Kite diagram through Symmetries of Feynman Integrals

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Review and References for Kite Diagram through SFI

- Work motivated from:
  1. arXiv:1507.01359:Symmetries of Feynman integrals and the Integration By Parts method. Barak Kol
  2. arXiv:1604.07827:The algebraic locus of Feynman integrals. Barak Kol
  3. arXiv:1606.09257:Bubble diagram through the Symmetries of Feynman Integrals method. Barak Kol
  4. arXiv:1704.02187:Vacuum seagull: Evaluating a three-loop Feynman diagram with three mass scales. Barak Kol et. al
  5. arXiv:1804.01175:Algebraic aspects of when and how a Feynman diagram reduces to simpler ones. Barak Kol
  6. arXiv:1807.07471:Two-loop vacuum diagram through the Symmetries of Feynman Integrals method. Barak Kol
- Talk based on : PhysRevD.99.045018 Kite diagram through Symmetries of Feynman Integrals. Barak Kol, Subhajit Mazumdar

Outline of the talk

- Motivation
- Brief Review of SFI
- Introduction to Kite Diagram
- G Group and SFI Equation Set for Kite
- Solution at the Algebraic Locus
- Tests at special cases

Motivation

- Feynman diagrams and the associated integrals are at the computational core of quantum field theory and their evaluation attracts considerable attention.
- It is necessary for experimental design and data analysis including at the LHC.
- Several important and well-known methods for their computation are Integration By Parts (IBP), Differential Equations (DE) method, Dimensional Recurrence, Canonical Basis for DE etc.

Brief Review: SFI method

- The Symmetries of Feynman Integrals (SFI) is a method for evaluating Feynman Integrals which exposes a novel continuous group associated with the diagram which depends only on its topology and acts on its parameters.
- It reduces the Feynman Integral to Line Integral over simpler diagrams.

Brief Review: Basic Set up

- An associated Feynman Integral with a diagram with L loops, P propagators and N external legs is given by

$$I(m_1^2, ..., m_P^2, p_i \cdot p_j) = \int \frac{d^d l_1 ... d^d l_L}{(k_1^2 - m_1^2)...(k_P^2 - m_P^2)}$$

- where

$$k_i = A_{ia} l_a + B_{ij} p_j$$

- Integral is function of mass squares and kinematical invarinats, which is known as "Parameter Space X" of the Diagram.

SFI Equation Set

- The change of integration variables $l'_a = l_a + \epsilon_{ab} q_b$ shouldn't affect the integral. Where q_a are linear combination of loop momenta and external momenta.
- So we have the identity

$$0 = \int dl \frac{\partial}{\partial l^\mu} k^\nu \tilde{I}$$

- This gives us a set of partial differential equations, namely,

$$c^\mu \, I + T x_j^\mu \, \partial^j \, I + J^\mu = 0$$

where J^μ depends on simpler diagrams.

- So the solution for the Integral is given by a line integral over sources.

$$\hat{I}(x) = \hat{I}(x_0) + \int_{x_0}^{x} J^\alpha(\xi) d\xi_\alpha$$

Algebraic Locus

- If one can find a left null vector for Tx matrix ,at a particular locus in parameter space, of the SFI equation set, it reduces it to an Algebraic Equation.
- Then the original Feynman Integral in study is given by linear combinations of simpler feynman integrals.
- This is known as Algebraic Locus and the linear combination is known as Algebraic Solution.

Systematic Procedure of getting Stabilizers and Invariants

- The Algebraic locus can be obtained systematically involving the maximal minors.
- We evaluate the maximal minors

$$M_A^I(x) = \epsilon_{Aa_1\dots a_r}\epsilon^{Ii_1\dots i_r}Tx_{i_1}^{a_1}\dots Tx_{i_r}^{a_r}$$

- One can show the factorization of the maximal minor gives

$$M_A^I(x) = S(x)Orb^I(x)Stb_A(x)$$

The Kite Diagram

- The Kite

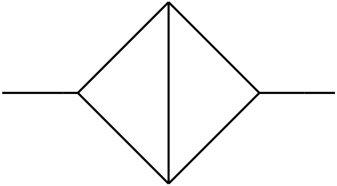


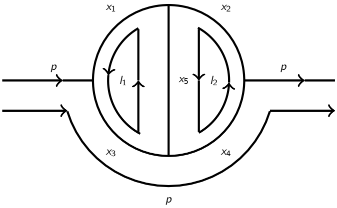
Figure: The kite diagram drawn in a way which explains its name.

Earlier works on kite

- This diagram appeared in the electron propagator renormalization of QED where a single mass scale was studied.
- The Integration By Parts method allowed a reduction of the massless case to a linear combination of simpler diagrams.
- QCD required two mass-scales and was studied in 4d in where the diagram was reduced to a line integral over logarithms and was called by the rather general name “the master diagram”.
- The diagram was also encountered while studying gluon splitting in QCD.
- The 3d massless version of the diagram was found to be essential in the second post-Newtonian approximation (2PN) of the two-body problem in Einstein’s gravity.

The Kite Integral

- The kite diagram with its parameters and a choice of currents.



- The associated integral

$$
\begin{aligned}
I \quad & (p^2; x_1, x_2, x_3, x_4, x_5) = \\
= \quad & \int \frac{d^d l_1 \, d^d l_2}{(l_1^2 - x_1)(l_2^2 - x_2)((l_1 + p)^2 - x_3)((p + l_2)^2 - x_4)((l_1 - l_2)^2 - x_5)}
\end{aligned}
$$

The Kite Integral

- The integral is a function of six parameters: five mass-squares and a single kinematical invariant, namely p^2. The parameter space X

$$X = \Big\{(x_1, ..., x_5, x_6) = (\,(m_1)^2, \ldots, (m_5)^2, p^2)\Big\}$$

- The figure defines our choice of loop currents l_1 and l_2 and the routing of p. We consider a general spacetime dimension d and the mass dimension of the integral is $2\,d - 10$.

Symmetry and Potential Numerators

- **The SFI group**. G is known to be a subset of certain triangular matrices

$$G \subseteq T_{L,n-1} \equiv T_{2,1}$$

 where $T_{L,n-1}$ represents the block upper triangular matrices such that the first block is of size L and the second one is $n-1$.
- The obstructions for $T_{2,1}$ generators are related to potential numerators of the diagram. The potential numerators are the quotient of the quadratics by the squares (of propagator currents)

$$\begin{aligned} Num &= Qd/Sq = \\ &= \mathrm{Sp}\left\{l_1^2, l_2^2, l_1\cdot l_2, p\cdot l_1, p\cdot l_2, p^2\right\}/\mathrm{Sp}\left\{l_1^2, l_2^2, (l_1+p)^2, (l_2+p)^2, (l_1-l_2)^2, p^2\right\} = \emptyset \end{aligned}$$

Symmetry and Potential Numerators

- So the kite does not have potential numerators

$$G = T_{2,1} \equiv \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$$

and the number of equations is

$$\dim(T_{2,1}) = 7 \ .$$

- More precisely the Lie algebra is $T_{2,1}$ and the group G consists of invertible upper triangular matrices.

SFI Equation Set

- **The SFI equation set.** As a basis for the space of generators we choose

$$\begin{pmatrix} E^1 \\ E^2 \\ E^3 \\ E^4 \\ E^5 \\ E^6 \\ E^7 \end{pmatrix} = \begin{pmatrix} l_1\,\partial_l^1 \\ -(l_1 - p)\partial_l^1 \\ l_1(\partial_l^2 - \partial_l^1) \\ l_2\,\partial_l^2 \\ -(l_2 - p)\partial_l^2 \\ l_2(\partial_l^1 - \partial_l^2) \\ l_1\,\partial_l^1 + l_2\partial_l^2 + p\partial_p \end{pmatrix}$$

where $\partial_l^a \equiv \partial/\partial_{l_a}$.

- The equations are given by the usual SFI form

$$c^a\,I + Tx_j^a\,\partial^j\,I + J^a = 0$$

SFI Equation Set

- where c^a, Tx^a_j and J^a shall be defined immediately within the above-mentioned basis. The vector of constants, c^a, is given by

$$c^a = \begin{pmatrix} d-4 \\ d-4 \\ d-4 \\ d-4 \\ d-4 \\ d-4 \\ 2\,d-10 \end{pmatrix} .$$

SFI Equation Set

- The generator matrix $Tx^a_j \partial^j$ is given by

$$Tx^a_j \partial^j = -2 \begin{pmatrix} x_1 & s^6_L & 0 & 0 & s^2 & 0 \\ s^6_L & x_3 & 0 & 0 & s^4 & 0 \\ s^2 & s^4 & 0 & 0 & x_5 & 0 \\ 0 & 0 & x_2 & s^6_R & s^1 & 0 \\ 0 & 0 & s^6_R & x_4 & s^3 & 0 \\ 0 & 0 & s^1 & s^3 & x_5 & 0 \\ x_1 & x_3 & x_2 & x_4 & x_5 & x_6 \end{pmatrix} \begin{pmatrix} \partial^1 \\ \partial^3 \\ \partial^2 \\ \partial^4 \\ \partial^5 \\ \partial^6 \\ \partial^7 \end{pmatrix} .$$

- The s variables are defined as follows

$$\begin{aligned} s^1 &:= (x_5 + x_2 - x_1)/2 & s^2 &:= (x_5 + x_1 - x_2)/2 \\ s^3 &:= (x_5 + x_4 - x_3)/2 & s^4 &:= (x_5 + x_3 - x_4)/2 \\ s^6_L &:= (x_1 + x_3 - x_6)/2 & s^6_R &:= (x_2 + x_4 - x_6)/2 \end{aligned}$$

Sources: Two Kinds

- The source vector J^a is given by

$$J^a = \begin{pmatrix} \partial^5 O_2 - (\partial^3 + \partial^5) O_1 \\ \partial^5 O_4 - (\partial^1 + \partial^5) O_3 \\ \partial^1 O^2 + \partial^3 O^4 - (\partial^1 + \partial^3) O_5 \\ \partial^5 O_4 - (\partial^4 + \partial^5) O_2 \\ \partial^5 O_3 - (\partial^2 + \partial^5) O_4 \\ \partial^2 O^1 + \partial^4 O^3 - (\partial^2 + \partial^4) O_5 \\ 0 \end{pmatrix}$$

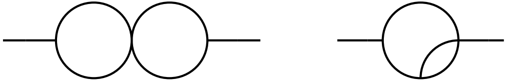
- O_i denote the diagram gotten by omitting, or contracting, the i'th propagator. Two possible topologies appear.
- 

Figure: The two source topologies (a) figure 8 (b) propagator seagull

Geometry of parameter space

- G-**orbit co-dimension and 6-minors.** The equation set consists of 7 equations in a 6 dimensional parameter space. The dimension of the G-orbit through any point $x \in X$ is given by the rank of Tx at that point.
- By the method of maximal minor M_a is found to be

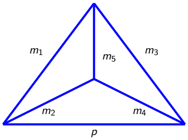
$$M_a = 4\,p^2\,B_3(x)\,K_a(x)$$

where the notation $B_3(x), K_a(x)$ will be defined now.
$B_3(x)$ is a cubic polynomial defined by

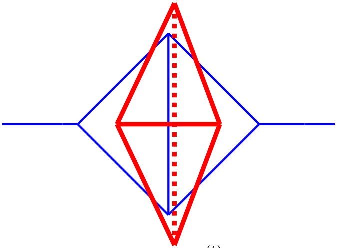
$$\begin{aligned} B_3 &= x_1\,x_4(x_1+x_4)+x_2\,x_3(x_2+x_3)+x_5\,x_6(x_5+x_6)+ \\ &+ x_1\,x_2\,x_5+x_1\,x_3\,x_6+x_2\,x_4\,x_6+x_3\,x_4\,x_5+ \\ &- (x_1\,x_4(x_2+x_3+x_5+x_6)+x_2\,x_3(x_1+x_4+x_5+x_6) \\ &+ x_5\,x_6(x_1+x_2+x_3+x_4)) \end{aligned}$$

Baikov Polynomial

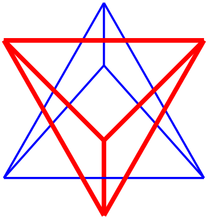
- According to the Cayley-Menger formula B_3 describes the squared volume of the dual tetrahedron.



(a)



(b)



Global Stabilizer

- B_3 appeared in the physics literature in the work of Baikov on the 3-loop vacuum diagram (tetrahedron)
  The vector K_a is given by

$$K = \begin{pmatrix} -\partial^2 B_3 \\ -\partial^4 B_3 \\ \lambda_L \\ \partial^1 B_3 \\ \partial^3 B_3 \\ -\lambda_R \\ 0 \end{pmatrix}^T$$

Definition of the Heron / Källén invariant

$$\lambda := x^2 + y^2 + z^2 - 2\,x\,y - 2\,x\,z - 2\,y\,z$$

$$\lambda_L = \lambda(p^2, x_1, x_3)$$
$$\lambda_R = \lambda(p^2, x_2, x_4)$$

Algebraic Constraint

- K_a is a global stabilizer, namely it satisfies

$$K_a\, Tx_j^a = 0$$

- Since $K_a\, c^a = 0$ multiplying the equation set by K_a generates a global constraint among the sources, namely,

$$K_a\, J^a = 0$$

- This is defined as Algebraic Constraint.

SFI maximally effective in Kite

- The dimension of the G-orbit is generically 6. Since $\dim(X) = 6$
$$\mathrm{codim}(G - \mathrm{orbit}) = 0 \ .$$
- This means that SFI is maximally effective for the kite diagram.

Homogeneous Solution

- **Homogeneous solution**. We found 5 such equations with zero constants and we switch the sources to zero.

$$0.I_0 + T\partial I_0 = 0$$

- Therefore I_0 is annihilated by G_{cf} and it must be a function of G_{cf} invariants.
- The dimension constant free Subgroup G_{cf} says that it has two independent invariants. As $dim(G_{cf} - orbit) = 4$. The invariants of G_{cf} turn out to be p^2, B_3.

Homogeneous Solution

- Substituting $I_0 = I_0(p^2, B_3)$ into the equation set (with J^a put to zero) we obtain the set

$$
\begin{aligned}
(d-4)I_0 - 2B_3\frac{\partial I_0}{\partial B_3} &= 0 \\
(d-5)I_0 - 3B_3\frac{\partial I_0}{\partial B_3} - p^2\frac{\partial I_0}{\partial p^2} &= 0\,.
\end{aligned}
$$

- Solving this we get that the homogeneous solution is

$$
I_0 = p^{2(1-\frac{d}{2})} B_3^{\frac{d-4}{2}}\,.
$$

Algebraic locus and solution

- At the singular locus, namely when $B_3(x) = 0$ or $p^2 = 0$ the dimension of the G orbit is reduced and accordingly an additional stabilizer appears.
- Given a stabilizer Stb_a, if the associated constant is non-zero, namely $Stb_a\, c^a \neq 0$ one can reduce the diagram to a linear combination of simpler ones by multiplying the equation set on the left by the stabilizer.

Algebraic locus and solution

- B_3 **locus**. At $B_3 = 0$ the global stabilizer K splits into a pair of stabilizers K^L, K^R as follows

$$K^L = \begin{pmatrix} -\partial^2 B_3 \\ -\partial^4 B_3 \\ \lambda_L \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T \qquad K^R = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\partial^1 B_3 \\ -\partial^3 B_3 \\ \lambda_R \\ 0 \end{pmatrix}^T$$

Algebraic Solution of the Kite

- **Algebraic solution**. The algebraic solution is now gotten by multiplying the equation set on the left by an arbitrary linear combination $\alpha_L K^L + \alpha_R K^R$. We notice that

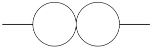
$$\left(\alpha_L K_a^L + \alpha_R K_a^R\right) c^a = (\alpha_L + \alpha_R)(d-4)\,\partial^5 B_3$$

- So the Algebraic Solution is

$$(4-d)\, I|_{B3=0} = \frac{\left(\alpha_L K_a^L + \alpha_R K_a^R\right) J^a}{(\alpha_L + \alpha_R)\,\partial^5 B_3}$$

Tests Special Cases

- In the massless case $m_1 = \cdots = m_5 = 0$ it was shown already in[Chetyrkin, Tkachov 1981] that the diagram can be reduced as follows

$$I_{\text{massless}} = \frac{2}{d-4}\Bigg($$

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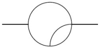
$$\Bigg)$$

- The algebraic solution generalizes the reduction of the massless case to the most general parameters, namely $B_3(m_1^2, \ldots, m_5^2, p^2) = 0$.
  The case $m_3 = m_4 = m_5 = 0$ is of special interest. In this case $B_3 = 0$ simplifies to the following two alternative forms

$$(4-d)I_{x_3=x_4=x_5=0} = \frac{(x_2 - x_1)J^2 + (x_1 - x_6)J^3}{x_2 - x_6} = \frac{(x_1 - x_2)J^5 + (x_2 - x_6)J^6}{x_1 - x_6}$$

This case falls into the applicability regime of the "diamond rule" (with $L = S = 1$)[Ruijl, Ueda and Vermaseren 2015].

$p^2 = 0$ locus

- p^2 **locus**. In this case we too we find a pair of stabilizers, they are given by

$$t^L = \begin{pmatrix} -2x_3 \\ 2x_1 \\ 0 \\ -2s_B^5 \\ 2s_T^5 \\ x_4 - x_2 \\ x_3 - x_1 \end{pmatrix}^T \qquad t^R = \begin{pmatrix} -2s_B^5 \\ 2s_T^5 \\ x_3 - x_1 \\ -2x_4 \\ 2x_2 \\ 0 \\ x_4 - x_2 \end{pmatrix}^T$$

- The global stabilizer is a linear combination given by

$$K|_{p^2=0} = (x_2 - x_4)\, t^L + (x_3 - x_1)\, t^R$$

Discussion

- The G-orbits were found to be 6-dimensional in our 6d parameter space X, namely the orbit co-dimension is zero. This means that for this diagram the SFI method would be maximally effective.
- On the surface $B_3 = 0$ the integral degenerates into a linear combination of simpler diagrams and is given by the algebraic soln., thereby providing a maximal generalization of the massless case. This is our central result.

THANK YOU