Functional reduction of Feynman integrals

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March 21st, Mainz, Germany

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Method for deriving FE

Three methods for deriving functional equations (FE) were proposed

- FE from recurrence relations
- FE from algebraic relations for propagators
- FE from algebraic relations for deformed propagators

The simplest is the method based on algebraic relations for propagators.
The following algebraic relation between the products of n propagators was discovered:

$$\prod_{r=1}^{n} \frac{1}{P_r} = \frac{1}{P_0} \sum_{r=1}^{n} x_r \prod_{\substack{j=1 \\ j \neq r}}^{n} \frac{1}{P_j},$$

where

$$P_j = (k_1 - p_j)^2 - m_j^2 + i\eta.$$

This equation can be fulfilled for arbitrary k_1 by imposing conditions on x_j, m_0, p_0.

Algebraic relation for propagators

Example:

$$\frac{1}{P_1 P_2} = \frac{x_1}{P_0 P_2} + \frac{x_2}{P_1 P_0}.$$

Multiplying both sides by $P_0 P_1 P_2$ we get:

$$(k_1 - p_0)^2 - m_0^2 = x_1[(k_1 - p_1)^2 - m_1^2] + x_2[(k_1 - p_2)^2 - m_2^2]$$

or

$$(1 - x_1 - x_2)k_1^2 + 2k_1(x_1 p_1 + x_2 p_2 - p_0) + p_0^2 - m_0^2 + x_1(m_1^2 - p_1^2) + x_2(m_2^2 - p_2^2) = 0.$$

Assuming that k_1 does not depend on p_j we obtain system of equations:

$$1 - x_1 - x_2 = 0, \qquad p_0 = x_1 p_1 + x_2 p_2, \qquad p_0^2 - m_0^2 + x_1(m_1^2 - p_1^2) + x_2(m_2^2 - p_2^2) = 0.$$

This system of 3 equations can be solved by appropriate choice of $x_j,\ m_0,\ p_0$.

Algebraic relation for propagators

For arbitrary n the following system of equations holds

$$\sum_{r=1}^{n} x_r = 1, \qquad p_0 = \sum_{j=1}^{n} x_j p_j .$$

$$m_0^2 - \sum_{k=1}^{n} x_k m_k^2 + \sum_{j=1}^{n} \sum_{l=1}^{j-1} x_j x_l s_{lj} = 0,$$

where

$$s_{ij} = (p_i - p_j)^2 .$$

Solutions of this system of equations will depend on $n-2$ *arbitrary parameters* x_i *and one arbitrary mass* m_0.

Considering p_j as external momenta and integrating w.r.t. k_1 we get functional equation .

Functional reduction of integrals

One can use also more general algebraic relation including several auxiliary propagators:

$$\frac{1}{P_1 P_2} = \frac{x_1}{Q_1 Q_2} + \frac{x_2}{Q_1 P_2} + \frac{x_3}{Q_1 P_1} + \frac{x_4}{Q_2 P_2} + \frac{x_5}{Q_2 P_1}, \qquad (11 \ parameters)$$

$$\frac{1}{P_1 P_2 P_3} = \frac{x_1}{Q_1 Q_2 P_1} + \frac{x_2}{Q_1 Q_2 P_2} + \frac{x_3}{Q_1 Q_2 P_3} + \frac{x_4}{Q_1 Q_3 P_1} + \frac{x_5}{Q_1 Q_3 P_2} + \frac{x_6}{Q_1 Q_3 P_3}$$
$$+ \frac{x_7}{Q_2 Q_3 P_1} + \frac{x_8}{Q_2 Q_3 P_2} + \frac{x_9}{Q_2 Q_3 P_3} + \frac{x_{10}}{Q_1 P_1 P_2} + \frac{x_{11}}{Q_1 P_1 P_3} + \frac{x_{12}}{Q_1 P_2 P_3} + \frac{x_{13}}{Q_2 P_1 P_2}$$
$$+ \frac{x_{14}}{Q_2 P_1 P_3} + \frac{x_{15}}{Q_2 P_2 P_3} + \frac{x_{16}}{Q_3 P_1 P_2} + \frac{x_{17}}{Q_3 P_1 P_3} + \frac{x_{18}}{Q_3 P_2 P_3} + \frac{x_{19}}{Q_1 Q_2 Q_3},$$

$$(28 \ parameters)$$

where

$$Q_j = (k_1 - q_j)^2 - M_j^2, \qquad q_j = \sum_{n=1}^{2} z_{jn} p_n.$$

The both sides of algebraic relations can be multiplied by other propagators raised to some powers or by loop integrals and the integration w.r.t. k_1 will give us functional equations for multileg/ multiloop integrals.

In mathematics integrals of algebraic functions are known as Abelian integrals. Abelian integral is an integral in the complex plane of the form

$$\int_{z_0}^{z} R(x,y)dx,$$

where $R(x,y)$ is an arbitrary *rational* function of the two variables x and y. These variables are related by the equation

$$F(x,y) = 0,$$

where $F(x,y)$ is an irreducible polynomial in y,

$$F(x,y) \equiv \phi_n(x)y^n + ... + \phi_1(x)y + \phi_0(x),$$

whose coefficients $\phi_j(x)$, $j = 0, 1, ...n$ are rational functions of x. Abelian integrals are natural generalizations of elliptic integrals, which arise when

$$F(x,y) = y^2 - P(x),$$

where $P(x)$ is a polynomial of degree 3 and 4. If degree of the polynomial is greater than 4 then we have *hyperelliptic integral*.

Abel's theorem

At the beginning of 19th century Abel proved addition theorem for these integrals. Let C and C' be plane curves given by the equations

$$\begin{aligned} C: \quad F(x,y) &= 0, \qquad \textit{fixed curve} \\ C': \quad \phi(x,y) &= 0. \qquad \textit{variable curve} \end{aligned}$$

These curves have n points of intersections $(x_1, y_1), \ldots \ (x_n, y_n)$, where n is the product of degrees of C and C'. Let $R(x,y)$ be a rational function of x and y where y is defined as a function of x by the relation $F(x,y) = 0$.

Consider the sum

$$I = \sum_{i=1}^{n} \int_{x_0,y_0}^{x_i,y_i} R(x,y)dx$$

Integrals being taken from a fixed point to the n points of intersections. If some of the coefficients $a_1, a_2, \ldots, a_k$ of $\phi(x,y)$ are regarded as continuous variables, the points (x_i, y_i) will vary continuously and hence I will be a function, whose form is to be determined, of the variable coefficients $a_1, a_2, \ldots, a_k$.

Abel's theorem

Abel's theorem:
The partial derivatives of the sum I , with respect to any of the coefficients of the variable curve $\phi(x, y) = 0$, is a *rational* function of the coefficients and hence I is equal to a *rational* function of the coefficients of $\phi(x, y) = 0$, plus a finite number of logarithms or arc tangents of such rational functions.

Important: integrals themselves can be rather complicated transcendental functions but their sum can be simple.

Atle Selberg
(Fields medal, Wolf Prize, honorary Abel Prize) about Abel's addition theorem:

It still stands for me as pure magic. Neither with Gauss nor Riemann, nor with anybody else, have I found anything that really measures up to this.

Abel's theorem

Addition formula for elliptic integral of the second type:

$$E(k,x) = \int_0^x \frac{(1-k^2x)dx}{\sqrt{x(1-x)(1-k^2x)}}.$$

$$\begin{aligned} C: &\quad y^2 = x(1-x)(1-k^2x), && \text{fixed curve} \\ C': &\quad y = ax+b. && \text{variable curve} \end{aligned}$$

The elimination of y between two equations will give us as the abscissae x_1, x_2, x_3 of the points of intersection the three roots of the equation:

$$\phi(x) = k^2x^3 - (1+k^2+a^2)x^2 + (1-2ab)x - b^2 = 0.$$

The corresponding sum will be

$$I(a,b) = \int_0^{x_1} R(x,y)dx + \int_0^{x_2} R(x,y)dx + \int_{1/k^2}^{x_3} R(x,y)dx,$$

where

$$R(x,y) = \frac{1-kx^2}{y}.$$

Abel's theorem

Finding derivatives

$$\frac{\partial I(a,b)}{\partial a}, \qquad \frac{\partial I(a,b)}{\partial b},$$

solving simple system of differential equations and fixing parameters a, b from the boundary conditions, yields

$$\int_0^{x_1} \frac{dx(1-k^2x)}{\sqrt{x(1-x)(1-k^2x)}}$$
$$= \int_0^{x_2} \frac{dx(1-k^2x)}{\sqrt{x(1-x)(1-k^2x)}} + \int_0^{x'} \frac{dx(1-k^2x)}{\sqrt{x(1-x)(1-k^2x)}} + 2k^2\sqrt{x_1x_2x'}$$

where

$$x' = \frac{1}{x_1x_2}\left(\frac{x_1y_2 + x_2y_1}{1-k^2x_1x_2}\right)^2.$$

This is addition formula for elliptic integrals of the second type in the Riemann normal form.

Functional reduction of integrals

Unfortunately there is no analog of the Abel addition theorem for the algebraic functions depending on several variables.

There is something in common between our derivation and Abel's addition theorem. Fixation of arguments of functions in the addition theorem was done by choosing variable curve.
In our case arguments of Feynman integrals in FE are determined from the addition of the variable sum of terms to the integrand.

The goal of my talk : to present a systematic, computer oriented method for reducing complicated integrals to simpler ones by using functional equations.

In fact the method is nothing but solution of the FE for Feynman integrals and these solutions represent complicated integrals in terms of simpler integrals, integrals with fewer variables.

These integrals can be considered as 'boundary integrals' or integrals living on some hypersurfices.

Analytic results for such 'boundary integrals' can be obtained by some other methods.

Functional reduction of integrals

A functional equation is an equation which involves independent variables, known functions, unknown functions and constants. Very frequently the equation relates the value of a function (or functions) at some point with its values at other points.

In general solving functional equations can be very difficult, but there are some common methods of solving them. Systematic enumeration of such methods with examples is given in

Castillo, E., Iglesias, A., Ruiz-Cobo, R.,
Functional Equations in Applied Sciences,
Elsevier Science, Mathematics in Science and Engineering, 2004.

For solving FE for Feynman integrals the following methods, described in this book, can be used :

1. Replacement of variables by given values
2. Transforming one or several variables
3. Using a more general equation
4. Iterative methods
5. Mixed methods

To some extent all these methods can be used in finding solutions of FE for Feynman integrals.

Functional reduction of integrals

Let's consider one-loop propagator type integral. Integrating algebraic relation

$$\frac{1}{P_1 P_2} = \frac{x_1}{P_0 P_2} + \frac{x_2}{P_1 P_0}$$

w.r.t. k_1 yields:

$$I_2^{(d)}(m_1^2, m_2^2, s_{12}) = x_1 I_2^{(d)}(m_2^2, m_0^2, s_{20}) + x_2 I_2^{(d)}(m_1^2, m_0^2, s_{10}),$$

where

$$I_2^{(d)}(m_1^2, m_2^2;\ s_{12}) = \int \frac{d^d k_1}{i\pi^{d/2}} \frac{1}{[(k_1-p_1)^2 - m_1^2 + i\eta][(k_1-p_2)^2 - m_2^2 + i\eta]}.$$

$$x_1 = \frac{m_2^2 - m_1^2 + s_{12}}{2s_{12}} \pm \frac{\sqrt{4s_{12}m_0^2 - \lambda_{12}}}{2s_{12}}, \qquad x_2 = 1 - x_1,$$

$$s_{10} = (p_1 - p_0)^2 = \frac{2s_{12}(m_1^2 + m_0^2) - \lambda_{12}}{2s_{12}} \pm \frac{m_2^2 - m_1^2 - s_{12}}{2s_{12}} \sqrt{4s_{12}m_0^2 - \lambda_{12}},$$

$$s_{20} = (p_2 - p_0)^2 = \frac{2s_{12}(m_2^2 + m_0^2) - \lambda_{12}}{2s_{12}} \pm \frac{m_2^2 - m_1^2 + s_{12}}{2s_{12}} \sqrt{4s_{12}m_0^2 - \lambda_{12}},$$

$$\lambda_{ij} = -s_{ij}^2 - m_i^4 - m_j^4 + 2s_{ij}m_i^2 + 2s_{ij}m_j^2 + 2m_i^2 m_j^2.$$

Functional reduction of integrals

We will consider this equation as a FE for the function $I_2^{(d)}(m_1^2, m_2^2, s_{12})$ and will solve it by the method similar to that was used for solving Sincov's functional equation

$$f(x,y) = f(x,z) - f(y,z).$$

Setting $z = 0$ in this equation, we get the general solution

$$f(x,y) = g(y) - g(x),$$

where

$$g(x) = f(x,0).$$

I.e. we expressed the function $f(x,y)$ in terms of its *' boundary values'* .

In the same way, setting $m_0 = 0$ in the FE for $I_2^{(d)}$ we get its solution

$$I_2^{(d)}(m_1^2, m_2^2, s_{12}) = \overline{x}_1 I_2^{(d)}(m_2^2, 0, \overline{s}_{20}) + \overline{x}_2 I_2^{(d)}(m_1^2, 0, \overline{s}_{10}),$$

where

$$\overline{x}_j = x_j\big|_{m_0=0}\,, \qquad \overline{s}_{ij} = s_{ij}\big|_{m_0=0}\,.$$

Integral with three variables was expressed in terms of integrals with two variables.

Functional reduction of integrals

One can check that the obtained solution satisfies initial FE for $I_2^{(d)}$ with arbitrary m_0. Substituting solution into both sides of the equation

$$I_2^{(d)}(m_1^2, m_2^2, s_{12}) = x_1 I_2^{(d)}(m_2^2, m_0^2, s_{20}) + x_2 I_2^{(d)}(m_1^2, m_0^2, s_{10}),$$

we obtain two terms on the left – hand side and four terms on the right – hand side of the equation. After simplifying arguments, we find that on the right hand side two terms with arguments depending on m_0 cancel and the remaining two terms cancel two terms on the left hand side. With arbitrary m_0^2 this check is not quite trivial because we must substitute into

$$\overline{x}_1 = \frac{m_2^2 - m_1^2 + s_{12}}{2s_{12}} \pm \frac{\sqrt{-\lambda_{12}}}{2s_{12}}, \qquad \overline{x}_2 = 1 - \overline{x}_1,$$

$\overline{s}_{10}, \overline{s}_{20}$ instead of s_{12}.

Functional reduction of integrals

Integrating extended algebraic relation

$$\frac{1}{P_1 P_2} = \frac{x_1}{Q_1 Q_2} + \frac{x_2}{Q_1 P_2} + \frac{x_3}{Q_1 P_1} + \frac{x_4}{Q_2 P_2} + \frac{x_5}{Q_2 P_1},$$

w.r.t. momentum k_1 and by fixing the remaining free parameters, we get

$$I_2^{(d)}(m_1^2, m_2^2, s_{12}) = -\frac{\sqrt{-\lambda_{12}}}{s_{12}} I_2^{(d)}(0, 0, S_3)$$
$$+\frac{s_{12} - m_1^2 + m_2^2 + \sqrt{-\lambda_{12}}}{2s_{12}} I_2^{(d)}(0, m_2^2, S_1) + \frac{s_{12} + m_1^2 - m_2^2 + \sqrt{-\lambda_{12}}}{2s_{12}} I_2^{(d)}(m_1^2, 0, S_2),$$

where

$$S_1 = \frac{-\lambda_{12} + (m_1^2 - m_2^2 - s_{12})\sqrt{-\lambda_{12}}}{2s_{12}}, \qquad S_2 = \frac{-\lambda_{12} + (m_2^2 - m_1^2 - s_{12})\sqrt{-\lambda_{12}}}{2s_{12}},$$

$$S_3 = \frac{-\lambda_{12}}{s_{12}}.$$

Functional reduction of integrals

Several parameters x_j, z_{ij} were fixed to fulfill algebraic relation.
The remaining 5 free parameters were chosen so that at least one mass in each integral was equal to zero, the integral with kinematic argument increasing at $s_{12} \to \infty$ was eliminated and the following conditions were satisfied:

$$S_1 \longrightarrow \frac{m_2^4}{s_{12}} + O\left(\frac{1}{s_{12}^2}\right), \qquad S_2 \longrightarrow \frac{m_1^4}{s_{12}} + O\left(\frac{1}{s_{12}^2}\right)$$

The formula also provides convergent expansion at $\lambda_{12} \longrightarrow 0$ because

$$S_1 \sim O(\sqrt{-\lambda_{12}}), \quad S_2 \sim O(\sqrt{-\lambda_{12}}).$$

Functional reduction of integrals

Solution of FE will be expressed in terms of ratios of modified Cayley and Gram determinants:

$$\Delta_n \equiv \Delta_n(\{p_1, m_1\}, \ldots \{p_n, m_n\}) = \begin{vmatrix} Y_{11} & Y_{12} & \ldots & Y_{1n} \\ Y_{12} & Y_{22} & \ldots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \ldots & Y_{nn} \end{vmatrix}, \qquad Y_{ij} = m_i^2 + m_j^2 - s_{ij}$$

$$G_{n-1} \equiv G_{n-1}(p_1, \ldots, p_n) = -2 \begin{vmatrix} S_{11} & S_{12} & \ldots & S_{1\ n-1} \\ S_{21} & S_{22} & \ldots & S_{2\ n-1} \\ \vdots & \vdots & \ddots & \vdots \\ S_{n-1\ 1} & S_{n-1\ 2} & \ldots & S_{n-1\ n-1} \end{vmatrix}, \qquad S_{ij} = s_{in} + s_{jn} - s_{ij},$$

We will use also an indexed notation for Δ_n and G_{n-1}

$$\lambda_{i_1 i_2 \ldots i_n} = \Delta_n(\{p_{i_1}, m_{i_1}\}, \{p_{i_2}, m_{i_2}\}, \ldots, \{p_{i_n}, m_{i_n}\}),$$
$$g_{i_1 i_2 \ldots i_n} = G_{n-1}(p_{i_1}, p_{i_2}, \ldots, p_{i_n}).$$

In what follows solutions of FE will be expressed in terms of

$$r_{ij\ldots k} = -\frac{\lambda_{ij\ldots k}}{g_{ij\ldots k}}, \qquad r^{(n)}_{ij\ldots k} = \frac{\partial\ r_{ij\ldots k}}{\partial m_n^2}$$

Functional reduction of one-loop integrals

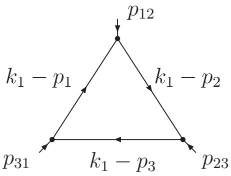
Vertex type integral

Let's consider integral with 3 massless propagators:

$$I_3^{(d)}(s_{23}, s_{13}, s_{12}) = \int \frac{d^d k_1}{i\pi^{d/2}} \frac{1}{P_1 P_2 P_3},$$

where

$$P_i = (k_1 - p_i)^2 + i\eta, \qquad s_{ij} = p_{ij}^2, \qquad p_{ij} = p_i - p_j.$$



Vertex type integral

In order to obtain FE for this integral we will use relationship for 3 propagators

$$\frac{1}{P_1P_2P_3} = \frac{x_1}{P_0P_2P_3} + \frac{x_2}{P_1P_0P_3} + \frac{x_3}{P_1P_2P_0},$$

where

$$P_j = (k_1 - p_j)^2 - m_j^2 + i\eta, \qquad p_0 = x_1p_1 + x_2p_2 + x_3p_3,$$

and parameters x_j, m_0 obey the following conditions:

$$x_1 + x_2 + x_3 = 1,$$
$$x_1x_2s_{12} + x_1x_3s_{13} + x_2x_3s_{23} - x_1m_1^2 - x_2m_2^2 - x_3m_3^2 + m_0^2 = 0.$$

Integration of this relationship with respect to k_1 yields FE:

$$\begin{aligned} I_3^{(d)}&(m_1^2, m_2^2, m_3^2;\ s_{23}, s_{13}, s_{12}) \\ &= x_1\ I_3^{(d)}(m_0^2, m_2^2, m_3^2;\ s_{23}, s_{30}, s_{20}) \\ &+ x_2\ I_3^{(d)}(m_1^2, m_0^2, m_3^2;\ s_{30}, s_{13}, s_{10}) \\ &+ x_3\ I_3^{(d)}(m_1^2, m_2^2, m_0^2;\ s_{20}, s_{10}, s_{12}). \end{aligned}$$

Vertex type integral

At $m_1 = m_2 = m_3 = m_0 = 0$ from this equation we get FE for the integral with massless propagators

$$
\begin{aligned}
I_3^{(d)}&(0,0,0;\ q_{23}, q_{13}, q_{12}) \\
&= z_1\ I_3^{(d)}(0,0,0;\ q_{23}, q_{30}, q_{20}) \\
&+ z_2\ I_3^{(d)}(0,0,0;\ q_{30}, q_{13}, q_{10}) \\
&+ z_3\ I_3^{(d)}(0,0,0;\ q_{20}, q_{10}, q_{12}).
\end{aligned}
$$

where

$$
\begin{aligned}
q_{10} &= q_{13} - q_{13}z_1 + (q_{12} - q_{13})z_2, \\
q_{20} &= q_{23} + (q_{12} - q_{23})z_1 - z_2 q_{23}, \\
q_{30} &= q_{13}z_1 + q_{23}z_2.
\end{aligned}
$$

and the following equations to be hold:

$$
\begin{aligned}
&z_1 + z_2 + z_3 = 1, \\
&z_1 z_2 q_{12} + z_1 z_3 q_{13} + z_2 z_3 q_{23} = 0.
\end{aligned}
$$

Vertex type integral

In order to obtain solution of this equation we use more general equation. Such an equation we get by setting $m_1 = m_2 = m_3 = 0$ and keeping arbitrary m_0^2 in the equation for the general mass case

$$
\begin{aligned}
I_3^{(d)}&(0,0,0;\ s_{23}, s_{13}, s_{12}) \\
&= x_1\ I_3^{(d)}(m_0^2, 0, 0;\ s_{23}, s_{30}, s_{20}) \\
&+ x_2\ I_3^{(d)}(0, m_0^2, 0;\ s_{30}, s_{13}, s_{10}) \\
&+ x_3\ I_3^{(d)}(0, 0, m_0^2;\ s_{20}, s_{10}, s_{12}).
\end{aligned}
$$

where

$$
\begin{aligned}
s_{10} &= m_0^2 + s_{13} - s_{13}x_1 + (s_{12} - s_{13})x_2, \\
s_{20} &= m_0^2 + s_{23} + (s_{12} - s_{23})x_1 - x_2 s_{23}, \\
s_{30} &= m_0^2 + s_{13}x_1 + s_{23}x_2.
\end{aligned}
$$

and the following conditions must be satisfied:

$$
\begin{aligned}
&x_1 + x_2 + x_3 = 1, \\
&x_1x_2\ s_{12} + x_1x_3\ s_{13} + x_2x_3\ s_{23} + m_0^2 = 0.
\end{aligned}
$$

Vertex type integral

To solve this FE means to express the integral $I_3^{(d)}(0,0,0;\ s_{23},s_{13},s_{12})$ in terms of functions with fewer variables.

By choosing arbitrary parameters we will try to express integrals in the right–hand side in terms of integrals with lesser number of variables. This means that we can try to impose some conditions on new variables $s_{j0},\ m_0^2$ like

$$
\begin{array}{llllll}
s_{10}=0, & s_{10}-s_{20}=0, & s_{10}-s_{30}=0, & s_{20}=0, & s_{20}-s_{30}=0, & s_{30}=0, \\
s_{10}-s_{12}=0, & s_{20}-s_{12}=0, & s_{30}-s_{12}=0, & s_{10}-s_{23}=0, & s_{20}-s_{23}=0, & \\
s_{30}-s_{23}=0, & s_{10}-s_{13}=0, & s_{20}-s_{13}=0, & s_{30}-s_{13}=0, & s_{10}-m_0^2=0, & \\
s_{20}-m_0^2=0, & s_{30}-m_0^2=0, & s_{10}+m_0^2=0, & s_{20}+m_0^2=0, & s_{30}+m_0^2=0. &
\end{array}
$$

We considered 1330 systems of equations, each consisting of 3 equations composed out of the above 21 equations

In 35 sec of CPU time, 7 solutions without square roots of Gram determinants were discovered.

Vertex type integral

Substituting these solutions into the FE we found that one of them gives the most compact and simple relationship:

$$I_3^{(d)}(0,0,0;\ s_{23},s_{13},s_{12}) \\ = r_{123}^{(1)}\ \xi_3^{(d)}(\mu_{123},s_{23}) + r_{123}^{(2)}\ \xi_3^{(d)}(\mu_{123},s_{13}) + r_{123}^{(3)}\ \xi_3^{(d)}(\mu_{123},s_{12}),$$

where

$$\xi_3^{(d)}(\mu_{123},s_{ij}) = I_3^{(d)}(0,0,\mu_{123};\ -\mu_{123},-\mu_{123},s_{ij}),$$

$$r_{ijk}^{(n)} = -\frac{\partial\ r_{ijk}}{\partial m_n^2}\Bigg|_{m_i=m_j=m_k=0} = \frac{s_{ij}-s_{ik}-s_{jk}}{s_{ik}s_{jk}}\ \mu_{ijk},$$

$$\mu_{ijk} = r_{ijk}\big|_{m_i=m_j=m_k=0} = \frac{s_{ij}s_{ik}s_{jk}}{s_{ij}^2+s_{ik}^2+s_{jk}^2-2s_{ij}s_{ik}-2s_{ij}s_{jk}-2s_{ik}s_{jk}},$$

Thus, we expressed integral depending on 3 variables in terms integrals depending on 2 variables.

Vertex type integral

We can check that the obtained solution is a solution of the initial FE:

$$
\begin{aligned}
I_3^{(d)}&(0,0,0;\ q_{23}, q_{13}, q_{12}) \\
&= z_1\ I_3^{(d)}(0,0,0;\ q_{23}, q_{30}, q_{20}) \\
&+ z_2\ I_3^{(d)}(0,0,0;\ q_{30}, q_{13}, q_{10}) \\
&+ z_3\ I_3^{(d)}(0,0,0;\ q_{20}, q_{10}, q_{12}).
\end{aligned}
$$

Substituting the solution into the left –hand side and the right –hand side of this equation we obtain 12 terms. After algebraic simplification, taking into account algebraic conditions on z- parameters, we found that 6 terms with one arbitrary parameters z on the right –hand side cancel. The remaining 3 terms cancel 3 terms on the left –hand side.

Vertex type integral

Integral $\xi_3^{(d)}(m^2, q^2)$ can be evaluated as a solution of simple dimensional recurrence relation:

$$(d-2)\ \xi_3^{(d+2)}(m^2, q^2) = -2\widetilde{m}^2\ \xi_3^{(d)}(m^2, q^2) - \xi_2^{(d)}(q^2),$$

where

$$\xi_2^{(d)}(q^2) = I_2^{(d)}(0, 0, q^2) = -\frac{\pi^{3/2}(-\widetilde{q}^2)^{\frac{d}{2}-2}}{2^{d-3}\Gamma\left(\frac{d-1}{2}\right)\sin\frac{\pi d}{2}},$$

and

$$\widetilde{q}^2 = q^2 + 4i\eta, \qquad \widetilde{m}^2 = m^2 - i\eta.$$

Solution of dimensional recurrence relation can be written as

$$\xi_3^{(d)}(m^2, q^2) = -\frac{1}{2m^2}\xi_2^{(d)}(q^2)\,{}_2F_1\left[\begin{matrix}1, \frac{d-2}{2}; \\ \frac{d-1}{2};\end{matrix}\ \frac{-\widetilde{q}^{\,2}}{4\widetilde{m}^2}\right] + \frac{(-\widetilde{m}^2)^{d/2}}{\Gamma\left(\frac{d-2}{2}\right)} C_3(q^2, d),$$

where $C_3(q^2, d)$ is a periodic function $C_3(q^2, d) = C_3(q^2, d+2)$.

Vertex type integral

To find it we use differential equation for $\xi_3^{(d)}(m^2, q^2)$:

$$\frac{\partial}{\partial q^2}\xi_3^{(d)}(m^2,q^2) = \frac{-(q^2+2m^2)}{q^2(q^2+4m^2)}\xi_3^{(d)}(m^2,q^2)$$
$$-\frac{(d-3)}{\widetilde{q}^{\,2}(q^2+4m^2)}\xi_2^{(d)}(q^2) + \frac{d-2}{2\widetilde{m}^2 q^2(q^2+4m^2)}\xi_1^{(d)}(m^2).$$

From this equation it follows that

$$q^2\frac{\partial C_3(q^2,d)}{\partial q^2} + \frac{(q^2+2m^2)}{q^2+4m^2}C_3(q^2,d) + \frac{\Gamma\left(\frac{d}{2}\right)}{(-\widetilde{m}^2)^{d/2+1}(q^2+4m^2)}\xi_1^{(d)}(m^2) = 0.$$

where

$$\xi_1(m^2) = -\frac{\pi(\widetilde{m}^2)^{d/2-1}}{\Gamma\left(\frac{d}{2}\right)\sin\frac{\pi d}{2}}.$$

Vertex type integral

Solution of this equation:

$$C_3(q^2,d) = \frac{-\Gamma\left(\frac{d}{2}\right)\xi_1^{(d)}(m^2)}{\sqrt{q^2(q^2+4m^2)}(-\widetilde{m}^2)^{d/2+1}} \ln\left(2m^2+q^2+\sqrt{(q^2+4m^2)q^2}\right) + \frac{K_3}{\sqrt{q^2(q^2+4m^2)}},$$

depends on an arbitrary constant K_3. $\xi_3^{(d)}(m^2,q^2)$ is finite at $q^2 \to 0$, therefore

$$K_3 = \frac{\Gamma\left(\frac{d}{2}\right)\xi_1^{(d)}(m^2)}{(-\widetilde{m}^2)^{d/2+1}} \ln(2m^2)$$

Finally

$$\xi_3^{(d)}(m^2,q^2) = -\frac{1}{2m^2}\xi_2^{(d)}(q^2)\ {}_2F_1\left[\begin{matrix}1, \frac{d-2}{2}; \\ \frac{d-1}{2};\end{matrix}\ \frac{-\widetilde{q}^{\,2}}{4\widetilde{m}^2}\right] + \frac{(d-2)\xi_1^{(d)}(m^2)}{2m^2\sqrt{q^2(q^2+4m^2)}} \ln\left(1+\frac{q^2+\sqrt{q^2(q^2+4m^2)}}{2m^2}\right).$$

Vertex type integral

Some Remarks:

To obtain result for $I_3^{(d)}(0,0,0;\ s_{23},s_{13},s_{12})$ in the whole kinematical domain we must know simpler integral $\xi_3^{(d)}(m^2,q^2)$ for arbitrary q^2 and m^2.

Our result for $I_3^{(d)}(0,0,0;\ s_{23},s_{13},s_{12})$ was compared numerically with the results obtained with the program SecDec version 3.0 by S. Borowka, G. Heinrich, S. Jones, M. Kerner, J. Schlenk, T. Zirke

Perfect agreement was found in Euclidean as well as in Minkowski and mixed regions of kinematical variables.

For some kinematical regions logarithmic terms cancel each other but in some regions they give contribution.

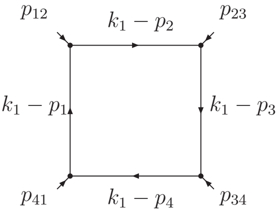
Box type integral

Let's consider integral with 4 propagators:

$$I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2;\ s_{12}, s_{23}, s_{34}, s_{14}, s_{24}, s_{13}) = \int \frac{d^d k_1}{i\pi^{d/2}} \frac{1}{P_1 P_2 P_3 P_4}.$$

where

$$P_i = (k_1 - p_i)^2 - m_i^2 + i\eta, \qquad s_{ij} = p_{ij}^2, \qquad p_{ij} = p_i - p_j.$$



Box type integral

To obtain FE we use relation for product of 4 propagators:

$$\frac{1}{P_1P_2P_3P_4} = \frac{x_1}{P_0P_2P_3P_4} + \frac{x_2}{P_1P_0P_3P_4} + \frac{x_3}{P_1P_2P_0P_4} + \frac{x_4}{P_1P_2P_3P_0},$$

where

$$p_0 = x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4,$$

and parameters x_j, m_0 obey the following conditions:

$$x_1 + x_2 + x_3 + x_4 = 1,$$
$$x_1x_2s_{12} + x_1x_3s_{13} + x_1x_4s_{14} + x_2x_3s_{23} + x_2x_4s_{24} + x_3x_4s_{34}$$
$$-x_1m_1^2 - x_2m_2^2 - x_3m_3^2 - x_4m_4^2 + m_0^2 = 0.$$

Integration of this relationship with respect to k_1 yields FE:

$$\begin{aligned} &I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2;\ s_{12}, s_{23}, s_{34}, s_{14}, s_{24}, s_{13}) \\ &\quad = x_1\ I_4^{(d)}(m_0^2, m_2^2, m_3^2, m_4^2;\ s_{20}, s_{23}, s_{34}, s_{40}, s_{24}, s_{30}) \\ &\quad + x_2\ I_4^{(d)}(m_1^2, m_0^2, m_3^2, m_4^2;\ s_{10}, s_{30}, s_{34}, s_{14}, s_{40}, s_{13}) \\ &\quad + x_3\ I_4^{(d)}(m_1^2, m_2^2, m_0^2, m_4^2;\ s_{12}, s_{20}, s_{40}, s_{14}, s_{24}, s_{10}) \\ &\quad + x_4\ I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_0^2;\ s_{12}, s_{23}, s_{30}, s_{10}, s_{20}, s_{13}). \end{aligned}$$

Box type integral

At $m_1 = m_2 = m_3 = m_4 = m_0 = 0$ we have FE for massless integral $I_4^{(d)}$

$$
\begin{aligned}
& I_4^{(d)}(0,0,0,0; q_{12}, q_{23}, q_{34}, q_{14}, q_{24}, q_{13}) \\
= \;& z_1 \; I_4^{(d)}(0,0,0,0; q_{20}, q_{23}, q_{34}, q_{40}, q_{24}, q_{30}) \\
& + z_2 \; I_4^{(d)}(0,0,0,0; q_{10}, q_{30}, q_{34}, q_{14}, q_{40}, q_{13}) \\
& + z_3 \; I_4^{(d)}(0,0,0,0; q_{12}, q_{20}, q_{40}, q_{14}, q_{24}, q_{10}) \\
& + z_4 \; I_4^{(d)}(0,0,0,0; q_{12}, q_{23}, q_{30}, q_{10}, q_{20}, q_{13}).
\end{aligned}
$$

In this case

$$
\begin{aligned}
q_{10} &= q_{14} - q_{14} z_1 + (q_{12} - z_{14}) z_2 + (q_{13} - q_{14}) z_3, \\
q_{20} &= q_{24} + (q_{12} - q_{24}) z_1 - z_2 s_{24} + (q_{23} - q_{24}) z_3, \\
q_{30} &= q_{34} + (q_{13} - q_{34}) z_1 + (q_{23} - q_{34}) z_2 - q_{34} z_3, \\
q_{40} &= q_{14} z_1 + q_{24} z_2 + q_{34} z_3,
\end{aligned}
$$

and the following conditions to be hold:

$$
\begin{aligned}
& z_1 + z_2 + z_3 + z_4 = 1, \\
& z_1 z_2 \; q_{12} + z_1 z_3 \; q_{13} + z_1 z_4 \; q_{14} + z_2 z_3 \; q_{23} + z_2 z_4 \; q_{24} + z_3 z_4 \; q_{34} = 0.
\end{aligned}
$$

Box type integral

To solve this FE we use more general equation. At $m_1 = m_2 = m_3 = m_4 = 0$ and $m_0 \neq 0$ from the general equation we get:

$$
\begin{aligned}
& I_4^{(d)}(0,0,0,0; s_{12}, s_{23}, s_{34}, s_{14}, s_{24}, s_{13}) \\
= \;\; & x_1 \; I_4^{(d)}(m_0^2, 0,0,0; s_{20}, s_{23}, s_{34}, s_{40}, s_{24}, s_{30}) \\
& + x_2 \; I_4^{(d)}(0, m_0^2, 0, 0; s_{10}, s_{30}, s_{34}, s_{14}, s_{40}, s_{13}) \\
& + x_3 \; I_4^{(d)}(0,0, m_0^2, 0; s_{12}, s_{20}, s_{40}, s_{14}, s_{24}, s_{10}) \\
& + x_4 \; I_4^{(d)}(0,0,0, m_0^2; s_{12}, s_{23}, s_{30}, s_{10}, s_{20}, s_{13}).
\end{aligned}
$$

where

$$
\begin{aligned}
s_{10} &= s_{14} + m_0^2 - s_{14}x_1 + (s_{12} - s_{14})x_2 + (s_{13} - s_{14})x_3, \\
s_{20} &= s_{24} + m_0^2 + (s_{12} - s_{24})x_1 - x_2 s_{24} + (s_{23} - s_{24})x_3, \\
s_{30} &= s_{34} + m_0^2 + (s_{13} - s_{34})x_1 + (s_{23} - s_{34})x_2 - s_{34}x_3, \\
s_{40} &= m_0^2 + s_{14}x_1 + s_{24}x_2 + s_{34}x_3,
\end{aligned}
$$

and the following conditions to be hold:

$$
\begin{aligned}
& x_1 + x_2 + x_3 + x_4 = 1, \\
& x_1 x_2 s_{12} + x_1 x_3 s_{13} + x_1 x_4 s_{14} + x_2 x_3 s_{23} + x_2 x_4 s_{24} + x_3 x_4 s_{34} + m_0^2 = 0.
\end{aligned}
$$

Box-type integral

We combined out of 42 equations

$$s_{i0}=0,\quad s_{i0}-s_{12}=0,\quad s_{i0}-s_{23}=0,\quad s_{i0}-s_{34}=0,\quad s_{i0}-s_{14}=0,$$
$$s_{i0}-s_{24}=0,\quad s_{i0}-s_{13}=0,\quad s_{i0}\pm m_0^2=0,\quad s_{10}-s_{20}=0,\quad s_{10}-s_{30}=0,$$
$$s_{10}-s_{40}=0,\quad s_{20}-s_{30}=0,\quad s_{20}-s_{40}=0,\quad s_{30}-s_{40}=0,$$

111930 systems, each consisting of 4 equations. Only 29 systems have nontrivial solutions without radicals. It took 2 hours 19 min CPU time to analyze 111930 systems of equations.

One of these solutions gives FE expressing our initial integral depending on 6 variables in terms of integrals depending on 4 variables:

$$\begin{aligned} I_4^{(d)}&(0,0,0,0;s_{12},s_{23},s_{34},s_{14},s_{24},s_{13})\\ &= r_{1234}^{(1)}\;B_4^{(d)}(\mu_4;s_{23},s_{34},s_{24})+r_{1234}^{(2)}\;B_4^{(d)}(\mu_4;s_{34},s_{14},s_{13})\\ &\quad + r_{1234}^{(3)}\;B_4^{(d)}(\mu_4;s_{12},s_{14},s_{24})+r_{1234}^{(4)}\;B_4^{(d)}(\mu_4;s_{12},s_{23},s_{13}), \end{aligned}$$

where

$$B_4^{(d)}(\mu_4;s_{ij},s_{jk},s_{ik})=I_4^{(d)}(0,0,0,\mu_4;s_{ij},s_{jk},-\mu_4,-\mu_4,-\mu_4,s_{ik}),$$

Box-type integral

where

$$\mu_4 = r_{1234}\big|_{m_1=m_2=m_3=m_4=0}$$

Further reduction is possible. Again we used more general equation and computer search of solutions of systems of equations. Second step of functional reduction gives:

$$B_4^{(d)}(\mu_4; s_{ij}, s_{jk}, s_{ik}) \\ = r_{ijk}^{(i)}\ \xi_4^{(d)}(\mu_{ijk}, \mu_4; s_{jk}) + r_{ijk}^{(j)}\ \xi_4^{(d)}(\mu_{ijk}, \mu_4; s_{ik}) + r_{ijk}^{(k)}\ \xi_4^{(d)}(\mu_{ijk}, \mu_4; s_{ij}),$$

where

$$\xi_4^{(d)}(\mu_{ijk}, \mu_4; s_{ij}) = I_4^{(d)}(0, 0, \mu_{ijk}, \mu_4; s_{ij}, -\mu_{ijk}, \mu_{ijk} - \mu_4, -\mu_4, -\mu_4, -\mu_{ijk}).$$

The function with 4 variables was expressed in terms of function with 3 variables. Two of them are 'effective masses'.

Box-type integral

Combining formulae obtained on the first and second steps of reductions we find that one-loop box type integral with massless propagators depending on 6 variables is a combination of 12 integrals depending on 3 variables.

$$
\begin{aligned}
& I_4(s_{12}, s_{23}, s_{34}, s_{14}, s_{24}, s_{13}) \\
& = r_{1234}^{(1)} \left[r_{234}^{(2)} \xi_4^{(d)}(\mu_{234}, \mu_4, s_{34}) + r_{234}^{(3)} \xi_4^{(d)}(\mu_{234}, \mu_4, s_{24}) + r_{234}^{(4)} \xi_4^{(d)}(\mu_{234}, \mu_4, s_{23}) \right] \\
& + r_{1234}^{(2)} \left[r_{134}^{(1)} \xi_4^{(d)}(\mu_{134}, \mu_4, s_{34}) + r_{134}^{(3)} \xi_4^{(d)}(\mu_{134}, \mu_4, s_{14}) + r_{134}^{(4)} \xi_4^{(d)}(\mu_{134}, \mu_4, s_{13}) \right] \\
& + r_{1234}^{(3)} \left[r_{124}^{(1)} \xi_4^{(d)}(\mu_{124}, \mu_4, s_{24}) + r_{124}^{(2)} \xi_4^{(d)}(\mu_{124}, \mu_4, s_{14}) + r_{124}^{(4)} \xi_4^{(d)}(\mu_{124}, \mu_4, s_{12}) \right] \\
& + r_{1234}^{(4)} \left[r_{123}^{(1)} \xi_4^{(d)}(\mu_{123}, \mu_4, s_{23}) + r_{123}^{(2)} \xi_4^{(d)}(\mu_{123}, \mu_4, s_{13}) + r_{123}^{(3)} \xi_4^{(d)}(\mu_{123}, \mu_4, s_{12}) \right].
\end{aligned}
$$

We checked that this formula is a solution of the FE containing $I_4^{(d)}$ only with massless propagators. Substituting the above solution into the left –hand side and the right– hand side of the FE with two arbitrary parameters gives 60 terms. After nontrivial algebraic simplification 36 terms on the right –hand side cancel and the remaining 12 terms cancel 12 terms in the left – hand side.

Box-type integral

To evaluate $\xi_4^{(d)}$ we used simple dimensional recurrence relation

$$(d-3)\xi_4^{(d+2)}(\mu_3,\mu_4,s_{ij}) = -2\widetilde{\mu}_4\xi_4^{(d)}(\mu_3,\mu_4,s_{ij}) - \xi_3^{(d)}(\mu_3,s_{ij}).$$

Exploiting the method similar to that used for $\xi_3^{(d)}$ we get

$$\begin{aligned}\xi_4^{(d)}(\mu_3,\mu_4,s_{ij}) = &-\frac{(d-2)}{2\widetilde{\mu}_3\widetilde{\mu}_4 R}\mathrm{Arth}\left(\frac{s_{ij}}{R}\right)\xi_1^{(d)}(\widetilde{\mu}_3)\,{}_2F_1\left[\begin{matrix}1, \frac{d-3}{2}; \\ \frac{d-2}{2};\end{matrix}\ \frac{\widetilde{\mu}_3}{\widetilde{\mu}_4}\right] \\ &-\frac{\pi^{3/2}\ \widetilde{\mu}_4^{d/2-3}}{\Gamma\left(\frac{d-3}{2}\right)KR\ \sin\frac{\pi d}{2}}\left[\mathrm{Arth}\left(\frac{s_{ij}}{R}\right) - \mathrm{Arth}\left(\frac{s_{ij}}{R}K\right)\right] \\ &+\frac{1}{2\widetilde{\mu}_3\widetilde{\mu}_4}\left(\frac{\widetilde{\mu}_3}{\widetilde{s}_{ij}+4\widetilde{\mu}_3}\right)^{\frac{1}{2}}\xi_2^{(d)}(s_{ij})F_1\left(\frac{d-3}{2},\frac{1}{2},1,\frac{d-1}{2};\frac{-\widetilde{s}_{ij}}{4\widetilde{\mu}_3},\frac{-\widetilde{s}_{ij}}{4\widetilde{\mu}_4}\right),\end{aligned}$$

where

$$R = \sqrt{s_{ij}(s_{ij}+4\mu_3)} \qquad K = \left(1 - \frac{\widetilde{\mu}_3}{\widetilde{\mu}_4}\right)^{1/2},$$

and F_1 is the Appell function. Numerical comparison of our result for $I_4^{(d)}(s_{12},s_{23},s_{34},s_{14},s_{24},s_{13})$ with the result obtained by using package SecDec reveals perfect agreement.

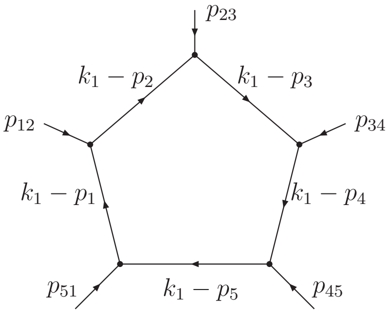
Definitions

Now let's consider pentagon type integral:

$$I_5^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2; s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, s_{13}, s_{14}, s_{24}, s_{25}, s_{35}) = \int \frac{d^d k_1}{i\pi^{d/2}} \frac{1}{P_1 P_2 P_3 P_4 P_5}.$$

where

$$P_i = (k_1 - p_i)^2 - m_i^2 + i\eta, \qquad s_{ij} = p_{ij}^2, \qquad p_{ij} = p_i - p_j.$$



Notations

To obtain FE we use relation for product of 5 propagators:

$$\frac{1}{P_1P_2P_3P_4P_5} = \frac{x_1}{P_0P_2P_3P_4P_5} + \frac{x_2}{P_1P_0P_3P_4P_5} + \frac{x_3}{P_1P_2P_0P_4P_5} + \frac{x_4}{P_1P_2P_3P_0P_5} + \frac{x_5}{P_1P_2P_3P_4P_0}$$

where p_0, x_j, m_0 obey the following conditions:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 1, \qquad p_0 = x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4 + x_5p_5,$$

$$x_1x_2s_{12} + x_1x_3s_{13} + x_1x_4s_{14} + x_1x_5s_{15}$$
$$+x_2x_3s_{23} + x_2x_4s_{24} + x_2x_5s_{25} + x_3x_4s_{34} + x_3x_5s_{35} + x_4x_5s_{45}$$
$$-x_1m_1^2 - x_2m_2^2 - x_3m_3^2 - x_4m_4^2 - x_5m_5^2 + m_0^2 = 0.$$

Integration of this relationship with respect to k_1 yields FE:

$$I_5^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2; s_{12}, s_{23}, s_{34}, s_{45}, s_{15}; s_{13}, s_{14}, s_{24}, s_{25}, s_{35})$$
$$= x_1 I_5^{(d)}(m_0^2, m_2^2, m_3^2, m_4^2, m_5^2; s_{20}, s_{23}, s_{34}, s_{45}, s_{50}, s_{30}, s_{40}, s_{24}, s_{25}, s_{35})$$
$$+x_2 I_5^{(d)}(m_1^2, m_0^2, m_3^2, m_4^2, m_5^2; s_{10}, s_{30}, s_{34}, s_{45}, s_{15}; s_{13}, s_{14}, s_{40}, s_{50}, s_{35})$$
$$+x_3 I_5^{(d)}(m_1^2, m_2^2, m_0^2, m_4^2, m_5^2; s_{12}, s_{20}, s_{40}, s_{45}, s_{15}, s_{10}, s_{14}, s_{24}, s_{25}, s_{50})$$
$$+x_4 I_5^{(d)}(m_1^2, m_2^2, m_3^2, m_0^2, m_5^2; s_{12}, s_{23}, s_{30}, s_{50}, s_{15}; s_{13}, s_{10}, s_{20}, s_{25}, s_{35})$$
$$+x_5 I_5^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2, m_0^2; s_{12}, s_{23}, s_{34}, s_{40}, s_{10}, s_{13}, s_{14}, s_{24}, s_{20}, s_{30}).$$

Pentagon type integral

Setting all masses $m_j^2 = 0$ we get FE for the integral $I_5^{(d)}$ with massless propagators

$$
\begin{aligned}
& I_5^{(d)}(0,0,0,0,0; q_{12}, q_{23}, q_{34}, q_{45}, q_{15}; q_{13}, q_{14}, q_{24}, q_{25}, q_{35}) \\
& = z_1 I_5^{(d)}(0,0,0,0,0; q_{20}, q_{23}, q_{34}, q_{45}, q_{50}, q_{30}, q_{40}, q_{24}, q_{25}, q_{35}) \\
& + z_2 I_5^{(d)}(0,0,0,0,0; q_{10}, q_{30}, q_{34}, q_{45}, q_{15}; q_{13}, q_{14}, q_{40}, q_{50}, q_{35}) \\
& + z_3 I_5^{(d)}(0,0,0,0,0; q_{12}, q_{20}, q_{40}, q_{45}, q_{15}, q_{10}, q_{14}, q_{24}, q_{25}, q_{50}) \\
& + z_4 I_5^{(d)}(0,0,0,0,0; q_{12}, q_{23}, q_{30}, q_{50}, q_{15}; q_{13}, q_{10}, q_{20}, q_{25}, q_{35}) \\
& + z_5 I_5^{(d)}(0,0,0,0,0; q_{12}, q_{23}, q_{34}, q_{40}, q_{10}, q_{13}, q_{14}, q_{24}, q_{20}, q_{30}).
\end{aligned}
$$

where

$$
\begin{aligned}
q_{10} &= q_{15} - q_{15} z_1 + (q_{12} - q_{15}) z_2 + (q_{13} - q_{15}) z_3 + (q_{14} - q_{15}) z_4, \\
q_{20} &= q_{25} + (q_{12} - q_{25}) z_1 - z_2 q_{25} + (q_{23} - q_{25}) z_3 + (q_{24} - q_{25}) z_4, \\
q_{30} &= q_{35} + (q_{13} - q_{35}) z_1 + (q_{23} - q_{35}) z_2 - z_3 q_{35} + (q_{34} - q_{35}) z_4, \\
q_{40} &= q_{45} + (q_{14} - q_{45}) z_1 + (q_{24} - q_{45}) z_2 + (q_{34} - q_{45}) z_3 - z_4 q_{45}, \\
q_{50} &= q_{15} z_1 + q_{25} z_2 + q_{35} z_3 + q_{45} z_4,
\end{aligned}
$$

Pentagon type integral

and the following conditions to be hold:

$$z_1 + z_2 + z_3 + z_4 + z_5 = 1,$$
$$z_1 z_2 q_{12} + z_1 z_3 q_{13} + z_1 z_4 q_{14} + z_1 z_5 q_{15}$$
$$+ z_2 z_3 q_{23} + z_2 z_4 q_{24} + z_2 z_5 q_{25} + z_3 z_4 q_{34} + z_3 z_5 q_{35} + z_4 z_5 q_{45} = 0.$$

To solve this FE we will use more general FE setting in the initial one $m_1^2 = m_2^2 = m_3^2 = m_4^2 = m_5^2 = 0$, keeping $m_0^2 \neq 0$

$$I_5^{(d)}(0,0,0,0,0; s_{12}, s_{23}, s_{34}, s_{45}, s_{15}; s_{13}, s_{14}, s_{24}, s_{25}, s_{35})$$
$$= x_1 I_5^{(d)}(m_0^2, 0, 0, 0, 0; s_{20}, s_{23}, s_{34}, s_{45}, s_{50}, s_{30}, s_{40}, s_{24}, s_{25}, s_{35})$$
$$+ x_2 I_5^{(d)}(0, m_0^2, 0, 0, 0; s_{10}, s_{30}, s_{34}, s_{45}, s_{15}; s_{13}, s_{14}, s_{40}, s_{50}, s_{35})$$
$$+ x_3 I_5^{(d)}(0, 0, m_0^2, 0, 0; s_{12}, s_{20}, s_{40}, s_{45}, s_{15}, s_{10}, s_{14}, s_{24}, s_{25}, s_{50})$$
$$+ x_4 I_5^{(d)}(0, 0, 0, m_0^2, 0; s_{12}, s_{23}, s_{30}, s_{50}, s_{15}; s_{13}, s_{10}, s_{20}, s_{25}, s_{35})$$
$$+ x_5 I_5^{(d)}(0, 0, 0, 0, m_0^2; s_{12}, s_{23}, s_{34}, s_{40}, s_{10}, s_{13}, s_{14}, s_{24}, s_{20}, s_{30}).$$

Pentagon type integral

If $m_1 = m_2 = m_3 = m_4 = m_5 = 0$ then

$$
\begin{aligned}
s_{10} &= m_0^2 + s_{15} - s_{15}x_1 + (s_{12} - s_{15})x_2 + (s_{13} - s_{15})x_3 + (s_{14} - s_{15})x_4, \\
s_{20} &= m_0^2 + s_{25} + (s_{12} - s_{25})x_1 - x_2 s_{25} + (s_{23} - s_{25})x_3 + (s_{24} - s_{25})x_4, \\
s_{30} &= m_0^2 + s_{35} + (s_{13} - s_{35})x_1 + (s_{23} - s_{35})x_2 - x_3 s_{35} + (s_{34} - s_{35})x_4, \\
s_{40} &= m_0^2 + s_{45} + (s_{14} - s_{45})x_1 + (s_{24} - s_{45})x_2 + (s_{34} - s_{45})x_3 - x_4 s_{45}, \\
s_{50} &= m_0^2 + s_{15}x_1 + s_{25}x_2 + s_{35}x_3 + s_{45}x_4.
\end{aligned}
$$

We will solve this more general FE if we will find m_0^2, x_i such that the number of variables for functions in the right hand side will be less than in the function in the left hand side. Again, to find such solutions we used computer. We combined out of 20 equations:

$$
\begin{aligned}
&s_{10} = 0, \quad s_{10} - s_{20} = 0, \quad s_{10} - s_{30} = 0, \quad s_{10} - s_{40} = 0, \quad s_{10} - s_{50} = 0, \\
&s_{10} + m_0^2 = 0, \quad s_{20} = 0, \quad s_{20} - s_{30} = 0, \quad s_{20} - s_{40} = 0, \quad s_{20} - s_{50} = 0, \\
&s_{20} + m_0^2 = 0, \quad s_{30} = 0, \quad s_{30} - s_{40} = 0, \quad s_{30} - s_{50} = 0, \quad s_{30} + m_0^2 = 0, \\
&s_{40} = 0, \quad s_{40} - s_{50} = 0, \quad s_{40} + m_0^2 = 0, \quad s_{50} = 0, \quad s_{50} + m_0^2 = 0.
\end{aligned}
$$

15504 systems, each consisting of 5 equations and found 36 solutions without square roots of Gram determinants.

Pentagon type integral

The most compact FE was found when $s_{10} = s_{20} = s_{30} = s_{40} = s_{50} = -m_0^2$:

$$
\begin{aligned}
& I_5^{(d)}(s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, s_{13}, s_{14}, s_{24}, s_{25}, s_{35}) \\
&= r_{12345}^{(1)} F^{(d)}(\mu_5; s_{23}, s_{34}, s_{45}, s_{24}, s_{25}, s_{35}) \\
&+ r_{12345}^{(2)} F^{(d)}(\mu_5; s_{13}, s_{34}, s_{45}, s_{14}, s_{15}, s_{35}) \\
&+ r_{12345}^{(3)} F^{(d)}(\mu_5; s_{12}, s_{24}, s_{45}, s_{14}, s_{15}, s_{25}) \\
&+ r_{12345}^{(4)} F^{(d)}(\mu_5; s_{12}, s_{23}, s_{35}, s_{13}, s_{15}, s_{25}) \\
&+ r_{12345}^{(5)} F^{(d)}(\mu_5; s_{12}, s_{23}, s_{34}, s_{13}, s_{14}, s_{24})
\end{aligned}
$$

where

$$
\begin{aligned}
& F^{(d)}(\mu_5; s_{ij}, s_{jk}, s_{kr}, s_{ik}, s_{ir}, s_{jr}) \\
&= I_5^{(d)}(-\mu_5, 0, 0, 0, 0; \mu_5, s_{ij}, s_{jk}, s_{kr}, \mu_5, \mu_5, \mu_5, s_{ik}, s_{ir}, s_{jr}),
\end{aligned}
$$

$$
\mu_5 = r_{12345}\big|_{m_1=m_2=m_3=m_4=m_5=0},
$$

The obtained FE represent integral depending on 10 variables in terms of integrals depending on 7 variables.

Pentagon with off-shell momenta

The one-loop massless pentagon integral depending on 10 variables was reduced to 60 integrals depending on 4 variables:

$$
\begin{aligned}
&I_5^{(d)}(s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, s_{13}, s_{14}, s_{24}, s_{25}, s_{35}) \\
&= [\overline{r}_{2345}^{(5)} \overline{r}_{234}^{(3)} \xi_5^{(d)}(\overline{r}_{234}, \overline{r}_{2345}, \overline{\mu}_5, s_{24}) + \overline{r}_{2345}^{(5)} \overline{r}_{234}^{(2)} \xi_5^{(d)}(\overline{r}_{234}, \overline{r}_{2345}, \overline{\mu}_5, s_{34}) \\
&+ \overline{r}_{2345}^{(4)} \overline{r}_{235}^{(5)} \xi_5^{(d)}(\overline{r}_{235}, \overline{r}_{2345}, \overline{\mu}_5, s_{23}) + \overline{r}_{2345}^{(4)} \overline{r}_{235}^{(3)} \xi_5^{(d)}(\overline{r}_{235}, \overline{r}_{2345}, \overline{\mu}_5, s_{25}) \\
&+ \overline{r}_{2345}^{(4)} \overline{r}_{235}^{(2)} \xi_5^{(d)}(\overline{r}_{235}, \overline{r}_{2345}, \overline{\mu}_5, s_{35}) + \overline{r}_{2345}^{(3)} \overline{r}_{245}^{(5)} \xi_5^{(d)}(\overline{r}_{245}, \overline{r}_{2345}, \overline{\mu}_5, s_{24}) \\
&+ \overline{r}_{2345}^{(3)} \overline{r}_{245}^{(4)} \xi_5^{(d)}(\overline{r}_{245}, \overline{r}_{2345}, \overline{\mu}_5, s_{25}) + \overline{r}_{2345}^{(3)} \overline{r}_{245}^{(2)} \xi_5^{(d)}(\overline{r}_{245}, \overline{r}_{2345}, \overline{\mu}_5, s_{45}) \\
&+ \overline{r}_{2345}^{(2)} \overline{r}_{345}^{(5)} \xi_5^{(d)}(\overline{r}_{345}, \overline{r}_{2345}, \overline{\mu}_5, s_{34}) + \overline{r}_{2345}^{(2)} \overline{r}_{345}^{(4)} \xi_5^{(d)}(\overline{r}_{345}, \overline{r}_{2345}, \overline{\mu}_5, s_{35}) \\
&+ \overline{r}_{2345}^{(2)} \overline{r}_{345}^{(3)} \xi_5^{(d)}(\overline{r}_{345}, \overline{r}_{2345}, \overline{\mu}_5, s_{45}) + \overline{r}_{2345}^{(5)} \overline{r}_{234}^{(4)} \xi_5^{(d)}(\overline{r}_{234}, \overline{r}_{2345}, \overline{\mu}_5, s_{23})] \overline{r}_{12345}^{(1)}
\end{aligned}
$$

$+$ 4 similar terms where

$$
\begin{aligned}
&\xi_5^{(d)}(\mu_3, \mu_4, \mu_5, u_{ij}) \\
&\quad = I_5^{(d)}(\mu_5, \mu_4, 0, 0, \mu_3; \mu_4 - \mu_5, -\mu_4, u_{ij}, -\mu_3, \mu_3 - \mu_5, -\mu_5, -\mu_5, -\mu_4, \mu_3 - \mu_4, -\mu_3),
\end{aligned}
$$

Pentagon type integral

We consider for simplicity the on-shell case when

$$s_{12} = 0, \quad s_{23} = 0, \quad s_{34} = 0, \quad s_{45} = 0, \quad s_{15} = 0$$

Substituting these values into FE obtained after the first iteration, yields:

$$\begin{aligned} I_5^{(d)}&(0,0,0,0,0,s_{13},s_{14},s_{24},s_{25},s_{35}) \\ &= \widetilde{r}^{(1)}_{12345} F^{(d)}(\widetilde{\mu}_5; 0,0,0,s_{24},s_{25},s_{35}) \\ &+ \widetilde{r}^{(2)}_{12345} F^{(d)}(\widetilde{\mu}_5; 0,0,0,s_{14},s_{13},s_{35}) \\ &+ \widetilde{r}^{(3)}_{12345} F^{(d)}(\widetilde{\mu}_5; 0,0,0,s_{14},s_{24},s_{25}) \\ &+ \widetilde{r}^{(4)}_{12345} F^{(d)}(\widetilde{\mu}_5; 0,0,0,s_{13},s_{35},s_{25}) \\ &+ \widetilde{r}^{(5)}_{12345} F^{(d)}(\widetilde{\mu}_5; 0,0,0,s_{13},s_{14},s_{24}) \end{aligned}$$

Integral with 5 variables is a combination of integrals with 4 variables.
In order to perform next step in the reduction we used again more general equation.

Pentagon-type integral

The result of the second step of functional reduction is:

$$F^{(d)}(\widetilde{\mu}_5; 0,0,0,s_{ik},s_{in},s_{jn})$$
$$= -s_{in}\rho_{ijkn}\widetilde{\xi}_5^{(d)}(s_{in},\mu_{ijkn},\widetilde{\mu}_5) + s_{jn}\rho_{ijkn}\widetilde{\xi}_5^{(d)}(s_{jn},\mu_{ijkn},\widetilde{\mu}_5) + s_{ik}\rho_{ijkn}\widetilde{\xi}_5^{(d)}(s_{ik},\mu_{ijkn},\widetilde{\mu}_5),$$

where

$$\widetilde{\xi}_5^{(d)}(s_{in},\mu_{ijkn},\widetilde{\mu}_5) = I_5^{(d)}(0,0,0,0,\widetilde{\mu}_5;\ 0,0,0,-\widetilde{\mu}_5,-\widetilde{\mu}_5,s_{in},\mu_{ijkn},\mu_{ijkn},-\widetilde{\mu}_5,-\widetilde{\mu}_5),$$

$$\mu_{ijkn} = s_{ik}s_{jn}\rho_{ijkn},$$
$$\rho_{ijkn} = \frac{1}{s_{ik}-s_{in}+s_{jn}},$$
$$\widetilde{\mu}_5 = -\frac{\delta_5}{g_4},$$

and δ_5, g_4 are the on-shell values of Δ_5, G_4:

$$\delta_5 = -2s_{13}s_{14}s_{24}s_{25}s_{35},$$
$$g_4 = 2s_{24}s_{35}(2s_{13}s_{24} - 2s_{13}s_{25} - s_{24}s_{35}) - 4(s_{25}+s_{35})s_{13}s_{14}s_{24}$$
$$-2s_{13}^2(s_{24}-s_{25})^2 + 2(s_{25}-s_{35})s_{14}[2s_{25}s_{13} - 2s_{24}s_{35} - (s_{25}-s_{35})s_{14}].$$

Pentagon type integral

After second step of functional reduction integrals with 4 variables are expressed in terms of integrals with 3 variables.

This means that *the on-shell massless pentagon type integral depending on 5 variables is a combination of 15 integrals depending on 3 variables*.

Pentagon type integral

Analytical result for the most elementary integral

$$\widetilde{\xi}_5^{(d)}(s_{13}, s_{14}, m^2) \equiv I_5^{(d)}(0,0,0,0,m^2;0,0,0,-m^2,-m^2,s_{13},s_{14},s_{14},-m^2,-m^2),$$

can be obtained as a solution of a simple dimensional recurrence relation:

$$(d-4)\widetilde{\xi}_5^{(d+2)}(s_{13},s_{14},m^2) = -2\widetilde{m}^2\widetilde{\xi}_5^{(d)}(s_{13},s_{14},m^2) - I_4^{(d)}(s_{13},s_{14}),$$

where

$$I_4^{(d)}(s_{13},s_{14}) \equiv I_4^{(d)}(0,0,0,0;0,0,0,s_{14},s_{14},s_{13})$$
$$= -\frac{4(d-3)}{s_{13}s_{14}(d-4)} I_2^{(d)}(s_{13})\, {}_2F_1\left[\begin{matrix} 1, \frac{d-4}{2}; \\ \frac{d-2}{2}; \end{matrix} \frac{\widetilde{s}_{13}}{\widetilde{s}_{14}}\right] - \frac{(d-3)}{s_{13}s_{14}} I_2^{(d)}(s_{14}) \ln\left(1 - \frac{\widetilde{s}_{13}}{\widetilde{s}_{14}}\right).$$

was obtained as a solution of dimensional recurrence relation:

$$I_4^{(d+2)}(s_{13},s_{14}) = \frac{\widetilde{s}_{14}}{2(d-3)} I_4^{(d)}(s_{13},s_{14}) + \frac{2}{\widetilde{s}_{13}(d-4)} I_2^{(d)}(s_{13}).$$

Pentagon type integral

In order to solve dimensional recurrence relation we redefine $\widetilde{\xi}_5^{(d)}(s_{13}, s_{14}, m^2)$

$$\widetilde{\xi}_5^{(d)}(s_{13}, s_{14}, m^2) = \frac{(-\widetilde{m}^2)^{d/2-2}}{\Gamma\left(\frac{d-4}{2}\right)}\, \overline{\xi}_5^{(d)}(s_{13}, s_{14}, m^2) \\ + b(d) I_4^{(d)}(s_{13}, s_{14}) + A_{13}(d) I_2^{(d)}(s_{13}).$$

By choosing arbitrary $b(d)$ and $A_{13}(d)$ we will obtain homogeneous equation for $\overline{\xi}_5^{(d)}(s_{13}, s_{14}, m^2)$.
Substituting this Ansatz into dimensional recurrence relation, equating to zero coefficient in front of $I_2^{(d)}(s_{13})$ yields equation for $b(d)$:

$$I_2^{(d)}(s_{13}): \qquad \widetilde{s}_{14}(d-4) b(d+2) + 4\widetilde{m}^2(d-3) b(d) + 2d - 6 = 0.$$

It's particular solution is:

$$b(d) = -\frac{1}{2m^2}\, {}_2F_1\left[\begin{array}{c} 1, \frac{d}{2}-2; \\ \frac{d-3}{2}; \end{array} \frac{-\widetilde{s}_{14}}{4m^2}\right].$$

Pentagon type integral

Equating to zero coefficient in front of $I_4^{(d)}$ gives equation for A_{13}:

$$I_4^{(d)}: \qquad A_{13}(d+2)+\frac{4(d-1)\widetilde{m}^2}{(d-4)\widetilde{s}_{13}}A_{13}(d)+\frac{4(d-1)}{\widetilde{s}_{13}^2(d-4)}b(d+2)=0.$$

Solution of the equation for $A_{13}(d)$:

$$A_{13}(d)=\frac{2}{\widetilde{s}_{13}\widetilde{m}^2(\widetilde{s}_{14}+4\widetilde{m}^2)}F_3\left(1,1,\frac{1}{2},\frac{d-4}{2},\frac{d-1}{2};\frac{\widetilde{s}_{14}}{\widetilde{s}_{14}+4\widetilde{m}^2},\frac{-\widetilde{s}_{13}}{4\widetilde{m}^2}\right),$$

where F_3 is Appell function:

$$F_3\left(1,1,\frac{1}{2},\frac{d-4}{2},\frac{d-1}{2};w,z\right)=\frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d-4}{2}\right)}\sum_{j,k=0}^{\infty}\frac{\Gamma\left(\frac{d-4}{2}+j\right)\left(\frac{1}{2}\right)_k}{\Gamma\left(\frac{d-1}{2}+j+k\right)}w^k z^j.$$

It can be expressed in terms of one-fold integral

$$F_3(\ldots,w,z)\to {}_2F_1+\int_0^1\frac{dv\ v^{\frac{d-4}{2}}}{(1-vz)}\ \ln\frac{1+\sqrt{w(1-v)}}{1+\sqrt{w(1-v)}},$$

convenient for the $\varepsilon=(4-d)/2$ expansion.

Pentagon type integral

Solution for homogeneous part was obtained as a solution of a first order differential equation:

$$(1-t)^2 m^6 R_{14}\, \overline{\xi}_5^{(d)}(s_{13}, s_{14}) = f(R_{14}) - f(-R_{14})$$

$$+\frac{t}{4}\left[\ln(m^2 t^2 - (2m^2+s_{14})t + m^2) - \ln t - \ln(-s_{14})\right],$$

where

$$f(R_{14}) = \frac{t}{4}\ln t \;\; \ln\left(1 - \frac{2m^2 t}{2m^2+s_{14}+R_{14}}\right) + \mathrm{Li}_2\left(\frac{2m^2}{2m^2+s_{14}-R_{14}}\right) - \mathrm{Li}_2\left(\frac{2m^2 t}{2m^2+s_{14}-R_{14}}\right)$$

and

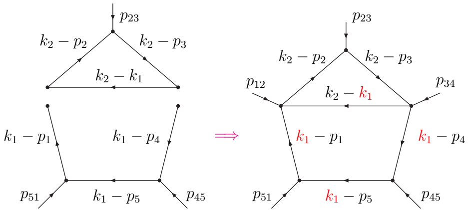
$$t = 1 + \frac{s_{13}+R_{13}}{2m^2},$$

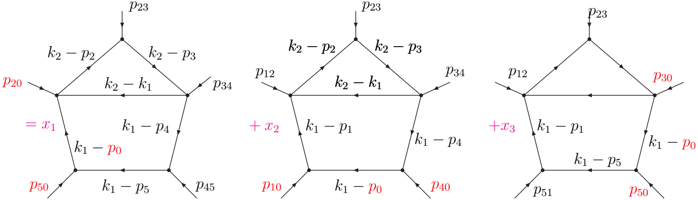
$$R_{13} = \sqrt{s_{13}(s_{13}+4m^2)},$$

$$R_{14} = \sqrt{s_{14}(s_{14}+4m^2)},$$

FE for multiloop integrals

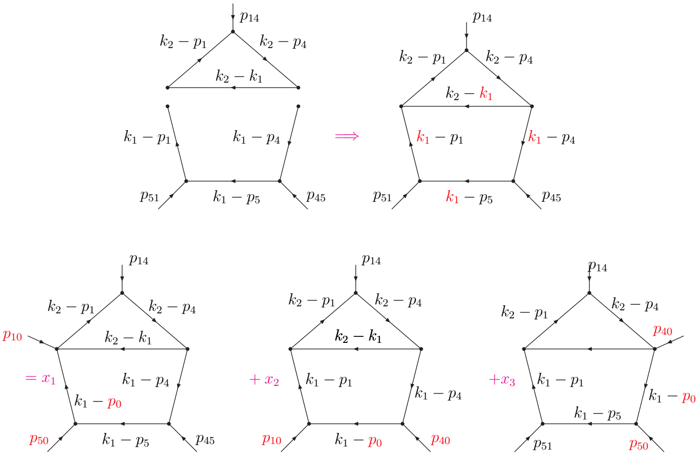
Integrating algebraic relations for products of propagators with loop integrals one can get FE for multi-loop integrals. For example, integrating three term relation with one-loop vertex type integral depending on k_1 one can get FE for two-loop pentagon integral:





FE for multiloop integrals

In some cases derivation of FE for one topology leads to integrals corresponding to diagrams with different topology. For example, integrating three term relation with one-loop vertex type integral depending on k_1 one can get FE for two-loop vertex integral:



FE for multiloop integrals

Integrals corresponding to diagrams of different topology looks like other functions. However such FE also can be solved, i.e. integrals can be expressed in terms of integrals with lesser number of variables.

One can find similarity with generalized Sincov's functional equation:

$$F(x,z) = G(x,y) + H(y,z).$$

General solution of this equation is:

$$F(x,z) = h(z) - f(x), \quad G(x,y) = g(y) - f(x), \quad H(y,z) = h(z) - g(y),$$

where f, g, h are arbitrary functions. Really, substituting $y = a$ in th FE and denoting $h(z) = H(a,z)$, $f(x) = -G(x,a)$, we get:

$$F(x,z) = G(x,a) + H(a,z) = h(z) - f(x).$$

Then setting $z = b$ and denoting $r(y) = H(y,b)$ we get:

$$G(x,y) = F(x,b) - H(y,b) = h(b) - f(x) - r(y),$$
$$H(y,z) = F(c,z) - G(c,y) = h(z) - h(b) + r(y).$$

Denoting $g(y) = h(b) - r(y)$ we get the above general solution.

The situation with FE for Feynman integrals is very similar. Investigation of FE for multiloop integrals is in progress.

Conclusions and outlook

- Master integrals are not the simplest integrals. By applying the method of functional reduction master integrals can be reduced to elementary integrals.
- Functional reduction of six and seven point one-loop integrals even with massive propagartors to elementary integrals is straightforward.
- The next direction of investigation will be functional reduction of multiloop integrals.
- Another direction of investigation will be computer search of functional equations for analytic continuation of integrals into specific kinematic regions.
- One more topic of investigation will be applcation of the functional reduction method for Bethe - Salpeter and Dyson - Schwinger equations.