Feynman integral relations from parametric annihilators

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joint work with Thomas Bitoun, Christian Bogner, René Pascal Klausen
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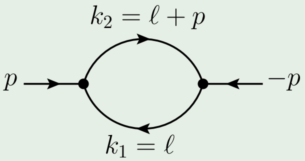
An integral family is defined by a set of denominators $\mathsf{D}_1, \ldots, \mathsf{D}_N$ that are quadratic (or linear) forms in loop momenta $\ell_1, \ldots, \ell_L$:

$$\mathcal{I}(a_1, \ldots, a_N; d) = \left(\prod_{k=1}^{L} \int \frac{\mathrm{d}^d \ell_k}{\mathrm{i}\pi^{d/2}} \right) \frac{1}{\mathsf{D}_1^{a_1} \cdots \mathsf{D}_N^{a_N}}$$

Example

$$\mathcal{I}(a_1, a_2; d) = \int \frac{\mathrm{d}^d \ell}{\mathrm{i}\pi^{d/2}} \frac{1}{(\ell^2)^{a_1}((\ell + p)^2)^{a_2}}$$



A family is also described by a matrix Λ, vectors Q_i and a scalar J such that

$$\sum_{k=1}^{N} x_k \mathsf{D}_k = -\sum_{i,j=1}^{L} \Lambda_{ij}(\ell_i \cdot \ell_j) + \sum_{i=1}^{L} 2(Q_i \cdot \ell_i) + J$$

Associated polynomials: $\mathcal{U} := \det \Lambda$, $\mathcal{F} := \mathcal{U} \left(Q^{\intercal} \Lambda^{-1} Q + J \right)$

In terms of $\omega := a_1 + \ldots + a_N - L\frac{d}{2}$ and $\mathcal{G} := \mathcal{U} + \mathcal{F}$ (Lee-Pomeransky),

$$\mathcal{I}(a) = \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d}{2} - \omega)} \left(\prod_{k=1}^{N} \int_0^\infty \frac{x_k^{a_k - 1} \mathrm{d}x_k}{\Gamma(a_k)} \right) \mathcal{G}^{-d/2}$$

Example

$$\mathcal{I}(a_1, a_2) = \frac{\Gamma(\frac{d}{2})}{\Gamma(d - a_1 - a_2)} \int_0^\infty \frac{x_1^{a_1 - 1} \mathrm{d}x_1}{\Gamma(a_1)} \int_0^\infty \frac{x_2^{a_2 - 1} \mathrm{d}x_2}{\Gamma(a_2)} \Big(\underbrace{x_1 + x_2}_{\mathcal{U}} \underbrace{- p^2 x_1 x_2}_{\mathcal{F}} \Big)^{-\frac{d}{2}}$$

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The (twisted) Mellin transform of a function $f \colon \mathbb{R}_+^N \longrightarrow \mathbb{C}$ is

$$\mathcal{M}\{f\}(a) := \left(\prod_{k=1}^{N} \int_0^\infty \frac{x_k^{a_k - 1} \mathrm{d}x_k}{\Gamma(a_i)} \right) f(x_1, \ldots, x_N),$$

whenever this integral exists. The Feynman integral is a special case:

$$\mathcal{I}(a) = \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d}{2} - \omega)} \widetilde{\mathcal{I}}(a) \quad \text{for} \quad \widetilde{\mathcal{I}}(a) = \mathcal{M}\Big\{ \mathcal{G}^{-d/2} \Big\}(a).$$

- Let $(q_1, \ldots, q_M) = (\ell_1, \ldots, \ell_L, p_1, \ldots, p_E)$ denote all loop- and external momenta ($M = L + E$).
- Assume that $\mathrm{D}_1, \ldots, \mathrm{D}_N$ are a basis of loop-momentum dependent quadratic forms. ("*full set of ISPs*")

Then the Gram determinant

$$\det \begin{pmatrix} q_1 \cdot q_1 & \cdots & q_1 \cdot q_M \\ \vdots & \ddots & \vdots \\ q_M \cdot q_1 & \cdots & q_M \cdot q_M \end{pmatrix} = \mathcal{P}(\mathrm{D}_1, \ldots, \mathrm{D}_N)$$

defines the Baikov polynomial $\mathcal{P}(y) \in \mathbb{C}[y_1, \ldots, y_N]$.

Theorem (Baikov representation)

$$\mathcal{I}(d) \propto \left(\prod_{k=1}^{N} \int \frac{\mathrm{d}y_k}{y_k^{a_k}} \right) \cdot \{\mathcal{P}(y)\}^{(d-N-1)/2}$$

- Again, this is a Mellin transform, this time of $\mathcal{P}^{(d-N-1)/2}$.
- Can also apply cuts, i.e. restrict $\mathcal{P}$ to $y_k = 0$ for any subset of k's.

Facts (Bernstein, Speer, ...)

- Such integrals converge in a non-empty, open domain wrt (d, a).
- They have a unique, meromorphic extension to $\mathbb{C}^{1+N}$.
- The poles are simple and located on integral hyperplanes.

$\Rightarrow$ To prove relations between regularized Feynman integrals, we may assume convergent values of the parameters.

Example

$$\mathcal{I}(a_1, a_2) = (-p^2)^{d/2-a_1-a_2} \frac{\Gamma(d/2-a_1)\Gamma(d/2-a_2)\Gamma(a_1+a_2-d/2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(d-a_1-a_2)}$$

Poles: $\{d/2-a_1=k\} \cup \{d/2-a_2=k\} \cup \{a_1+a_2-d/2=k\}$; $k \in \mathbb{Z}_{\leq 0}$

(If d is the only regulator ($a \in \mathbb{Z}^N$), poles coalesce and cease to be simple.)

Properties of the Mellin transform

1. $\mathcal{M}\{\alpha f + \beta g\}(a) = \alpha \mathcal{M}\{f\}(a) + \beta \mathcal{M}\{g\}(a) \qquad (\alpha, \beta \in \mathbb{C})$
2. $\mathcal{M}\{x_i f\}(a) = a_i \mathcal{M}\{f\}(a + \mathrm{e}_i)$

$$\int_0^\infty \frac{x_i^{a_i-1}\mathrm{d}x_i}{\Gamma(a_i)}(x_i f) = \int_0^\infty \frac{a_i x_i^{a_i}\mathrm{d}x_i}{a_i\Gamma(a_i)} f = \int_0^\infty \frac{a_i x_i^{(a_i+1)-1}\mathrm{d}x_i}{\Gamma(a_i+1)} f$$

3. $\mathcal{M}\{-\partial_i f\}(a) = \mathcal{M}\{f\}(a - \mathrm{e}_i)$

$$\int_0^\infty \frac{x_i^{a_i-1}\mathrm{d}x_i}{\Gamma(a_i)}(-\partial_i f) = -\left[\frac{x_i^{a_i-1}}{\Gamma(a_i)} f\right]_{x_i=0}^\infty + \int_0^\infty \frac{x_i^{a_i-2}\mathrm{d}x_i}{\Gamma(a_i-1)} f$$

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Shift operators:

$$(\mathbf{i}^- F)(a) := F(a - \mathrm{e}_i) \qquad (\mathbf{n}_i F)(a) = a_i F(a) \quad \text{for}$$

$$(\hat{\mathbf{i}}^+ F)(a) := a_i F(a + \mathrm{e}_i) \qquad \mathbf{n}_i := \hat{\mathbf{i}}^+ \mathbf{i}^-$$

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Given any differential operator $P \in A^N[d]$ in the Weyl algebra

$$A^N[d] := \mathbb{C}[d] \langle x_1, \ldots, x_N, \partial_1, \ldots, \partial_N \mid \ [\partial_i, x_j] = \delta_{i,j} \rangle$$

such that $P \bullet \mathcal{G}^{-d/2} = 0$ (annihilator), the substitutions

$$x_i \mapsto \hat{\mathbf{i}}^+, \quad \partial_i \mapsto -\mathbf{i}^-, \quad x_i \partial_i \mapsto -\mathbf{n}_i$$

define a shift operator $\mathcal{M}\{P\} \in S^N[d]$ in the shift algebra

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Example ($\mathcal{G} = x_1 + x_2 - p^2 x_1 x_2$)

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2. $(-p^2)(-d/2 + \mathbf{n}_1 + 1)\hat{\mathbf{1}}^+ \widetilde{\mathcal{I}} = -(-d/2 + \mathbf{n}_1 + \mathbf{n}_2)\widetilde{\mathcal{I}}$

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2. $(-p^2)(-d/2 + \mathbf{n}_1 + 1)\hat{\mathbf{1}}^+ \widetilde{\mathcal{I}} = -(-d/2 + \mathbf{n}_1 + \mathbf{n}_2)\widetilde{\mathcal{I}}$
3. $(-p^2) a_1 \widetilde{\mathcal{I}}(a_1 + 1, a_2) = -\dfrac{-d/2 + a_1 + a_2}{-d/2 + a_1 + 1} \widetilde{\mathcal{I}}(a_1, a_2)$

$$\left\{P \in A^N[d] \colon \; P \bullet \mathcal{G}^{-d/2} = 0\right\} \quad \xrightarrow{\mathcal{M}\{\cdot\}} \quad \left\{\tilde{P} \in S^N[d] \colon \; \tilde{P} \bullet \widetilde{\mathcal{I}} = 0\right\}$$

(annihilator) (shift relation)

$$\left\{P \in A^N[d] \colon \; P \bullet \mathcal{G}^{-d/2} = 0\right\} \quad \xleftarrow{\mathcal{M}^{-1}\{\cdot\}} \quad \left\{\tilde{P} \in S^N[d] \colon \; \tilde{P} \bullet \widetilde{\mathcal{I}} = 0\right\}$$

The inverse Mellin transform of $f^{\star}(a) := \mathcal{M}\{f\}(a)$ is

$$f(x) = \mathcal{M}^{-1}\{f^{\star}\}(x) = \left(\prod_{k=1}^{N} \int_{\sigma_k + \mathrm{i}\mathbb{R}} \frac{\Gamma(a_k)\,\mathrm{d}a_k}{(2\pi\mathrm{i})\, x_k^{a_k}} \right) f^{\star}(a).$$

Therefore, **every** shift relation comes from an annihilator.

Open problems

The annihilator ideal is always finitely generated, in simple cases accessible by Gröbner based computer algebra, e.g. SINGULAR.

Question 1

Can the structure of graph- or Baikov polynomials be exploited to gain annihilators more efficiently?

For a full set of ISPs, the action of $\frac{\partial}{\partial q_i} \cdot q_j$ on the *momentum space* integrand leads to IBP relations that map to linear annihilators $\widetilde{\mathbf{O}}^i_j$.

Question 2

Is the annihilator of $\mathcal{G}^{-d/2}$ linearly generated? What about $\mathcal{P}^{(d-N-1)/2}$?

$\Rightarrow$ if yes, (commutative) Syzygies are enough!

Question 3

Do the momentum space IBP's generate all annihilators (shift relations)?

We define the number of master integrals of $\widetilde{\mathcal{I}}(a) = \mathcal{M}\left\{\mathcal{G}^{-d/2}\right\}(a)$ as

$$\mathfrak{C}\left(\mathcal{G}\right) := \dim_{\mathbb{C}(d,a)}\left(\sum_{n\in\mathbb{Z}^N} \mathbb{C}(d,a)\cdot\widetilde{\mathcal{I}}(a+n)\right)$$

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- no symmetries
- not “modulo subtopologies”
- **computable exactly and efficiently**

Using the Mellin transform, $\theta_i := x_i \partial_i = -\mathcal{M}^{-1}\{\mathbf{n}_i\}$,

$$\mathfrak{C}(\mathcal{G}) = \dim_{\mathbb{C}(d,\theta)} \Big(\underbrace{\mathbb{C}(d,\theta) \otimes_{\mathbb{C}[d,\theta]} A^N[d]\mathcal{G}^{-d/2}}_{\mathfrak{M}} \Big)$$

Here, $A^N(d)\mathcal{G}^{-d/2}$ is a holonomic D-module, and $\mathfrak{M}$ is a holonomic system of finite difference equations [Loeser & Sabbah '91].

Theorem

$$(-1)^N \mathfrak{C}(\mathcal{G}) = \chi\left(\mathbb{C}^N \setminus \{x_1 \cdots x_N \mathcal{G} = 0\}\right) = \chi\left((\mathbb{C}^*)^N \setminus \{\mathcal{G} = 0\}\right)$$

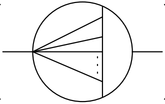
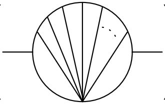
$\Rightarrow$ implies finiteness [Smirnov & Petukhov]

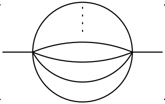
The Euler characteristic $\chi(X) = \sum_i (-1)^i \dim H^i(X)$ is a fundamental invariant and can be computed with many different tools, for example:

- $\chi(X) = \chi(X \setminus Z) + \chi(Z)$
- $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$
- $\chi(E) = \chi(B) \cdot \chi(F)$ for fibrations $F \to E \to B$
- D-modules and Groebner bases (e.g. SINGULAR) [Oaku & Takayama]
- algorithms by M. Helmer (`CharacteristicClasses` in *Macaulay2*)
- Kouchnirenko/Khovanskii's theorem: For *non-degenerate* $\mathcal{G}$,

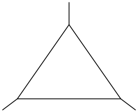
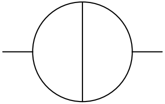
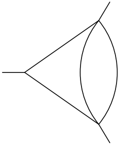
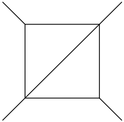
$$\mathfrak{C}(\mathcal{G}) = N! \cdot \operatorname{Vol} \operatorname{NP}(\mathcal{G})$$

For some infinite families one can prove explicit formulas:

$\mathfrak{C}\Bigg($  $\Bigg) = \mathfrak{C}\Bigg($  $\Bigg) = \dfrac{L(L+1)}{2}$

$\mathfrak{C}\Bigg($  $\Bigg) = 2^{L+1} - 1$ [Kalmykov & Kniehl]

Plenty of further computations agreed with predictions by AZURITE, e.g.

Graph G				
$\mathfrak{C}(G)$ massless	4	3	4	20
$\mathfrak{C}(G)$ massive	7	30	19	55

Massive one-loop sunrise

$$\mathcal{U} = x_1 + x_2 \qquad \mathcal{F} = (x_1 + x_2)^2 + x_1 x_2$$

In *Macaulay2*, the Euler characteristic $\mathfrak{C}(\mathcal{G}) = 3$ can be computed with

```
load "CharacteristicClasses.m2"
R=QQ[x0,x1,x2]
I=ideal(x0*x1*x2*((x1+x2)*x0+(x1+x2)^2+x1*x2))
Euler(I)
```

The individual cohomology groups can also be obtained with

```
load "Dmodules.m2"
R=QQ[x1,x2]
f=x1*x2*(x1+x2+(x1+x2)^2+x1*x2)
deRham f
```

$\Rightarrow H^0(X) \cong \mathbb{Q}, \quad H^1(X) \cong \mathbb{Q}^3, \quad H^2(X) \cong \mathbb{Q}^5 \quad \Rightarrow \chi(X) = 1-3+5 = 3$

The same can be done in SINGULAR.

IBP certificates and relations by Ansatz

Suppose we are given $n = \pm\chi(X)$ master integrands in a monomial basis:

$$\vec{B}^{\intercal} = (B_1, \ldots, B_n) \quad \text{where} \quad B_i = \mathcal{M}\left\{x^{b_i}\right\} \quad \text{for} \quad b_i \in \mathbb{N}_0^N.$$

To reduce an integral $\mathcal{M}\left\{x^{b_0}\right\}$, we are looking for a non-zero solution of

$$\left(\sum_{i=0}^{n} P_i(s,\theta) x^{b_i}\right) \cdot \mathcal{G}^s = 0, \quad \text{where} \quad P_0, \ldots, P_n \in \mathbb{C}[s,\theta].$$

Put differently, we want to compute the intersection

$$\mathrm{Ann}_{A^N(s)}\left(\mathcal{G}^s\right) \cap \left(\sum_{i=0}^{n} \mathbb{C}(s)[\theta] \cdot x^{b_i}\right).$$

In practice, an Ansatz for the P_i turns this into a sparse linear system!
$\Rightarrow$ might be an efficient way to compute individual relations

Thoughts on integral reduction

In a basis $\vec{B}^\intercal = (\mathcal{M}\{x^{b_1}\}, \ldots, \mathcal{M}\{x^{b_n}\})$, the shift operators are matrices

$$\hat{\mathbf{i}}^+ \vec{B} = \mathbf{M}_i^+ \vec{B} \quad \text{where} \quad \mathbf{M}_i^+(\vec{a}) \in \mathrm{GL}_n(\mathbb{C}(d, a_1, \ldots, a_N)).$$

Once these matrices are computed, and $B_1 = \mathcal{M}\{1\}$, an arbitrary integral

$$\mathcal{M}\{x^a\} \propto \vec{e}_1^\intercal \cdot \left(\prod_{k=1}^{N} (\mathbf{M}_k^+)^{a_k} \right) \cdot \vec{B}$$

can be reduced simply by multiplying and shifting matrices using

$$\hat{\mathbf{i}}^+ \mathbf{M}_j^+(a) = \mathbf{M}_j^+(a + \vec{e}_i)\hat{\mathbf{i}}^+$$

- different from Laporta and Mincer/Bicer/Forcer/LiteRed
- explicit examples (in simple cases though) [Tarasov,Panzer]
- potentially efficient (parallelizable)
- works for arbitrary a and d, i.e. not restricted to integer a
- compute $\mathbf{M}_k^\pm$ once and for all, then reduction of anything is in reach

- The Mellin transform translates IBP relations to annihilators [Tkachov, Baikov, Lee, Pomeransky].
- Algorithms for computations with D-modules are available.
- Application: The number of master integrals, for free a's, is

$$\mathfrak{C}(\mathcal{G}) = (-1)^N \chi((\mathbb{C}^*)^N \setminus \{\mathcal{G} = 0\}) < \infty$$

- Goal: Extend IBP reduction from $a \in \mathbb{Z}^N$ to free a.

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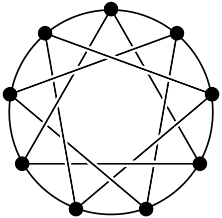


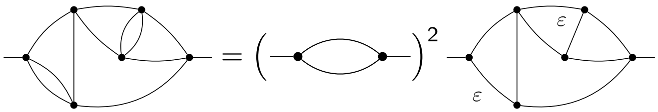
Parametric representations

$$\omega := a_1 + \cdots + a_N - L\frac{d}{2}$$

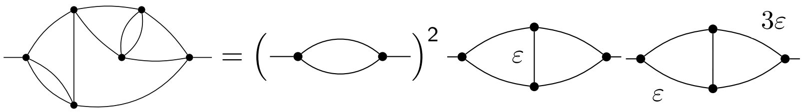
$$\mathcal{I}(a_1, \ldots, a_N) = \left(\prod_{i=1}^{N} \int_0^\infty \frac{x_i^{a_i - 1} \mathrm{d}x_i}{\Gamma(a_i)} \right) \frac{e^{-\mathcal{F}/\mathcal{U}}}{\mathcal{U}^{d/2}},$$

$$\mathcal{I}(a_1, \ldots, a_N) = \Gamma(\omega) \left(\prod_{i=1}^{N} \int_0^\infty \frac{x_i^{a_i - 1} \mathrm{d}x_i}{\Gamma(a_i)} \right) \frac{\delta\left(1 - \sum_{j=1}^{N} x_j\right)}{\mathcal{U}^{d/2-\omega} \mathcal{F}^{\omega}}$$

$$\mathcal{I}(a_1, \ldots, a_N) = \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2} - \omega\right)} \left(\prod_{i=1}^{N} \int_0^\infty \frac{x_i^{a_i - 1} \mathrm{d}x_i}{\Gamma(a_i)} \right) \mathcal{G}^{-d/2}.$$



2
ε
ε



=
2
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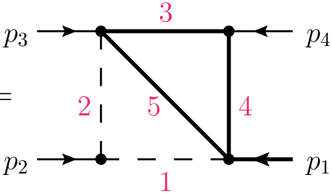
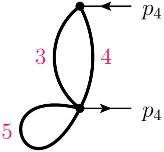
3ε

Issues with top-level integrals vs. subtopology

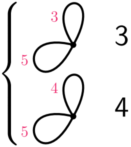
$G =$



$G/\{1,2\} =$



$\mapsto$



$G' =$