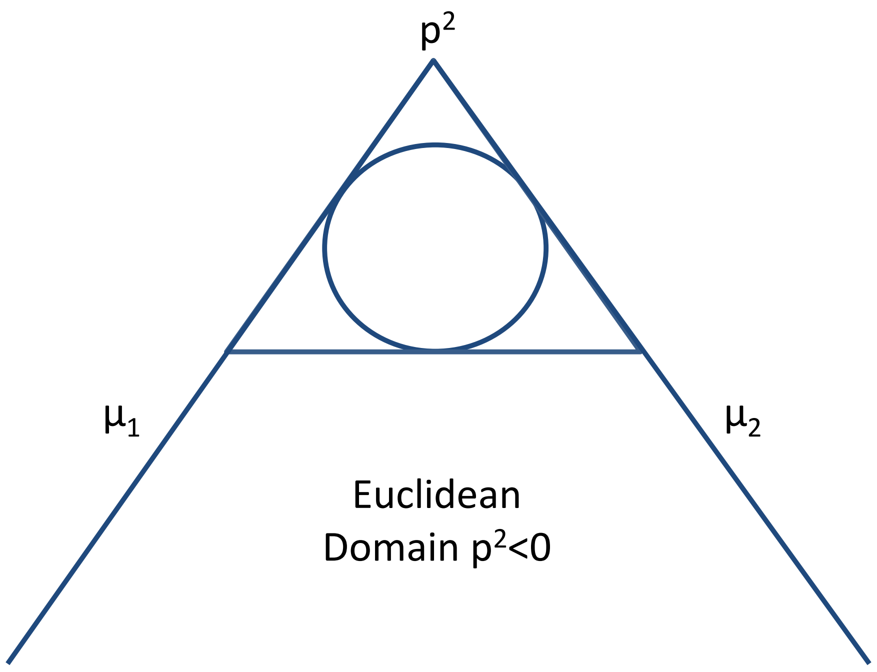
SFI equation system

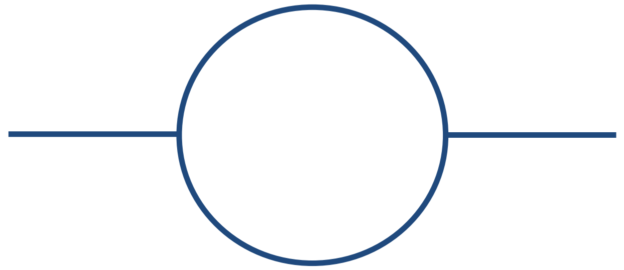
$$c^a\, I + (T^a)^i_j\, x_i\, \frac{\partial}{\partial x_j} I = J^a$$

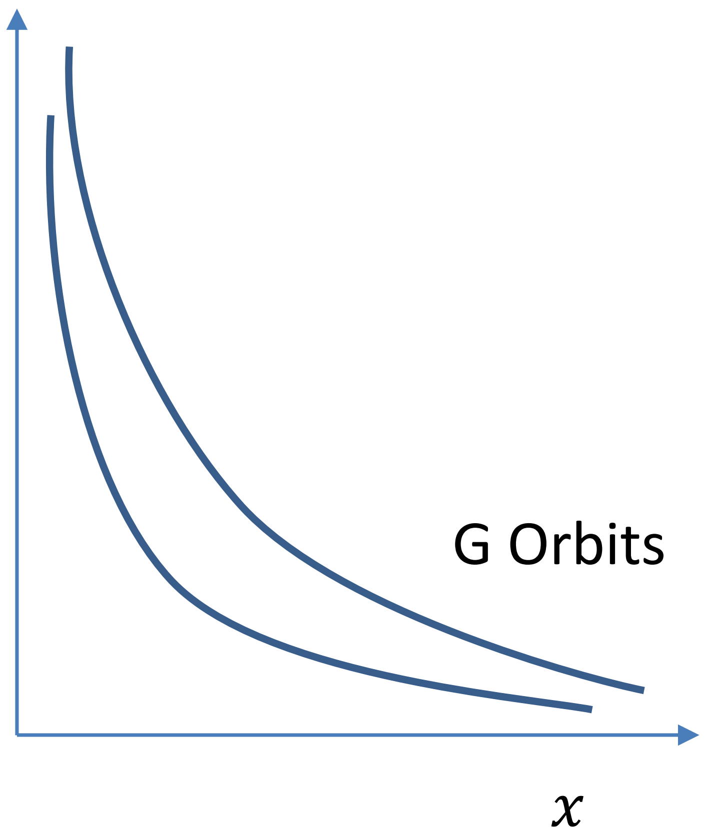
$$a = 1, \ldots, \dim G$$

- c^a - x-indep. const's
- T^a - group generators, generate no irreducible numerators (ISP's)
- Act on X
- J^a - simpler diagrams

Foliation of X into G orbits







Reduction

Reduction $$\hat{I}(x) = \hat{I}(x_0) + \int_{x_0}^{x} J^\alpha(\xi) d\xi_\alpha$$

Leading singularity normalization $\hat{I} := I/I_0$

I_0 - a homogeneous soln., incl. maximal cut.

x_0 - conveniently chosen base point
within G orbit (e.g. m=0 or $m_1=m_2$).

Integrand depends on simpler diagrams,
namely, with an edge contracted.

Singularity locus

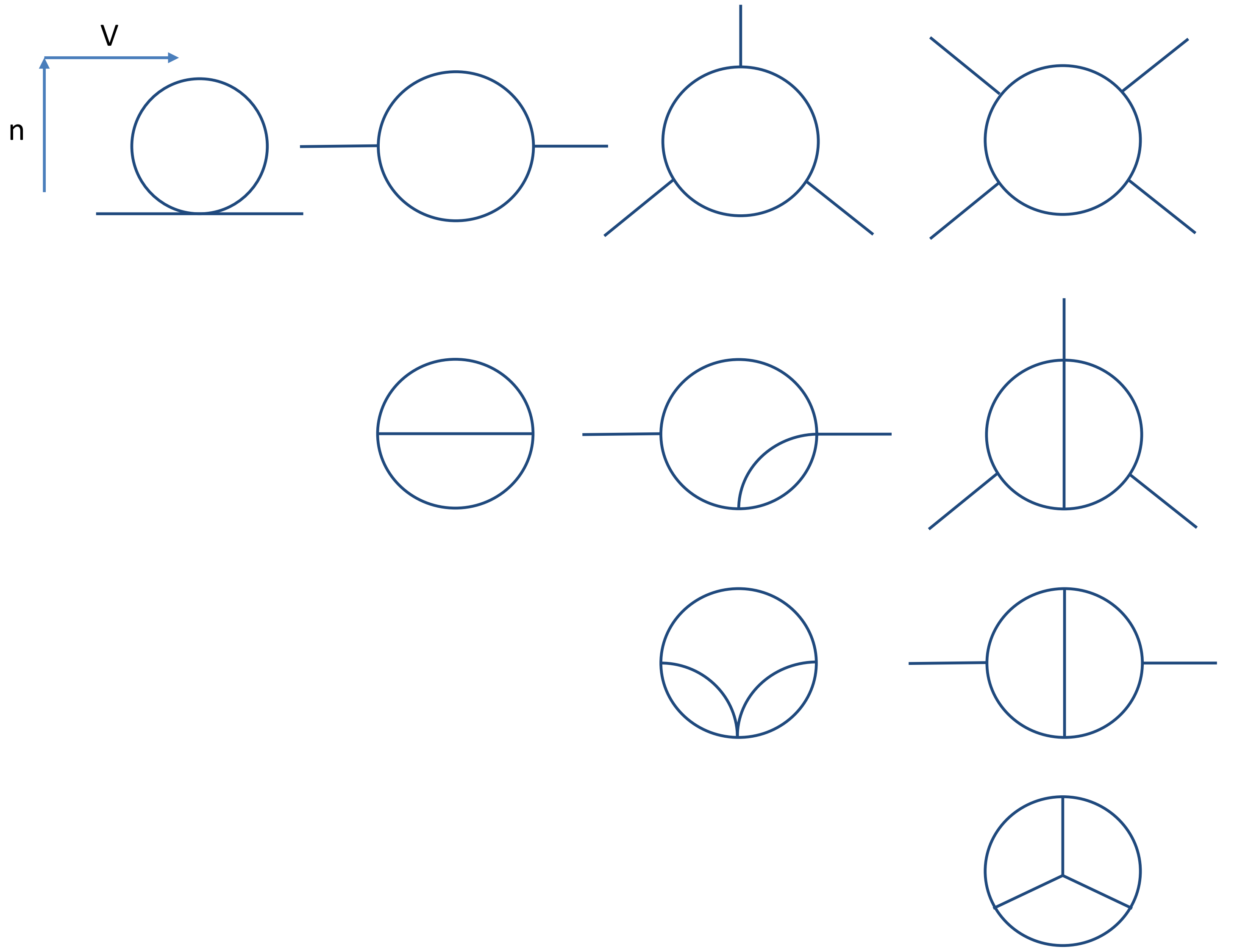
$$I(x)|_S = \sum_a k_a(x) J^a(x)$$

$$@ \; S(x) = 0$$

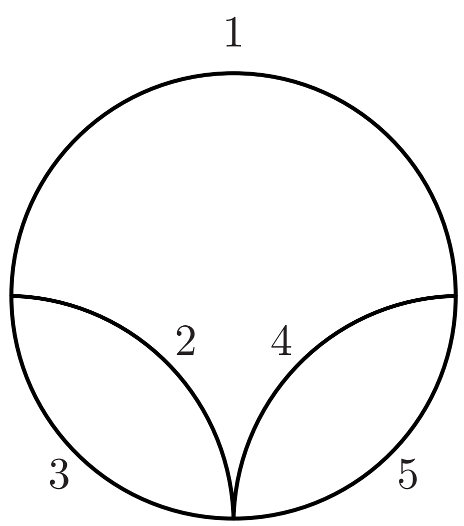
A linear combination of simpler diagrams.

Applications

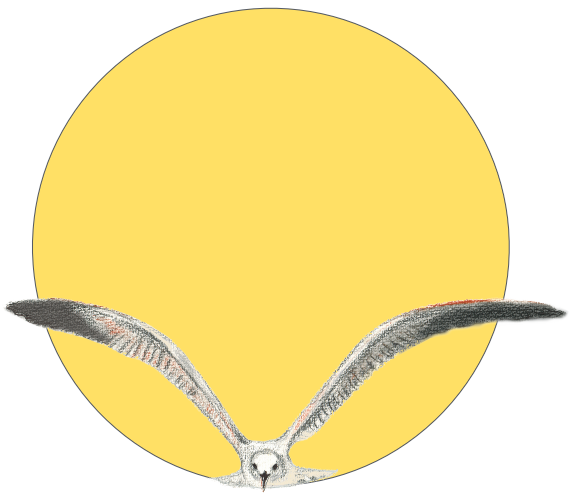
Hierarchy of diagrams



Vacuum seagull



(a)



(b)

evaluated a sector w. 3 mass scales (novel).

$$I(m^2,m^2x,0,m^2y,0) = (m^2)^{\frac{3d}{2}-5}I_0(x,y,d)$$
$$(I_1(x,y,d)+I_1(y,x,d)+I_2(x,d)+I_2(y,d)$$
$$+I_3(x,y,d)+c_4(d)) \tag{53a}$$

where

$$I_0(x,y,d) = i\pi^{\frac{3d}{2}}((1-x)(1-y))^{d-3}, \tag{53b}$$

$$I_1(x,y,d) = \left[c_{1a}(d)xy^{\frac{3d}{2}-5}{}_2F_1\left(5-\frac{3d}{2},4-d,3-\frac{d}{2}\middle|\frac{x}{y}\right)\right.$$
$$\left.+c_{1b}(d)x^{\frac{d}{2}-1}y^{d-3}{}_2F_1\left(3-d,2-\frac{d}{2},\frac{d}{2}-1\middle|\frac{x}{y}\right)\right]$$
$$\times F_1(3d/2-4,d-2,d-3,3d/2-3|x,y), \tag{53c}$$

$$I_2(x,d) = c_2(d)x^{\frac{d}{2}-1}{}_2F_1(d/2-1,d-2,d/2|x), \tag{53d}$$

$$I_3(x,y,d) = c_3(d)(xy)^{\frac{d}{2}-1}{}_2F_1(d/2-1,d-2,d/2|x)$$
$$\times{}_2F_1(d/2-1,d-2,d/2|y), \tag{53e}$$

$$c_{1a}(d) = -3\Gamma\left(3-\frac{3d}{2}\right)\Gamma(4-d)\Gamma\left(\frac{d}{2}-2\right)\Gamma\left(\frac{d}{2}-1\right), \tag{53f}$$

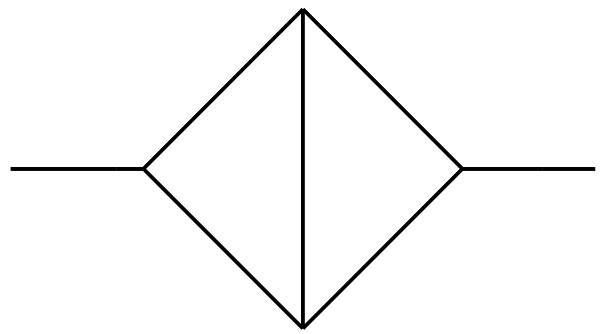
$$c_{1b}(d) = -\frac{4\pi\mathrm{Csc}(\frac{\pi d}{2})\Gamma(2-d)\Gamma(2-\frac{d}{2})}{3d-8}, \tag{53g}$$

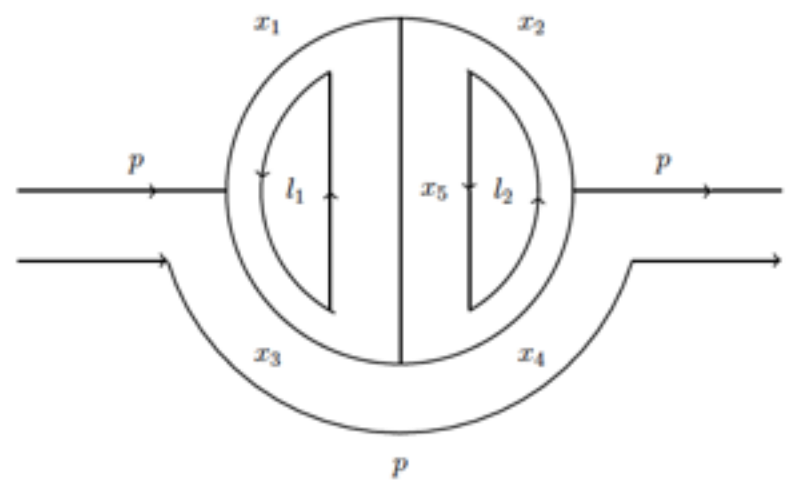
$$c_2(d) = \pi d\mathrm{Csc}\left(\frac{\pi d}{2}\right)\Gamma(2-d)\Gamma\left(-\frac{d}{2}\right), \tag{53h}$$

$$c_3(d) = \Gamma\left(1-\frac{d}{2}\right)^3. \tag{53i}$$

$c_4(d)$ was defined in (48e).

Kite - singularity locus





$$B_3(x) = 0$$

$$\begin{aligned} B_3 = {} & x_1\, x_4 (x_1 + x_4) + x_2\, x_3 (x_2 + x_3) + x_5\, x_6 (x_5 + x_6) + \\ & + x_1\, x_2\, x_5 + x_1\, x_3\, x_6 + x_2\, x_4\, x_6 + x_3\, x_4\, x_5 + \\ & - (x_1\, x_4 (x_2 + x_3 + x_5 + x_6) + x_2\, x_3 (x_1 + x_4 + x_5 + x_6) + x_5\, x_6 (x_1 + x_2 + x_3 + x_4)) \end{aligned} \tag{3.2}$$

$$I|_{B_3} = -\frac{1}{d-4} \frac{\vec{u} \cdot \vec{J}}{\partial^5 B_3} = \ldots$$

Generalizes the massless case [ChetyrkinTkachov1981], and the more general “diamond rule” [Ruijl, Ueda and Vermaseren 2015].

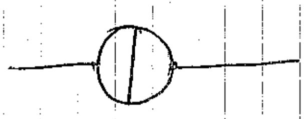
After this overview of results we proceed to a detailed blackboard presentation of the method

Outline

* Introduction and overview of results (done)
* Definitions
* Current freedom variations
* SFI group & equation system
* Solving equations
  - general solution
  - singular locus
* Conclusions

Definitions

Given a topology of a Feynman diagram (mathematically, a graph)

e.g. 

and assuming tensor reduction to a scalar integral has taken place, we denote

loop currents $l_1^\mu, l_2^\mu, \ldots, l_L^\mu$

independent external currents $p_1^\mu, \ldots p_{n-1}^\mu$ (n ext. legs)

Collective notation $(q_1, q_2, \ldots) = (l_1 \ldots l_L, p_1 \ldots p_{n-1})$

Parameter space $X = (\mu_1, \ldots \mu_P, \{p_r \cdot p_s\}_{r,s=1}^{n-1}) =$
$\equiv (x_1, x_2, \ldots \quad)$

Integral

$$I \equiv I(\{X\}; d) := \int \frac{d^d l_1 \ldots d^d l_L}{\prod_{i=1}^{P} (k_i^2 - \mu_i)}$$

where $\mu \equiv m^2$ squared mass

$k_i = A_i^{\ r} q_r$ propagator currents, $i = 1, \ldots, P$.
lin. combination of indep. currents.

Discussion

- I has no numerators in this definition. Extension to numerators is in progress.
- No need for indices/powers a: $\frac{1}{(k^2-\mu)^a}$ - can be obtained by $\partial/\partial\mu$.

Differently put $\frac{1}{p^2-\mu-\Delta\mu} = \frac{1}{p^2-\mu} + \frac{\Delta\mu}{(p^2-\mu)^2} + \frac{(\Delta\mu)^2}{(p^2-\mu)^3} + \ldots$

hence $\Delta\mu$ can be interpreted as formal parameter of

a generating function.
This is closely related to the Mellin transform
which was mentioned in some previous talks.

A common criticism:
Assigning a free mass to all propagators is
unuseful and is contained in current approach.

Replies

- Since μ and a are related through a transform
considering general μ is as useful as considering general a.

- Considering μ has the advantage of unifying:

instead of IBP recursion relations + DE diff. eq.
all equations are differential.
And there are additional advantages including
coupling of equations through commutation relations.

Current freedom variations

* Consider an IBP operator $\partial_\ell q$.
It generates a diff. eq. as follows

$$0 = \int \partial_\ell \, q \underset{\substack{\uparrow \\ \text{integrand}}}{\tilde{I}} \, d\ell = \underset{\substack{\uparrow \\ \text{from } \partial_\ell q}}{c} I + \sum_j \frac{\partial k_j}{\partial \ell} \cdot 2 k_j \cdot q \frac{\partial}{\partial k_j^2} \tilde{I}$$

Decompose

$$2 k_j \cdot q = T^i{}_j \, k_i^2 + T^{rs}{}_j \, p_r \cdot p_s + R^t{}_j \, n_t$$ irreducible numerators

we can substitute back and use

$$k_i^2 = k_i^2 - \mu_i + \mu_i = \mathcal{O}_i^2 + \mu_i$$ Omit propagator i operator

to obtain

$$0 = c I - T^i{}_j \, x_i \frac{\partial}{\partial x_j} I - R^t{}_j \frac{\partial}{\partial x_j} \int n_t \tilde{I} \, d\ell$$

This is a diff. eq. for I, coupled to numerator integrals.

* DE operators $p \frac{\partial}{\partial p}$ generate diff. eq. by operating either on the integral or on the integrand:

$$p \frac{\partial}{\partial p} I = \int p \frac{\partial}{\partial p} \tilde{I} \, d\ell = \text{(as before)}$$

* Equivalently these operators can be described by variations   operators $\to$ variations

$$\begin{cases} \delta \ell = \ell + p \\ \delta p = p \end{cases}$$

These variations belong to the triangular Lie algebra

$$T_{L,n-1} = \begin{array}{cc} \overset{L}{\longleftrightarrow} & \overset{n-1}{\longleftrightarrow} \\ \end{array}\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$$

known as <u>Lee Lie algebra</u>

<u>SFI group</u>

In order to obtain diff. eq. for I (decoupled from num. integ.) we define

$$\boxed{G \subset T_{L,n-1} \quad \text{preserves} \quad Sp\{k_i^2, p_r \cdot p_s\}}$$

namely $\forall i \quad \delta k_j^2 = 2k_j \delta k_j = \sum_i T^i{}_j k_i^2 + \sum_{rs} T^{rs}{}_j p_r \cdot p_s$

G is defined naturally by the graph.

<u>SFI eq. system</u>

Each generator in $T^a \in G$ generates an equation

$$0 = c^a I - (T^a)^i{}_j x_i \partial^j I + J^a \qquad a = 1, \ldots, \dim G$$

where all x dependence is shown, and J^a stands for simpler diagrams — those with an omitted propagator.

<u>Representation of G on X</u>

$(T^a)^i{}_j$ define a rep. of G on param space X.

This is the special property of G.

The following pages were not included in the talk for lack of time.

Solving the equation system

Comment: The SFI equation system is a

1^{st} order linear PDE.

It is special: it is the first time that I encountered such a system.

Homogeneous solution

Comment: The maximal $\overbrace{\text{cut}}^{\text{of } I}$ is known to be a homog. soln.

G may have invariants $P_3(x), P_4(x), \ldots$

the number of an indep. set of inv. = codim (G orbit)

Consider the constant free subgroup $G_{c.f} \subset G$

defined by linear comb. of equations such that $c^2 = 0$.

Since c is indep of x and of degree 1 in d, the space time dimension (e.g. $c = d-3$)

$G_{c.f}$ is codim 2 or codim 1 in G (generically 2).

Accordingly $G_{c.f}$ has 1 or 2 additional invariants

$$P_1(x), \; P_2(x)$$

Typically

$$P_1 = \det(p_r \cdot p_s) \qquad \text{Gram determinant}$$
$$P_2 = \det(q_r \cdot q_s) \qquad \text{Cayley} \quad ''$$

the homogeneous solution:

$$I_0 = I_0(P_1, P_2) \cdot f(P_3, P_4 \dots)$$

$\longrightarrow$

$$I_0(P_1, P_2) = P_1^{a_1 d + b_1} \cdot P_2^{a_2 d + b_2}$$

typic.

and $f(P_3, P_4, \dots)$ is an arbitrary function.

<u>How to find invariants?</u>

Define $(Tx)^a{}_j = (T^a)^i{}_j \, x_i$

Compute maximal minors $M_A{}^I(x)$

they decompose into

$A = (a_1, a_2, \dots)$
$I = (i_1, i_2, \dots)$
multi indices.

$$M_A{}^I = S(x) \; \mathrm{Inv}(x)^I \; \mathrm{Stb}(x)_A$$

A form in X related to <u>invariants</u>

An antisymm. multi-vector over the algebra, related to the stabilizer of G at x.

General solution

Define $\hat{I}(x) = I(x)/I_0(x)$ "leading singularity normalization".

The general reduction reads

$$\hat{I}(x) = \hat{I}(x_0) + \int_x^{x_0} S^\alpha(\xi)\, d\xi_\alpha$$

depends on simpler diagrams.

conveniently chosen base point on same G-orbit as x.

e.g. set mass to zero $m^2=0$ or identify masses $m_1^2 = m_2^2$.

Any path connecting x & x_0 within the G-orbit can be chosen.

A single 1st order linear PDE defines characteristic curves.

The SFI equation system defines characteristic manifolds, which are nothing but the G-orbits.

The general reduction is derived from the eq. system through "variation of constants".

Singular locus

For some $x \in X$ $\exists k_a(x)$ s.t.

$$k^a(x)\, T^{ai}{}_j\, x_i = 0$$

k^a is called stabilizer.

Multiplying the SFI eq. sys. by k_a we get

$$k_a c^a I = -k_a J^a(x) \underset{c\neq 0}{\Rightarrow} I = -\sum_a k_a J^a(x)/C$$

if $C \equiv k_a c^a \neq 0$ we obtain I as a linear combination of simpler diagrams.

The singularity locus is typically the Landau singularity locus.

To obtain singular locus & solution on it, again decompose

$$M_A{}^I = S(x)\; \mathrm{Inv}^I\; Stb_A$$

$S(x)=0$ is the locus

has information on k_a.

Conclusions

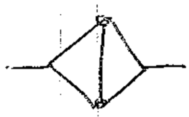
* Symmetries of Feynman Integral (SFI) - a new formulation to evaluate Feynman Integrals.

* Unifies IBP & DE - a larger system of diff. eq.

* Uncovers underlying geometry - the group G and the rep. on X

* Novel evaluations achieved

  -  Vacuum seagull w. 3 mass scales

  -  The kite - most general solution on singular locus.

  - Continued w. Mazumdar's talk.