Integrand reduction for particles with spins

1710.10208, 1802.06761 and 1904.XXXX
with R. Boels and Q. Jin

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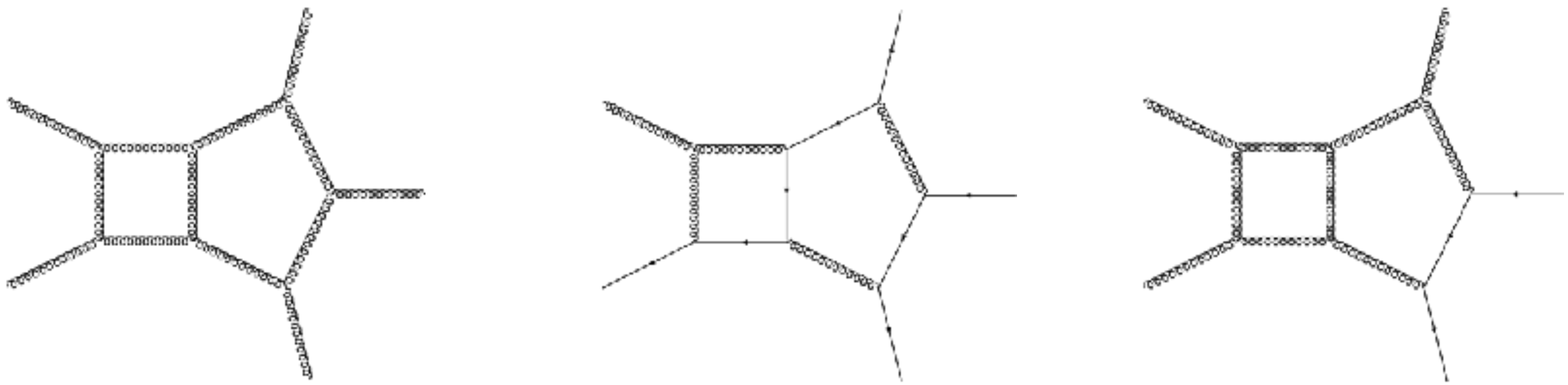
Objects to be attacked

High multiplicity multi-loop amplitudes (CP even), eg.



[Badger et al., 13’, 15’, 16’, 17’, 18’;   Abreu et al., 17’*2, 18’ *2, 19’ ;
Gehrmann et al., 15’, 18’; Dunbar et al. 16; Papadopoulos et al. 15’;
Boels et al., 18’;   Chawdhry et al., 18’]

[Vogt et al., 04’; Gehrmann et al., 10’; Almeid et al., 15’; Henn et al., 13’, 16’;]

Physical Properties of Amplitudes

Color stripped scattering amplitude

$$\mathcal{A}_n(\{p_i,\lambda_i\}) = \overline{v}_{\dot{a}_1}(p_{f_1})u_{a_1}(p_{f_{M+1}})\cdots\overline{v}_{\dot{a}_M}(p_{f_M})u_{a_M}(p_{f_{2M}})\,\xi_1^{\mu_1}\cdots\xi_N^{\mu_N}\,\hat{A}(\{\eta_{\mu\nu},\gamma_\rho,p_{k,\mu}\})$$

- A Lorentz scalar and multilinear of spin variables, satisfying
  - Momentum conservation $\sum_i p_i^\mu = 0$
  - Dirac eq.

    $$(\not{p}-m)u(p) = \bar{u}(p)(\not{p}-m) = (\not{k}+m)v(k) = \bar{v}(k)(\not{k}+m) = 0$$
  - Transversality of polarization vector $p_i^\mu \xi_{i,\mu} = 0$
  - On-shell gauge invariance $\mathcal{A}_n(\xi_i \to p_i) = 0$
  - Unitarity: branch cuts and poles

Amplitude Decomposition

Color stripped amplitude decomposes into

$$\mathcal{A}_n = \sum \alpha_i(\{p, l\}) B_i(\{p, \lambda\})$$

- A linear combination of a group of kinematic basis B_i
- Kinematic basis: external particle informations, multilinear of spins satisfying all physical properties except unitarity, but with locality

$$B_i(\{p, \lambda\}) = \overline{v}_{\dot{a}_1}(p_{f_1}) u_{a_1}(p_{f_{M+1}}) \cdots \overline{v}_{\dot{a}_M}(p_{f_M}) u_{a_M}(p_{f_{2M}})\, \xi_1^{\mu_1} \cdots \xi_N^{\mu_N} f_B(\{\eta_{\mu\nu}, \gamma_\rho, p_k\})$$

- Coefficients of kinematic basis:

$$\alpha_i(\{p, l\}) = \sum f_\alpha(\{p_j \cdot p_k\}) \int (d^D l)^L\, I(\{l \cdot l, p \cdot l\})$$

[Glover et al. , 03', 04', 12'; Boels et al., 16'; Arkarni-Hamed et al., 16'; Bern et al., 17']

Kinematic Basis Construction

Brute-force construction by solving physical constraints $\mathcal{A}_n = \sum \alpha_i B_i$

[R. Boels & R. Medina, 16'; R. Boels & HL, 17']

- Application: up to 6-pt tree and 4-pt 2-loop pure-YM amplitudes

- Shortcomings: complicated for (>=) 5-pt, ie. $P_{ij} = \sum_{\text{spin}} B_i B_j$

$$\sum_s u(p)\bar{u}(p) = \not{p} + m, \qquad \sum_s v(k)\bar{v}(k) = \not{k} - m$$

$$\sum_{\text{helicities}} \xi_\mu \xi_\nu = \eta_{\mu\nu} - \left(\frac{p_\mu q_\nu + p_\nu q_\mu}{q \cdot p}\right) \qquad \sum_{\text{helicities}} \xi \cdot \xi = d - 2$$

eg. 5 gluons {142, 142}
2 fermions + 3 gluons {144, 144}
4 fermions + 1 gluon + 5 pure gammas {336, 336}
6 gluon {2364, 2364} full matrix, impossible to inverse

- This construction way is kind of arbitrary, linear combinations of bases are still on-shell gauge invariant kinematic bases

Kinematic Basis Construction for Pure-YM

[R. Boels, Q. Jin and HL,18']

"Canonical" kinematic basis construction

- A-type building block: $A_i(j,k) = (p_k \cdot p_i)\, p_j \cdot \xi_i - (p_j \cdot p_i)\, p_k \cdot \xi_i$

$$\{A_i(j) = A_i(i+j, i+j+1) | j \in \{1, \ldots, n-3\}\}$$

  - Solutions for 1 gluon (n-1) scalar scattering [R. Boels and HL,17']
  - For n-gluon scattering, n copies A form a basis

- C-type building block: $C_{i,j} = (\xi_i \cdot \xi_j)(p_i \cdot p_j) - (p_i \cdot \xi_j)(p_j \cdot \xi_i)$
  - One solution for 2-gluon (n-2)-scalar (Another from 2-copies of A-type building blocks) [R. Boels and HL,17']
  - Proportional to two contracted linearized field strength tensor

$$F_{\mu\nu}(\xi_1) F^{\mu\nu}(\xi_2)$$

A &C-type building blocks: on-shell gauge invariant

Kinematic Basis Construction for Pure-YM

[R. Boels, Q. Jin and HL,18']

"Canonical" kinematic basis construction

- D-type building block: $$D_{i,j} = C_{i,j} - \sum_{k,l=1}^{n-3} X_{ij}(k,l) A_i(k) A_j(l)$$

Require orthogonality $$\sum_{h_i} A_i(k) D_{i,j} = 0 = \sum_{h_j} A_j(k) D_{i,j}, \ \forall k$$

Fix the constructions with $$P_i^A(k,l) = \sum_{h_i} A_i(k) A_i(l)$$

$$A^i(k) \equiv \sum_l (P_i^A)^{-1}(k,l) A_i(l) \qquad A^i(k) A_i(l) \equiv \sum_{\text{helicities},i} A^i(k) A_i(l) = \delta(k,l)$$

$$D_{i,j} = C_{i,j} - \sum_{k,l=1}^{n-3} A_i(k) A_j(l) \left(A^m(k) A^n(l) C_{m,n} \right)$$

$$\sum_{\text{helicities}} D_{i,j} D_{i,j} = (p_i \cdot p_j)^2 (d - n + 1) \qquad \sum_{\text{helicities},i} D_{i,j} D_{i,k} = \frac{(p_i \cdot p_j)(p_i \cdot p_k)}{(p_j \cdot p_k)} D_{j,k}$$

Kinematic Basis Construction for Pure-YM

[R. Boels, Q. Jin and HL,18']

"Canonical" kinematic basis construction

- Given >=3 gluon particles in the process, kinematic basis B can be constructed from multi-copies of all possible A and C/D types

  Conjecture: linearly independent and complete in general dimensions

- The total number of basis elements with n gluons and no scalars is

$$N_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!(n-2)^{(n-2k)}}{2^k k!(n-2k)!}$$

Compare with brute-force derivation up to seven gluons

Example: 4 gluon kinematic basis

[R. Boels, Q. Jin and HL, in prep.]

"Canonical" kinematic basis: 10 in total

- Expressed in terms of A and C:

$$
\begin{aligned}
B_1 &= A_1 A_2 A_3 A_4, & B_2 &= C_{13} C_{24}, \\
B_3 &= C_{13} A_2 A_4, & B_4 &= C_{24} A_1 A_3, \\
B_5 &= C_{12} C_{34}, & B_6 &= C_{23} C_{14}, \\
B_7 &= C_{12} A_3 A_4, & B_8 &= C_{23} A_4 A_1, \\
B_9 &= C_{34} A_1 A_2, & B_{10} &= C_{41} A_2 A_3,
\end{aligned}
$$

$$
\begin{aligned}
&A_i(j,k) = (p_k \cdot p_i)\, p_j \cdot \xi_i - (p_j \cdot p_i)\, p_k \cdot \xi_i \\
&\{A_i(j) = A_i(i+j, i+j+1) | j = 1, 2\} \\
&C_{i,j} = (\xi_i \cdot \xi_j)(p_i \cdot p_j) - (p_i \cdot \xi_j)(p_j \cdot \xi_i)
\end{aligned}
$$

- Under cyclic permutation:

$$
p_i \to p_{i+1}
\qquad
\begin{aligned}
&B_1 \to B_1,\ B_2 \to B_2, \\
&B_3 \leftrightarrow B_4,\ B_5 \leftrightarrow B_6, \\
&B_7 \to B_8,\ B_8 \to B_9,\ B_9 \to B_{10},\ B_{10} \to B_7
\end{aligned}
$$

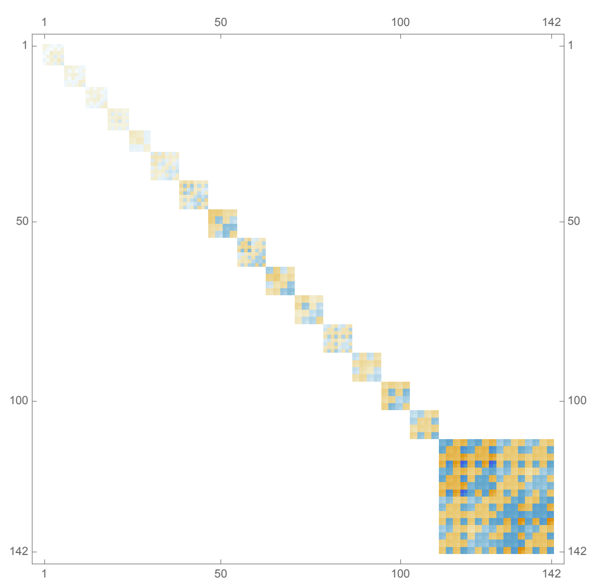
- Inner product with rank 10 in D-dim and rank 8 in 4-dim, eg.

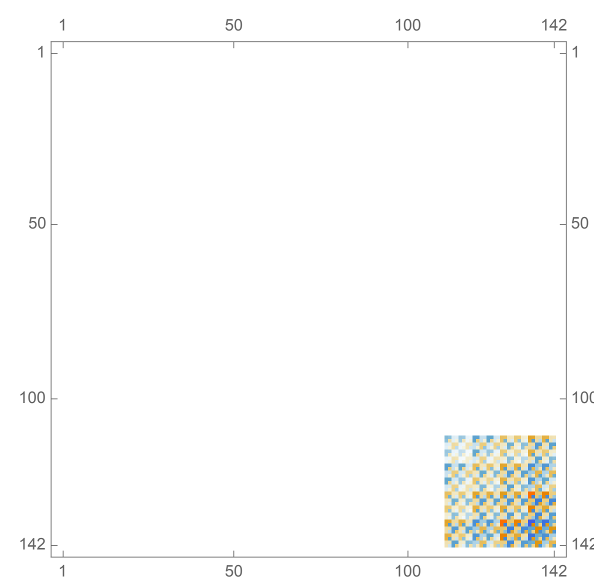
$$
\begin{pmatrix}
\frac{1}{256} s^4 t^4 u^4 & \frac{1}{64} s^2 t^2 u^4 & \frac{1}{128} s^3 t^3 u^4 & \frac{1}{128} s^3 t^3 u^4 & \frac{1}{64} s^4 t^2 u^2 & \frac{1}{64} s^2 t^4 u^2 & -\frac{1}{128} s^4 t^3 u^3 & -\frac{1}{128} s^3 t^4 u^3 & -\frac{1}{128} s^4 t^3 u^3 & -\frac{1}{128} s^3 t^4 u^3 \\
\frac{1}{64} s^2 t^2 u^4 & \frac{1}{16} (-2+D)^2 u^4 & \frac{1}{32} (-2+D) s t u^4 & \frac{1}{32} (-2+D) s t u^4 & \frac{1}{16} (-2+D) s^2 u^2 & \frac{1}{16} (-2+D) t^2 u^2 & -\frac{1}{32} s^2 t u^3 & -\frac{1}{32} s t^2 u^3 & -\frac{1}{32} s^2 t u^3 & -\frac{1}{32} s t^2 u^3 \\
\frac{1}{128} s^3 t^3 u^4 & \frac{1}{32} (-2+D) s t u^4 & \frac{1}{64} (-2+D) s^2 t^2 u^4 & \frac{1}{64} s^2 t^2 u^4 & \frac{1}{32} s^3 t u^2 & \frac{1}{32} s t^3 u^2 & -\frac{1}{64} s^3 t^2 u^3 & -\frac{1}{64} s^2 t^3 u^3 & -\frac{1}{64} s^3 t^2 u^3 & -\frac{1}{64} s^2 t^3 u^3 \\
\frac{1}{128} s^3 t^3 u^4 & \frac{1}{32} (-2+D) s t u^4 & \frac{1}{64} s^2 t^2 u^4 & \frac{1}{64} (-2+D) s^2 t^2 u^4 & \frac{1}{32} s^3 t u^2 & \frac{1}{32} s t^3 u^2 & -\frac{1}{64} s^3 t^2 u^3 & -\frac{1}{64} s^2 t^3 u^3 & -\frac{1}{64} s^3 t^2 u^3 & -\frac{1}{64} s^2 t^3 u^3 \\
\frac{1}{64} s^4 t^2 u^2 & \frac{1}{16} (-2+D) s^2 u^2 & \frac{1}{32} s^3 t u^2 & \frac{1}{32} s^3 t u^2 & \frac{1}{16} (-2+D)^2 s^4 & \frac{1}{16} (-2+D) s^2 t^2 & -\frac{1}{32} (-2+D) s^4 t u & -\frac{1}{32} s^3 t^2 u & -\frac{1}{32} (-2+D) s^4 t u & -\frac{1}{32} s^3 t^2 u \\
\frac{1}{64} s^2 t^4 u^2 & \frac{1}{16} (-2+D) t^2 u^2 & \frac{1}{32} s t^3 u^2 & \frac{1}{32} s t^3 u^2 & \frac{1}{16} (-2+D) s^2 t^2 & \frac{1}{16} (-2+D)^2 t^4 & -\frac{1}{32} s^2 t^3 u & -\frac{1}{32} (-2+D) s t^4 u & -\frac{1}{32} s^2 t^3 u & -\frac{1}{32} (-2+D) s t^4 u \\
-\frac{1}{128} s^4 t^3 u^3 & -\frac{1}{32} s^2 t u^3 & -\frac{1}{64} s^3 t^2 u^3 & -\frac{1}{64} s^3 t^2 u^3 & -\frac{1}{32} (-2+D) s^4 t u & -\frac{1}{32} s^2 t^3 u & \frac{1}{64} (-2+D) s^4 t^2 u^2 & \frac{1}{64} s^3 t^3 u^2 & \frac{1}{64} s^4 t^2 u^2 & \frac{1}{64} s^3 t^3 u^2 \\
-\frac{1}{128} s^3 t^4 u^3 & -\frac{1}{32} s t^2 u^3 & -\frac{1}{64} s^2 t^3 u^3 & -\frac{1}{64} s^2 t^3 u^3 & -\frac{1}{32} s^3 t^2 u & -\frac{1}{32} (-2+D) s t^4 u & \frac{1}{64} s^3 t^3 u^2 & \frac{1}{64} (-2+D) s^2 t^4 u^2 & \frac{1}{64} s^3 t^3 u^2 & \frac{1}{64} s^2 t^4 u^2 \\
-\frac{1}{128} s^4 t^3 u^3 & -\frac{1}{32} s^2 t u^3 & -\frac{1}{64} s^3 t^2 u^3 & -\frac{1}{64} s^3 t^2 u^3 & -\frac{1}{32} (-2+D) s^4 t u & -\frac{1}{32} s^2 t^3 u & \frac{1}{64} s^4 t^2 u^2 & \frac{1}{64} s^3 t^3 u^2 & \frac{1}{64} (-2+D) s^4 t^2 u^2 & \frac{1}{64} s^3 t^3 u^2 \\
-\frac{1}{128} s^3 t^4 u^3 & -\frac{1}{32} s t^2 u^3 & -\frac{1}{64} s^2 t^3 u^3 & -\frac{1}{64} s^2 t^3 u^3 & -\frac{1}{32} s^3 t^2 u & -\frac{1}{32} (-2+D) s t^4 u & \frac{1}{64} s^3 t^3 u^2 & \frac{1}{64} s^2 t^4 u^2 & \frac{1}{64} s^3 t^3 u^2 & \frac{1}{64} (-2+D) s^2 t^4 u^2
\end{pmatrix}
$$

Example: 5 gluon kinematic basis

[R. Boels, Q. Jin and HL,18']

- "Canonical" kinematic basis: 142 in total $D_{i,j} = C_{i,j} - \sum_{k,l=1}^{n-3} A_i(k)A_j(l)\left(A^m(k)A^n(l)C_{m,n}\right)$
  - 1 A + 2 D s: in total $5 \times 2 \times C_4^2/2! = 30$, eg. $A_1(1)D_{2,3}D_{4,5}$
  - 3 A s + 1 D: in total $2^3 \times C_5^2 = 80$, eg. $A_1(1)A_2(1)A_3(1)D_{4,5}$
  - 5 A s: in total $2^5 = 32$, eg. $A_1(1)A_2(1)A_3(1)A_4(1)A_5(1)$
- Inner product matrix





$$\sum_{h_i} A_i(k)D_{i,j} = 0 = \sum_{h_j} A_j(k)D_{i,j}, \ \forall k$$

$$\sum_{\text{helicities},i} D_{i,j}D_{i,k} = \frac{(p_i \cdot p_j)(p_i \cdot p_k)}{(p_j \cdot p_k)} D_{j,k}$$

$$\sum_{\text{helicities}} D_{i,j}D_{i,j} = (p_i \cdot p_j)^2(d - n + 1)$$

$$\{d = 4, n = 5\} \quad \Rightarrow \sum_{\text{helicities}} D_{i,j}D_{i,j} = 0$$

- Could be used for 5 gluon any loop planar/non-planar amplitude decomposition

Coefficients of Kinematic Basis

Determine coefficients of kinematic basis $\mathcal{A}_n = \sum \alpha_i(\{p,l\}) B_i(\{p,\lambda\})$

- $\alpha_i(\{p,l\}) = \sum f_\alpha(\{p_j \cdot p_k\}) \int (d^D l)^L \, I(\{l \cdot l, p \cdot l\})$

- Given any form of $\mathcal{A}_n$, eg. derived from unitarity cuts

$$\sum_{\text{spins}} B_j \mathcal{A}_n = \sum_i \alpha_i \left(\sum_{\text{spins}} B_j B_i \right) \equiv \sum_i P_{ji} \alpha_i$$

$$\sum_{\text{helicities}} \xi_\mu \xi_\nu = \eta_{\mu\nu} - \left(\frac{p_\mu q_\nu + p_\nu q_\mu}{q \cdot p} \right)$$

$$\sum_{\text{helicities}} \xi \cdot \xi = d - 2$$

$$\alpha_i = \left(P^{-1}\right)_{ij} \sum_{\text{spins}} B_j \mathcal{A}_n$$

$$\sum_s u(p)\bar{u}(p) = \not{p} + m, \qquad \sum_s v(k)\bar{v}(k) = \not{k} - m$$

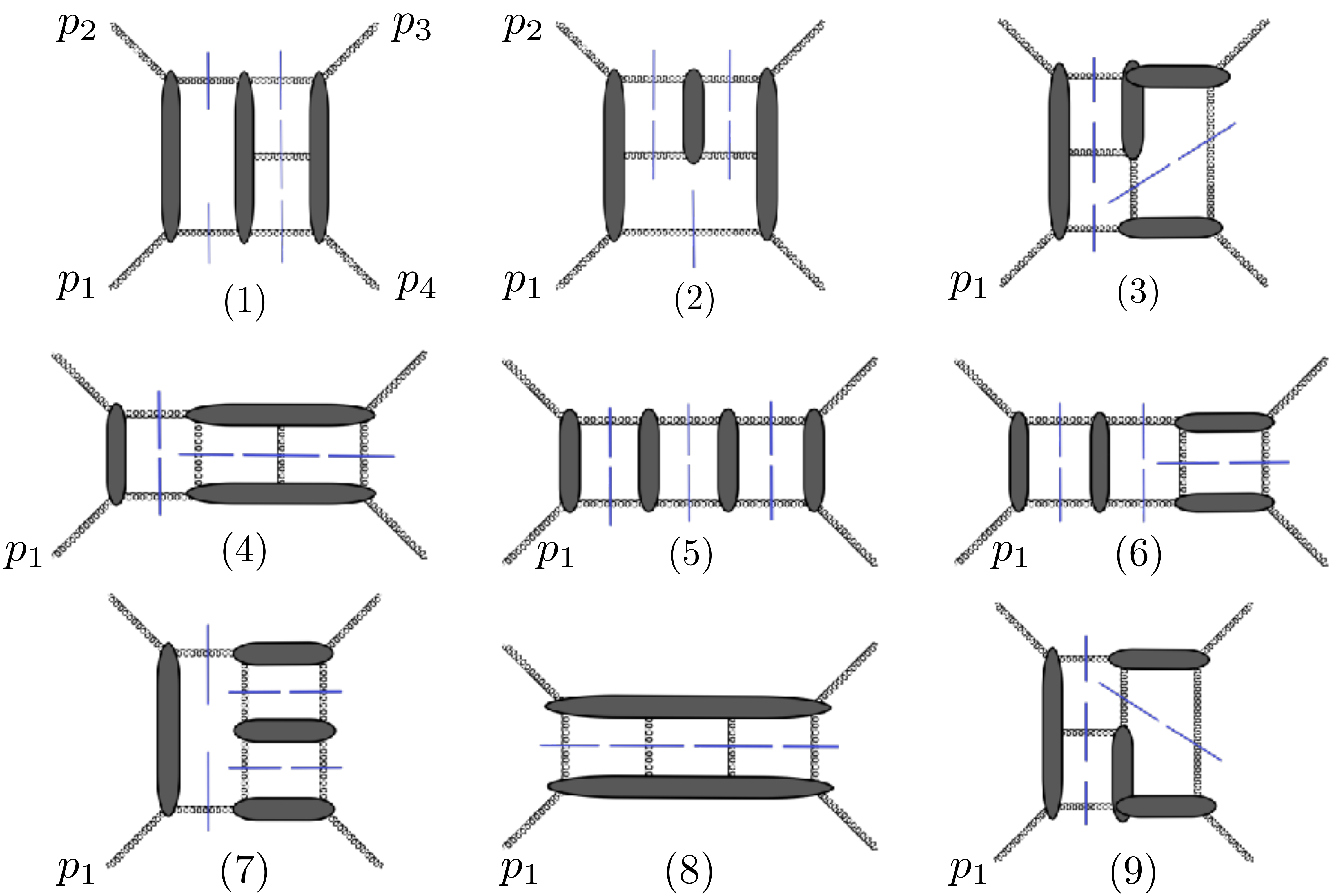
- 1) Run IBPs on the cut integrands and merge all cuts
  2) Or merge all cuts to integrands and run IBPs

$$\mathcal{A}_n = \sum_i \left(\sum_j c_{ij} \, \mathrm{MI}_j \right) B_i$$

Example: 4 gluon 3 loop planar

[R. Boels, Q. Jin and HL, in prep.]

Cuts for 4 gluon 3 loop planar



Example: 4 gluon 3 loop planar

[R. Boels, Q. Jin and HL, in prep.]

- Given a set of internal lines to be on-shell $\{L_1, \cdots, L_m\}$:

$$\sum_{\text{helicity}} \left(\lim_{L_k \to 0} L_1^2 \cdots L_m^2 A_n^L \right) B_i = \sum_{\text{helicity}} \left(\sum A_1^{\text{tree}}(\{L_{A_1}\}) A_2^{\text{tree}}(\{L_{A_2}\}) \cdots A_{m-2}^{\text{tree}}(\{L_{A_{m-2}}\}) \right) B_i$$

$$\downarrow$$

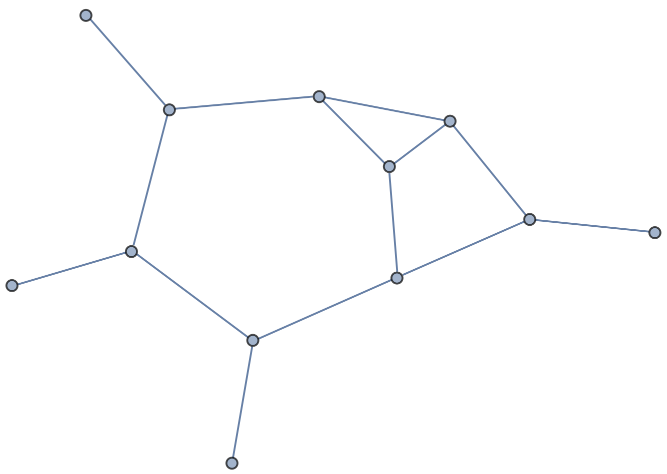
$$\{L_{A_1}\} \cup \cdots \cup \{L_{A_{m-2}}\} = \{p_1, p_2, p_3, p_4, L_1, \cdots, L_m\}$$

Project cut amplitude onto kinematic basis in CDR scheme

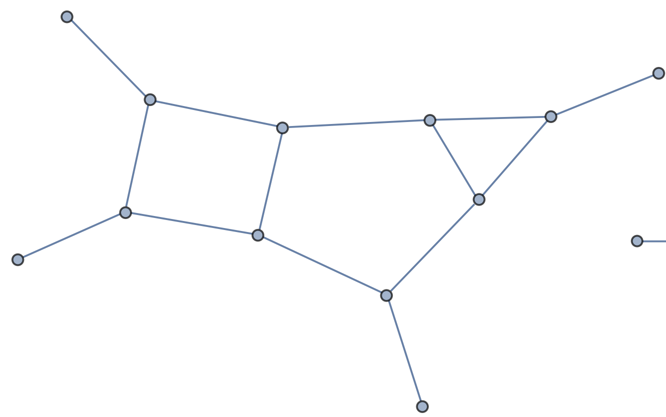
- Run over all possible cuts could be reduced by
  - Permutations among $\{l_1,\ l_2,\ l_3\}$: 3!
  - Cyclic rotations of external momenta $p_i \to p_{i+1}$: 4
  - Note double counting of integrand of symmetric topology

Example: 4 gluon 3 loop planar

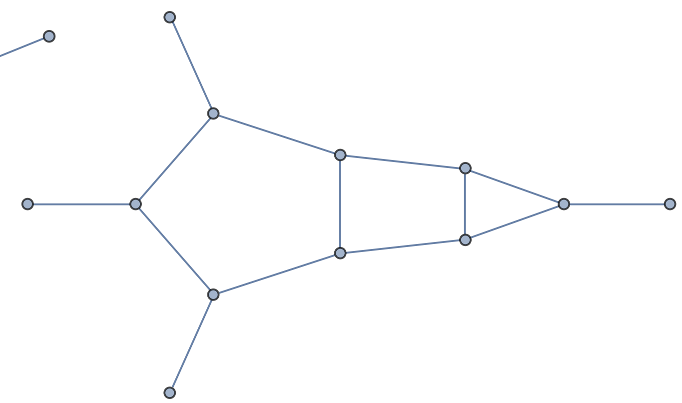
Highest integrand topologies (without bubbles)



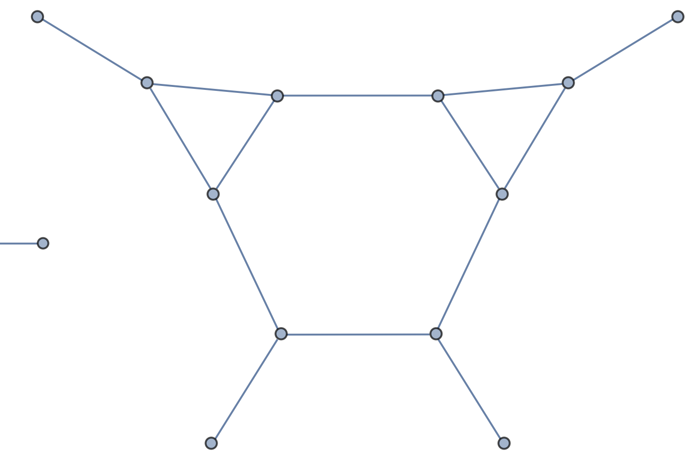
Deg. 3 -> 24 MIs



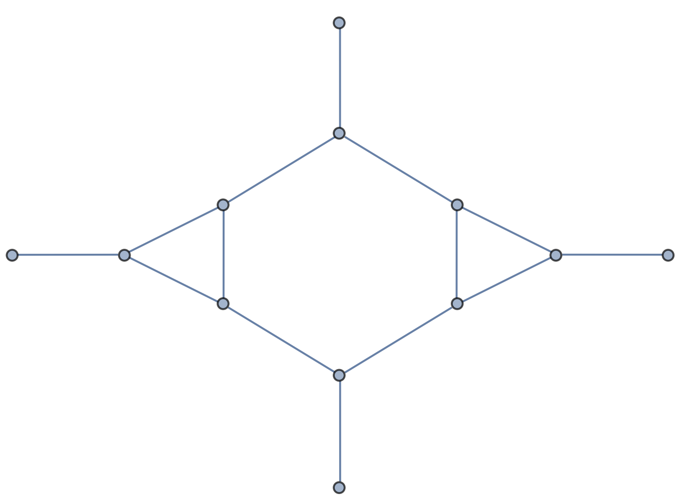
Deg. 6 -> 27 MIs



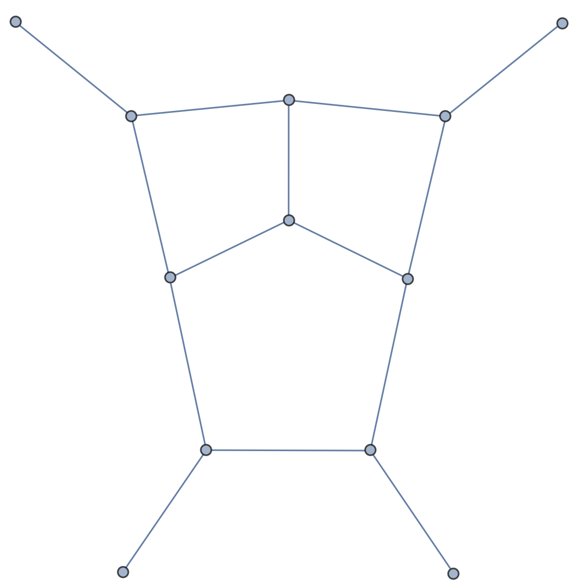
Deg. 3 -> 15 MIs



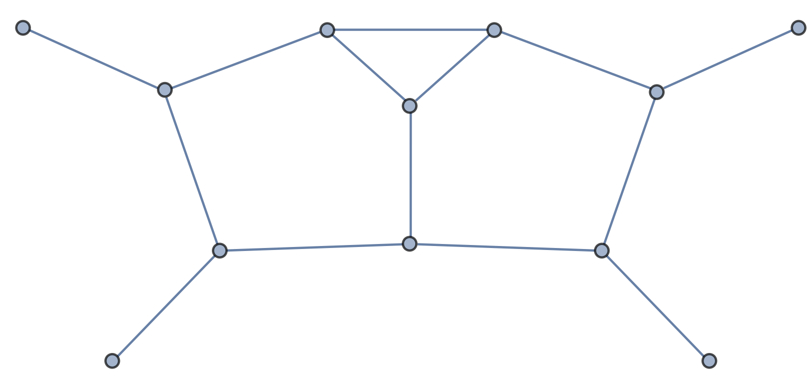
Deg. 5 -> 13 MIs



Deg. 6 -> 12 MIs

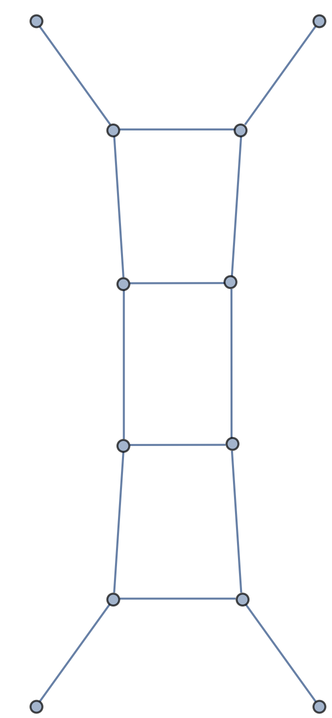


Deg. 6 -> 41 MIs



Deg. 6 -> 52 MIs

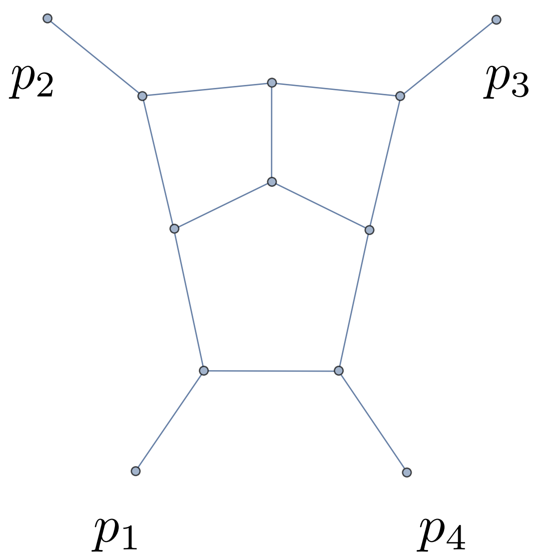
Most difficult one in IBP reduction



Deg. 6 -> 26 MIs

Example: 4 gluon 3 loop planar

81 MIs in total if including cyclic permutations



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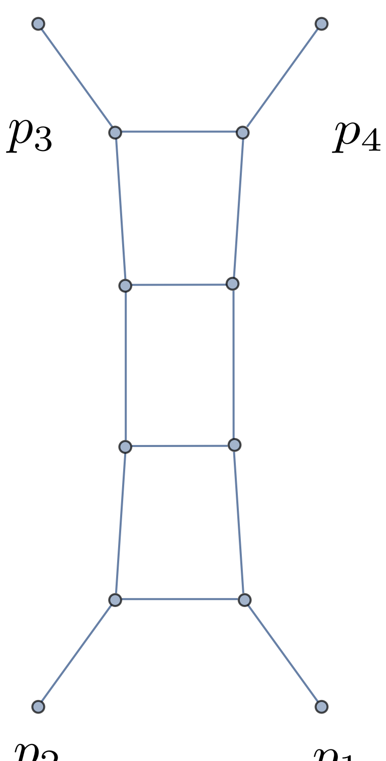
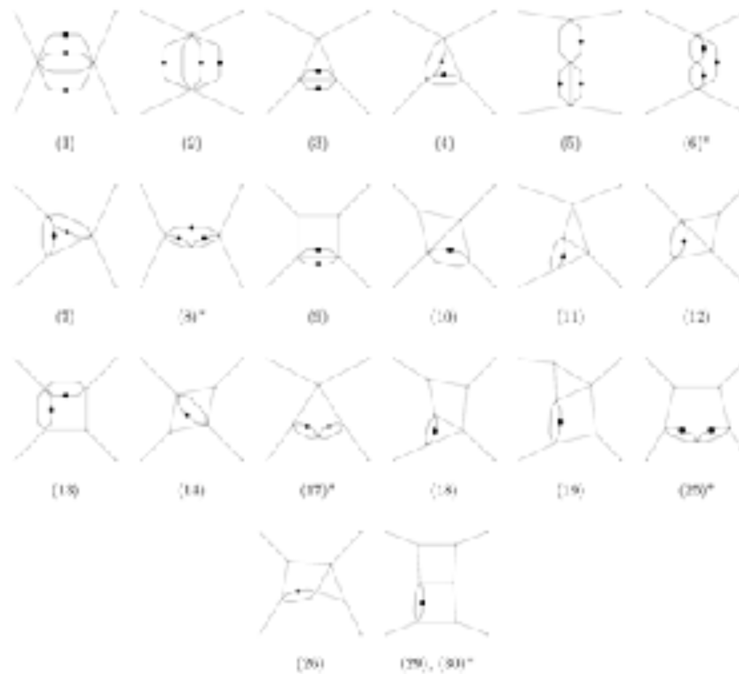
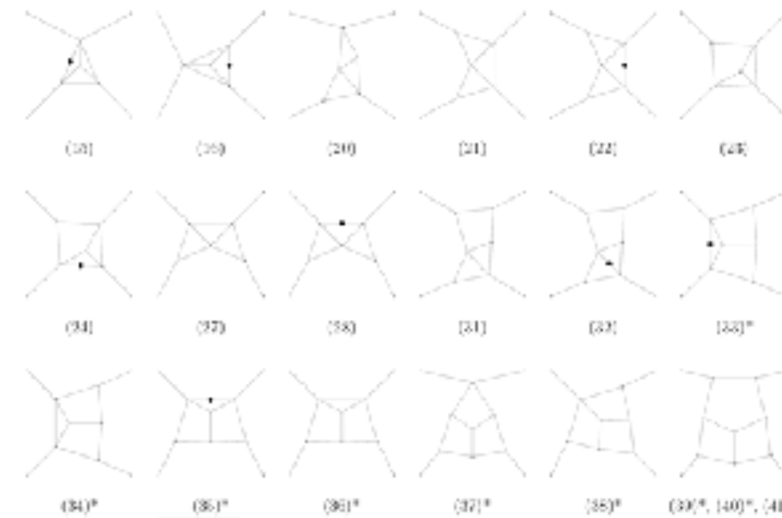




[J. Henn, A.V. Smirnov and V. A. Smirnov,13']

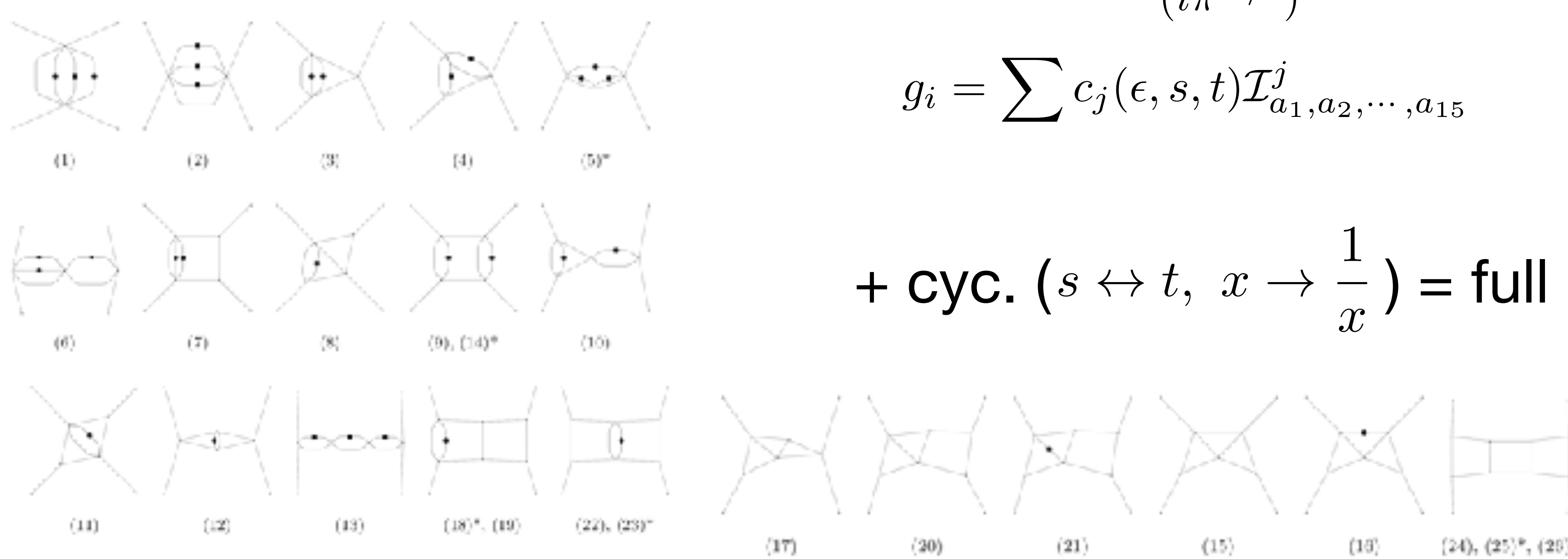


$$\partial_x f(x,\epsilon) = \epsilon\left[\frac{a}{x} + \frac{b}{1+x}\right] f(x,\epsilon)$$

$$f_i = \epsilon^3(-s)^{3\epsilon}\frac{e^{3\epsilon\gamma_E}}{(i\pi^{D/2})^3} g_i \qquad x = \frac{t}{s}$$

$$g_i = \sum c_j(\epsilon, s, t)\mathcal{I}^j_{a_1,a_2,\cdots,a_{15}}$$

+ cyc. ($s \leftrightarrow t,\ x \to \frac{1}{x}$) = full MIs



Conclusions

- Amplitudes as linear combinations of kinematic bases
- "Canonical" kinematic basis constructions for gauge bosons
- High-multiplicity high-loop amplitudes: Kinematic bases + unitarity cuts + IBP
- Implementations for 4 gluons planar 3-loop
  - Done: (1) Integrands from different cuts in CDR scheme
    (2) IBP reductions to MIs
  - Todo: (1) Merge into a readable results
    (2) Subtract UV and IR divergence into finite remainder
    (3) Results in HV scheme

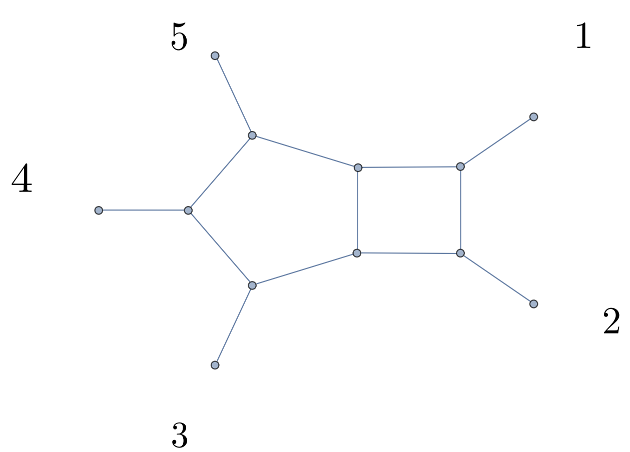
Thank you for your attention !

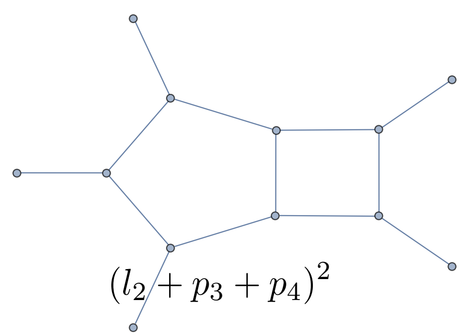
- Choose full propagators

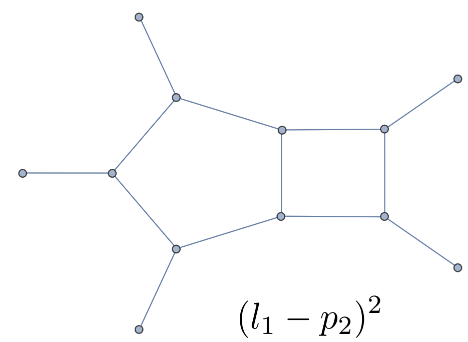
  IBPs up to power 5 done

$$\{(l_1-l_2)^2,(l_1)^2,(l_1-p_2)^2,(l_1-p_1-p_2)^2,\\ (l_1+p_3+p_4)^2,(l_1+p_3)^2,(l_2)^2,(l_2-p_2)^2,\\ (l_2-p_1-p_2)^2,(l_2+p_3+p_4)^2,(l_2+p_3)^2\}$$

- Maximal cuts for 5pt planar 2-loop







$$\{(l_1-l_2)^2,(l_1)^2,(l_1-p_1-p_2)^2,(l_1+p_3+p_4)^2,(l_1+p_3)^2,(l_2)^2,(l_2-p_2)^2,(l_2-p_1-p_2)^2\}$$

  Coefficients of highest MIs about 300M with unphysical singularities

- Integrand reductions for other cuts done, without substituting IBPs

- Comparison with known numerical results in HV scheme (in progress)

  - Number of kinematic basis reduces from 142 to 32

    Only kinematic basis from 5 As (eg. $A_1(2)A_2(3)A_3(4)A_4(5)A_5(1)$) remains

  - No $\sum_{\text{hels}\, i,j} (\xi_i^{\text{EX}} \cdot \xi_j^{\text{EX}})(\xi_i^{\text{EX}} \cdot l_a)(\xi_j^{\text{EX}} \cdot l_b) \sim l_a^{[4]} \cdot l_b^{[4]}$

    Only establish in 5pt case !

  - Relatively straight forward for HV scheme

  - Cross checked with numerical results [Badger et al., 17'; Abreu et al., 17']

[Gehrmann et al., 15', Badger et al., 18'; Abreu et al., 18']