Reduction of multi-loop Feynman integrals: an O(N) algorithm

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Based on: 1711.09572, 1801.10523
and works in preparation

The Mathematics of Linear Relations between Feynman Integrals
2019/03/18-22, MITP, Germany

Outline

Feynman loop integrals

➢ Definition in dimensional regularization

$$\lim_{\eta \to 0^+} \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{1}{(q_\alpha^2 - m_\alpha^2 + \mathrm{i}\eta)^{\nu_\alpha}}$$

- q_α: linear combination of loop momenta and external momenta
- Taking $\eta \to 0^+$ before taking $D \to 4$

➢ Theorem:

Smirnov, Petukhov, 1004.4199

For a given set of propagators, Feynman integrals form a finite-dimensional linear space

MY philosophy

- Reducing/evaluating FIs analytically may not be possible for sufficiently complicated problems

- A general solution for FIs calculation, if exists, should be a numerical method

  Only numerical numbers are needed to compare with experimental data

Evaluation of FIs

- **Sufficient conditions for a good solution:**
  1. Systematic: can be applied to any problem
  2. Efficient: the amount of computation is linearly dependent on the number of FIs and the number of effective digits, and it is insensitive to the number of mass scales involved
  3. "Analytical": knows all singularities, and can calculate coefficients of asymptotic expansion at any given singular point

This talk: A method may satisfy these conditions

Outline

I. Introduction

II. A New Representation

III. Reduction

IV. Remarks

Modified FIs

➢ **Modify Feynman loop integral by keeping finite η**

$$\mathcal{M}(D,\vec{s},\eta) \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{1}{(\mathcal{D}_\alpha + \mathrm{i}\eta)^{\nu_\alpha}} \qquad \mathcal{D}_\alpha \equiv q_\alpha^2 - m_\alpha^2$$

- **Take it as an analytical function of η**
- **Physical result is defined by**

$$\mathcal{M}(D,\vec{s},0) \equiv \lim_{\eta\to 0^+} \mathcal{M}(D,\vec{s},\eta)$$

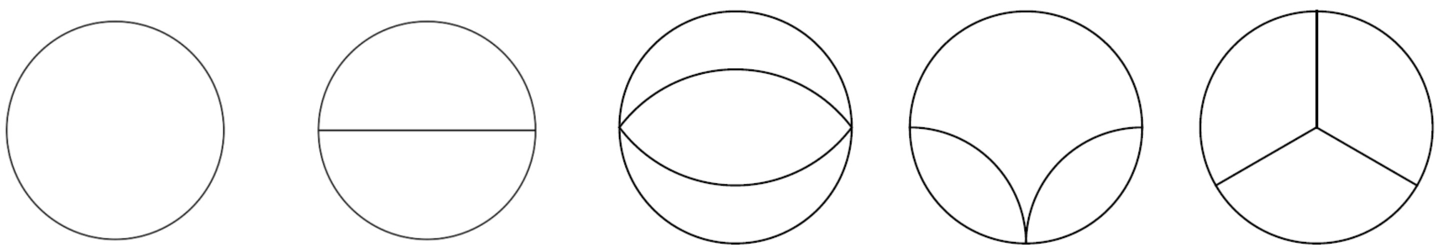
Expansion at infinity

- **Expansion of propagators around $\eta = \infty$**

$$\frac{1}{[(\ell+p)^2 - m^2 + \mathrm{i}\eta]^\nu} = \frac{1}{(\ell^2+\mathrm{i}\eta)^\nu}\sum_{n=0}^{\infty}\frac{(\nu)_n}{n!}\left(\frac{-2\ell\cdot p - p^2 + m^2}{\ell^2+\mathrm{i}\eta}\right)^n$$

  - Only one region: $l^\mu \sim |\eta|^{1/2}$
  - No external momenta in denominator, vacuum integrals
  - Simple enough to deal with

- **Vacuum MIs with equal internal masses**



  - Analytical results are known up to 3-loop
  - Numerical results are known up to 5-loop

Davydychev,Tausk, NPB(1993)
Broadhurst, 9803091
Kniehl, Pikelner, Veretin, 1705.05136

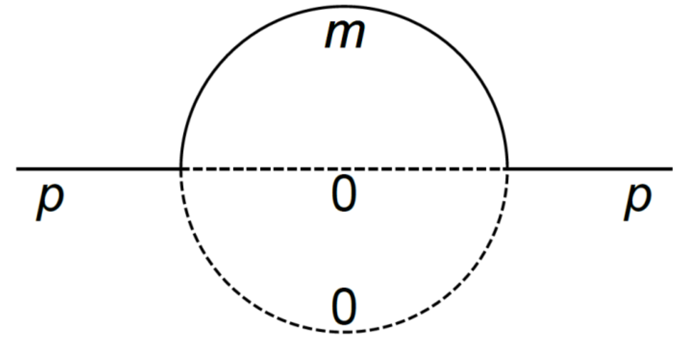
Schroder, Vuorinen, 0503209
Luthe, PhD thesis (2015)
Luthe, Maier, Marquard, Ychroder, 1701.07068

See R. Lee's talk for the possibility of even higher loops

Example

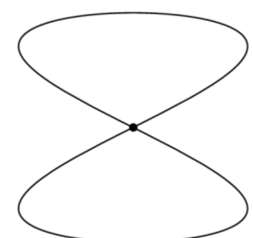
- **Sunrise integral**

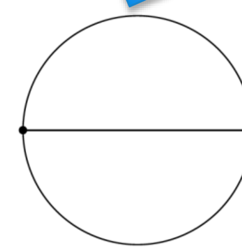
$$\hat{I}_{\nu_1\nu_2\nu_3} \equiv \int \prod_{i=1}^{2} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{1}{\mathcal{D}_1^{\nu_1}\mathcal{D}_2^{\nu_2}\mathcal{D}_3^{\nu_3}}$$



$$\mathcal{D}_1 = (\ell_1 + p)^2 - m^2,\ \mathcal{D}_2 = \ell_2^2,\ \mathcal{D}_3 = (\ell_1 + \ell_2)^2$$

$$I_{111} = \eta^{D-3}\left\{\left[1 - \frac{D-3}{3}\frac{m^2}{\mathrm{i}\eta} + \frac{(D+4)(D-3)}{9D}\frac{p^2}{\mathrm{i}\eta}\right] I_{2,2}^{\mathrm{bub}} - \mathrm{i}\left[\frac{(D-2)^2}{3D}\frac{p^2}{\mathrm{i}\eta}\right] I_{2,1}^{\mathrm{bub}} + \mathcal{O}(\eta^{-2})\right\}$$





A new representation

➢ Asymptotic expansion

$$\mathcal{M}(D,\vec{s},\eta) = \eta^{LD/2-\sum_\alpha \nu_\alpha} \sum_{\mu_0=0}^{\infty} \eta^{-\mu_0} \mathcal{M}^{\text{bub}}_{\mu_0}(D,\vec{s})$$

$$\mathcal{M}^{\text{bub}}_{\mu_0}(D,\vec{s}) = \sum_{k=1}^{B_L} I^{\text{bub}}_{L,k}(D) \sum_{\vec{\mu}\in\Omega^r_{\mu_0}} C_k^{\mu_0\cdots\mu_r}(D) s_1^{\mu_1}\cdots s_r^{\mu_r}$$

- $I^{\text{bub}}_{L,k}(D)$: k-th master vacuum integral at L-loop order
- $C_k^{\mu_0\ldots\mu_r}(D)$: rational functions of D

➢ A new representation

- Uniqueness theorem of analytical functions: physical FI is uniquely determined by this asymptotic series via analytical continuation
- A new (series) representation of FIs
- All FIs are determined by equal-mass vacuum integrals

Outline

What is reduction

➢ Reduction

- Find relations between loop integrals
- Use them to express all loop integrals as linear combinations of MIs

➢ Relations among $G \equiv \{M_1, M_2, \ldots, M_n\}$

$$\sum_{i=1}^{n} Q_i(D, \vec{s}, \eta)\, \mathcal{M}_i(D, \vec{s}, \eta) = 0$$

- $Q_i(D, \vec{s}, \eta)$: homogeneous polynomials of $\vec{s}, \eta$ of degree d_i

➢ Constraints from mass dimension

$$2d_1 + \mathrm{Dim}(\mathcal{M}_1) = \cdots = 2d_n + \mathrm{Dim}(\mathcal{M}_n)$$

- Only 1 degree of freedom in $\{d_i\}$, chosen as $d_{\max} \equiv \mathrm{Max}\,\{d_i\}$

Find relations

- **Decomposition of $Q_i(D,\vec{s},\eta)$**

$$\sum_{i=1}^{n} Q_i(D,\vec{s},\eta)\,\mathcal{M}_i(D,\vec{s},\eta)=0$$

$$Q_i(D,\vec{s},\eta)=\sum_{(\lambda_0,\vec{\lambda})\in\Omega_{d_i}^{r+1}} Q_i^{\lambda_0\cdots\lambda_r}(D)\,\eta^{\lambda_0}s_1^{\lambda_1}\cdots s_r^{\lambda_r}$$

$$\Rightarrow \sum_{k,\rho_0,\vec{\rho}} f_k^{\rho_0\cdots\rho_r}\,\mathcal{I}_{L,k}^{\text{bub}}(D)\;\eta^{\rho_0}s_1^{\rho_1}\cdots s_r^{\rho_r}=0$$

- **Linear equations:** $f_k^{\rho_0\cdots\rho_r}=0$
  - With enough constraints $\Rightarrow Q_i^{\lambda_0\ldots\lambda_r}(D)$
  - With finite field technique, only integers in a finite field are involved, equations can be efficiently solved

- **Relations among $G\equiv\{M_1,M_2,\ldots,M_n\}$ with a fixed $d_{\max}$ are fully determined**

Reduction

- **With $G = G_1 \cup G_2$, satisfy**
  - G_1 is more complicated than G_2
  - G_1 can be reduced to G_2
- **Algorithm** ***Search for simplest relations***
  1. Set $d_{\max} = 0$
  2. Find out all reduction relations among G with fixed $d_{\max}$
  3. If obtained relations are enough to determine G_1 by G_2, stop;
     else, $d_{\max} = d_{\max} + 1$ and go to step 2
- **Conditions for G_1 and G_2**
  1. Relations among G_1 and G_2 are not too complicated: easy to find
  2. $\# G_1$ is not too large: numerically diagonalize relations easily

Reduction scheme with only dots

- **FIs:** $\vec{\nu} = (\nu_1, \dots, \nu_N), \nu_i \geq 0$
  - $\mathbf{0}^{\pm} \equiv \textbf{Identity}, \ \mathbf{m}^{\pm} \equiv (\mathbf{m}-\mathbf{1})^{\pm}\mathbf{1}^{\pm}$
  - $\mathbf{1}^{+}(5,1,0,3) = \{(6,1,0,3), (5,2,0,3), (5,1,0,4)\}$
  - $\mathbf{1}^{-}(5,1,0,3) = \{(4,1,0,3), (5,0,0,3), (5,1,0,2)\}$
- **1-loop:** $G_1 = \mathbf{1}^{+}\vec{\nu}, G_2 = \mathbf{1}^{-}\mathbf{1}^{+}\vec{\nu}$ Duplancic and Nizic, 0303184
- **Multi-loop:**

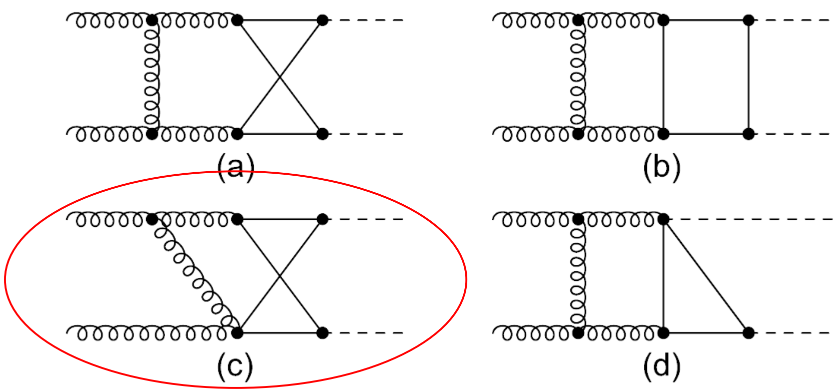
$$G_1 = \mathbf{m}^{+}\vec{\nu}, G_2 = \{\mathbf{1}^{-}\mathbf{m}^{+}, \mathbf{1}^{-}(\mathbf{m}-\mathbf{1})^{+}, \dots, \mathbf{1}^{-}\mathbf{1}^{+}\}\vec{\nu}$$

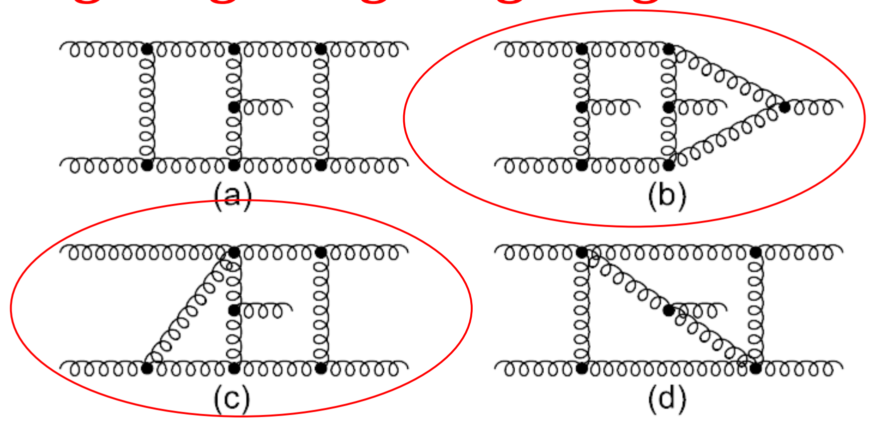
  - $m = 2,3$ in examples, $\#G_1$ is not too large, include dozens of integrals
  - Relations among G_1 and G_2 are not too complicated, see examples

A step-by-step reduction is realized!

Examples

- 2-loop $g+g\rightarrow H+H$ and $g+g\rightarrow g+g+g$





$g+g\rightarrow H+H$				$g+g\rightarrow g+g+g$			
Sector	Type	$d_{\max}$	$\mathbf{m}^+$	Sector	Type	$d_{\max}$	$\mathbf{m}^+$
1(a)	7-NP	1	$\mathbf{3}^+$	2(a)	8-NP	1	$\mathbf{3}^+$
1(b)	7-P	1	$\mathbf{3}^+$	2(b)	8-NP	3	$\mathbf{3}^+$
1(c)	6-NP	5	$\mathbf{3}^+$	2(c)	7-NP	4	$\mathbf{3}^+$
1(d)	6-P	4	$\mathbf{2}^+$	2(d)	6-NP	2	$\mathbf{3}^+$

Difficulty:

- More legs > less legs
- Nonplanar > Planar
- $\mathbf{m}^+\vec{e} > \mathbf{m}^+\vec{\nu}$

- Relations can be obtained by a single-core laptop in a few hours
- Diagonalizing at each phase space point (floating number): 0.01 second
- Results checked numerically by FIRE

Reduction of numerators

- **Method similar to the reduction of denominators, work in progress**

➢ Set up and solve DEs of MIs

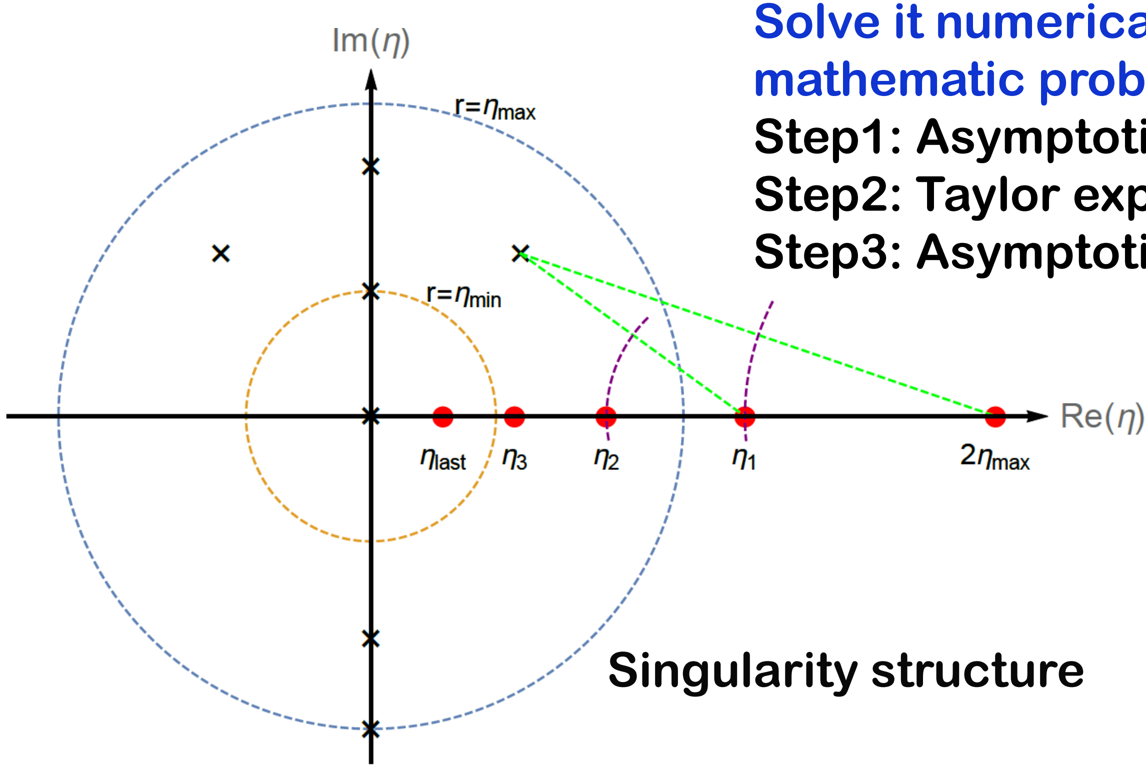
$$\frac{\partial}{\partial \eta}\vec{I}(D;\eta) = A(D;\eta)\vec{I}(D;\eta)$$ **with known** $\vec{I}(D;\infty)$

Solve it numerically: a well-studied mathematical problem

Step1: Asymptotic expansion at $\eta=\infty$

Step2: Taylor expansion at analytical points

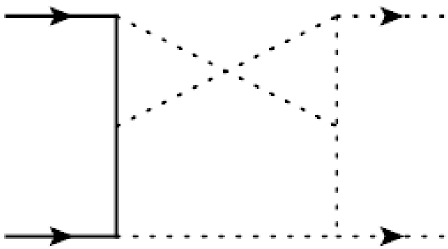
Step3: Asymptotic expansion at $\eta=0$



Singularity structure

Example

- **2-loop non-planar sector for $\mathrm{Q} + \overline{\mathrm{Q}} \rightarrow g + g$**



- **168 master integrals**
- **Traditional method sector decomposition: $O(10^4)$ CPU core-hour**
- **Our method: a few minutes**

Feng, Jia, Sang, 1707.05758

MIs can be thought as special functions, and DEs tell us how to evaluate these special functions

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Remarks: generality

- As η presents in all FIs, our method can be used for any problem

- We can find out any existing relation between FIs
  - Relations due to IBP, LI, symmetries, accidental relations, non-linear relations …

Remarks: efficiency of reduction

- **Cost of setting up analytical reduction relations: linear in the number of target FIs**
  - Set up one reduction relation for each FI
  - Each reduction relation can be obtained in a short time
  - The cost for each relation is insensitive to the number of scales: two-loop $gg \to t\bar{t}H$ is similar to $5g$
- **Cost of numerically diagonalizing reduction relations: linear in the number of target FIs**
  - Reduction relations are block-diagonalized
  - # of equations equals to # of target FIs
  - Do it at each phase space point (floating numbers)

Remarks: efficiency of evaluating MIs

- **Our strategy is to numerically solve DEs w.r.t. η and kinematic variables**
  - Increase the efficiency
  - Determine analytical structure
  - Cost is linearly dependent on the required number of effective digits

Remarks: infrared divergences

- **No IR divergence when η is finite**
  - η plays the role as an IR regulator

- **IR divergences come out as $\eta \to 0^+$**
  - ϵ becomes the IR regulator after taking this limit

Remarks: number of MIs

- **Number of MIs at finite η is larger than the number of MIs at $\eta \to 0^+$**
  - It is not a problem become the number is still small, and much smaller than the number of target FIs
- **DEs w.r.t. η provide constraints as $\eta \to 0^+$**
  - Number of MIs at $\eta \to 0^+$ can be minimized

Remarks: effect of η

- **Do reduction relations become more complicated with η?**
  - No! Just the opposite!
  - The mass dimension of reductions relations becomes smaller

Remarks: series representations

- **Quantities present in all Feynman integrals:**
  - Space-time dimension $D \to 4$ and Feynman prescription $\eta \to 0^+$
- **Baikov's series representation:**
  - Asymptotic expansion of FIs at $D \to \infty$ — Baikov 0507053
  - According our test: calculation of the series is expensive
  - We still try hard to see if it is possible to improve the speed
- **Our series representation:**
  - Asymptotic expansion of FIs at $\eta \to \infty$
  - Calculation is cheaper, due to all coefficients are polynomials

See also S. Laporta's talk for the idea of "numerical representation"

Summary

- A new (series) representation for Feynman integrals, translates loop integration to the problem of performing analytical continuations
- A general strategy to do reduction
- A general strategy to evaluate MIs
- Two–loop examples: our method is correct and efficient

Future plan

- A package to do systematic reduction within our method
  - Express all FIs as linear combinations of MIs
- A package to calculate MIs within our method
  - Can be thought as a multi-loop version of "looptools"

Thank you!