Direct Solutions of IBP Systems & Conjugate Polynomials

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@ *Mathematics of Linear Relations between Feynman Integrals* workshop

MITP, March 18–22, 2019

[1804.00131] & work in progress

Integrals

$$I^{(L)}[Poly(\{\ell_i\},\{k_j\})] = \int \prod_{i=1}^{L} \frac{d^D \ell_i}{(2\pi)^D} \, \frac{Poly}{\prod_{j=1}^{N} d_j}$$

- Many integrals on line 1, mostly differing in *Poly*
- Polylogarithmic content determined by denoms
- Expect linear relations & a small number of master integrals

Integration by Parts

Chetyrkin & Tkachov (1981)

- Technique for finding all linear relations
  - Sufficient: Petukhov & Smirnov (2011)

$$0 = \int \prod_{i=1}^{L} \frac{d^D \ell_i}{(2\pi)^D} \; \frac{\partial}{\partial \ell_a^\mu} \frac{v^\mu \, Poly}{\prod_{j=1}^{N} d_j}$$

- No boundary term thanks to dimensional regularization
- Choose all possible vectors v^μ, sufficient number of *Poly*s of sufficient dimension; system will close
- Solve system by Gaussian elimination

Laporta (2000)

- Nontrivial beyond one loop because of *irreducible numerators*
- Implementations: AIR [Anastasiou & Lazopoulos (2004)]; FIRE [Smirnov (2008–15)]; Reduze [Studerus & van Manteuffel (2010–12)]; Kira [Maierhoefer, Usovitch, & Uwer (2017)]; …
- Non-Laporta: LiteRed [Lee (2012)]; Azurite [Georgoudis, Larsen, Zhang (2016)]

Generic Reduction

$$I[\text{General Irreducible}] = \sum_{j \in basis} c_j(s_{ij}, \epsilon)\, I_j + \text{simpler graphs}$$

Problems

- Introduces unwanted integrals
  - In intermediate stages, lots of integrals with doubled propagators which never arise from Feynman diagrams & ultimately cancel

$$\frac{\partial}{\partial \ell_i^\mu} \frac{1}{d_j(\ell_i)} = -\frac{1}{d_j^2(\ell_i)} \frac{\partial}{\partial \ell_i^\mu} d_j(\ell_i)$$

- Need to solve huge systems of equations
  - Careful ordering, programming, disk management, …
- Hard to do arbitrary powers

General parameter, not fixed value

$$I[Irred^n] = \sum_{j \in basis} c^{(n)}(s_{ij}, \epsilon) I_j + \text{simpler graphs}$$

Resolve these problems

IBP-Generating Vectors

Gluza, Kajda, & DAK (2011)

- Basic idea: choose *special* vectors s.t.

$$v_i^\mu \frac{\partial}{\partial \ell_i^\mu} d_j \propto d_j \; \forall j$$

This eliminates new doubled propagators

$$v_i^\mu \frac{\partial}{\partial \ell_i^\mu} \frac{1}{d_j} = -\frac{1}{d_j^2} v_i^\mu \frac{\partial}{\partial \ell_i^\mu} d_j$$

$$= -\frac{p(\ell_i)}{d_j}$$

Generating-Vector Equations

- Vectors come in L-tuples; focusing on two loops, we have pairs (v_1, v_2), with equations

$$\sum_{i=1}^{2} v_i^\mu \frac{\partial}{\partial \ell_i^\mu} d_j = c_j d_j \; \forall j$$

where each c_j is a polynomial in the variables

$$\{\ell_1^2, \ell_1 \cdot \ell_2, \ell_2^2, \{\ell_1 \cdot b\}_{b \in B}, \{\ell_2 \cdot b\}_{b \in B}, s_{12}\}$$

with B a basis of external momenta; coefficients of terms in the polynomials are rational functions of $\chi_{ij} \equiv s_{ij}/s_{12}$

- Parametrize the vectors

$$v_i^\mu = c_i^{\ell_1} \ell_1^\mu + c_i^{\ell_2} \ell_2^\mu + \sum_{b \in B} c_i^b b^\mu$$

Generating-Vector Equations

- Organize the cs into a row vector $\tilde{c}$

- Doing all the algebra leads to an equation

$$\tilde{c}\, E = 0$$

where each column corresponds to a different denominator

This is a syzygy equation.

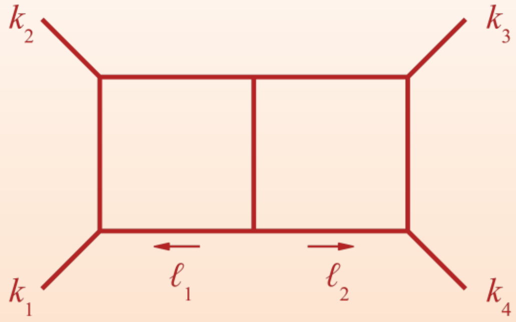
Solving

- We *cannot* solve this by linear algebra without using ansätze for the polynomials, because that would give rational and not polynomial solutions

- Find all syzygies
- Find independent basis set
  - Vector pairs that differ only by reducible invariants are not considered independent
- Standard problem in computational algebraic geometry
- Method uses Gröbner bases of tuples

[Linear algebra approach Schabinger (2012), von Manteuffel (2018)]

Example 1



- All-massless planar double box

- 3 vector pairs

$$v^{\mu}_{1;1} = -2k_4\cdot\ell_1 k_1^{\mu} + \ell_1^2 k_2^{\mu} + (2k_1\cdot\ell_1 - \ell_1^2)k_4^{\mu} - (2k_2\cdot\ell_1 - 2k_4\cdot\ell_1 - s_{12})\ell_1^{\mu}$$
$$v^{\mu}_{1;2} = -2k_4\cdot\ell_2 k_1^{\mu} - \ell_2^2 k_2^{\mu} + (2k_1\cdot\ell_2 + \ell_2^2)k_4^{\mu} - (2k_4\cdot\ell_2 - 2k_2\cdot\ell_2 - s_{12})\ell_2^{\mu}$$

other two a bit longer & of engineering dimension 5

$$0 = c_0[V_1, V_2] - c_1 V_1 - c_2 V_2 - c_3 V_3 + \text{purely reducible}\,,$$

Define a compact notation for derivatives $\partial_A v_A \equiv \sum_{i=1}^{2} \frac{\partial}{\partial \ell_i^{\mu}} v_i^{\mu}$

Compact Variables

- Use variables to simplify structure: r, u are purely reducible; t are irreducible

$$
\begin{aligned}
r_{11} &= \ell_1^2\,, \\
r_{12} &= \ell_1 \cdot \ell_2\,, \\
r_{22} &= \ell_2^2\,, \\
u_{11} &= \ell_1 \cdot k_1\,, \\
u_{12} &= \ell_1 \cdot k_2 - s_{12}/2\,, \\
u_{23} &= \ell_2 \cdot k_3 - s_{12}/2\,, \\
u_{24} &= \ell_2 \cdot k_4\,, \\
t_{14} &= \ell_1 \cdot k_4\,, \\
t_{21} &= \ell_2 \cdot k_1
\end{aligned}
$$

Conjugate Polynomials

- Goal: diagonalize systems a priori
- Make choices of the *Poly* to obtain IBPs for target numerators

$$0 = \int \prod_{i=1}^{L} \frac{d^D \ell_i}{(2\pi)^D} \, \partial_A \frac{v_A \, Poly}{\prod_{j=1}^{N} d_j}$$

- General expression for numerator

Requires more than Gröbner bases

ME: $$\sum_{r=1}^{n_v} Poly_r \, Denom \, \partial_A \frac{v_{rA}}{Denom} + \sum_{r=1}^{n_v} v_{rA} \partial_A Poly_r .$$

Universal prefactor

- Universal prefactor for v_1

$$Denom\,\partial_A \frac{v_{1A}}{Denom} = -2\,\epsilon\,(t_{14} - t_{21} - u_{12} - u_{23} - 2\,u_{24})$$

- Simplest example: $Poly = 1$

$$\begin{aligned} 0 &= -2\epsilon\, I_{\rm DB}[t_{14} - t_{21} - u_{12} - u_{23} - 2\,u_{24}] \\ &= 2\epsilon I_{\rm DB}[t_{21} - t_{14} + \text{purely reducible}] \\ &= 2\epsilon I_{\rm DB}[t_{21} - t_{14}] + \text{simpler integrals} \end{aligned}$$

- Here, the simpler integrals cancel, and we can solve for $I_{\rm DB}[t_{14}]$ in terms of one of the masters $I_{\rm DB}[t_{21}]$

Motivation

- Multiply the first vector pair by

$$a_1 t_{14} + a_2 t_{21}$$

and the second pair by $b_1(1+\chi)$, where $\chi = t/s$

We obtain the IBP

$$\begin{aligned} 0 = & I_{\mathrm{DB}}[-\tfrac{1}{2}\, b_1\, \chi\, \epsilon\, s_{12}^2 + \tfrac{1}{2}\, (-a_1\, \chi + 2\, b_1\, \epsilon)\, s_{12}\, t_{14} \\ & + a_1\, (1 - 2\, \epsilon)\, t_{14}^2 + \tfrac{1}{2}\, (a_2\, \chi + b_1\, \chi + 4\, b_1\, \epsilon)\, s_{12}\, t_{21} \\ & + 2\, (a_1 - a_2 + b_1)\, \epsilon\, t_{14}\, t_{21} - (a_2 + b_1)\, (1 - 2\, \epsilon)\, t_{21}^2 \\ & + \text{purely reducible}]\, . \end{aligned}$$

If we make the 'canonical' choices

$$(a_1, a_2, b_1) = (1,0,0); (0,1,0); (0,0,1)$$

we obtain three equations which still need to be diagonalized

Example 2

- If instead we make the choice

$$a_1 = \frac{1}{(1-2\epsilon)}\,, \qquad a_2 = \frac{1}{2(1-2\epsilon)}\,, \qquad b_1 = -\frac{1}{2(1-2\epsilon)}\,,$$

we obtain the direct IBP for $I_{\rm DB}[t_{14}^2]$

$$I_{\rm DB}[t_{14}^2] = \frac{(\chi+3\epsilon)}{2\,(1-2\,\epsilon)}\, s_{12}\, I_{\rm DB}[t_{21}] - \frac{\chi\,\epsilon}{4\,(1-2\,\epsilon)}\, s_{12}^2 I_{\rm DB}[1]$$
$$+ \text{ simpler integrals}$$

We can obtain similar direct equations for the other quadratic irreducibles

Finding Master Integrals

- Write out all monomials of a given degree in a vector; for example, degree 3

$$\begin{pmatrix} t_{14}^3 \\ t_{14}^2 t_{21} \\ t_{14} t_{21}^2 \\ t_{21}^3 \\ t_{14}^2 s_{12} \\ t_{14} t_{21} s_{12} \\ t_{21}^2 s_{12} \\ t_{14} s_{12}^2 \\ t_{21} s_{12}^2 \\ s_{12}^3 \end{pmatrix}$$

- One such monomial vector for each IBP-generating tuple of vectors, with different degrees

Finding Master Integrals

- Independently take *Poly* to be each entry
- Compute numerator using the Master Equation
- Set reducibles to zero

$\Rightarrow$ Reduced numerators, which we can regard as a linear transformation of irreducibles

- Organize these into a matrix
  - Columns correspond to monomials
  - Rows correspond to IBP equations
- Independent reductions $\Leftrightarrow$ range of the matrix
- Redundant reductions $\Leftrightarrow$ master integrals $\Leftrightarrow$ kernel of matrix

Masters Example: Double Box

- Take first vector pair, with degree-0 vector

$$M_1 = \begin{pmatrix} -2\epsilon & 2\epsilon & 0 \end{pmatrix}$$

with columns corresponding to t_{14}, t_{21}, s_{12}

Corresponding IBP is

$$0 = I_{\rm DB}[M_1 \begin{pmatrix} t_{14} \\ t_{21} \\ s_{12} \end{pmatrix}] + \text{simpler topologies}$$

Kernel is two-dimensional, generated by

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Masters Example

- Use non-trivial IBP to reduce first vector to get $I_{\text{DB}}[t_{21}]$ and $I_{\text{DB}}[1]$ as masters

- Might worry that we're missing equations
- Try polynomials of higher degree: degree-1 for 1st pair, degree-0 for 2nd & 3rd pairs

$$M_2 = \begin{pmatrix} 2(1-2\epsilon) & 4\epsilon & 0 & -\chi & 0 & 0 \\ 0 & -4\epsilon & -2(1-2\epsilon) & 0 & \chi_{14} & 0 \\ 0 & 0 & 0 & -2\epsilon & 2\epsilon & 0 \\ 0 & \frac{2\epsilon}{1+\chi} & -\frac{1-2\epsilon}{1+\chi_{14}} & \frac{\epsilon}{1+\chi} & \frac{\chi+4\epsilon}{2(1+\chi)} & -\frac{\chi\epsilon}{2(1+\chi)} \\ 0 & -8\epsilon & 0 & -6\epsilon & 0 & \chi\epsilon \end{pmatrix}$$

with columns corresponding to $t_{14}^2, t_{14}t_{21}, t_{21}^2, s_{12}t_{14}, s_{12}t_{21}, s_{12}^2$

Masters Example

- Again a kernel of dimension 2, with same masters as before
- Repeat with one higher dimension $\Rightarrow$ same result

- Pedestrian procedure: assume convergence

- More complete solution will presumably require *D*-algebras
- Should also provide connections to algebraic geometry

Doubled Propagators

- IBP-generating vectors ensure that no doubled propagators are newly introduced
- What about those already present?
- IBP-generating vectors can be used with them too: ensure no higher powers are introduced
- For double box, usual integrals require only first two vector pairs
- Doubled propagators also need third pair, but reduce to usual masters (no new masters)

General Powers

- Multiply the 1st vector pair by

$$a_1\, t_{14}^{n-1} + a_2\, t_{14}^{n-2}\, t_{21} + a_3\, t_{14}^{n-2}\, s_{12}$$

and the 2nd pair by

$$b_1(1+\chi)t_{14}^{n-2}$$

- Feed it through the Master Equation to obtain the IBP

$$\begin{aligned}0 = I_{\text{DB}}\big[&a_1\,(1+2\,\epsilon-n)\,t_{14}^{n}\\ &-(2\,a_1\,\epsilon+(b_1-a_2)\,(2+2\,\epsilon-n))\,t_{14}^{n-1}\,t_{21}\\ &+(a_2+b_1)\,(1-2\,\epsilon)\,t_{14}^{n-2}\,t_{21}^{2}\\ &+\text{lower irreducible degree}\big]\\ &+\text{simpler integrals}\end{aligned}$$

General Powers

- Choose
$$a_1 = \frac{1}{n-1-2\epsilon}\,,$$
$$a_2 = -\frac{\epsilon}{(n-1-2\epsilon)(n-2-2\epsilon)}\,,$$
$$a_3 = \frac{1}{2(n-1-2\,\epsilon)}\,,$$
$$b_1 = \frac{\epsilon}{(n-1-2\epsilon)(n-2-2\epsilon)}$$

to obtain an IBP equation for $I_{\rm DB}[t_{14}^n]$

$$0 = I_{\rm DB}\Big[t_{14}^n + \tfrac{(2+(n-1)\chi+3\,\epsilon-n)}{2\,(1+2\,\epsilon-n)}\,t_{14}^{n-1}\,s_{12} - \tfrac{\chi\,(2+\epsilon-n)}{4\,(1+2\,\epsilon-n)}\,t_{14}^{n-2}\,s_{12}^2\Big]$$
$$+\ \text{simpler integrals}$$

n could even be non-integer (with additional masters)

Recurrence Relations

- This is a recurrence relation: $w_n \equiv s_{12}^{-n}\, I_D[t_{14}^n]$

$$4(1+2\epsilon-n)w_n + 2(2+(n-1)\chi+3\epsilon-n)\, w_{n-1} - \chi(2+\epsilon-n)\, w_{n-2} \doteq 0$$

- How can we solve it?
- First thought: use *Mathematica*
- Solution in terms of DifferenceRoot objects
  - Possibly efficient, but useless analytically

Solving: Generating Function

- Instead, build a generating function

$$\sum_{n=0}^{\infty} Rec_n x^n$$

- Substitute

$$\sum_{n=0}^{\infty} c_n a_{n+r} x^n \to x^{-r}\left(\sum_{n=0}^{\infty} c_{n-r} a_n x^n - x^{-r} \sum_{n=0}^{r-1} c_{n-r} a_n x^n\right)$$

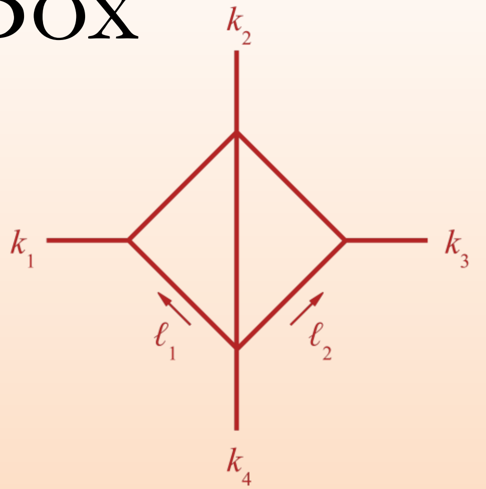
- Replace $\sum_{n=0}^{\infty} n^p a_n x^n \to D_x^p f(x)$ $\quad (D_x \equiv x\partial_x)$

to obtain a differential equation; differentiate it again to make it homogeneous

Solving: Differential Equation

- Solve differential equation
- Boundary conditions given by $x \to 0$ behavior as well as master integrals
- Extract n^{th} power in x to obtain desired coefficient

Example: Slashed Box



- One master integral $I_{\mathrm{SB}}[1]$
- Four irreducible invariants
- Seven pairs of generating vectors ($\check{r}_{12} = \ell_1 \cdot \ell_2 + t_{14} + t_{24}$, $\check{u}_{23} = \ell_2 \cdot k_3$)

$$
\begin{aligned}
v^\mu_{1;1} =& -k_1^\mu\, r_{11} - k_2^\mu\, r_{11} - \ell_1^\mu\,(s_{12} - 2\,t_{12} - 2\,u_{11})\,,\\
v^\mu_{1;2} =& k_1^\mu\,(r_{22} + 2\,t_{24}) + k_2^\mu\,(r_{22} + 2\,t_{24}) + 2\,\ell_2^\mu\,(t_{24} + \check{u}_{23}) + 2\,k_4^\mu\,(t_{24} + \check{u}_{23})\,,\\
v^\mu_{2;1} =& \tfrac{1}{2}\,k_2^\mu\, r_{11} + \tfrac{1}{2}\,k_4^\mu\, r_{11} - k_1^\mu\,(\check{r}_{12} - t_{14} - t_{24}) + \ell_2^\mu\, u_{11}\\
&+ \tfrac{1}{2}\,\ell_1^\mu\,(s_{12} + \chi\, s_{12} - 2\,t_{12} - 2\,t_{14} - 2\,t_{22} - 2\,t_{24} - 2\,\check{u}_{23})\,,\\
v^\mu_{2;2} =& \tfrac{1}{2}\,k_2^\mu\,(2\,\check{r}_{12} + r_{22} - 2\,t_{14} - 2\,t_{24}) + \tfrac{1}{2}\,k_4^\mu\,(2\,\check{r}_{12} + r_{22} - 2\,t_{14} - 2\,t_{24}) + \ell_1^\mu\,\check{u}_{23}\\
&+ k_1^\mu\,(\check{r}_{12} - t_{14} - t_{24}) + \ell_2^\mu\,(\tfrac{1}{2}s_{12} + \tfrac{1}{2}\chi\, s_{12} - t_{12} - t_{14} - t_{22} - t_{24} - u_{11})
\end{aligned}
$$

$$\vdots$$

Example: Slashed Box

- Study $\breve{w}_n \equiv s_{12}^{-n}\, I_{\rm SB}[t_{12}^n]$

- Recurrence

$$0 \doteq (1+n)\,\breve{w}_n + 2\,\big((3+2\,\chi)(1+\epsilon) - (\chi+2)\,(3+n)\big)\,\breve{w}_{n+1} + 4\,(1+\chi)\,(n+2-3\,\epsilon)\,\breve{w}_{n+2}$$

- Differential equation

$$0 = -2\,f(x) + 2\,\big(2(2-\epsilon) + 2(1+\chi)\,(1-2\epsilon) - 5\,x\big)\,f'(x) - \big(4(1+\chi)((2-\epsilon)(1-x) - 2\epsilon) + x(7x - 10 + 2\epsilon)\big)\,f''(x) - (x-2)\,x\,\big(x - 2(1+\chi)\big)\,f^{(3)}(x)$$

Example: Slashed Box

- Raw solution to differential equation

$$f(x) = \frac{x^{3\epsilon} c_1}{(2-x)^{\epsilon} \big(2(1+\chi)-x\big)^{1+2\epsilon}} - \frac{2^{3\epsilon}(1+\chi)^{2\epsilon} c_2}{6\epsilon (2-x)^{\epsilon} \big(2(1+\chi)-x\big)^{1+2\epsilon}} \times F_1\left(-3\epsilon, -\epsilon, -2\epsilon, 1-3\epsilon; \tfrac{x}{2}, \tfrac{x}{2(1+\chi)}\right) + \frac{2^{3\epsilon}(1+\chi)^{2\epsilon} x (c_2 + 2c_3)}{4(1-3\epsilon)(2-x)^{\epsilon}\big(2(1+\chi)-x\big)^{1+2\epsilon}} \times F_1\left(1-3\epsilon, 1-\epsilon, -2\epsilon, 2-3\epsilon; \tfrac{x}{2}, \tfrac{x}{2(1+\chi)}\right)$$

Appell function

- $x \to 0$ behavior forces $c_1 = 0$
- $f(0) = \breve{w}_0$ fixes c_2
- $f'(0) = \breve{w}_1 = c\,\breve{w}_0$ fixes $c_3 = 0$

Example: Slashed Box

- Solution to differential equation with desired boundary behavior

$$f(x) = -\frac{3\,2^{3\,\epsilon}\,(1+\chi)^{1+2\,\epsilon}\,\epsilon\,x}{(1-3\,\epsilon)\,(2-x)^{\epsilon}\,(2(1+\chi)-x)^{1+2\,\epsilon}}$$
$$\times F_1\left(1-3\,\epsilon, 1-\epsilon, -2\,\epsilon, 2-3\,\epsilon; \frac{x}{2}, \frac{x}{2(1+\chi)}\right)\check{w}_0$$
$$+\frac{2^{1+3\,\epsilon}\,(1+\chi)^{1+2\,\epsilon}}{(2-x)^{\epsilon}\,(2(1+\chi)-x)^{1+2\,\epsilon}}$$
$$\times F_1\left(-3\,\epsilon, -\epsilon, -2\,\epsilon, 1-3\,\epsilon; \frac{x}{2}, \frac{x}{2(1+\chi)}\right)\check{w}_0$$

Example: Slashed Box

- Brute-force expansion to extract x^n term yields a triple sum

$$\check{w}_n = \frac{6\,\epsilon^3\,(1+\chi)\,\check{w}_0}{2^n\,\Gamma(1-2\,\epsilon)\,\Gamma(1-\epsilon)\,\Gamma(1+\epsilon)\,\Gamma(1+2\,\epsilon)}\Bigg[\sum_{n_1=1}^{n}\frac{1}{(1+\chi)^{n_1}\,(n-n_1+1-3\,\epsilon)}$$
$$\times\sum_{n_3=0}^{n-n_1}\frac{\Gamma(n-n_1-n_3+2-\epsilon)\,\Gamma(n_3-2\,\epsilon)}{(1+\chi)^{n_3}\,\,(n-n_1-n_3+1)!\,n_3!}$$
$$\times\sum_{n_4=0}^{n_1}\frac{(1+\chi)^{n_4}\,\Gamma(n_1-n_4+2\,\epsilon)\,\Gamma(n_4+\epsilon)}{(n_1-n_4-1)!\,n_4!}$$
$$+\Gamma(1-\epsilon)(1+\chi)^{-n-1}\sum_{n_1=1}^{n+1}\frac{\Gamma(n-n_1+1-2\,\epsilon)}{(n-n_1+1-3\,\epsilon)\,(n-n_1+1)!}$$
$$\times\sum_{n_4=0}^{n_1}\frac{(1+\chi)^{n_4}\,\Gamma(n_1-n_4+2\,\epsilon)\,\Gamma(n_4+\epsilon)}{(n_1-n_4-1)!\,n_4!}\Bigg]$$

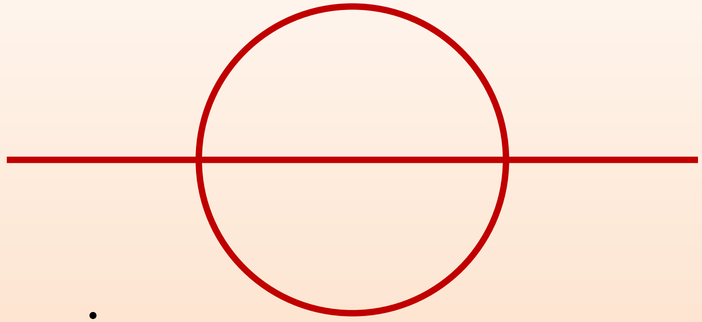
Example: Slashed Box

- A little sprinkling of magic simplifies the expression

$$\check{w}_n = -\frac{2^{1-n}\epsilon\Gamma(n-2\epsilon)\Gamma(1-3\epsilon)}{\Gamma(1-2\epsilon)\Gamma(n+1-3\epsilon)}\,{}_2F_1\left(1-\epsilon,-n;1-n+2\epsilon;(1+\chi)^{-1}\right)\check{w}_0$$

- Still manifestly rational in χ and ϵ

Sunrise at Noon



- Equal-mass case
- 2 irreducible invariants
- 4 IBP-generating vector pairs
- Naively, four masters; but symmetries & iterated integration reduce this to 2: $\text{Sun}_{2:m}[1]$ and $\text{Sun}_{2:m}[(\ell_1 \cdot K)^2]$
- Conjugate polynomial: degrees (3,2,2,2)
- $y_n \equiv s^{-n}\, \text{Sun}_{2:m}[(\ell_1 \cdot K)^n]$
- Recurrence

$$
\begin{aligned}
0 = {} & n\tau(1+\tau)(-1+3\tau)y_{n-1} + 2\tau(-2+\epsilon-2n+\epsilon\tau+2n\tau)y_n \\
& + (3-2\epsilon+n-14\tau+8\epsilon\tau-6n\tau-9\tau^2+6\epsilon\tau^2-3n\tau^2)y_{n+1} \\
& + 2(8-5\epsilon+2n-6\tau+3\epsilon\tau-2n\tau)y_{n+2} - 4(-5+3\epsilon-n)y_{n+3} + c_n \text{tadpole}
\end{aligned}
$$

Summary

- New approach to integration-by-parts systems
- IBP-generating vectors to remove unwanted doubled propagators & block-diagonalize system
- Conjugate polynomials to fully diagonalize, and target specific numerators
- Derive recurrence relations for arbitrary powers of irreducibles