Algorithmic approaches to syzygies for no-dot or no-numerator relations

Andreas v. Manteuffel

Michigan State University

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Multi-Loop Feynman integrals

$$I(\nu_1,\ldots,\nu_N) = \int \mathrm{d}^d k_1 \cdots \mathrm{d}^d k_L \frac{1}{D_1^{\nu_1}\cdots D_N^{\nu_N}} \qquad \nu_i \in \mathbb{Z}, \quad D_1 = k_1^2 - m_1^2 \text{ etc.}$$

family of loop integrals:

- fulfill linear relations: integration-by-parts identities, ...
- systematic reduction to master integrals possible
- linear vector space with some finite basis
- specific basis choices:
  - ϵ-basis (uniform weight) for differential equations [Kotikov '10 , Henn '13]
  - basis of finite integrals for direct integration (analyt., numeric.) [A.v.M., Panzer, Schabinger '16]
  - uniform weight + finite: [Schabinger '18]
- depending on application, ν_i will be different from 0 and 1:
  - $\nu_i < 0$: occur naturally in scattering amplitudes, ...
  - $\nu_i > 1$: occur naturally for finite integrals, ...

Singularity resolution for $d \to 4$

consider Feynman parameter representation of multi-loop integral

$$I(\nu_1, \ldots, \nu_N) = \mathcal{N}\left[\prod_{i=1}^{N} \int_0^\infty \mathrm{d}x_i x_i^{\nu_i - 1}\right] \delta(1 - x_N)\, \mathcal{U}^{\nu-(L+1)\frac{d}{2}}\, \mathcal{F}^{-\nu+L\frac{d}{2}}$$

where

- $\nu = \sum_i \nu_i$, ν_i denotes propagator multiplicity
- $\mathcal{U}$ and $\mathcal{F}$ are Symanzik polynomials in x_i

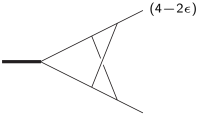
problem:

- can't directly expand integrand around $\epsilon = (4-d)/2 = 0$: divergencies from x_i integrations
- no straight-forward analytical or numerical integration

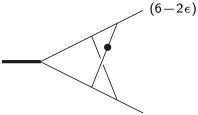
generic approaches to singularity resolution:

1. sector decomposition [Hepp '66, Binoth, Heinrich '00]
2. polynomial exponent raising [Bernstein '72, Tkachov '96, Passarino '00]
3. Mellin-Barnes representations
4. analytic regularisation [Panzer '14]
5. basis of finite Feynman integrals [AvM, Schabinger, Panzer '14]

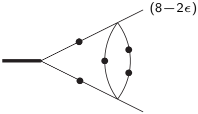
observation: always possible to decompose wrt basis of finite integrals



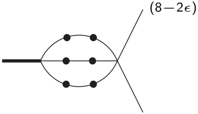
$$= -\frac{4(1-4\epsilon)}{\epsilon(1-\epsilon)q^2}$$



$$-\frac{2(2-3\epsilon)(5-21\epsilon+14\epsilon^2)}{\epsilon^4(1-\epsilon)^2(2-\epsilon)^2q^2}$$



$$+\frac{4(2-3\epsilon)(7-31\epsilon+26\epsilon^2)}{\epsilon^4(1-2\epsilon)(1-\epsilon)^2(2-\epsilon)^2q^2}$$



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basis consists of standard Feynman integrals, but

- in shifted dimensions
- with additional dots (propagators taken to higher powers)

Practical algorithm for basis construction

Algorithm: construction of finite basis

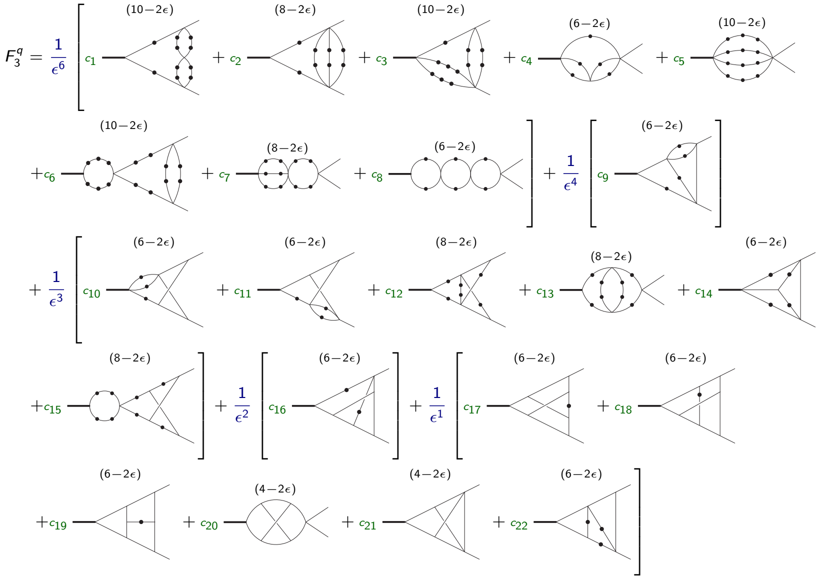
- systematic scan for finite integrals with dim-shifts and dots (with Reduze 2)
- IBP + dimensional recurrence for actual basis change

remarks:

- computationally expensive part shifted to IBP solver
- efficient, easy to automate
- any dim-shift good, e.g. shifts by [Tarasov '96], [Lee '10]
- see [Bern, Dixon, Kosower '93] for dim-shifted one-loop pentagon

Quark form factor @ 3-loops [AvM, Panzer, Schabinger '15]

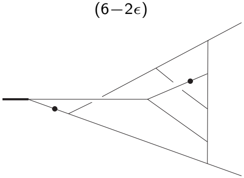
original ff@3l: [Baikov, Chetyrkin, Smirnov, Smirnov, Steinhauser '09], [Gehrmann, Glover, Huber, Ikizlerli, Studerus '10, '10]



Analytical integration @ 4-loops

[AvM, Panzer, Schabinger '15]

a non-planar 12-line topology @ 4-loops:



$$= \frac{18}{5}\zeta_2^2\zeta_3 - 5\zeta_2\zeta_5 + \Big(24\zeta_2\zeta_3 + 20\zeta_5 - \frac{188}{105}\zeta_2^3 - 17\zeta_3^2 + 9\zeta_2^2\zeta_3$$

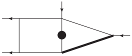
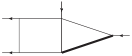
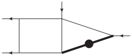
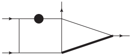
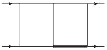
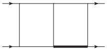
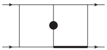
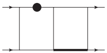
$$- 47\zeta_2\zeta_5 - 21\zeta_7 + \frac{6883}{2100}\zeta_2^4 + \frac{49}{2}\zeta_2\zeta_3^2 + \frac{1}{2}\zeta_3\zeta_5 - 9\zeta_{5,3}\Big)\epsilon + \mathcal{O}\left(\epsilon^2\right)$$

- only shallow ϵ expansion needed
- numerical result with `Fiesta` [A. Smirnov]: straight-forward confirmation
- if leading order high weight, might not contribute to specific physics quantities (cusp anomalous dimension)

Numerical performance

[AvM, Schabinger '17]

basis of finite integrals renders problematic double boxes numerically accessible

finite	time	rel. err.	conventional	time	rel. err.
$(6-2\epsilon)$  (s,t)	201 s	2.34×10^{-4}	$(4-2\epsilon)$  (s,t)	384 s	8.12×10^{-4}
$(6-2\epsilon)$  (s,t)	150 s	4.83×10^{-4}	$(4-2\epsilon)$  (s,t)	56538 s	1.67×10^{-2}
$(6-2\epsilon)$  (s,t)	280 s	1.00×10^{-3}	$(4-2\epsilon)$  (s,t)	214135 s	8.29×10^{-3}
$(6-2\epsilon)$  (s,t)	294 s	1.21×10^{-3}	$(4-2\epsilon)$  (s,t)	3484378 s	30.9

timings with SecDec 3 in physical region

Integration-by-parts (IBP) identities

in dimensional regularisation, integral over total derivative vanishes:

$$0 = \int \mathrm{d}^d k_1 \cdots \mathrm{d}^d k_L \frac{\partial}{\partial k_i^\mu} \left(k_j^\mu \frac{1}{D_1^{\nu_1} \cdots D_N^{\nu_N}} \right)$$

$$0 = \int \mathrm{d}^d k_1 \cdots \mathrm{d}^d k_L \frac{\partial}{\partial k_i^\mu} \left(p_j^\mu \frac{1}{D_1^{\nu_1} \cdots D_N^{\nu_N}} \right)$$

where p_j are external momenta, $\nu_i \in \mathbb{Z}$, $D_1 = k_1^2 - m_1^2$ etc.

integral reduction:

- express arbitrary integral for given problem via few basis integrals
- integration-by-parts (IBP) reductions [Chetyrkin, Tkachov '81], [Laporta '00]
- public codes: `Air` [Anastasiou], `Fire` [Smirnov], `Reduze 1` [Studerus], `Reduze 2` [AvM, Studerus], `LiteRed` [Lee], `Kira` [Maierhöfer, Usovitsch, Uwer]

Shift relations

- assigning loop momenta to propagators not unique:

$$\int \mathrm{d}^d k \frac{1}{(k^2)^{a_1}((k+p)^2)^{a_2}} \xrightarrow{k\to -k+p} \int \mathrm{d}^d k \frac{1}{((k+p)^2)^{a_1}(k^2)^{a_2}}$$

- shift induces relations between indexed integrals of same or different sectors:

$$D_1 := k^2$$
$$D_2 := (k+p)^2$$
$$I_{a_1,a_2} := \int \mathrm{d}^d k \frac{1}{D_1^{a_1} D_2^{a_2}}$$

shift relation:

$$\Rightarrow \quad I_{a_1,a_2} = I_{a_2,a_1}$$

- also needed to map diagrams to sectors of families

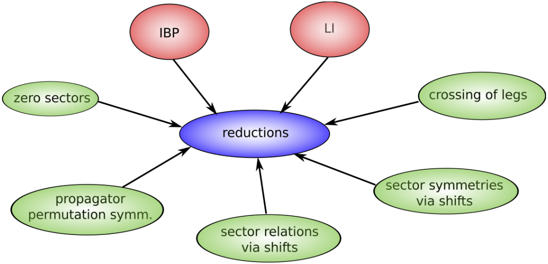
Shift relations in Reduze 2

- additional info from non-infinitesimal linear transformation of loop momenta

$$k_i \to \sum_{j=1}^{l} M_{ij} k_j + \sum_{j=1}^{m} N_{ij} p_j \qquad \text{with } |\det M| = 1$$

- different shift finders in Reduze 2 [AvM, Studerus '12]:
  1. linear algebra for momenta
  2. graph matroid isomorphism
  3. minimal graph construction

- handling of different types of shifts:
  - permutation symmetries: for all sectors of a family, very efficient
  - sector relations: eliminate complete sectors
  - sector symmetries: relates integrals of same sector (sometimes important !)



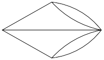
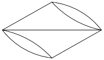
Reduze 2: sectors, graphs and matroids

integral A (propagators) ⟵ ∃shift ? ⟶ integral B (propagators)

↓ *construct* ↓ *construct*

graph A ⟵ isomorphic ? ⟶ graph B

- problem: graphs not unique !

e.g.:  ⟵ same propagators but not isomorphic ⟶ 

- solution: select unique representative by allowing for twists

based on: first Symanzik polynomials $\mathcal{U}$ isomorphic ⇔ graphs isomorphic up to twists

theorem by [Bogner, Weinzierl (2010)], proof based on [Whitney's theorem] for isomorphisms of graph matroids

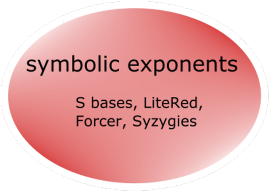
for non-vacuum graphs with masses: join external legs, color edges

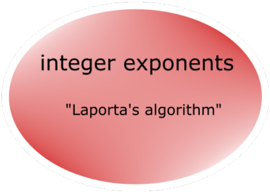
Numbers for a practical example: four-loop form factors

- $O(50000)$ diagrams
- just one overall scale
- use R_ξ gauge (power one for props + ext pol)
- $O(10^9)$ integrals in diagrams, up to 6 irreducible scalar products
- $O(100)$ different 12-line topologies
- 10 integral families (18 indices)
- $O(300)$ master integrals
- $O(25000)$ sectors in reduction, $O(2000)$ non-shiftable
- up to $O(10^8 - 10^9)$ equations per sector

$\Rightarrow$ need improved IBP solver

Approaches to IBP reduction





- in real life: typically a combination of both approaches
- new focus: syzygy constructions pioneered by Gluza, Ita, Kadja, Kosower, Larsen, Lee, Schabinger, Zhang, Zeng and others
- structure of IBPs with non-integer exponents [Bitoun, Bogner, Klausen, Panzer '18]
- this talk: improve integer part, combine it with syzygy approach

IBP reductions from finite field samples

A Finite Field Approach to IBPs [AvM, Schabinger '14]

1. finite field sampling
   - set variables to integer numbers
   - consider coefficients modulo a prime field $\mathbb{Z}_p$
2. solve finite field system
3. reconstruct rational solution from many such samples

finite field techniques:

- no intermediate expression swell by construction
- early discard of redundant and auxiliary quantities
- great potential for parallelisation

established in computer sciences, more and more popular in physics:

- dense solver: [Kauers]
- filtering: `Ice` [Kant '13]
- tensor reduction: [Heller]
- QCD integrand construction: [Peraro '16]
- Kira [Maierhöfer, Usovitsch, Uwer]
- Fire [Smirnov]
- and more ...

A fast rational solver

INPUT: $I_{\mathbb{Q}}$ unreduced rational matrix

OUTPUT: $O_{\mathbb{Q}}$ row reduced rational matrix

$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}} & \xrightarrow[\text{image}]{\text{homomorphic}} & I_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{reduction}]{\text{Gauss row}} & O_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{remaindering}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1 \cdot p_2 \cdot p_3 \cdots}} & \xrightarrow[\text{reconstruction}]{\text{rational}} & O_{\mathbb{Q}} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}} & \longrightarrow & O_{\mathbb{Z}_{p_2}} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}} & \longrightarrow & O_{\mathbb{Z}_{p_3}} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

Function reconstruction

univariate rational function $\mathbb{Q}[d]$ reconstruction:

- works similar to the case $\mathbb{Q}$
- Chinese remaindering becomes Lagrange polynomial interpolation:

$$p_1 \cdots p_N \to (d - p_1) \cdots (d - p_N)$$

- rational reconstruction becomes Pade approximation:

$$\text{interpolating polynomial} \to \text{rational function}$$

multivariate rational function $\mathbb{Q}[d, s, t, \ldots]$ reconstruction:

- by iteration

A fast univariate solver

rational solver: reduce matrix $I_{\mathbb{Q}}$ of rational numbers

$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}} & \xrightarrow[\text{image}]{\text{homomorphic}} & I_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{reduction}]{\text{Gauss row}} & O_{\mathbb{Z}_{p_1}} & \xrightarrow[\text{remaindering}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1 \cdot p_2 \cdot p_3 \cdots}} & \xrightarrow[\text{reconstruction}]{\text{rational}} & O_{\mathbb{Q}} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}} & \longrightarrow & O_{\mathbb{Z}_{p_2}} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}} & \longrightarrow & O_{\mathbb{Z}_{p_3}} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

univariate solver: reduce matrix $I_{\mathbb{Q}[x]}$ of rational functions in x

$$
\begin{array}{ccccccccc}
I_{\mathbb{Q}[x]} & \xrightarrow[\text{img.}]{\text{hom.}} & I_{\mathbb{Z}_{p_1}}[x] & \xrightarrow[\text{(see below)}]{\text{aux solver}} & O_{\mathbb{Z}_{p_1}[x]} & \xrightarrow[\text{remaind.}]{\text{Chinese}} & O_{\mathbb{Z}_{p_1 \cdot p_2 \cdot p_3 \cdots}[x]} & \xrightarrow[\text{rec.}]{\text{rat.}} & O_{\mathbb{Q}[x]} \\
 & \longrightarrow & I_{\mathbb{Z}_{p_2}[x]} & \longrightarrow & O_{\mathbb{Z}_{p_2}[x]} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_{p_3}[x]} & \longrightarrow & O_{\mathbb{Z}_{p_3}[x]} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

aux solver: reduce matrix $I_{\mathbb{Z}_p[x]}$ of polynomials in x with finite field coefficients

$$
\begin{array}{ccccccccc}
I_{\mathbb{Z}_p}[x] & \xrightarrow[\text{by number } x_i]{\text{sample } x} & I_{\mathbb{Z}_p, x_1} & \xrightarrow[\text{reduction}]{\text{row}} & O_{\mathbb{Z}_p, x_1} & \xrightarrow[\text{interpolation}]{\text{polynomial}} & O_{\mathbb{Z}_p}[x] & \xrightarrow[\text{reconstruction}]{\text{rational function}} & O_{\mathbb{Z}_p}[x] \\
 & \longrightarrow & I_{\mathbb{Z}_p, x_2} & \longrightarrow & O_{\mathbb{Z}_p, x_2} & \longrightarrow & & & \\
 & \longrightarrow & I_{\mathbb{Z}_p, x3} & \longrightarrow & O_{\mathbb{Z}_p, x3} & \longrightarrow & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & &
\end{array}
$$

note: massively parallisable

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Package: finred
Author:  Andreas v. Manteuffel
```

features:

- C++11 implementation for multivariate sparse matrices
- links `flint` library
- parallelisation: SIMD, threads, batch
- custom file formats with compression and checksums
- equation filtering: eliminate redundant rows
- pivot optimization
- plus lots of IBP specific features
- much faster than Reduze 2
- employed for construction and application of syzygies

Integration-by-parts (IBP) identities

in dimensional regularisation, integral over total derivative vanishes:

$$0 = \int \mathrm{d}^d k_1 \cdots \mathrm{d}^d k_L \frac{\partial}{\partial k_i^\mu} \left(k_j^\mu \frac{1}{D_1^{\nu_1} \cdots D_N^{\nu_N}} \right)$$

$$0 = \int \mathrm{d}^d k_1 \cdots \mathrm{d}^d k_L \frac{\partial}{\partial k_i^\mu} \left(p_j^\mu \frac{1}{D_1^{\nu_1} \cdots D_N^{\nu_N}} \right)$$

where p_j are external momenta, $\nu_i \in \mathbb{Z}$, $D_1 = k_1^2 - m_1^2$ etc.

problems of above construction:

- introduces many auxiliary integrals with additional dots and/or numerators
- sparse but still rather coupled system of equations

Syzygy Based IBPs Without Numerators

Lee-Pomeransky representation:

$$I(\nu_1,\ldots,\nu_N) = \mathcal{N}\left[\prod_{i=1}^{N}\int_0^\infty \mathrm{d}x_i x_i^{\nu_i-1}\right] G^{-d/2} \qquad \text{with } G = \mathcal{U}+\mathcal{F}$$

linear relations from "annihilators" [Lee '14; Bitoun, Bogner, Klausen, Panzer '17]:

$$\left[c_0 + \sum_{i=1}^{N} c_i \frac{\partial}{\partial x_i} + \sum_{i,j=1}^{N} c_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \ldots\right] G^{-d/2} = 0$$

new in this talk: applications of 2nd order diff.op.

Syzygy Based IBPs Without Numerators

Lee-Pomeransky representation:

$$I(\nu_1,\dots,\nu_N)=\mathcal{N}\left[\prod_{i=1}^{N}\int_0^\infty \mathrm{d}x_i x_i^{\nu_i-1}\right] G^{-d/2} \qquad \text{with } G=\mathcal{U}+\mathcal{F}$$

linear relations from "annihilators" [Lee '14; Bitoun, Bogner, Klausen, Panzer '17]:

$$\left[c_0+\sum_{i=1}^{N} c_i\frac{\partial}{\partial x_i}+\sum_{i,j=1}^{N} c_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}+\dots\right] G^{-d/2}=0$$

new in this talk: applications of 2nd order diff.op.
determine $c_0(x_1,\dots,x_N),\dots$ via syzygy equations:

$$c_0\left[-\frac{2}{d}G^2\right]+\sum_{i=1}^{N} c_i\left[G\frac{\partial G}{\partial x_i}\right]+\sum_{i,j=1}^{N} c_{ij}\left[G\frac{\partial^2 G}{\partial x_i\partial x_j}+\left(-\frac{d}{2}-1\right)\frac{\partial G}{\partial x_i}\frac{\partial G}{\partial x_j}\right]+\dots=0$$

Syzygies generate linear relations for Feynman integrals:

$$\left(\left[c_0(\hat{1}^+,\dots,\hat{N}^+)-\sum_{i=1}^{N} c_i(\hat{1}^+,\dots,\hat{N}^+)\hat{i}^- +\sum_{i,j=1}^{N} c_{ij}(\hat{1}^+,\dots,\hat{N}^+)\hat{i}^-\hat{j}^- +\dots\right]\tilde{I}\right)(\nu_1,\dots,\nu_N)=0$$

with $\tilde{I}(\nu)=\Gamma[(L+1)d/2-\nu]I(\nu)$ and shift operators (Weyl algebra)

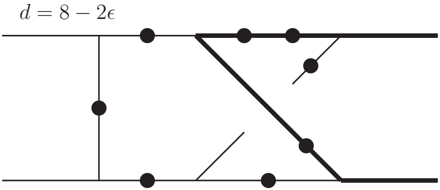
$$(\hat{i}^+ I)(\nu_1,\dots,\nu_N)=\nu_i I(\nu_1,\dots,\nu_i+1,\dots,\nu_N),$$
$$(\hat{i}^- I)(\nu_1,\dots,\nu_N)=I(\nu_1,\dots,\nu_i-1,\dots,\nu_N)$$

Computation

- problem:
  - computation of syzygy modules challenging
  - for complicated Feynman integrals I was not successful with Singular
- strategy:
  - observation: need only generating syzygies of low degree
  - use custom syzygy finder with `Finred`
  - induced syzygies: via seed integrals
  - followed by "Laporta's reduction"
- basic linear algebra method: `LASyz` [Carbacas, Ding '11], see also [Schabinger '11]
  - compute generating syzygies for homogenous system up to some degree
  - speeds up Gröbner basis calculations by avoiding reductions to zero
  - related ideas in Faugère's work
  - experiments by authors: competitive with `F4` (and faster than `F5`) for challenging systems
  - $\rightarrow$ show idea at blackboard
- result:
  - reductions of integrals with large number of dots possible
  - no auxiliary numerators need to be introduced

Example at 2-loops

reductions like this easily accessible now:

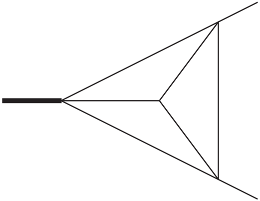
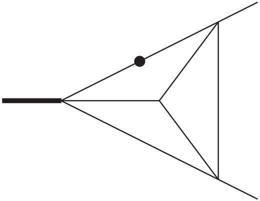


finite integral with improved convergence properties

Question: are first order annihilators sufficient ?

Example: planar three-point function, 3 loops

- Relations with 1 derivative: leave two integrals unreduced

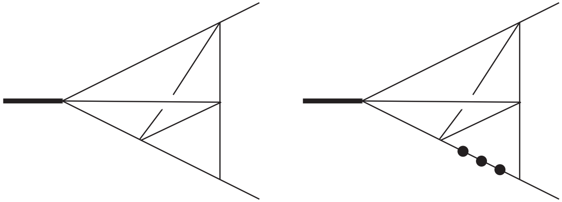
  - counting for integer powers, ignoring subsectors
  - tested up to 15 dots (degree of syzygies) in relations

- Relations with 2 derivatives: reduce one of the two integrals
  - degree 3 syzygies + seeds sufficient
  - in the same sector, in general with a different number of dots
  - in subsectors, possibly with a numerator (inverse propagator)
  - both, inverse propagators and sub-sub-sectors are key to completeness

Question: are first order annihilators sufficient ?

Example: non-planar three-point function, 4 loops

- Relations with 1 derivative: a tower of integrals is not reduced:



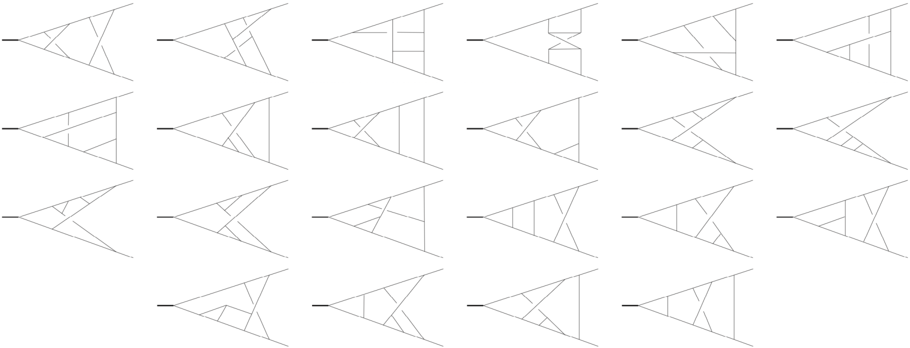
  - dots on a specific propagator unreduced
  - tested up to degree four syzygies + 17 dots for seeds)

- Relations with 2 derivatives: reduce all but one integral
  - degree 3 syzygies + seeds sufficient

Recent results at four loops

Four loop form factors in QCD: [Henn, Smirnov, Smirnov, Steinhauser '16; Henn, Lee, Smirnov, Smirnov, Steinhauser '16, '17, '19; AvM, Schabinger '16, '19, '19]

New: contributions with two fermion loops to quark and gluon form factors:



46 twelve-line topologies, up to 6 ISPs, 163 master integrals

analytic results in terms of multiple zeta values [AvM, Schabinger '19]

Syzygy Based IBPs Without Dots

Baikov's parametric representation of Feynman integrals:

$$I(\nu_1,\ldots,\nu_N) = \mathcal{N}\int \mathrm{d}z_1\cdots\mathrm{d}z_m P^{\frac{d-L-E-1}{2}}\frac{1}{z_1^{\nu_1}\cdots z_N^{\nu_N}}$$

[Böhm, Georgoudis, Larsen, Schulze, Zhang '18]: useful for IBPs without dots

$$0 = \int \mathrm{d}z_1\cdots\mathrm{d}z_m \sum_{i=1}^{m}\frac{\partial}{\partial z_i}\left(a_i P^{\frac{d-L-E-1}{2}}\frac{1}{z_1^{\nu_1}\cdots z_N^{\nu_N}}\right)$$

$$= \int \mathrm{d}z_1\cdots\mathrm{d}z_m \sum_{i=1}^{N}\left(\frac{\partial a_i}{\partial z_i} + \frac{d-L-E-1}{2P}a_i\frac{\partial P}{\partial z_i} - \frac{\nu_i a_i}{z_i}\right)P^{\frac{d-L-E-1}{2}}\frac{1}{z_1^{\nu_1}\cdots z_N^{\nu_N}}$$

explicit solutions to constraint:

$$\left(\sum_{i=1}^{N} a_i\frac{\partial P}{\partial z_i}\right) + bP = 0 \qquad \text{(absence of dim. shifts)}$$

in addition, require for denominators of sector:

$$a_i = b_i z_i \qquad \text{(absence of dots)}$$

need intersection of two syzygy modules
fast algorithmic approach [work with Agarwal]: linear algebra with `Finred` → idea at blackboard

Conclusions

basis of finite integrals:

- simple and efficient method for singularity resolution in multi-loop integrals
- analytical integrations: finite integrals are Feynman integrals (dim-shifted, dotted)
- numerical integrations: faster and more stable evaluations (see also HH, Hj)

reductions via finite field sampling:

- speeds up integration-by-parts reductions
- useful also in other contexts

reductions based on syzygies:

- custom syzygy finder in `Finred` based on linear algebra
- low degree generators for syzygies sufficient
- no numerators: need 2nd order derivatives
- no dots: fast method for module intersection

some questions for syzygy-based approaches:

- need beyond 2nd order differential operators ?
- how to include "inter-sector" IBPs ?
- (less urgent:) how to include discrete symmetries ?