Constructing multi-loop scattering amplitudes with manifest singularity structure

Robert M. Schabinger

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Outline

Gauge Theory Amplitude Singularity Structure

S. Catani, Phys. Lett. **B427** (1998) 161; S. Mert Aybat *et. al.*, Phys. Rev. **D74** (2006) 074004

T. Becher and M. Neubert, JHEP **0906** (2009) 081; E. Gardi and L. Magnea, JHEP **0903** (2009) 079

The IR divergences of the simplest non-Abelian gauge theory, planar $SU(N_c)$ $\mathcal{N}=4$ super Yang-Mills, are believed to be of the form:

$$\mathcal{A}_1^{\mathcal{N}=4}(p_1,\ldots,p_n) = \exp\left\{-\frac{1}{2}\sum_{L=1}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^L \mu_\epsilon^{2L\epsilon}\int_0^{\mu_\epsilon^2} \mathrm{d}\mu^2\left(\mu^2\right)^{-1-L\epsilon}\right.$$
$$\left.\sum_{\substack{\mathrm{i,j}=1\\ \mathrm{i<j}}}^{n}\left(\Gamma_{1;L}^{\mathcal{N}=4}\ln\left(\frac{\mu^2}{-s_{ij}}\right)+\mathcal{G}_{1;L}^{\mathcal{N}=4}\right)\frac{\mathbf{T}_\mathrm{i}\cdot\mathbf{T}_\mathrm{j}}{N_c}\right\}\sum_{L=0}^{\infty}\mathbf{H}_{1;L}^{\mathcal{N}=4}(\epsilon;p_1,\ldots,p_n)$$

At four points, this structure has been realized explicitly at strong coupling (L. F. Alday and J. Maldacena, JHEP **0706** (2007) 064). In a nutshell, the dipole conjecture is the suggestion that, with minor modifications, the above structure could hold for more general gauge theories like QCD.

What Can We Hope To See In A Two-Loop Integrand?

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In the spirit of Catani's work, let's start with QCD form factors!

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In the spirit of Catani's work, let's start with QCD form factors! The QCD form factors in dimensional regularization satisfy a renormalization group equation which was understood long ago:

L. Magnea and G. Sterman, Phys. Rev. **D42** (1990) 4222

$$q^2 \frac{\partial}{\partial q^2} \ln\left(\mathcal{F}\left(-q^2/\mu^2, \alpha_s, \epsilon\right)\right) = 1/2\mathcal{K}(\alpha_s) + 1/2\mathcal{G}\left(-q^2/\mu^2, \alpha_s, \epsilon\right)$$

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s)\frac{\partial}{\partial \alpha_s}\right)\mathcal{G}\left(-q^2/\mu^2, \alpha_s, \epsilon\right) = \sum_{L=1}^{\infty} \Gamma_L \alpha_s^L$$

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s)\frac{\partial}{\partial \alpha_s}\right)\mathcal{K}\left(\alpha_s\right) = -\sum_{L=1}^{\infty} \Gamma_L \alpha_s^L$$

At L loops, Γ_L characterizes the leading $1/\epsilon^2$ IR divergences which cannot be understood as exponentiated lower-loop contributions.

Pole Predictions For One- And Two-Loop Form Factors

From the renormalization group equation one can show that:

S. Moch *et. al.*, JHEP **0508** (2005) 049

$$\mathcal{F}_1^q = -\frac{1}{2}\frac{1}{\epsilon^2}\Gamma_1 - \frac{1}{2}\frac{1}{\epsilon}G_1(\epsilon)$$

$$\mathcal{F}_2^q = \frac{1}{8}\frac{1}{\epsilon^4}\Gamma_1^2 + \frac{1}{8}\frac{1}{\epsilon^3}\Gamma_1\Big(2G_1(0) - \beta_0\Big)$$

$$-\frac{1}{8}\frac{1}{\epsilon^2}\Bigg(\Gamma_2 + G_1(0)\Big(2\beta_0 - G_1(0)\Big) - 2\Gamma_1 G_1'(0)\Bigg) + \cdots$$

where we have $\mathcal{G}\left(1, \alpha_s, \epsilon\right) = G_1(\epsilon)\alpha_s + \cdots$

$$\Gamma_1 = 4C_F \qquad \Gamma_2 = -\tfrac{40}{9}C_F N_f + \Big(\tfrac{268}{9} - 8\zeta_2\Big)C_A C_F$$

$$G_1(0) = 6C_F \qquad G_1'(0) = \Big(16 - 2\zeta_2\Big)C_F \qquad \beta_0 = \tfrac{11}{3}C_A - \tfrac{2}{3}N_f$$

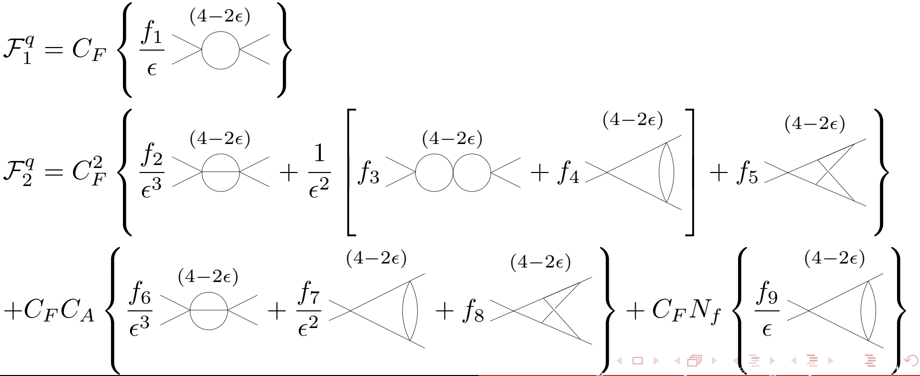
Explicit Results For One- And Two-Loop Form Factors

F. Tkachov, Phys. Lett. **B100** (1981) 65; K. Chetyrkin and F. Tkachov, Nucl. Phys. **B192** (1981) 159;

S. Laporta, Int. J. Mod. Phys. **A15** (2000) 5087; A. von Manteuffel and C. Studerus, arXiv:1201.4330

T. Matsuura *et. al.*, Z. Phys. **C38** (1988) 623; T. Gehrmann *et. al.*, Phys. Lett. **B622** (2005) 295

With respect to the conventional basis of integrals:

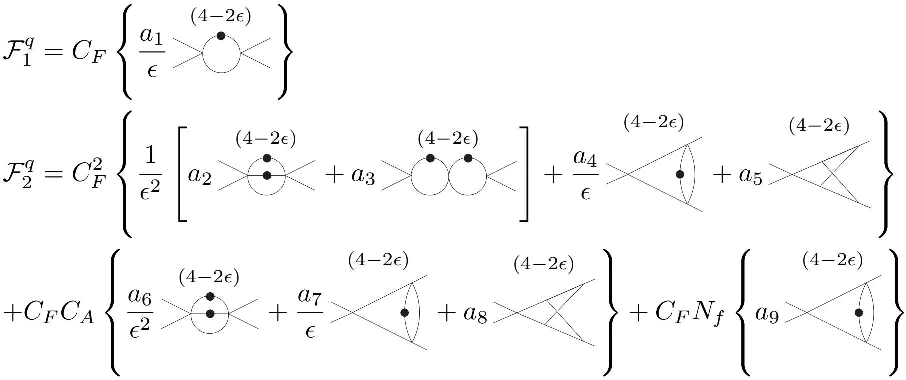


Explicit Results For One- And Two-Loop Form Factors

G. Heinrich *et. al.*, Phys. Lett. **B678** (2009) 359;

R. N. Lee *et. al.*, Nucl. Phys. Proc. Suppl. **205-206** (2010) 308; JHEP **1102** (2011) 102

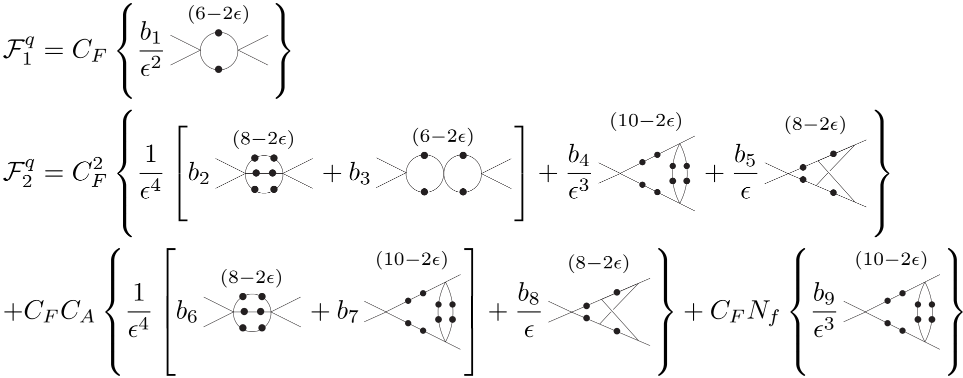
With respect to a uniform weight basis of integrals:



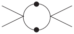
Explicit Results For One- And Two-Loop Form Factors

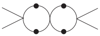
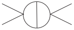
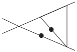
A. von Manteuffel *et. al.*, Phys. Rev. **D93** (2016) no. 12, 125014

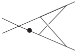
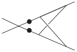
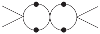
With respect to a finite basis of integrals:

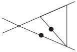
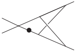
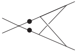
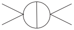


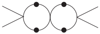
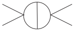
Explicit Results For One- And Two-Loop Form Factors

$\mathcal{F}_1^q = C_F \Big\{ \frac{c_1}{\epsilon^2}(1-2\epsilon)$ $(6-2\epsilon)$  $\Big\}$

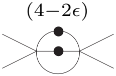
$\mathcal{F}_2^q = C_F^2 \Big\{ \frac{c_2}{\epsilon^4}(1-2\epsilon)^2$ $(6-2\epsilon)$  $+ \frac{1}{\epsilon}\Big[c_3(1-2\epsilon)$ $(4-2\epsilon)$  $+ c_4(1-2\epsilon)(1-3\epsilon)$ $(6-2\epsilon)$ 

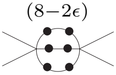
$+ c_5(1-4\epsilon)\Big(2(1-3\epsilon)$ $(6-2\epsilon)$  $-$ $(6-2\epsilon)$  $\Big)\Big]\Big\} + C_F C_A \Big\{ \frac{c_6}{\epsilon^3}(1-2\epsilon)^2$ $(6-2\epsilon)$ 

$+ \frac{c_7}{\epsilon^2}(1-2\epsilon)(1-3\epsilon)$ $(6-2\epsilon)$  $+ \frac{c_8}{\epsilon}(1-4\epsilon)\Big(2(1-3\epsilon)$ $(6-2\epsilon)$  $-$ $(6-2\epsilon)$  $\Big) + c_9(1-2\epsilon)$ $(4-2\epsilon)$  $\Big\}$

$+ C_F N_f \Big\{ \frac{c_{10}}{\epsilon^3}(1-2\epsilon)^2$ $(6-2\epsilon)$  $+ \frac{c_{11}}{\epsilon}(1-2\epsilon)(1-3\epsilon)$ $(6-2\epsilon)$  $+ c_{12}(1-2\epsilon)$ $(4-2\epsilon)$  $\Big\}$

If One Works Strictly With The Conventional Master Topologies, It Appears That Nothing Is Gained

$-\epsilon^2$ 

$= -8(1-3\epsilon)(2-3\epsilon)$ 

$= \frac{e^{2\gamma_E \epsilon}\Gamma^3(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)}$

$$= 1 - \zeta_2\epsilon^2 - \frac{32}{3}\zeta_3\epsilon^3 - \frac{57}{10}\zeta_2^2\epsilon^4 + \left(\frac{32}{3}\zeta_2\zeta_3 - \frac{272}{5}\zeta_5\right)\epsilon^5 + \mathcal{O}\left(\epsilon^5\right)$$

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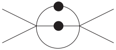
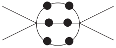
$(4-2\epsilon)$ $-\epsilon^2$  $= -8(1-3\epsilon)(2-3\epsilon)$ $(8-2\epsilon)$  $= \dfrac{e^{2\gamma_E \epsilon}\Gamma^3(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)}$

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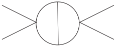
$\Longrightarrow$ To construct better candidate master integrals with higher leading weights, one should consider reducible topologies as well!

$(4-2\epsilon)$ $-\dfrac{1}{6}(1-2\epsilon)$  $= \zeta_3 + \dfrac{3}{5}\zeta_2^2\epsilon + \left(-\zeta_2\zeta_3 + 7\zeta_5\right)\epsilon^2 + \left(\dfrac{99}{35}\zeta_2^3 - \dfrac{23}{3}\zeta_3^2\right)\epsilon^3$

$$+ \left(-\frac{17}{2}\zeta_2^2\zeta_3 - 7\zeta_2\zeta_5 + 49\zeta_7\right)\epsilon^4 + \left(\frac{3777}{350}\zeta_2^4 + \frac{23}{3}\zeta_2\zeta_3^2 - \frac{1306}{15}\zeta_3\zeta_5\right)\epsilon^5 + \mathcal{O}\left(\epsilon^6\right)$$

What Uniformly-Finite Integrals Have To Offer

A uniformly-finite integral basis simultaneously realizes all the advantages of uniform weight and finite integral bases.

- Multi-scale uniformly-finite integrals satisfy especially simple differential equations, which usually makes their analytical evaluation easier.
  A. V. Kotikov, "Subtleties in quantum field theory," 150, (2010);
  J. M. Henn, Phys. Rev. Lett. **110** (2013) 251601; J. Broedel *et. al.*, JHEP **1901** (2019) 023
- Single-scale uniformly-finite integrals may be evaluated analytically using `HyperInt` (E. Panzer, Comput. Phys. Commun. **188** (2015) 148) in many cases.
  E. Panzer, JHEP **1403** (2014) 071; A. von Manteuffel *et. al.*, JHEP **1502** (2015) 120
- Numerical evaluations of uniformly-finite integrals are possible in situations where integrals with subdivergences (*i.e.* conventional or uniform weight integrals) cause problems for `FIESTA 4` and `pySecDec`.
  P. Marquard *et. al.*, Phys. Rev. **D94** (2016) no. 7, 074025;
  S. Borowka *et. al.*, JHEP **1610** (2016) 107; A. von Manteuffel and RMS, JHEP **1704** (2017) 129;
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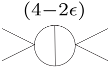
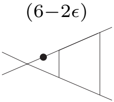
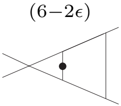
  S. P. Jones *et. al.*, Phys. Rev. Lett. **120** (2018) no.16, 162001
- Integrands for virtual corrections expressed in terms of a uniformly-finite basis may be fixed such that one can see immediately that the deeper poles in ϵ have their origin in lower-loop contributions. (conjecturally for now)

The relation:

$-\frac{1}{6}(1-2\epsilon)$ 

$= -\frac{1}{3}(1-2\epsilon)^2 \Bigg($ 

$+$ 

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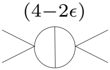
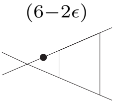
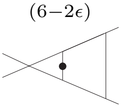
$-\frac{1}{6}(1-2\epsilon)$ 

$= -\frac{1}{3}(1-2\epsilon)^2 \Bigg($ 

$+$ 

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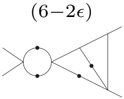
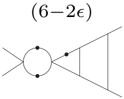
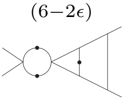
immediately suggests that, if the conjecture holds, all higher-order poles of the three-loop form factors ($\epsilon^{-6} - \epsilon^{-3}$) are generated by:

$-(1-2\epsilon)^2(1-3\epsilon)$ 

$-\frac{1}{3}(1-2\epsilon)^3 \Bigg($ 

$+$ 

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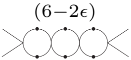
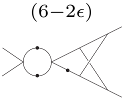
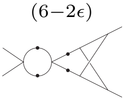
$(1-2\epsilon)^3$ 

$-\frac{1}{6}(1-2\epsilon)(1-4\epsilon)\Bigg(2(1-3\epsilon)$ 

$-$ 

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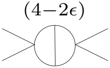
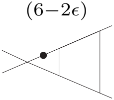
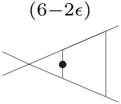
$-\frac{1}{6}(1-2\epsilon)$ 

$= -\frac{1}{3}(1-2\epsilon)^2 \Bigg($ 

$+$ 

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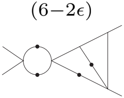
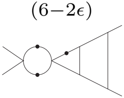
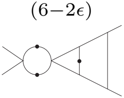
$-(1-2\epsilon)^2(1-3\epsilon)$ 

$-\frac{1}{3}(1-2\epsilon)^3 \Bigg($ 

$+$ 

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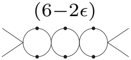
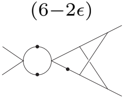
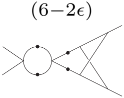
$(1-2\epsilon)^3$ 

$-\frac{1}{6}(1-2\epsilon)(1-4\epsilon)\Bigg(2(1-3\epsilon)$ 

$-$ 

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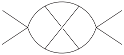
Dovetails nicely with Remark 60 of T. Bitoun *et. al.*, Lett. Math. Phys. 109 (2019) no. 3, 497

Selected Results For Three-Loop Form Factor Integrals

R. N. Lee *et. al.*, Nucl. Phys. Proc. Suppl. **205-206** (2010) 308; JHEP **1102** (2011) 102

$$\frac{1}{20}(1-2\epsilon)$$

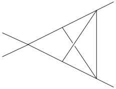
(4−2ε)



$$= \zeta_5 + \left(\frac{4}{7}\zeta_2^3 + \frac{17}{5}\zeta_3^2\right)\epsilon + \left(\frac{102}{25}\zeta_2^2\zeta_3 - \frac{3}{2}\zeta_2\zeta_5 + \frac{45}{2}\zeta_7\right)\epsilon^2 + \left(\frac{175149}{4375}\zeta_2^4 - \frac{51}{10}\zeta_2\zeta_3^2 - \frac{647}{5}\zeta_3\zeta_5 - \frac{2268}{25}\zeta_{5,3}\right)\epsilon^3 + \mathcal{O}\left(\epsilon^4\right)$$

$$-\frac{1}{12}(1-6\epsilon)$$

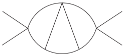
(4−2ε)



$$= \zeta_2\zeta_3 + \frac{5}{6}\zeta_5 + \left(\frac{11}{9}\zeta_2^3 + \frac{3}{2}\zeta_3^2\right)\epsilon + \left(\frac{13}{10}\zeta_2^2\zeta_3 - \frac{7}{12}\zeta_2\zeta_5 + \frac{307}{24}\zeta_7\right)\epsilon^2 + \left(-\frac{30239}{15750}\zeta_2^4 - \frac{185}{4}\zeta_2\zeta_3^2 - \frac{199}{6}\zeta_3\zeta_5 - \frac{62}{15}\zeta_{5,3}\right)\epsilon^3 + \mathcal{O}\left(\epsilon^4\right)$$

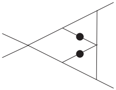
Selected Results For Three-Loop Form Factor Integrals

$(4-2\epsilon)$



$$-\frac{1}{20}(1-2\epsilon)\,[\text{diagram}]^{(4-2\epsilon)} = \zeta_5 + \left(\frac{4}{7}\zeta_2^3 - \frac{1}{5}\zeta_3^2\right) + \left(-\frac{6}{25}\zeta_2^2\zeta_3 - \frac{3}{2}\zeta_2\zeta_5 + \frac{359}{20}\zeta_7\right)\epsilon^2 + \left(\frac{2712}{625}\zeta_2^4 + \frac{3}{10}\zeta_2\zeta_3^2 - \frac{49}{5}\zeta_3\zeta_5 + \frac{162}{25}\zeta_{5,3}\right)\epsilon^3 + \mathcal{O}\left(\epsilon^4\right)$$

$(6-2\epsilon)$



$$-(1-2\epsilon)^2(1-4\epsilon)\,[\text{diagram}]^{(6-2\epsilon)} = \zeta_2^2 + 4\zeta_2\zeta_3\epsilon + \left(\frac{59}{10}\zeta_2^3 + 4\zeta_3^2\right)\epsilon^2 + \left(-\frac{41}{5}\zeta_2^2\zeta_3 + 28\zeta_2\zeta_5\right)\epsilon^3 + \left(\frac{27621}{1400}\zeta_2^4 - 86\zeta_2\zeta_3^2 + 56\zeta_3\zeta_5\right)\epsilon^4 + \mathcal{O}\left(\epsilon^5\right)$$

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- Even if a natural generalization of the pole-term conjecture fails to hold in general, it is still quite possible that a uniformly-finite basis of integrals may be constructed for an arbitrary hard scattering process. If such a basis does exist in general, how canonical is the construction and how cost-effective is it?