TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

Polynomials associated to graphs, matroids, and configurations

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joint work with

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FB Mathematik, TU Kaiserslautern

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"The Mathematics of Linear Relations between Feynman Integrals"

workshop at MITP

Graphs and spanning trees

We consider (connected) graphs $G = (V, E)$ with vertex set V and edge set E.

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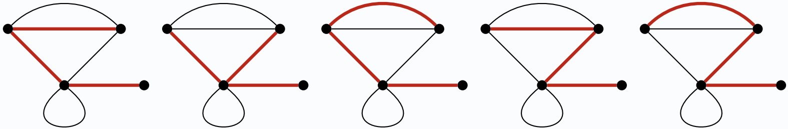

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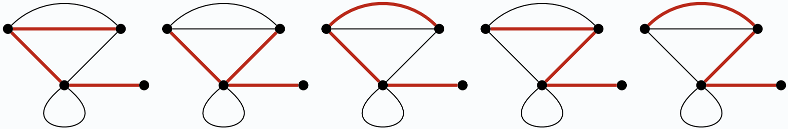

- Write $\mathcal{T}_G$ for the set of all spanning trees of G.

Kirchhoff's theorem

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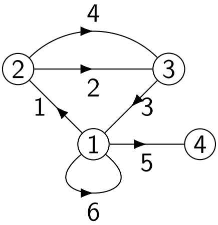

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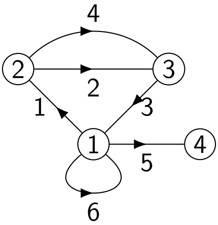

$$\delta = \begin{pmatrix} -1 & 0 & 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

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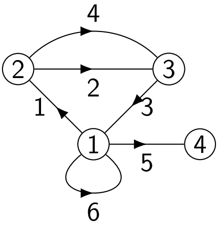

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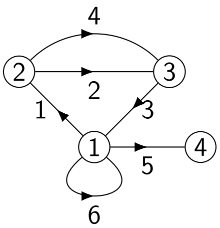

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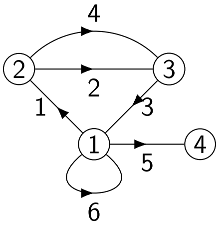

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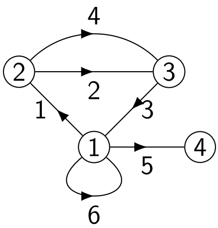

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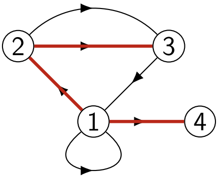
Kirchhoff's matrix-tree theorem

The number of spanning trees of G equals $\det AA^t$.

Graph polynomial

To see the individual spanning trees



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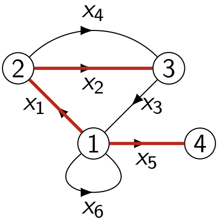
To see the individual spanning trees, associate variables with the edges.



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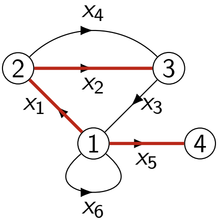

$$X := \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_6 \end{pmatrix}$$

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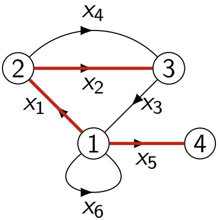

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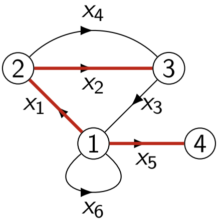

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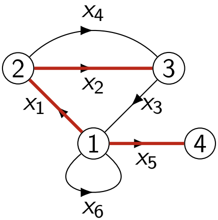

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$$Q := AXA^t = \begin{pmatrix} x_1 + x_2 + x_4 & -x_2 - x_4 & 0 \\ -x_2 - x_4 & x_2 + x_3 + x_4 & 0 \\ 0 & 0 & x_5 \end{pmatrix}$$

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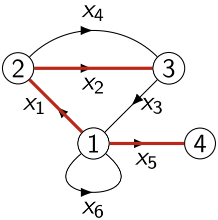

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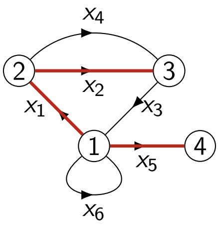

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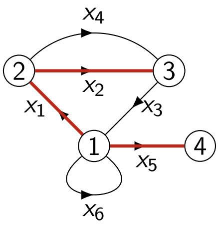

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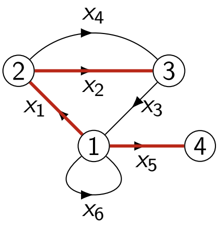

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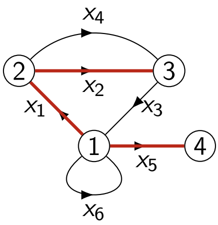

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Definition

ψ_G is the (first) graph polynomial or Kirchhoff polynomial of G.

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Graph polynomials are important in the case of Feynman diagrams.

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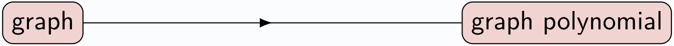
Marcolli:

"Graph hypersurfaces tend to have singularity loci of small codimension."

Approach

Generalization



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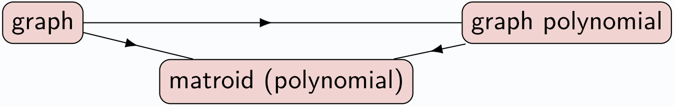

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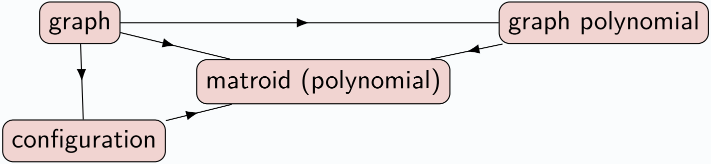

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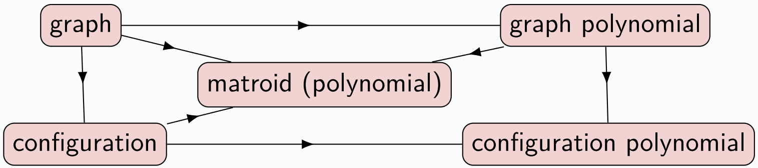

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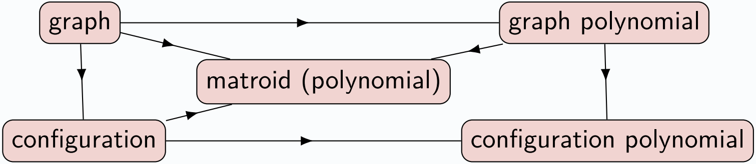

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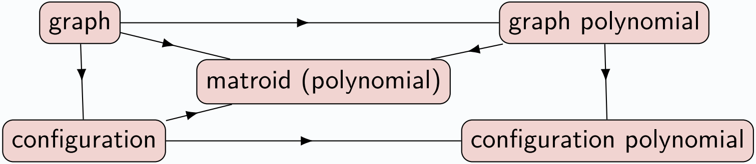

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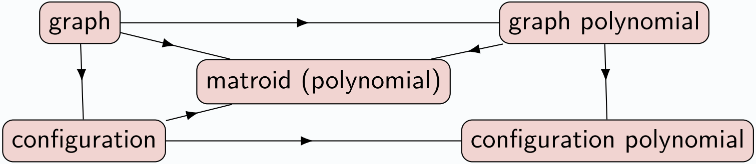

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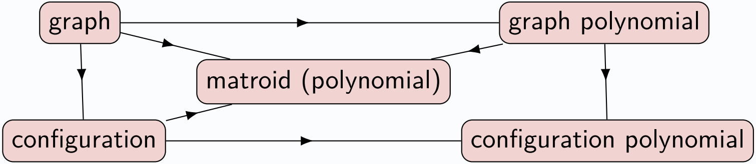

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- The second "graph polynomial" is only a configuration polynomial.

From graphs to configurations

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 & & \| & & \Cup & & \Cup & & \| & & \\
 & & \langle 1_C \mid C \text{ cycle}\rangle_{\mathbb{K}} & & (s \to t) & \longmapsto & t - s & & \mathbb{K} & & \\
 & & & & & & & & \Cup & & \\
 & & & & & & v & \longmapsto & 1 & &
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 & & & & \nwarrow & & \swarrow & & & & \\
 & & & & & W_G & & & & &
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$W_G \hookrightarrow \mathbb{K}^E$, $\quad \mathbb{K}^V \twoheadrightarrow W_G$

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Matroids axiomatize finite sets of vectors (configurations) in a vector space.

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- $\psi_G = \psi_{\mathrm{M}_G} = \psi_{W_G}$ is both a matroid and configuration polynomial.
- More generally, $\psi_W = \psi_{\mathrm{M}}$ for regular M (up to scaling variables).

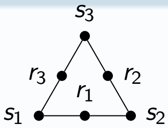
Definition

- $X_{\mathrm{M}} := \mathrm{Spec}(\mathbb{K}[x]/\langle \psi_{\mathrm{M}} \rangle)$ is the matroid hypersurface of M.
- $X_W := \mathrm{Spec}(\mathbb{K}[x]/\langle \psi_W \rangle)$ is the configuration hypersurface of W.

Configuration polynomial vs. matroid polynomial

Example

The 3-whirl matroid W^3 is realized by 6 points in $\mathbb{P}^2$ with depicted collinearities.



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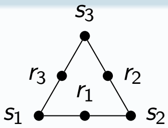

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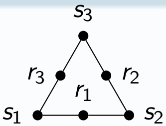

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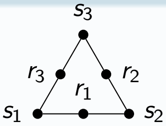

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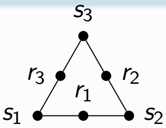

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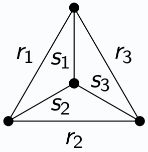

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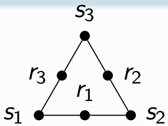

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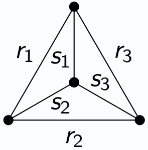

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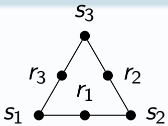

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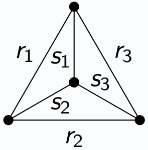

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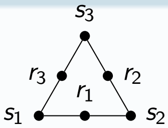

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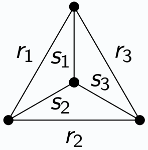

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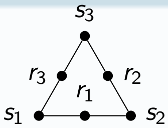

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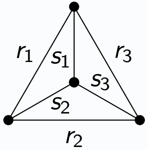

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X_W has (maximal) codimension 3 singular locus in $\mathbb{K}^6$, for X_{M} it is 4.

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Proposition (Delphine Pol)

$\operatorname{rk} \mathrm{M}$	1	2	3	4
$\#\{X_W\}/\sim$	1	2	6	∞?

where $X \sim X'$ if $X' = X \times \mathbb{K}$.

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$\operatorname{codim}_{\mathbb{K}^E} \Delta_W \leq 3$ *and equality makes* Δ_W *Cohen–Macaulay.*

Cohen–Macaulay rings/schemes

Hierarchy of rings (commutative local/graded Noetherian)

regular → hypersurface → complete intersection → Gorenstein → Cohen–Macaulay

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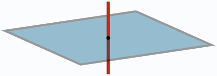

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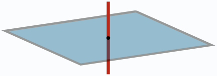

Cohen–Macaulayness $\Longrightarrow$ purity

Cohen–Macaulay rings/schemes

Hierarchy of rings (commutative local/graded Noetherian)

regular → hypersurface → complete intersection → Gorenstein → Cohen–Macaulay

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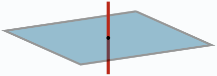

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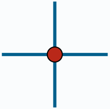

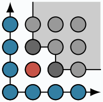

The main result

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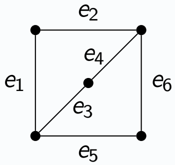
- The intersection of two different components of X_W yields components of Σ_W of codimension 2.

Embedded points in the Jacobian scheme

Example

The prism matroid is associated to the graph



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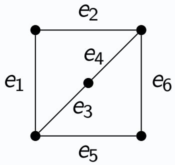

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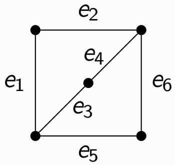

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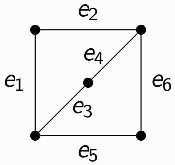

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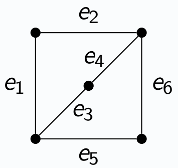

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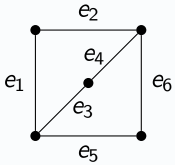

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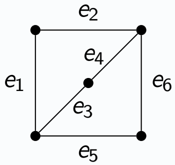

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Thank you for your attention!

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