Differential Reduction

Mikhail Kalmykov

Dubna/Hamburg

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Historical note

1999 (ONSHELL2) hep-ph: 9905379

$$J_{011}(\sigma,\beta,\alpha) = \int \frac{d^n k_1 d^n k_2}{[(p-k_1)^2]^\sigma [(k_1-k_2)^2+m^2]^\alpha [k_2^2+m^2]^\beta}\Bigg|_{p^2=-m^2},$$

$$J_{011}(1,2,2) = \frac{2}{3}\zeta_2 - \varepsilon\frac{2}{3}\zeta_3 + \varepsilon^2 3\zeta_4 - \varepsilon^3\left[\zeta_{\mathbf{5}}\right] + \varepsilon^4\left[\zeta_{\mathbf{6}}\right] + \varepsilon^5\left[\zeta_{\mathbf{7}}\right]$$

where $n = 4 - 2\varepsilon$ is the dimensionality of space time.

Davydychev-Kalmykov: hep-th/0012189

$$
\begin{aligned}
&(3n-8)J_{011}(1,1,1)+6m^2J_{011}(1,1,2)\\
&=2(m^2)^{n-3}\Gamma\left(\frac{n}{2}-1\right)\Gamma\left(3-n\right)\Gamma\left(2-\frac{n}{2}\right) \ .
\end{aligned}
$$

- Tarasov (1997): failed
- Czakon (2006): failed

Kalmykov-Kniehl: math-ph/1105.5319

$$J_{012}(\sigma,\beta,\alpha) = \int \frac{d^n k_1 d^n k_2}{[(p-k_1)^2]^\sigma [(k_1-k_2)^2+M^2]^\alpha [k_2^2+m^2]^\beta} \Bigg|_{p^2=-m^2} ,$$

where $n = 4 - 2\varepsilon$ is the dimensionality of space time.

$$\begin{aligned}(3n-8)J_{012}(1,1,1) + 4m^2 J_{012}(1,2,1) + 2M^2 J_{012}(1,1,2) \\ = \ 2(M^2)^{n-3}\Gamma\left(\frac{n}{2}-1\right)\Gamma\left(3-n\right)\Gamma\left(2-\frac{n}{2}\right) \ .\end{aligned}$$

- TARCER:
- LiteRed:
- Azurite:

Kniehl-Tarasov: hep-th-1602.00115

$$J_{123}(\sigma,\beta,\alpha) = \int \frac{d^n k_1 d^n k_2}{[(p-k_1)^2 - M_1^2]^\sigma [(k_1-k_2)^2 - M_2^2]^\alpha [k_2^2 - M_3^2]^\beta} \bigg|_{p^2=}$$

where $n = 4 - 2\varepsilon$ is the dimensionality of space time.

$$\begin{aligned} & 2M_1^2 J_{123}(2,1,1) + 4M_2^2 J_{123}(1,2,1) + 2M_3^2 J_{123}(1,1,2) \\ & +(3n-8) J_{123}(1,1,1) \\ & = 2(M_1)^{n-3}(M_3)^{n-3}\Gamma\left(\frac{n}{2}-1\right)\Gamma(3-n)\Gamma\left(2-\frac{n}{2}\right) \ . \end{aligned}$$

Is there something beyond IBP?

1-loop example

$$J_{01}(\sigma,\alpha) = \int \frac{d^n k_1}{[(p-k_1)^2]^\sigma [(k_1^2 - M^2]^\alpha}$$

where $n = 4 - 2\varepsilon$ is the dimensionality of space time.

$$J_{01}(\sigma,\alpha) = C(\sigma,\alpha) {}_2F_1\left(\sigma, \sigma+\alpha-\frac{n}{2}; \frac{n}{2}; \frac{p^2}{M^2}\right) \ ,$$

where $C(\sigma,\alpha)$ is the ratio of Gamma-functions.

Contiguous relations versus IBP

F. Gauss,1823

$$
\begin{aligned}
& P_1(a,b,c,z)\,{}_2F_1\left(\begin{array}{c} a+I_1, b+I_2 \\ c+I_3 \end{array} \middle| z \right) \\
+ \; & P_2(a,b,c,z)\,{}_2F_1\left(\begin{array}{c} a+J_1, b+J_2 \\ c+J_3 \end{array} \middle| z \right) \\
+ \; & P_3(a,b,c,z)\,{}_2F_1\left(\begin{array}{c} a+K_1, b+K_2 \\ c+K_3 \end{array} \middle| z \right) = 0 \; ,
\end{aligned}
$$

where $\{I, K, L\}$ are integer and a, b, c are generic.

Feynman Diagram as Mellin-Barnes Integral

Our staring point is the multiple Mellin-Barnes representation for Feynman Diagram

$$\Phi(\mathbf{A}, \vec{B}; \mathbf{C}, \vec{D}; \vec{z}) = \int_{-i\infty}^{+i\infty} \phi(\vec{t}) d\vec{t}$$

$$= \quad \mathrm{Const} \int_{-i\infty}^{+i\infty} \prod_{a,b,c,r} \frac{\Gamma(\sum_{i=1}^{m} A_{ai} t_i + B_a)}{\Gamma(\sum_{j=1}^{r} C_{bj} t_j + D_b)} dt_c z_k^{\sum_l \alpha_{kl} t_l} \quad ,$$

where z_k are the ratio of Mandelstam variables and A, B, C, D are some matrices and vectors depending in a linear way on the dimension of space-time n and the powers of the propagators.

From Mellin-Barnes Integral to PDE

Let us define the polynomials P_i and Q_i as

$$\frac{P_i(\vec{t})}{Q_i(\vec{t})} = \frac{\phi(\vec{t} + e_i)}{\phi(\vec{t})} \ ,$$

where e_j is a unit vector with nonzero element at j-place. Then, the function $\Phi(\mathbf{A}, \vec{B}; \mathbf{C}, \vec{D}; \vec{z})$ satisfy to the system of linear differential equations

$$L_j \Phi := \left[Q_i(\vec{t})\Big|_{t_j \to \theta_j} \frac{1}{z_i} - P_i(\vec{t})\Big|_{t_j \to \theta_j} \right] \Phi(\mathbf{A}, \vec{B}; \mathbf{C}, \vec{D}; \vec{z}) \ ,$$

where $\theta_i = z_i \frac{d}{dz_i}$.

Prolongation procedure

Let us consider the set of PDE (L_i) operators

$$L_j := P_j(\theta) - z_j Q_j(\theta)$$

We can prolong this system of equations by adding the new equations

$$\theta_{\vec{a}} L_j = 0.$$

In accordance with CartanKuranishi prolongation theorem after a finite number of prolongations the system is either in involution form or incompatible.

$$L_j \Phi \Rightarrow \left\{ \vec{d}\Phi(\vec{z}) = \Omega^r(\vec{z})\Phi(\vec{z}) \;, \quad \vec{d}\left[\vec{d}\Phi(\vec{z})\right] = 0 \right\} .$$

StepUp perators

$$\Phi(\vec{A};\vec{B};\vec{C};\vec{D};z) \sim \int_{-i\infty}^{+i\infty} \prod_{a,b,c,r} \frac{\Gamma(\sum_{i=1}^m A_{ai}t_i + B_a)}{\Gamma(\sum_{j=1}^r C_{bj}t_j + D_b)} dt_c z_k^{\sum_l \alpha_{kl}t_l} \ ,$$

The function Φ satisty to differential contiguous relations:

$$\begin{aligned} \Phi(\mathbf{A},\vec{B}+e_a;\mathbf{C},\vec{D};\vec{z}) &= \left(\sum_{i=1}^m A_{ai}\theta_i + B_a\right)\Phi(\mathbf{A},\vec{B};\mathbf{C},\vec{D};\vec{z}) \ , \\ \Phi(\mathbf{A},\vec{B};\mathbf{C},\vec{D}-e_b;\vec{z}) &= \left(\sum_{j=1}^r C_{bj}\theta_j + D_b\right)\Phi(\mathbf{A},\vec{B};\mathbf{C},\vec{D};\vec{z}) \ . \end{aligned}$$

Construction of inverse differential operators

- There are linear differential operators R changing the value of the discrete variables by one unit:

$$R_K(\vec{z})\frac{\partial^K}{\partial\vec{z}}\Phi(\vec{J};\vec{z}) = \Phi(\vec{J} \pm e_K;\vec{z}) \ ,$$

where $R_K(\vec{z})$ are polynomial functions.

- There is a closed system of differential equations:

$$\vec{d}\Phi(\vec{J};\vec{z}) = \Omega(\vec{z})\Phi(\vec{J};\vec{z}) \ , \quad \vec{d}\left[\vec{d}\Phi(\vec{J};\vec{z})\right] = 0 \ .$$

- There is an algorithmic solution for the construction of inverse linear differential operators B:

$$B_L(\vec{z})\frac{\partial^L}{\partial\vec{z}}\left(R_K(\vec{z})\frac{\partial^K}{\partial\vec{z}}\right)\Phi(\vec{J};\vec{z}) \equiv \Phi(\vec{J};\vec{z}) \ .$$

Literature

M. Saito, B. Sturmfels, N. Takayama,
Gröbner Deformations of Hypergeometric Differential Equations,

Differential Reduction

Applying the direct or inverse differential operators to the function Φ, the value of parameters can be changed by an arbitrary integer number:

$$S(\vec{z})\Phi(\vec{J}+\vec{m};\vec{z}) = \sum_{j=0}^{r} S_j(\vec{z})\frac{\partial^j}{\partial \vec{z}}\Phi(\vec{J};\vec{z}) \ ,$$

where $\vec{m}$ is a set of integers, S and S_j are polynomials and r is the holonomic rank (the number of linearly independent solutions) of the system of differential equations. In the end, the dirivatives are converted into finite difference:

$$\sum_{j=0}^{r} S_j(\vec{z})\frac{\partial^j}{\partial \vec{z}}\Phi(\vec{J};\vec{z}) = \sum_{j=0}^{r} S_j(\vec{z})\Phi(\vec{J}+j;\vec{z})$$

Example: one-fold Mellin-Barnes integral

Let us consider the function

$$\Phi(\vec{A};\vec{B};\vec{C};\vec{D};z) =$$
$$C_\Phi \int_{\gamma+iR} dt \frac{\Pi_{i=0}^{K}\Gamma\left(A_i+t\right)\Pi_{j=0}^{L}\Gamma\left(C_j-t\right)}{\Pi_{k=0}^{R}\Gamma\left(B_k+t\right)\Pi_{l=0}^{J}\Gamma\left(D_l-t\right)}\Gamma(-t)z^t$$

Then:

$$\left\{\theta(\theta+\vec{B}-1)(\theta-\vec{C})(-1)^{L+1-J} - z(\theta+\vec{A})(\theta-\vec{D}+1)\right\}\Phi = 0 \ ,$$

where $\theta = z\frac{d}{dz}$

Let us consider the non-confluent differential equation, when the order of differential operator in l.h.s. and in r.h.s. are equal to each other, so that the function Φ satisfy to the differential equation of order p. In this case there are p-linearly independent solutions of differential equation.

Factorization

- The following factorization is possible

$$\int dt \frac{\Gamma(A+m+t)}{\Gamma(A+t)} z^t F(t) =$$

$$(A+m-1+\theta)\cdots(A+\theta)\int dt \Gamma(A+t) z^t F(t) \; .$$

where m is positive integer.

- Factorization of the differential equation

$$z\,(\theta+1+a)\,f(z) = (\theta+a)\,(zf(z)) \;\; ,$$

where a is an arbitrary parameter.

Example: one-loop massless box (Tarasov)

$$\int \frac{d^n k}{[k^2]^{a_1}[(k+p_1)^2]^{a_2}[(k+p_1+p_2)]^{a_3}[(k+p_1+p_2+p_3)]^{a_4}}$$

$$\sim \int_{-i\infty}^{i\infty} dt \left(\frac{s}{t}\right)^t \Gamma(-t)\Gamma(a_1+t)\Gamma(a_3+t)$$

$$\Gamma\left(a-\frac{n}{2}+t\right)\Gamma\left(\frac{n}{2}-a_{134}-t\right)\Gamma\left(\frac{n}{2}-a_{123}-t\right)$$

$$\Rightarrow$$

$$\theta\left(\theta+a_{134}-\frac{n}{2}\right)\left(\theta+a_{123}-\frac{n}{2}\right)+z\left(\theta+a-\frac{n}{2}\right)(\theta+a_1)(\theta+a_3) \quad .$$

Example: J_{012}

$$J_{012} \equiv \int \frac{d^n(k_1 k_2)}{[(k_1-p)^2]^\sigma [k_1^2 - M^2]^\alpha [(k_1-k_2)^2 - m^2]^\beta} \Bigg|_{p^2=m^2}$$

This integral corresponds to differential equation of order 4.
For integer σ, α, β is could be rewritten as

$$\left(\theta - \frac{n}{2} + I_1\right)(\theta - n + I_2)\left[\theta\left(\theta - n + \frac{1}{2} + I_3\right) + z\left(\theta - \frac{3n}{2} + I_4\right)\right]$$

Sunrise: Mellin-Barnse

Let us consider L-loop sunrise

$$J(\vec{M}_j^2; \vec{\alpha_j}; p^2) = \int \Pi_{j=1}^{L} \frac{d^n k_j}{[k_j^2 - M_j^2]^{\alpha_j}} \times \frac{1}{[(p - \cdots - k_L)^2 - M_{L+1}^2]^{\alpha_{L+1}}} \,,$$

Its MB representation is the following:

$$J(\vec{M}_j^2; \vec{\alpha_j}; p^2) = (p^2)^{\frac{n}{2}L - \alpha} [i^{1-n} \pi^{n/2}]^L$$
$$\times \int \left\{ \Pi_{j=1}^{L+1} dt_j \frac{\Gamma(-t_j) \Gamma\left(\frac{n}{2} - \alpha_j - t_j\right)}{\Gamma(\alpha_j)} \left(- \frac{M_j^2}{p^2} \right)^{t_j} \right\}$$
$$\times \frac{\Gamma\left(\alpha - \frac{n}{2} L + \vec{t}\right)}{\Gamma\left(\frac{n}{2}(L+1) - \alpha - \vec{t}\right)} \,,$$

Sunrise:contiguous relations

$$\Phi(A,B;\{C_k\};\{z_1\}) = \int \Pi_{j=1}^{K} dt_j \Gamma(-t_j) \Gamma\left(C_j - t_j\right) z_j^{t_j} \frac{\Gamma\left(A + \vec{t}\right)}{\Gamma\left(B - \vec{t}\right)} .$$

$$\begin{aligned}
\Phi(A,B;\{\},C_j+1,\{\};\vec{z}) &= (C_j - \theta_j)\Phi(A,B;\{\},C_j,\{\};\vec{z}) , \\
\Phi(A+1,B;\{C_k\};\vec{z}) &= (A + \sum_{j=1}^{K} \theta_j)\Phi(A,B;\{C_k\};\vec{z}) , \\
\Phi(A,B-1;\{C_k\};\vec{z}) &= (B - 1 - \sum_{j=1}^{K} \theta_j)\Phi(\vec{A};B;\{C_k\};\vec{z}) ,
\end{aligned}$$

where

$$\theta_j = z_j \frac{d}{dz_j} , \quad j = 1, \cdots, m.$$

Sunrise: PDE

A linear system of partial differential equations (PDE) for the function Φ can be derived in two steps: (I) at the first step we define the polynomials P and Q as:

$$\frac{P_j}{Q_j} = \frac{\phi(t_j+1)}{\phi(t_j)} \; .$$

These polynomials define the corresponding system of differential equations (step II):

$$\frac{P_j^\Phi}{Q_j^\Phi} = -\frac{\left(A + \sum_{j=1}^K t_j\right)\left(1 - B + \sum_{j=1}^K t_j\right)}{(1-C_j+t_j)(1+t_j)} \Rightarrow$$

$$L_j^\Phi : (\theta_j - C_j)\,\theta_j \Phi = -z_j \left(\sum_{j=1}^K \theta_j + A\right)\left(\sum_{j=1}^K \theta_j + (1-B)\right)\Phi \; ,$$

Sunrise

- holonomic rank (2^K): Lauricella (1893):

$$d\Phi(\vec{z_K}) = \Omega^{(2K)}(\vec{z})\Phi(\vec{z}) \ ,$$

- zero of b-functio: Saito (1995)
  when parameters $A, B, \{C_i\}$ satisfy some linear combinations,

$$f_a(A, B, C_j) = 0, \quad a = 1, ...p$$

  the additional differential operators are generated so that the Puiseux type solution appear.
- How to find the corresponding linear combination of parameters and how to define the minimal set of additional differential equations.

Zero of b-function

- Our "old" approach to these problems is based on studying the inverse differential operators: the exceptional case of parameters, when dimension of the solution space is reduced, corresponds to the condition that the denominators of functions are equal to zero for an arbitrary values of z_i
  However, the inverse differential operators have a very complicated structure, which gives rise to technical problems in the analysis of the number of independent PDEs.
- New algorihtm: evaluation of the dimension of the invariant subspace of the differential contiguous operators

GKZ: criteria of reducibility

M. Saito,
F. Beukers,
A. Dickenstein, L. Matusevich, E. Miller,
M. Schulze, U. Walther,
T. Sadykov,
F. Beukers, G. Heckman,
K. Mimachi, T. Sasaki,

What is happen when monodromy is reducible

Let Φ be the set of solutions for the system of linear differential operators L_j^Φ:

$$L_j^\Phi \Phi = 0 \; .$$

Let $S(A, B, \vec{C})$ denote the local solution space of operators $L_j^\Phi(A, B, \vec{C})$ around some point z_0.
The contiguous differential operators $B_{A,B,C}$ map the solution space $S(A, B, \vec{C})$ to the solution space $S(A \pm I_1, B \pm I_2, \vec{C} \pm \vec{I}; \vec{z})$, where $\{I_a\}$ are set of integers.
When monodromy is reducible there is monodromy invariant subspace (invariant under action of monodromy) in the space of solutions. In this case the contiguous differential operators has a non-trivial kernel and it is necessary to evaluate their dimension.

Practical evaluation: I

$$L_j^{\Phi}(A,B,\vec{C}) : \left[(\theta_j - C_j)\,\theta_j + z_j \left(\sum_{j=1}^{K} \theta_j + A \right) \left(\sum_{j=1}^{K} \theta_j + (1-B) \right) \right] \;,$$

Let us consider the equation $B_A^+ \Phi =$, where B_A^+ is defined by

$$B_A^+ : \quad (A + \sum_{j=1}^{K} \theta_j)\Phi = 0$$

Then the system of equations reduces to

$$L_j^{\Phi}(A,B,\vec{C})\Phi \equiv (\theta_j - C_j)\,\theta_j \Phi \equiv 0 \;, \quad j = 1, \cdots, K.$$

Practical evaluation: II

$$L_j^{\Phi}(A,B,\vec{C})\Phi \equiv (\theta_j - C_j)\,\theta_j \Phi \equiv 0\,, \quad j=1,\cdots,K.$$

When $C_j \neq 0$ the solution Φ_0 of this equation has the following form:

$$\Phi_0 = c_0 + \sum_{i=1}^{K} c_i z_i^{C_i} + \sum_{\substack{i,j=1 \\ i<j}}^{K} c_{i,j} z_i^{C_i} z_j^{C_j} + \cdots + \text{Const.} \times \Pi_{i=1}^{K} z_i^{C_i}\ .$$

Applying operator B_A^+ to function Φ_0 we get:

$$(A + \sum_{j=1}^{K} \theta_j)\Phi_0 \equiv 0 = Ac_0 + \sum_{i=1}^{K} c_i (A + C_i) z_i^{C_i}$$

$$+ \sum^{K} c_{i,j}(A + C_i + C_j) z_i^{C_i} z_j^{C_j} + \cdots$$

Solution

Under conditions that

$$A, C_j, \notin \mathbb{Z}\ , \quad A + \sum_{j=1}^{K} C_j - C_a = 0\ , \quad a = 1, \cdots, K,$$

there is an invariant subspace of dimension K for operator B_A^+. Similar consideration can be done also for operator B_B^+. In particular, it is easy to show that under conditions that

$$B, C_j \notin \mathbb{Z}\ , \quad B + \sum_{j=1}^{K} C_j = 0\ ,$$

there is one-dimensional invariant subspace.

L-loop sunrise diagram

For sunrise

$$A = \alpha + \alpha_{L+1} - \frac{n}{2} L \,, \quad D = \alpha - \frac{n}{2}(L-1) \,, \quad C_j = \frac{n}{2} - \alpha_j \,.$$

To find the dimension of the invariant subspace, it is necessary to find all solutions of the following system of algebraic equations:

$$\frac{n}{2} L - \sum_{S_1} \frac{n}{2} = 0 \pmod{\mathbb{Z}} \,, \tag{2a}$$

$$\frac{n}{2}(L+1) - \sum_{S_2} \frac{n}{2} = 0 \pmod{\mathbb{Z}} \,, \tag{2b}$$

where S_1 and S_2 are any subsets of $1, \ldots, L+1$,
$\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$, and n is non-integer.

L-loop sunrise diagram

$$\frac{n}{2}L - \sum_{S_1} \frac{n}{2} = 0 \pmod{\mathbb{Z}} \ ,$$

The subset S_1 can be constructed in $L+1$ different ways (it includes all possible combinations of L out of the $L+1$ massive lines).

$$\frac{n}{2}(L+1) - \sum_{S_2} \frac{n}{2} = 0 \pmod{\mathbb{Z}} \ ,$$

there is only one solution of the second equation (the subset S_2 includes all the lines).

L-loop sunrise diagram

The number N_J of irreducible master integrals of the L-loop sunrise diagram with generic values of masses and momenta is equal to

$$\boxed{N_J = 2^{L+1} - L - 2} \,.$$

For example, for $L = 1, 2, 3, 4, 5, 6,$ we have $N_J = 1, 4, 11, 26, 57, 120,$ respectively.

Bubble: MB

The massive bubble integrals corresponds to the following function:

$$\Psi(A, D; \{C_k\}; z_K) = \int \Pi_{j=1}^{K} dt_j \Gamma(-t_j) z_j^{t_j}$$
$$\times \Gamma\left(C_j - t_j\right) \Gamma\left(A + \vec{t}\right) \Gamma\left(D + \vec{t}\right) \ .$$

In this case we have:

$$\frac{P_j^\Psi}{Q_j^\Psi} = \frac{\left(A + \sum_{j=1}^{K} t_j\right)\left(D + \sum_{j=1}^{K} t_j\right)}{(1 - C_j + t_j)(1 + t_j)} \Rightarrow$$

$$L_j^\Psi : \left(\theta_j - C_j\right)\theta_j \Psi = z_j \left(\sum_{j=1}^{K} \theta_j + A\right)\left(\sum_{j=1}^{K} \theta_j + D\right) \Psi \ ,$$

Conclusion

- 1-fold MB: counting, reduction, analytical evalution of ε-expansion. Factorization play an important role.
- k-fold MB: depends on the complexity. Typically limited by $k = 2, 3$.
- evaluation of singular locus of Feynman Diagram (Oaku, 1994).
- evaluation of zero of b-functions related with Feynman Diagram.
- construction of coefficients of ε-expansion - mainly in application to multivariable hypergeometric functions.