L-loop watermelon and sunrise graphs

Differential equations, index and dimension shifting relations, and quadratic constraints.

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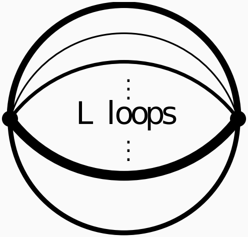
Introduction

Motivation

- Massive sunrise and watermelon graphs:
  - appear in many calculations,
  - can not be expressed via polylogarithms,
  - probably received the most attention among other loop integrals.
- As a rule, for L-loop diagrams with unspecified L
  - no reduction rules known,
  - no differential equations known,
  - no dimensional shifts known.



L-loop watermelon and sunrise graphs

Our main object of investigation is L-loop watermelon graph with arbitrary masses $m_0, \ldots m_L$, powers of denominators $\alpha_0, \ldots, \alpha_L$ and dimension d:

$$\mathcal{T} = 2^{D-1}\Gamma\left(\tfrac{D}{2}\right) i\pi^{D/2} \int \prod_{k=0}^{L} \left[\frac{d^D p_k}{i\pi^{D/2}} \frac{m_k \Gamma(\alpha_k)}{\left(m_k^2 - p_k^2 - i0\right)^{\alpha_k}} \right] \delta\left(\sum p_k\right)$$

$$= \{p_{k0} \to i p_{k0}\} = 2^{D-1}\Gamma\left(\tfrac{D}{2}\right) \pi^{D/2} \int \prod_{k=0}^{L} \left[\frac{d^D p_k}{\pi^{D/2}} \frac{m_k \Gamma(\alpha_k)}{\left(m_k^2 + p_k^2\right)^{\alpha_k}} \right] \delta\left(\sum p_k\right)$$

The blue factors are introduced for further convenience. The sunrise graph is obtained by 'cutting' 0-th line, i.e., by putting $\alpha_0 = 1$ and replacing $\frac{1}{m_0^2 - p_0^2 - i0} \to \frac{1}{m_0^2 - p_0^2 - i0} - \frac{1}{m_0^2 - p_0^2 + i0} = 2\pi i \delta\left(m_0^2 - p_0^2\right)$, with m_0^2 being the square of the incoming momentum.

Coordinate representation

In coordinate space we have

$$\mathcal{T}\,(D,\boldsymbol{\mu},\boldsymbol{m}) = \int\limits_0^\infty dx\,x^{D-1}\prod_{k=0}^{L}\int\frac{d^Dp}{\pi^{D/2}}e^{ipx}\frac{m_k\Gamma(\alpha_k)}{(p^2+m_k^2)^{\alpha_k}}$$

$$= \int\limits_0^\infty dx\,x^{D-1}\prod_{k=0}^{L}P_0(\mu_k,m_k,x)\,,$$

where in the last transition we have used that the integral $\int\frac{d^Dp}{\pi^{D/2}}e^{ipx}\frac{m\Gamma(\alpha)}{(p^2+m^2)^{\alpha}}$ depends on D and α only via combination $\mu = D - 2\alpha + 1$ and introduced

$$P_0(\mu,m,x) = \int\frac{\mathrm{d}^Dp}{\pi^{D/2}}e^{ipx}\frac{m\Gamma(\alpha)}{(p^2+m^2)^{\alpha}} = 2m\left(\frac{x}{2m}\right)^{\frac{1-\mu}{2}}K_{\frac{\mu-1}{2}}\left(mx\right)\,,$$

where $K(x)$ is the Macdonald function.

Workflow

- Introduce suitable auxiliary functions

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- Obtain the solution basis

Differential equations, recurrences, constraints

Coming next:

Introduce suitable auxiliary functions

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Introduce suitable auxiliary functions

Since we want to proceed for arbitrary loop order L, our goal (and only hope) is to obtain the first-order differential system rather than one high-order equation. Then we might hope to be able to write down the matrix in the right-hand side in a compact form. Therefore, we want to have a column of functions with the first entry being our original function.

Auxiliary functions i

In order to derive the Pfaffian system we have to introduce auxiliary functions. We first define

$$P_1(\mu, m, x) = \frac{1}{m}\partial_x P_0(\mu, m, x) = -2m\left(\frac{x}{2m}\right)^{\frac{1-\mu}{2}} K_{\frac{\mu+1}{2}}(mx)$$

and then introduce a column with 2^{L+1} entries, numbered by an $(L+1)$-digit binary number $\boldsymbol{a} = a_0 a_1 \ldots a_L$:

$$T_{\boldsymbol{a}}(D, \boldsymbol{\mu}, \boldsymbol{m}) = \int_0^\infty \mathrm{d}x\, x^{D-1} \prod_{k=0}^{L} P_{a_k}(\mu_k, m_k, x)\,, \qquad (1)$$

where $T_{00\ldots0}(D, \boldsymbol{\mu}, \boldsymbol{m}) = \mathcal{T}(D, \boldsymbol{\mu}, \boldsymbol{m})$ and other $2^{L+1} - 1$ components $T_{10\ldots0},\ T_{110\ldots0}, \ldots,\ T_{11\ldots1}$ being the auxiliary functions.

Auxiliary functions ii

Remark

Auxiliary functions are in fact tadpoles in shifted dimensions with shifted indices:

$$T_{\boldsymbol{a}}(D, \boldsymbol{\mu}) = \frac{\mathcal{T}(D \to D + |\boldsymbol{a}|, \mu_k \to \mu_k + 2a_k)}{\prod_k (-2m_k)^{a_k}} = \frac{\mathcal{T}(D \to D + |\boldsymbol{a}|, \alpha_k \to \alpha_k + |\boldsymbol{a}|/2 - a_k)}{\prod_k (-2m_k)^{a_k}} .$$

Note that the components with $|\boldsymbol{a}| \overset{def}{=} \sum_k a_k$ being an odd integer have indices shifted by half-integers.

Useful notations i

Think of $T_{\boldsymbol{a}}$ as an $a_0 a_1 \dots a_L$-th component of the function T of $L+1$ variables $\boldsymbol{m}$ with values in

$$\left(\mathbb{C}^2\right)^{\otimes(L+1)} \stackrel{def}{=} \underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{L+1}$$

Then we will need linear operators in $\left(\mathbb{C}^2\right)^{\otimes(L+1)}$. As usual we can use Pauli matrices in each $\mathbb{C}^2$ as building blocks. We denote operators acting on k-th factor by subscript k, in particular

$$\sigma^i_k = 1^{\otimes k} \otimes \sigma^i \otimes 1^{\otimes(L-k)}$$

Useful notations ii

We will use notations

$$\sigma = \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad n = \tfrac{1}{2}(1-\sigma^z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \bar{n} = 1 - n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$N = \sum_k n_k$$

Here and below we denote by blue color the parity-even operators and by red the parity-even operators, i.e., any operator S/A which commutes/anticommutes with 'parity' $(-1)^N$:

$$S(-1)^N = (-1)^N S\,, \qquad A(-1)^N = -(-1)^N A$$

NB: $(-1)^N = \prod_k \sigma_k^z$.

Coming next:

Differential system

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Differential system

We differentiate with respect to m_k and use integration by part to express the result via the same set of functions.

Derivatives with respect to m_k

Now we differentiate $T_{\boldsymbol{a}} = \int dx\, x^{D-1} \prod P_{a_k}$ wrt m_k. The derivative acts only on the factor P_{a_k}. Using the differential equation for Macdonald functions we have

$$m\partial_m P_0 = mxP_1 + \mu P_0\,,$$
$$m\partial_m P_1 = mxP_0\,,$$

which can be written as

$$m_m\partial_m P = (m\; x\; \sigma + \mu\bar{n})P\,,$$

where $P = \binom{P_0}{P_1}$.

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which can be written as

$$m_m \partial_m P = (m\ x\ \sigma + \mu \bar{n})P\,,$$

where $P = \binom{P_0}{P_1}$. Note the nasty x popping out in the above formula. We need to eliminate it in order to have a closed system for $T_{\boldsymbol{a}}$.

IBP identity

We use the only IBP identity we can construct:

$$0 = \int dx\, \partial_x (x^D \prod_k P_{a_k}(\mu_k, m_k, x))$$

Explicitly differentiating and using the identity

$$x\partial_x P = (m\partial_m - \mu)P = (mx\sigma - \mu n)P\,,$$

we obtain

$$\int_0^\infty dx\, x^{D-1}\,(x\textcolor{red}{M} - \textcolor{blue}{W}) \bigotimes_k P(\mu_k, m_k, x) = 0\,,$$

where

$$\textcolor{red}{M} = \sum_{k=0}^{L} m_k \textcolor{red}{\sigma_k}\,, \quad \textcolor{blue}{W} = \sum_{k=0}^{L} \mu_k \textcolor{blue}{n_k} - D\,,$$

Therefore,

$$\int_0^\infty dx\, x^{D-1}\, \boxed{x} \bigotimes_k P(\mu_k, m_k, x) = \textcolor{red}{M}^{-1} \textcolor{blue}{W}\, T_{\boldsymbol{a}} \tag{2}$$

Differential equations

Finally, we obtain the system of PDE

$$\partial_{m_k} T = A_k T, \qquad A_k = \sigma_k M^{-1} W + \frac{\mu_k \bar{n}_k}{m_k}.$$

Equivalently, the above equations can be written as

Differential system

$$dT = A\, T\,, \quad A = \sum_{k=0}^{L} A_k dm_k$$

One can check explicitly that the integrability condition

$$dA = A \wedge A$$

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Note that A is parity-even operator and therefore, the above system decouples into two pieces, each for 2^L components with $|\boldsymbol{a}|$ being either even or odd. $\mathcal{T}$ belong to the system with even $|a|$.

Coming next:

Index and dimension shifting operators

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We will construct now the matrices R_k and R which relate functions with shifted index or dimension.

$$T(D, \mu_0, \ldots, \mu_k - 2 \ldots, \mu_L) = R_k \, T(D, \mu_0, \ldots, \mu_k, \ldots, \mu_L) \, ,$$

$$T(D+1, \mu_0, \ldots, \mu_L) = R \, T(D, \mu_0, \ldots, \mu_L)$$

Index shifting identities

We can check explicitly that

$$P_0(\mu-2,m,x) = -\frac{1}{2m^2}(m\partial_m - 1)P_0(\mu,m,x)\ ,$$
$$P_1(\mu-2,m,x) = -\frac{1}{2m}\partial_m P_1(\mu,m,x)\ .$$

Then using the obtained differential equations we have the index shifting identity

$$T(\boldsymbol{\mu}-2\boldsymbol{e}_k) = R_k(D,\boldsymbol{\mu})\,T(\boldsymbol{\mu})\,,$$

Index shifting operator

$$R_k(D,\boldsymbol{\mu}) = -\frac{1}{2m_k}\left(A_k - \frac{\bar{n}_k}{m_k}\right) = -\frac{1}{2m_k}\left(\frac{\sigma_k}{M}W + \frac{(\mu_k-1)\,\bar{n}_k}{m_k}\right)$$

Dimension shifting identities

Let us first derive dimension shifts for fixed $\boldsymbol{\mu}$. Note that dimension shift corresponds to extra power of x in the integrand. Therefore

$$T(D+1,\boldsymbol{\mu}) = R(D,\boldsymbol{\mu})\,T(D,\boldsymbol{\mu})\,,$$

Dimension shifting operator for fixed μ

$$R(D,\boldsymbol{\mu}) = M^{-1}W\,,$$

In order to obtain the dimensional recurrence relations which shift D at fixed $\boldsymbol{\alpha}$, we should first use the operators R_k^{-1} to shift μ and then shift the dimension by $+2$ at fixed μ:

$$T\left(D-2,\boldsymbol{\mu}-\mathbf{2}\right) = \mathcal{R}(D,\boldsymbol{\mu})\,T(D,\boldsymbol{\mu})\,,$$

Dimension shifting operator for fixed α

$$\mathcal{R}(D,\boldsymbol{\mu}) = R^{-1}(D-2,\boldsymbol{\mu}-\mathbf{2})R^{-1}(D-1,\boldsymbol{\mu}-\mathbf{2})\prod_{k=0}^{L} R_k\left(D,\boldsymbol{\mu}-2\sum_{l=k+1}^{L}\boldsymbol{e}_l\right)$$

Compatibility conditions

One can check explicitly that operators R, R_k satisfy appropriate compatibility conditions:

$$R_j(\mu_k - 2)R_k = R_k(\mu_j - 2)R_j\,,$$
$$dR_k + R_k A = A(\mu_k - 2)R_k\,,$$
$$dR + RA = A(D+1)R\,,$$
$$R_k(D+1)R = R(\mu_k - 2)R_k\,,$$

where we have suppressed unchanged arguments for brevity.

Coming next:

Basis of solution space

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We will fix the basis of solution space. Each solution will be expressed via Lauricella functions $F_C^{(L)}$ with unconstrained indices and argument [Berends et al., 1994]. Thus we will discover that we have obtained a Pfaffian differential system for $F_C^{(L)}$.

Basis of solution space i

Assume that $m_0 > \sum_{k=1}^{L} m_k$. Then we can define the solution space basis as

$$V^{(\boldsymbol{\rho})} = \int_0^\infty dx\, x^{D-1} P(\mu_0, m_0, x) \otimes \bigotimes_{k=1}^{L} Q^{(\rho_k)}(\mu_k, m_k, x)\,,$$

$$Q^{(\rho)}(\mu, m, x) = \frac{\pi(-1)^\rho}{2\cos\frac{\pi\mu}{2}} 2m \left(\frac{x}{2m}\right)^{\frac{1-\mu}{2}} \begin{pmatrix} I_{(-1)^\rho \frac{\mu-1}{2}}(mx) \\ I_{(-1)^\rho \frac{\mu+1}{2}}(mx) \end{pmatrix},$$

where $\boldsymbol{\rho} = \rho_1 \ldots \rho_L$ is an L-digit binary number which enumerates the solutions. Loosely speaking, we have replaced K_ν functions by $I_{\pm\nu}$ functions in the original definition of the tadpole integral, but kept one K function for convergence. Since the differential equations for K and I are the same, these are obviously solutions.

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Let check how many of those we have. Since we have kept one K, we have only 2^L solutions $V^{\boldsymbol{\rho}}$, not 2^{L+1}. Fortunately we know that even and odd components are separately solutions, therefore we have exactly 2^{L+1} solutions

Basis of solution space ii

Thus we introduce

$$V^{(\varrho_0 \boldsymbol{\rho})} = \frac{1}{2}\left[1 + (-1)^{\varrho_0 + \sum_{k=1}^{L} \rho_k}(-1)^{N}\right] V^{(\boldsymbol{\rho})}\,, \quad \varrho_0 = 0, 1, \qquad (3)$$

which are both the solutions of the differential system. Note that we have introduced the additional factor $(-1)^{\sum_{k=1}^{L} \rho_k}$ for further convenience. We have chosen notation ϱ_0 rather than ρ_0 to underline a somewhat different meaning of this bit compared to others $(\rho_1, \rho_2, \ldots)$.

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NB: Note that 2^L solutions with

$$\varrho_0 = |\rho| \quad \text{(mod2)}$$

are the most relevant for us as they have nonzero zeroth component. This number agrees with $2^L - L - 1$ obtained in [Kalmykov and Kniehl, 2017] as $L + 1$ is the number of trivial clover-leaf tadpoles arizing from contraction of one of $L + 1$ lines.

Solution basis via Lauricella functions Berends et al. [1994]

We substitute the power series expansion for all I functions and use $\int_0^\infty dx\, x^{\beta-1} K_\nu(x) = 2^{\beta-2}\Gamma\left[\frac{\beta+\nu}{2}\right]\Gamma\left[\frac{\beta-\nu}{2}\right]$. Then we obtain the series for $F_C^{(L)}$ and finally have[1]

$$V^{(\rho)_{\mathbf{a}}} = \frac{(-)^{a_0}\,\Gamma\left[b_1, b_2\right]}{2^{1-D} m_0^{D-\sum_{k=0}^{L}\mu_k}} \prod_{k=1}^{L} (-)^{a_k}\,\Gamma\left[1-c_k\right]\left(\frac{m_k}{m_0}\right)^{c_k+\frac{\mu_k-1}{2}} F_C^{(L)}\left(b_1, b_2; c_1 \dots c_L; \frac{m_1^2}{m_0^2} \dots \frac{m_L^2}{m_0^2}\right),$$

where $c_k = 1 + (-1)^{\rho_k}\frac{\mu_k - (-1)^{a_k}}{2}$, $b_1 = \frac{D+a_0}{2} + \frac{1}{2}\sum_{k=1}^{L}\left[c_k - \frac{\mu_k+1}{2}\right]$, $b_2 = b_1 - a_0 + \frac{1-\mu_0}{2}$.

NB

Both indices and arguments are free in F_C. Therefore we just obtained a Pfaffian system for the Lauricella function $F_C^{(L)}$.
For the first time for arbitrary L?

[1] Disclaimer: this formula is not checked yet, typos are possible.

Polylogarithmic solutions near odd D

If we look carefully at our differential system we will see that it can easily be put to canonical $(\epsilon,\boldsymbol{\delta})$-form when $D = 1 - 2\epsilon$ and $\alpha_k = 1 + \delta_k$. For this purpose we pass to "canonical" masters

$$Y = MT$$

Then we have

$$dY = -2\sum_k dm_k \left[\left(\sum_{l=0}^{L} (\epsilon + \delta_l) n_l - \epsilon \right) \sigma_k M^{-1} + \frac{(\epsilon + \delta_l)\bar{n}_k}{m_k} \right] Y$$

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The boundary conditions can be deduced from the asymptotics $m_{k>0}/m_0 \to 0$. The result is expressed via Goncharov's polylogarithms $G(a_1, a_2, \ldots, 1)$ where each letter is either zero or of the form

$$a(\boldsymbol{\eta}) = \frac{m_0}{\sum_{k=1}^{L} \eta_k m_k},$$

where L-tuple $\{\eta_1, \ldots, \eta_L\}$ has elements ± 1.

Coming next:

Quadratic constraints

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Now we are after nontrivial quadratic constraints between solutions of our differential system. We will construct such a rational matrix $B_0(D, \boldsymbol{\mu}, \boldsymbol{m})$ that for any two solutions $T_1(D, \boldsymbol{\mu}, \boldsymbol{m})$ and $T_2(D, \boldsymbol{\mu}, \boldsymbol{m})$ the constraint will have a form

$$T_1^{\intercal}(-D, -\boldsymbol{\mu}, \boldsymbol{m}) B_0(D, \boldsymbol{\mu}, \boldsymbol{m}) T_2(D, \boldsymbol{\mu}, \boldsymbol{m}) = \mathrm{const},$$

where const in the r-h.s. denotes a matrix which depends on D and $\boldsymbol{\mu}$, but not $\boldsymbol{m}$.

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Using the index and dimension shifting identities we will then obtain constraints of the form

$$T_1^{\intercal}(D^\star + 2\epsilon, \mu^\star + 2\delta, \boldsymbol{m}) B(D^\star, \mu^\star | \epsilon, \delta, \boldsymbol{m}) T_2(D^\star - 2\epsilon, \mu^\star - 2\delta, \boldsymbol{m}) = \mathrm{const}\,,$$

which is perfectly adjusted for expansion in ϵ and δ and gives an infinite number of constraints. Here $D^\star, \mu_0^\star, \ldots, \mu_n^\star$ are arbitrary integers.

Symmetry of the matrix A in $dT = AT$

By thoroughly examining the form of the matrix A,

$$A = \sum_k \left[\sigma_k M^{-1} W + \frac{\mu_k \bar{\mathbf{n}}_k}{m_k} \right] dm_k$$

we find the following symmetry property:

$$A^\intercal W = WA\,.$$

Note that W is a diagonal invertible matrix independent of $\boldsymbol{m}$ and we could have passed to $\tilde{A} = W^{1/2} A W^{-1/2}$ which is symmetric, $\tilde{A}^\intercal = \tilde{A}$ (cf. [RL,2018]). But we prefer to avoid square roots and will work with A.

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We also have a simple property

$$A(-D, -\boldsymbol{\mu}) = -A(D, \boldsymbol{\mu}) \,.$$

Quadratic constraint near $D = \boldsymbol{\mu} = 0$

Using these properties, it is easy to prove that

$$T_1^{\mathsf{T}}(-D, -\boldsymbol{\mu}, \boldsymbol{m}) W(D, \boldsymbol{\mu}) T_2(D, \boldsymbol{\mu}, \boldsymbol{m}) = \mathrm{const}\,.$$

Quadratic constraint near $D=\boldsymbol{\mu}=0$

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If we choose two basis solutions, $V^{(\tilde{\varrho}_0\tilde{\boldsymbol{\rho}})\mathsf{T}}$ and $V^{(\varrho_0\boldsymbol{\rho})}$, then we can fix the right-hand side by considering asymptotics $m_{k>0}/m_0\to 0$. We find

$$V^{(\tilde{\varrho}_0\tilde{\boldsymbol{\rho}})\mathsf{T}}\left(-D,-\boldsymbol{\mu}\right)W\left(D,\boldsymbol{\mu}\right)V^{(\varrho_0\boldsymbol{\rho})}\left(D,\boldsymbol{\mu}\right)=\frac{1}{2}\delta_{\tilde{\varrho}_0\varrho_0}\delta_{\tilde{\boldsymbol{\rho}}\boldsymbol{\rho}}\left[\prod_{k=1}^{L}\frac{\pi}{\cos\frac{\pi\mu_k}{2}}\right]$$
$$\times\frac{\pi}{\sin\frac{\pi}{2}\left(D-\mu_0\varrho_0-\sum_{k=1}^{L}\mu_k\rho_k\right)}\frac{\pi}{\cos\frac{\pi}{2}\left(D-\mu_0\bar{\varrho}_0-\sum_{k=1}^{L}\mu_k\rho_k\right)}$$

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$$T_1^\intercal(-D, -\boldsymbol{\mu}, \boldsymbol{m}) W(D, \boldsymbol{\mu}) T_2(D, \boldsymbol{\mu}, \boldsymbol{m}) = \mathrm{const}\,.$$

If we choose two basis solutions, $V^{(\tilde{\varrho}_0 \tilde{\boldsymbol{\rho}})\intercal}$ and $V^{(\varrho_0 \boldsymbol{\rho})}$, then we can fix the right-hand side by considering asymptotics $m_{k>0}/m_0 \to 0$. We find

$$V^{(\tilde{\varrho}_0 \tilde{\boldsymbol{\rho}})\intercal}(-D, -\boldsymbol{\mu})\, W(D, \boldsymbol{\mu})\, V^{(\varrho_0 \boldsymbol{\rho})}(D, \boldsymbol{\mu}) = \frac{1}{2} \delta_{\tilde{\varrho}_0 \varrho_0} \delta_{\tilde{\boldsymbol{\rho}} \boldsymbol{\rho}} \left[\prod_{k=1}^{L} \frac{\pi}{\cos \frac{\pi \mu_k}{2}} \right]$$
$$\times \frac{\pi}{\sin \frac{\pi}{2} \left(D - \mu_0 \varrho_0 - \sum_{k=1}^{L} \mu_k \rho_k \right)} \frac{\pi}{\cos \frac{\pi}{2} \left(D - \mu_0 \bar{\varrho}_0 - \sum_{k=1}^{L} \mu_k \rho_k \right)}$$

Summing over $\tilde{\varrho}_0, \tilde{\boldsymbol{\rho}}, \varrho_0, \boldsymbol{\rho}$ we obtain

$$T^\intercal(-D, -\boldsymbol{\mu})\, W(D, \boldsymbol{\mu})\, T(D, \boldsymbol{\mu}) = \left[\prod_{k=0}^{L} \frac{\pi}{\cos \frac{\pi \mu_k}{2}} \right]$$
$$\times \sum_{\rho_0 = 0,1} \cdots \sum_{\rho_L = 0,1} \frac{\pi}{\sin \pi \left(D - \sum_{k=0}^{L} \mu_k \rho_k \right)}\,.$$

Quadratic constraints near $D = D^\star,\ \boldsymbol{\mu} = \boldsymbol{\mu}^\star$

Had we been interested in the expansion near $D = \boldsymbol{\mu} = 0$, we would have had alot of constraints directly from this equation. However, $D = \boldsymbol{\mu} = 0$ corresponds to $\alpha_k = 1/2$ (propagators with half-integer indices). Therefore, we use dimension and index shifts to obtain

$$T_1^\intercal(D^\star + 2\epsilon, \boldsymbol{\mu}^\star + 2\boldsymbol{\delta}, \boldsymbol{m}) B T_2(D^\star - 2\epsilon, \boldsymbol{\mu}^\star - 2\boldsymbol{\delta}, \boldsymbol{m}) = \mathrm{const}\,,$$

where

$$\begin{aligned} B &= B(D^\star, \boldsymbol{\mu}^\star |\, \epsilon, \boldsymbol{\delta}, \boldsymbol{m}) \\ &= \prod_{l=0}^{L} \left(\prod_{\nu=1}^{\mu_l^\star} R_L^\intercal (D^\star + 2\epsilon, -\boldsymbol{\mu}^\star + 2\sum_{k=l}^{L} \mu_k^\star \boldsymbol{e}_k - 2\nu \boldsymbol{e}_l + 2\boldsymbol{\delta}) \right) \\ &\qquad \times \prod_{d=1}^{2D^\star} R^{-1\intercal}(D^\star - d + 2\epsilon, -\boldsymbol{\mu}^\star + 2\boldsymbol{\delta}) W(D^\star - 2\epsilon, \boldsymbol{\mu}^\star - 2\boldsymbol{\delta})\,. \end{aligned}$$

Special cases

Repeating lines

Let us consider what happens when m_k and μ_k coincide for $n > 1$ lines.

$$\underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{n} \otimes \ldots \otimes \mathbb{C}^2$$

We can consider only symmetrical in the first n indices tensors. This corresponds to passing from n copies of spin-1/2 (2^n states $|\frac{1}{2}, \pm\frac{1}{2}\rangle \otimes |\frac{1}{2}, \pm\frac{1}{2}\rangle \ldots |\frac{1}{2}, \pm\frac{1}{2}\rangle$) to states with maximal spin $s = n/2$ ($2s + 1 = n + 1$ states $|s, s\rangle, \ldots, |s, -s\rangle$). Then the holonomic rank drops from $2^L = 2^n \cdot 2^{L-n}$ to $(n+1) \cdot 2^{L-n}$.

Repeating lines

$$\underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{n_0} \otimes \underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{n_1} \otimes \ldots$$

In general, if we have n_k lines with mass m_k and index μ_k ($k = 0, \ldots, N$), then the holonomic rank is $\prod_{k=0}^{N}(n_k + 1)/2$ (to be compared with $2^{\sum_k n_k}/2$ for non-symmetric case). The $/2$ part corresponds to a parity-even subsystem. This expression is integer iff at least one spin is half-integer. If all spins are integer, we are sitting on the singularity of the differential system: one has to be careful. This case is not yet fully analysed so far, but it looks that we have rank $\lfloor \prod_{k=0}^{N}(n_k + 1)/2 \rfloor$ (while a naive treatment would give $\lceil \ldots \rceil$).

Note that for sunrise graphs with generic incoming momentum we don't run into trouble.

Coming next:

Removing analytical regularization

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Removing analytical regularization corresponds to putting $\mu = D - 1$. It is relatively easy. The system triangularizes a bit because the equations for $L + 1$ clover-leaf tadpoles decouple. The dimension and index-shifting operators R and R_k have regular limit $\mu = D - 1$.

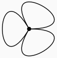
Removing analytical regularization

We again pass to

$$Y(\mu, \boldsymbol{m}) = M\, T(D, \boldsymbol{\mu}, \boldsymbol{m})\Big|_{\substack{D=\mu+1 \\ \mu_k=\mu}}$$

The differential equation for this new function has the form

$$dY = \mu H Y\,, \qquad H = (N-1)\,\frac{dM}{M} + \sum_k \bar{n}_k \frac{dm_k}{m_k}\,.$$

Thanks to left factor $(N-1)$ in the first term, we see that equation for each of $L+1$ components $Y_{10\ldots0}, Y_{010\ldots0}, \ldots Y_{0\ldots01}$ decouples. These components correspond to trivial clover-leaf tadpoles .

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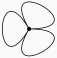
Thanks to left factor $(N-1)$ in the first term, we see that equation for each of $L+1$ components $Y_{10\ldots0}, Y_{010\ldots0}, \ldots Y_{0\ldots01}$ decouples. These components correspond to trivial clover-leaf tadpoles .

The clover-leaf tadpoles enter as inhomogeneous terms in the rest of equations and thus we define the homogeneous solutions Y as the ones that have

$$Y_{10\ldots0} = Y_{010\ldots0} = \ldots Y_{0\ldots01} = 0$$

.

Basis of homogeneous solutions

Previously we have had a basis of solution space $V^{(\varrho_0 \boldsymbol{\rho})}$ Let's define

$$Y^{(\varrho_0 \boldsymbol{\rho})}(\mu, \boldsymbol{m}) = M\, V^{(\varrho_0 \boldsymbol{\rho})}(D, \boldsymbol{\mu}, \boldsymbol{m})\bigg|_{\substack{D=\mu+1 \\ \mu_k=\mu}}.$$

This is a full basis of 2^{L+1} solutions of which there is only a half (2^L) we are mainly interested, defined by

$$\varrho_0 = |\boldsymbol{\rho}| \pmod 2. \qquad (\star)$$

Luckily (but not obviously), the subspace of homogeneous solutions is spanned by the subset of basis vectors $Y^{(\varrho_0 \boldsymbol{\rho})}(\mu, \boldsymbol{m})$ subject to condition

$$\varrho_0 \neq |\boldsymbol{\rho}|$$

This sorts out $1_{\varrho_0=|\boldsymbol{\rho}|=0} + L_{\varrho_0=|\boldsymbol{\rho}|=1}$ solutions which all belong to 2^L-dimensional space we are interested in (as ($\star$) holds when $\varrho_0 = |\boldsymbol{\rho}|$). Then we have holonomy rank $2^L - L - 1$ for the homogeneous system in agreement with [Kalmykov and Kniehl, 2017]. For repeating lines we have instead $[\prod_{k=0}^{N}(n_k+1)]/2 - N - 1$ for the homogeneous system.

Quadratic constraint

If we substitute $T = M^{-1}Y$ into the constraint $T_1^{\intercal}(-D, -\boldsymbol{\mu}) W T_2(D, \boldsymbol{\mu})$ we obtained before, we will immediately run into troubles. The problem is that $V^{\varrho_0, \boldsymbol{\rho}}(D = \mu + 1, \mu)$ has divergencies if $\varrho_0 = |\boldsymbol{\rho}|$. So, once we have removed analytic regularization, the quadratic constraints make sense only for homogeneous solutions. It then reads

$$Y^{(\tilde{\varrho}_0 \tilde{\boldsymbol{\rho}})\intercal}(-\mu)\,(N-1)^{-1}\,Y^{(\varrho_0 \boldsymbol{\rho})}(\mu) = \frac{\mu}{2}\delta_{\tilde{\varrho}_0 \varrho_0}\delta_{\tilde{\boldsymbol{\rho}}\boldsymbol{\rho}}\left(\frac{\pi}{\cos\frac{\pi\mu}{2}}\right)^L \times \frac{\pi}{\sin\frac{\pi\mu}{2}\left(\varrho_0 - \sum_{k=1}^{L}\rho_k\right)}\frac{\pi}{\cos\frac{\pi\mu}{2}\left(\bar{\varrho}_0 - \sum_{k=1}^{L}\rho_k\right)}\,.$$

Here $\tilde{\varrho}_0 \neq |\tilde{\boldsymbol{\rho}}|$ and $\varrho_0 \neq |\boldsymbol{\rho}|$, i.e. we take only homogeneous solutions. Note that $(N-1)$ is invertible precisely in the subspace of homogeneous solutions.

Coming next:

Story of suffering: removing dimensional regularization at $D = 2$

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Removing dimensional regularization is extremely tricky. It appears that all basis of $2^L + 2^L$ solutions becomes dependent at $D = 2$. If we look at the integrand we might even think that all 2^L solutions shrink to one (so $2^L + 2^L$ becomes $1 + 1$).

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BOTTOM LINE: SO FAR WE CAN NOT WRITE CLOSED EXPRESSIONS FOR $D = 2$ AND ARBITRARY L. WE CAN ONLY PROCEED IF L IS SET TO ARBITRARY BUT SPECIFIED INTEGER.

Broadhurst&Roberts identities

In 2017-2018 Broadhurst&Roberts conjectured a set of mysterious quadratic relations between moments of the product of I_0 and K_0 functions

$$\mathrm{IKM}(i,k,m) = \int dx I_0^i(x) K_0^k(x) x^m \tag{5}$$

We seem now to have a systematic tool to prove these identities for any specified L (we have done this for $L = 10$).

We even can write many generalizations. One example:

$$1 = -36m^4 \int dx \left(K I_0 I_1^2 x\right) \int dx \left(K I_0^2 K_0 x\right)$$
$$+12m^4 \int dx \left(K I_0^3 x\right) \int dx \left(K I_1^2 K_0 x\right) - 24m^4 \int dx \left(K I_0^3 x\right) \int dx \left(K I_0 I_1 K_1 x\right)$$
$$+24m^3 \int dx \left(K I_1^2 K_1\right) \int dx \left(K I_0^3 x\right) + 24m^3 \int dx \left(K I_1^3\right) \int dx \left(K I_0^2 K_0 x\right)$$
$$+6\left(3m^2-1\right) m^2 \int dx \left(K I_0^3 x\right) \int dx \left(K I_0 I_1^2 x\right) + \left(3m^2-1\right)^2 \int dx \left(K I_0^3 x\right)^2$$

where, e.g., $\int dx \left(K I_0 I_1 K_1 x\right) = \int dx \left(K(x) I_0(mx) I_1(mx) K_1(mx) x\right)$.

Conclusion

Conclusion

- For L-loop watermelon tadpole and sunrise graphs with generic parameters (masses, indices and dimension) we have obtained
  - Differential equations in Pfaffian form
  - Index and dimension shifting identities
  - Quadratic constraints near any integer D and $\boldsymbol{\mu}$
  - Solution in terms of Goncharov's polylogarithms near $D = 1$ (and, thanks to dimension shift, near any odd d).
- By-product: Pfaffian system for the Lauricella $F_C^{(L)}$ functions
- For the interesting case $D = 2$ quadratic identities found can lead to a systematic proof of Broadhurst&Roberts quadratic relations for Bessel momenta integrals IKM.
- Special cases are numerous, many of them are extremely nasty. We have considered them only partly (work in progress).

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