Feynman Integrals & Intersection Theory

Pierpaolo Mastrolia

The mathematics of linear relations between Feynman Integrals

MITP Mainz, 20.3.2018

Based on:

- **PM, Mizera**, *Feynman Integrals and Intersection Theory,* **JHEP 1902 (2019) 139** **arXiv: 1810.03818**

- **Frellesvig, Gasparotto, Laporta, Mandal, PM, Mattiazzi, Mizera**, *Decomposition of Feynman Integrals on the Maximal Cut by Intersection Numbers* **arXiv: 1901.1151**

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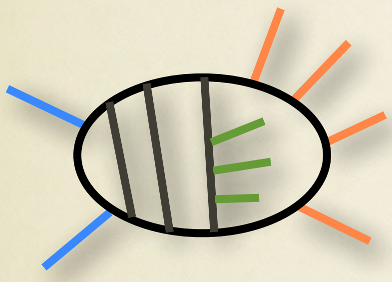
Outline

- **Introduction**
- **A few facts about Feynman Integrals**
  - Integration-by-parts Identities
- **Basics of Intersection Theory**
- **Intersection Numbers for 1-forms**
- **Integral Relations by Intersection Numbers**
  - Special Functions
  - Feynman Integrals (on maximal cuts)
- **Intersection Numbers for 2-forms**
- **Conclusions**

Feynman Integrals

- **Momentum-space Representation**



$$= \quad I_{a_1,a_2,\ldots,a_N} \equiv \int \prod_{i=1}^{L} \frac{d^d k_i}{\pi^{d/2}} \prod_{j=1}^{N} \frac{1}{D_j^{a_j}}$$

N-denominator
generic Integral

L loops, $E{+}1$ external momenta,

$N = LE + \frac{1}{2}L(L{+}1)$ (generalised) denominators

total number of *reducible* and *irreducible* scalar products

't Hooft & Veltman

- **Integration-by-parts Identites** Tkachov; Chetyrkin & Tkachov

Laporta, Remiddi, Baikov, Smirnov, Tarasov, Grozin, …; Kuehn & the Karlsruhe school

$$\int \prod_{i=1}^{L} \frac{d^d k_i}{\pi^{d/2}} \frac{\partial}{\partial k_j^{\mu}} \left(v_\mu \prod_{n=1}^{N} \frac{1}{D_n^{a_n}} \right) = 0$$

$v_\mu = v_\mu(p_i, k_j)$ arbitrary

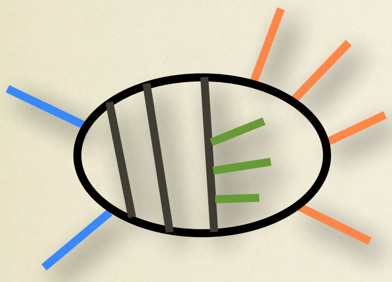
The role of the Integration Domain is hidden

Feynman Integrals :: *Baikov Representation*

- **Denominators as integration variables** Baikov

$$\{D_1, \ldots, D_N\} \to \{z_1, \ldots, z_N\} \equiv \mathbf{z}$$

Volume



$$= \quad I_{a_1,\ldots,a_N} \equiv K(d, s_{ij}) \int_{\mathcal{C}} d\mathbf{z}\; B(\mathbf{z})^{\gamma} \prod_{i=1}^{N} \frac{1}{z_i^{a_n}}$$

$$B(\mathbf{z}) = \det(q_i \cdot q_j)$$

$$\gamma \equiv (d - E - L - 1)/2$$

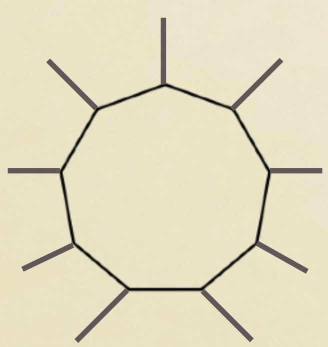
$$q = \{p_i, k_j\} \qquad s_{ij} = p_i \cdot p_j$$

$$B(\partial\mathcal{C} = 0)$$

Fundamental property

N-denominator
generic Integral

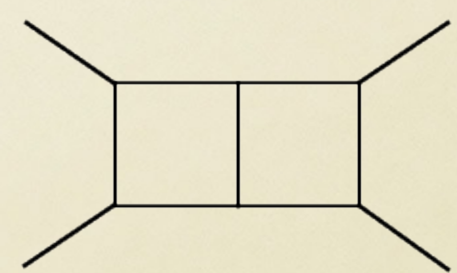
- **1-loop Nonagon**



$$N = LE + \frac{1}{2}L(L+1)$$

$$\int_{\mathcal{C}} dz_1 \wedge \cdots \wedge dz_9 \; \frac{B(\mathbf{z})^{\gamma}}{z_1^{n_1} \cdots z_9^{n_9}}$$

- **2-loop Box**



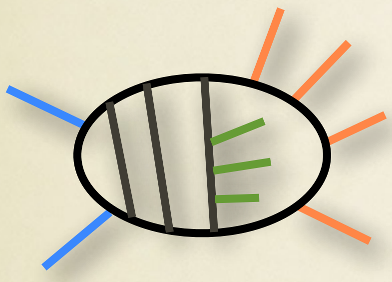
$B(\mathbf{z})$, $\mathcal{C}$, γ depend on the graph.

Feynman Integrals :: *Baikov Representation*

- **Denominators as integration variables** Baikov

$\{D_1, \ldots, D_N\} \to \{z_1, \ldots, z_N\} \equiv \mathbf{z}$

Volume



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$\gamma \equiv (d - E - L - 1)/2$

$q = \{p_i, k_j\} \qquad s_{ij} = p_i \cdot p_j$

$B(\partial\mathcal{C} = 0)$

Fundamental property

N-denominator
generic Integral

- **Integration-by-parts Identites** Zhang, Larsen; Lee;

$$\int_{\mathcal{C}} d\left(h(\mathbf{z})\ B(\mathbf{z})^\gamma \prod_{i=1}^{N} \frac{1}{z_i^{a_n}} \right) = 0$$

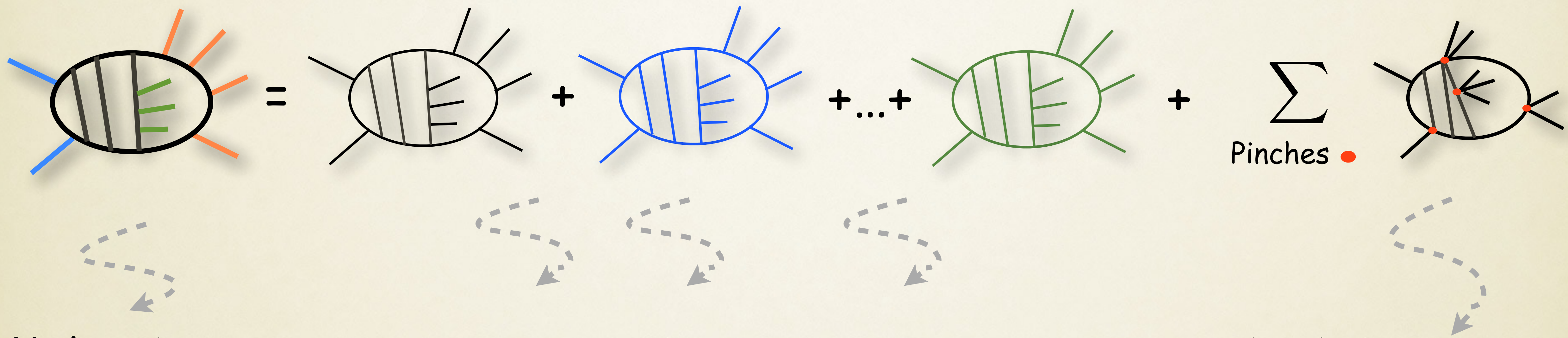
$h(\mathbf{z})$ arbitrary rational function

$B(\partial\mathcal{C} = 0)$

Fundamental property

Integration-by-parts identities

- Relations among Integrals in dim. reg.



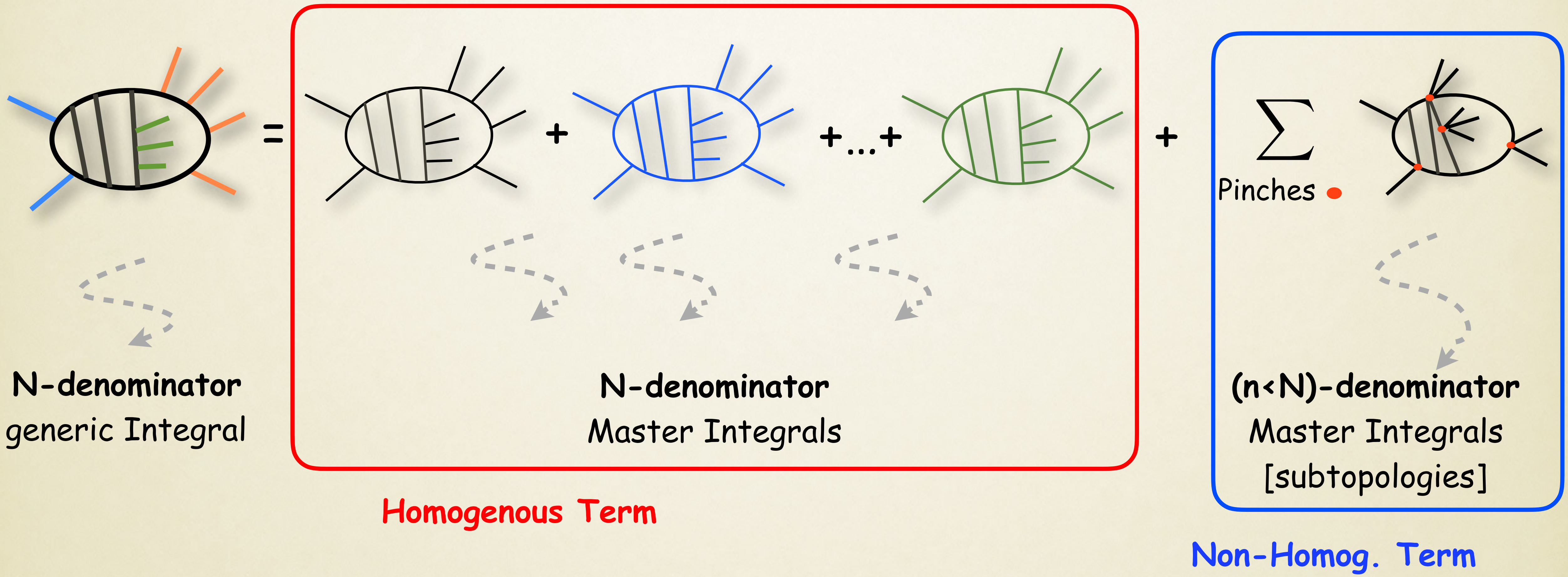
N-denominator
generic Integral

N-denominator
Master Integrals

(n<N)-denominator
Master Integrals
[subtopologies]

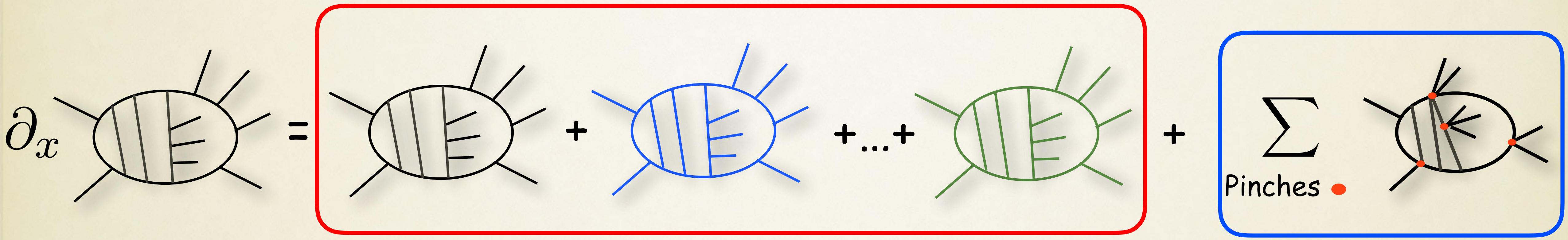
Integration-by-parts identities

- **Relations among Integrals in dim. reg.**



Integration-by-parts identities :: *byproducts*

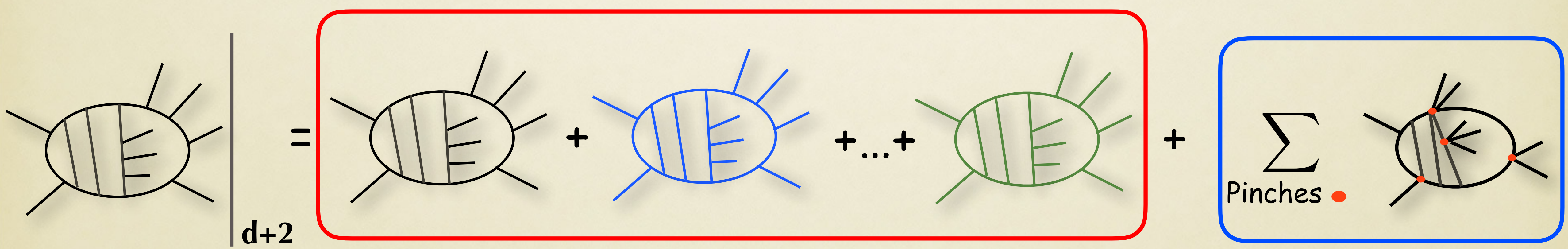
- **1st order Differential Equations for MIs** Barucchi, Ponzano; Kotikov; Remiddi, & Gerhmann; …Weinzierl, Adams, Bogner … Henn; Lee; Argeri, diVita, Mirabella, Schubert, Tancredi, Schlenck **& P.M.**; …



Homogenous Term

Non-Homog. Term

- **Dimension-Shift relations** Tarasov; Lee; Bernstein-Tkachov; + related work by Gluza, Kajda, Kosower; Remiddi, Tancredi

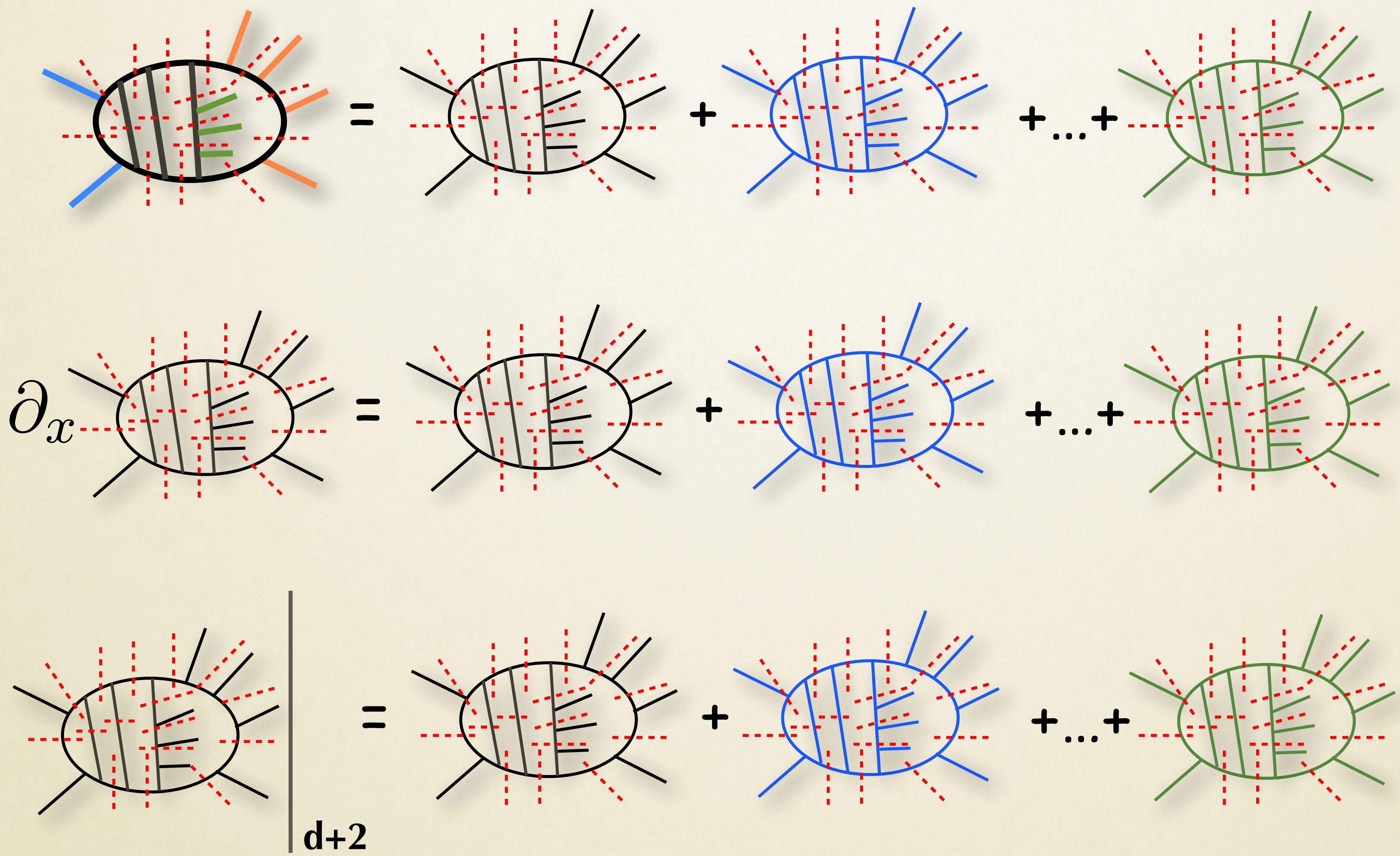


Homogenous Term

Non-Homog. Term

Integral Relations on *Maximal Cuts*

- **cutting m (*all*) internal lines** $D_1 = \ldots, D_m = 0 \iff z_1 = \ldots, z_m = 0\ , \qquad m \leq N$



Larsen & Kosower;
+ CaronHuot;

Remiddi & Tancredi;
Primo & Tancredi;

Lee & Smirnov;

Homogenous Term only

Vanishing Non-Homog. Term

- Maximal cuts are solutions of the Homogenous Differential Equations
- Maximal cuts are solutions of the Homogenous Dimensional Recurrence Relations

A few facts :: *smoking guns*

- **Equivalent statements about (Master) Integrals**
  - Two integrals may give the same result if:
    - i) have the **same integration** domain, but the integrands differ for a term whose primitive vanishes on the integration boundaries.
    - ii) or have the **same integrand**, but the integration domain differ for a contour on which the primitive vanishes.
  - A sector [topology] has a **number** ν of **Master Integrals**
  - Each Master Integral obeys a **Dimensional Recurrence Relation** of order ($d+2\nu$)
  - Each Master Integral obeys a **Differential Equation** of order ν

A few facts :: *smoking guns*

Equivalent statements about (Master) Integrals

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- Each Master Integral obeys a **Dimensional Recurrence Relation** of order (d+2ν)
- Each Master Integral obeys a **Differential Equation** of order ν
- ν = number of **critical points of B(z)** (=?= Euler $\chi(\mathcal{G})$)

Lee & Pomeranski;

Bitoun, Bogner, Klausen & Panzer;

- The **homogenous solutions** of **Differential Equation** play a role in the construction of **epsilon-factorised** systems of differential equations

Argeri, diVita, Schubert, Schlenck, Tancredi & **PM**;
Remiddi & Tancredi; Tancredi & Primo;

- The **independent homogeneous solutions** admit integral representations that differ for the integration contour :: there are ν independent integration contours

Bosma, Sogaard & Zhang; Tancredi & Primo;

A few questions :: *that kept me busy for a while*

- About the. ${}_2F_1$ **Hypergeometric Function** ::
  - It obeys a 2nd ODE ==> **2 independent solutions**:
    - One is proportional to: $\int_0^1 z^{b-1}(1-z)^{c-b-1}(1-xz)^{-a}\,dz$
    - Where is the other one?
    - Has it a different integrand? or a different integration domain?
    - Is there a relations between the DEQ and the integral representation?
  - ${}_2F_1$ obeys a 2nd ODE ==> there must be **2 basic integrals**
  - **Gauss contiguity** relations
    - How are they derived? Can I derive them **by IBP**?
    - Can they be used to find the 2 basic integrals?

- About **IBP relations** and the role of the **integration domain** ::
  - Is there a simple class of IBP of the form: $\int_{\mathcal{C}} d(B(z)^\gamma) = 0$ with $B(\partial\mathcal{C} = 0)$?
  - Answer: $\mathcal{C} = [0,1]$, $B(z) = z(1-z)$

A Novel Method for Feynman Calculus

- Direct **decomposition** into a Integral Basis?
- Direct construction of system of **differential equations** for the Integral Basis?
- Direct construction of **finite difference equations** for the Integral Basis?
- NO intermediate relation required ?

Basics of Intersection Theory

Aomoto, Cho, Kita , Mazumoto, Mimachi, Mizera, Yoshida,…

Consider an integral I over the variables $\mathbf{z} = (z_1, z_2, \ldots, z_m)$

$$I = \int_{\mathcal{C}} u(\mathbf{z})\, \varphi(\mathbf{z})$$

$\varphi(\mathbf{z}) = \hat{\varphi}(\mathbf{z}) d^m\mathbf{z}$ is a differential m-form.

$u(\mathbf{z})$ is a multi-valued function $\qquad u(\partial\mathcal{C}) = 0$

Equivalence classes for *I* and Integration-by-parts Identities

there could exist many forms φ that integrate to give the result I.

$(m-1)$-differential form ξ

$$0 = \int_{\mathcal{C}} d\,(u\,\xi) = \int_{\mathcal{C}} (du \wedge \xi + u\, d\xi) = \int_{\mathcal{C}} u \left(\frac{du}{u} \wedge + d\right) \xi \equiv \int_{\mathcal{C}} u\, \nabla_\omega \xi$$

$$\int_{\mathcal{C}} u\, \varphi = \int_{\mathcal{C}} u\, (\varphi + \nabla_\omega \xi)$$

- Twisted cocycle $\qquad {}_\omega\langle\varphi| : \; \varphi \sim \varphi + \nabla_\omega \xi.$

- **Covariant derivative**

$$\nabla_\omega \equiv d + \omega\wedge$$

$$\omega \equiv d \log u$$

Special role!

Integrals [pairing :: cocycle + cycle]

$$I = {}_\omega\langle\varphi|\mathcal{C}] \equiv \int_{\mathcal{C}} u\, \varphi$$

Vector spaces of differential forms

ν = # of independent forms (Twisted cocycle)

- Basis of Twisted cocycle $\langle e_i|$ $\quad i = 1, 2, \ldots, \nu$
- dual-Basis of Twisted cocycle $|h_j\rangle$ $\quad j = 1, 2, \ldots, \nu$

dual space

$|\varphi\rangle_\omega : \ \varphi \sim \varphi + \nabla_{-\omega}\xi$

$\nabla_{-\omega} \equiv d - \omega\wedge$

- **Metric-Matrix** $\quad \mathbf{C}_{ij} = \langle e_i|h_j\rangle \qquad$ *intersection number*

- **Master Decomposition Formula**

projecting $\langle\varphi|$ onto a basis of $\langle e_i|$

The key formula!

$$\langle\varphi| = \sum_{i,j=1}^{\nu} \langle\varphi|h_j\rangle \left(\mathbf{C}^{-1}\right)_{ji} \langle e_i|$$

- *Proof*

for an arbitrary $|\psi\rangle$

$$\mathbf{M} = \begin{pmatrix} \langle\varphi|\psi\rangle & \langle\varphi|h_1\rangle & \langle\varphi|h_2\rangle & \ldots & \langle\varphi|h_\nu\rangle \\ \langle e_1|\psi\rangle & \langle e_1|h_1\rangle & \langle e_1|h_2\rangle & \ldots & \langle e_1|h_\nu\rangle \\ \langle e_2|\psi\rangle & \langle e_2|h_1\rangle & \langle e_2|h_2\rangle & \ldots & \langle e_2|h_\nu\rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle e_\nu|\psi\rangle & \langle e_\nu|h_1\rangle & \langle e_\nu|h_2\rangle & \ldots & \langle e_\nu|h_\nu\rangle \end{pmatrix} \equiv \begin{pmatrix} \langle\varphi|\psi\rangle & \mathbf{A}^\intercal \\ \mathbf{B} & \mathbf{C} \end{pmatrix}$$

$(\nu{+}1) \times (\nu{+}1)$ matrix $\mathbf{M}$

$$\det\mathbf{M} = \det\mathbf{C}\left(\langle\varphi|\psi\rangle - \mathbf{A}^\intercal\,\mathbf{C}^{-1}\,\mathbf{B}\right) = 0$$

$$\langle\varphi|\psi\rangle = \mathbf{A}^\intercal\,\mathbf{C}^{-1}\,\mathbf{B} \;\; = \sum_{i,j=1}^{\nu} \langle\varphi|h_j\rangle\,(\mathbf{C}^{-1})_{ji}\,\langle e_i|\psi\rangle$$

Intersection Numbers :: *1-forms*

- **1-forms** $\langle\varphi| \equiv \hat{\varphi}(z)\; dz$ $\hat{\varphi}(z)$ rational function

- **Zeroes and Poles of** ω

$$\omega \equiv d\log u$$

$$\nu = \{\text{the number of solutions of } \omega = 0\}$$

$$\mathcal{P} \equiv \{\, z \mid z \text{ is a pole of } \omega \,\}$$

$\mathcal{P}$ can also include the pole at infinity if $\mathrm{Res}_{z=\infty}(\omega) \neq 0$.

- **Intersection Numbers**

[pairing :: *cocycle* + *dual-cycle*]

The key operation!

1-forms φ_L and φ_R

$$\langle\varphi_L|\varphi_R\rangle_\omega = \sum_{p\in\mathcal{P}} \mathrm{Res}_{z=p}\Big(\psi_p\, \varphi_R\Big)$$

ψ_p is a function (0-form), solution to the differential equation $\nabla_\omega\psi = \varphi_L$, around p

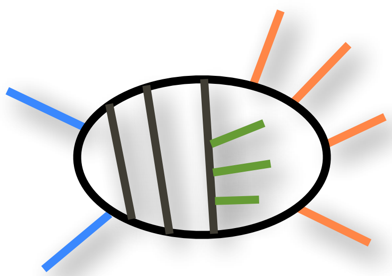
$$\nabla_{\omega_p}\psi_p = \varphi_{L,p}$$

- **Laurent expansions**

known $\varphi_{L,p}$

ansatz $\psi_p = \sum_{j=\min}^{\max} \psi_p^{(j)}\tau^j + \mathcal{O}\left(\tau^{\max+1}\right)$

Feynman Integrals & Intersection Theory



$$= \quad I_{a_1,a_2,\ldots,a_N} \equiv \int \prod_{i=1}^{L} \frac{d^d k_i}{\pi^{d/2}} \prod_{j=1}^{N} \frac{1}{D_j^{a_j}}$$

$$\equiv K \int_{\mathcal{C}} u\,\varphi \quad \equiv K \, \langle \varphi | \mathcal{C}]_\omega$$

Mizera **& P.M.** (2018)

- **Baikov representation**

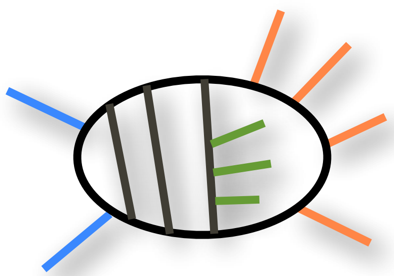
$$u = B^\gamma \,, \qquad \gamma \equiv (d{-}E{-}L{-}1)/2$$

$$\omega \equiv d\log(u) = \gamma d\log(B)$$

$$\varphi \equiv \hat{\varphi}\, d^N\mathbf{z} \,, \qquad \hat{\varphi} \equiv \frac{1}{z_1^{a_1} z_2^{a_2} \cdots z_N^{a_N}} \,,$$

$$d^N\mathbf{z} \equiv dz_1 \wedge dz_2 \wedge \cdots \wedge dz_N$$

Feynman Integrals & Intersection Theory



$$= \quad I_{a_1,a_2,\ldots,a_N} \equiv \int \prod_{i=1}^{L} \frac{d^d k_i}{\pi^{d/2}} \prod_{j=1}^{N} \frac{1}{D_j^{a_j}}$$

$$\equiv K \int_{\mathcal{C}} u\,\varphi \quad \equiv K\,\langle \varphi | \mathcal{C}]_\omega$$

Mizera **& P.M.** (2018)

- **Loop-by-Loop (LBL) Baikov repr'n**
  Frellesvig, Papadopoulos (2017)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

$$u = B_1^{\gamma_1} B_2^{\gamma_2} \cdots B_m^{\gamma_m},$$

$$\omega \equiv d\log(u) = \sum_{i=1}^{m} \gamma_i \, d\log(B_i)$$

$$\varphi \equiv \hat{\varphi}\, d^M\mathbf{z}, \qquad \hat{\varphi} \equiv \frac{f(z_1,\ldots,z_M)}{z_1^{a_1} z_2^{a_2} \cdots z_M^{a_M}},$$

$$d^M\mathbf{z} \equiv dz_1 \wedge dz_2 \wedge \cdots \wedge dz_M$$

$(N-M)$ ISPs
integrated out

f rational function

Cut Integrals

- **m-Cut Integrals**

$$z_1 = \ldots, z_m = 0$$

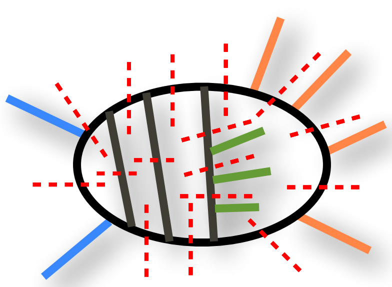
$$I_{a_1,a_2,\ldots,a_N}\Big|_{m\text{-cut}} \equiv K \int_{\mathcal{C}_{m\text{-cut}}} u\,\varphi \qquad \mathcal{C}_{m\text{-cut}} = \circlearrowleft_1 \wedge \circlearrowleft_2 \wedge \ldots \wedge \circlearrowleft_m \wedge \mathcal{C}'$$

$$I_{a_1,a_2,\ldots,a_N}\Big|_{m\text{-cut}} = K' \int_{\mathcal{C}'} u'\,\varphi' \,, \qquad \mathcal{C}' \equiv \bigcap_{i=1}^{m} \{z_i = 0\} \cap \mathcal{C}_{\cdot} = \bigcup_j \mathcal{C}_j{}' \,,$$

only ν of them can be independent

$$K'u' = (Ku)\Big|_{z_1=\ldots=z_m=0} \,, \qquad \varphi' \equiv \hat{\varphi}'\, d^{N-m}\mathbf{z}' \,,$$

$$\hat{\varphi}' \equiv \frac{f(z_{m+1},\ldots,z_N)}{z_{m+1}^{a_{m+1}} \cdots z_N^{a_N}} \left(\frac{\mathcal{D}_m(u)}{u}\right)\Bigg|_{z_1=\ldots=z_m=0} \,, \qquad \mathcal{D}_m \equiv \prod_{i=1}^{m} \frac{\partial_{z_i}^{(a_i-1)}}{(a_i-1)!} \,, \qquad d^{N-m}\mathbf{z}' \equiv dz_{m+1} \wedge \cdots \wedge dz_N$$



$$= \; I_{a_1,a_2,\ldots,a_N}\Big|_{m\text{-cut}} = I_{a_{m+1},\ldots,a_N} = K'\,{}_{\omega'}\langle \varphi' | \mathcal{C}']$$

Integrals reduction and Master Integrals

Mizera **& P.M.** (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

$$\nu = \{\text{the number of solutions of } \omega = 0\}$$

- **Basis of Master Forms** $\quad \langle e_i| \qquad i = 1, 2, \ldots, \nu$

- **Master Integrals** $\quad J_i \equiv K\, E_i \ , \quad \text{with} \quad E_i \equiv \langle e_i|\mathcal{C}]$

- **Integral Decomposition**

$$I = K\langle\varphi|\mathcal{C}] = \sum_{i=1}^{\nu} c_i\, J_i$$

$$\langle\varphi| = \sum_{i,j=1}^{\nu} \langle\varphi|h_j\rangle \, \left(\mathbf{C}^{-1}\right)_{ji} \, \langle e_i|$$

$$c_i \equiv \sum_{j=1}^{\nu} \langle\varphi|h_j\rangle \, \left(\mathbf{C}^{-1}\right)_{ji}$$

Basis choices

for $i = 1, 2, \ldots, \nu$

- **dLog Basis** $\langle e_i| = \langle \varphi_i| \equiv \dfrac{dz}{z - z_i}$ z_i are *poles* of ω

- **Monomial Basis** $\langle e_i| = \langle \phi_i| \equiv z^{i-1} dz$

- **Orthonormal Basis**

$\mathcal{P} = \{z_1, z_2, \ldots, z_{\nu+1}, z_{\nu+2}\}$ pick two special ones, say $z_{\nu+1}$ and $z_{\nu+2}$

$$\langle e_i| \equiv d\log \frac{z - z_i}{z - z_{\nu+1}}, \qquad |h_i\rangle \equiv \mathrm{Res}_{z=z_i}(\omega)\, d\log \frac{z - z_i}{z - z_{\nu+2}}$$

$$\mathbf{C}_{ij} = \delta_{ij} \qquad \langle \varphi| = \sum_{i=1}^{\nu} \langle \varphi | h_i \rangle \langle e_i|$$

- **…or any arbitrary rational basis…**

Dimensional Recurrence Relation

- **MIs in (d+2n) dimensions**

$$J_i^{(d+2n)} \equiv K(d+2n)\, E_i^{(d+2n)} \qquad E_i^{(d+2n)} \equiv \langle B^n e_i | \mathcal{C}] = \int_{\mathcal{C}} u\,(B^n\, e_i)\ , \qquad i = 1, 2, \ldots, \nu$$

- **Master Decomposition Formula**

$$\langle B^\nu e_i| = \sum_{n=0}^{\nu-1} c_n\, \langle B^n e_i| \qquad n = 0, 1, \ldots, \nu - 1$$

- **Recurrence Relations for Master Forms**

$$\sum_{n=0}^{\nu} c_n\, \langle B^n e_i| = 0\ , \qquad c_\nu \equiv -1$$

- **Recurrence Relations for Master Integrals**

$$\sum_{n=0}^{\nu} \alpha_n\, J_i^{(d+2n)} = 0 \qquad \alpha_n \equiv c_n/K(d+2n)$$

System of Differential Equations

- **External Derivative**

$$\partial_x I = \partial_x \langle \varphi | \mathcal{C}] = \partial_x \int_{\mathcal{C}} u \varphi = \int_{\mathcal{C}} u \left(\frac{\partial_x u}{u} \wedge + \partial_x \right) \varphi = \langle (\partial_x + \sigma) \varphi | \mathcal{C}] \qquad \sigma = \partial_x \log u$$

$$\partial_x \langle e_i | = \langle (\partial_x + \sigma \wedge) e_i | \equiv \langle \Phi_i |$$

- **Master Decomposition Formula**

$$\langle \Phi_i | = \langle \Phi_i | h_k \rangle \left(\mathbf{C}^{-1} \right)_{kj} \langle e_j | = \mathbf{\Omega}_{ij} \langle e_j |$$

$$\mathbf{\Omega} \equiv \mathbf{F} \mathbf{C}^{-1} \qquad \mathbf{F}_{ik} \equiv \langle \Phi_i | h_k \rangle$$

The C-matrix is important!

- **System of DEQ for Master Forms**

$$\partial_x \langle e_i | = \mathbf{\Omega}_{ij} \langle e_j | \ , \qquad \mathbf{\Omega} = \mathbf{\Omega}(d, x)$$

System of Differential Equations

- **System of DEQ for Master Integrals**

$$J_i \equiv K\, E_i \;, \quad \text{with} \quad E_i \equiv \langle e_i | \mathcal{C}],$$

$$\partial_x J_i = \mathbf{A}_{ij}\, J_j \qquad \mathbf{A} \equiv \mathbf{\Omega} + \mathbf{K} \qquad \mathbf{K} = \partial_x \log(K)\, \mathbb{I}$$

- **(Homogenous) Solutions**

For each i, the ν independent solutions

$$\mathbf{P}_{ij} = \langle e_i | \mathcal{C}_j] = \int_{\mathcal{C}_j} u\, e_i \;, \qquad i, j = 1, 2, \ldots, \nu \;,$$

$\nu \times \nu$ matrix $\mathbf{P}$

- math :: Resolvent matrix
- int. th. :: (Riemann) Twisted Period matrix

- Example :: Derivative basis

ν-dimensional basis formed by $\langle e_i|$ and its derivatives up the $(\nu-1)^{\text{th}}$-order

$$\mathbf{P} \;=\; \text{Wronski matrix}$$

Decomposition in 4 steps:

- **1. Integrals** $\int_{\mathcal{C}} u\,\varphi \;=\; \langle\varphi|\mathcal{C}]$ Compute $\omega \equiv d\log(u)$

- **2.1 Master (forms) Integrals** $\nu = \{\text{the number of solutions of } \omega = 0\}$

- **2.2 Choose a basis** $\langle e_i| = \langle\varphi_i| \equiv \dfrac{dz}{z - z_i}$ $\quad \langle e_i| = \langle\phi_i| \equiv z^{i-1}dz$

- **3. Build the "Metric" Matrix** $\mathbf{C}_{ij} = \langle e_i|h_j\rangle$

- **4. Master Decomposition Formula** $\langle\varphi| = \sum_{i,j=1}^{\nu} \langle\varphi|h_j\rangle \left(\mathbf{C}^{-1}\right)_{ji} \langle e_i|$

$$\int_{\mathcal{C}} u\,\varphi \;=\; \langle\varphi|\mathcal{C}] = \sum_{i=1}^{\nu} c_i\, J_i \qquad c_i \equiv \sum_{j=1}^{\nu} \langle\varphi|h_j\rangle \left(\mathbf{C}^{-1}\right)_{ji}$$

Master Integrals

Contiguity relations for Special Functions

Euler Beta Integrals

$$I_n \equiv \int_{\mathcal{C}} u\ z^n dz\ , \qquad u \equiv B^\gamma\ , \qquad B \equiv z(1-z)\ , \qquad \mathcal{C} \equiv [0,1]$$

- **Direct Integration**

$$I_n = \frac{\Gamma(1+\gamma)\Gamma(1+\gamma+n)}{\Gamma(2+2\gamma+n)}$$

  - Integral relation

    a relation between I_n and I_0

$$I_n = \frac{\Gamma(1+\gamma+n)\Gamma(2+2\gamma)}{\Gamma(1+\gamma)\Gamma(2+2\gamma+n)}\, I_0$$

  - Special case $n = 1$

$$I_1 = \frac{1}{2} I_0$$

Euler Beta Integrals

$$I_n \equiv \int_{\mathcal{C}} u\ z^n dz\ , \qquad u \equiv B^\gamma\ , \qquad B \equiv z(1-z)\ , \qquad \mathcal{C} \equiv [0,1]$$

- **IBP identities**

$$\int_{\mathcal{C}} d(B^{\gamma+1} z^{n-1}) = 0$$

$$(\gamma+n) I_{n-1} - (1+2\gamma+n) I_n = 0$$

$$I_n = \frac{(\gamma+n)}{(1+2\gamma+n)}\, I_{n-1}$$

  - Special case $n = 1$ $$I_1 = \frac{1}{2} I_0$$

Euler Beta Integrals

- **Intersection Theory**

$$I_n \equiv \int_{\mathcal{C}} u\, \phi_{n+1} \equiv {}_\omega\langle \phi_{n+1} | \mathcal{C}] \ , \qquad \phi_{n+1} \equiv z^n dz$$

$$u = B^\gamma \qquad B = z(1-z) \ , \qquad \omega = d\log u = \gamma\left(\frac{1}{z} + \frac{1}{z-1}\right) dz \qquad \nu = 1 \ , \qquad \mathcal{P} = \{0, 1, \infty\}$$

- **Monomial Basis**

1 master integral $\qquad I_0 = {}_\omega\langle \phi_1 | \mathcal{C}]$

- **Integral relation**

$$I_1 = c_1 \ I_0 \qquad \Longleftrightarrow \qquad {}_\omega\langle \phi_2 | \mathcal{C}] = c_1 \ {}_\omega\langle \phi_1 | \mathcal{C}]$$

$$\Longleftrightarrow \qquad \langle \phi_2 | = c_1 \langle \phi_1 |$$

- **Master Decomposition Formula**

$\mathbf{C}_{ij}$ has just one element $\mathbf{C}_{11} = \langle \phi_1 | \phi_1 \rangle$ $\qquad c_1 = \langle \phi_2 | \phi_1 \rangle \langle \phi_1 | \phi_1 \rangle^{-1}$

Intersection Numbers

$$\langle \varphi_L | \varphi_R \rangle_\omega = \sum_{p \in \mathcal{P}} \mathrm{Res}_{z=p} \Big(\psi_p \, \varphi_R \Big) \qquad \nabla_{\omega_p} \psi_p = \varphi_{L,p}$$

We need to evaluate $\langle \phi_1 | \phi_1 \rangle$, and $\langle \phi_2 | \phi_1 \rangle$

- **Laurent expansions**

  For each pole $p \in \mathcal{P}$

  $\tau = z - p$

  Known $\quad \phi_{i,p} = \sum_{k=\mathrm{min}-1} \phi_{i,p}^{(k)} \tau^k$

  Known $\quad \omega_p = \sum_{k=-1} \omega_p^{(k)} \tau^k$

  Ansatz $\quad \psi_p = \sum_{k=\mathrm{min}}^{\mathrm{max}} \alpha_k \, \tau^k \,, \qquad (\alpha_k \text{ unknown})$

- **Solve:**

$$\nabla_\omega \, \psi_{i,p} = \phi_{i,p} \qquad \Longleftrightarrow \qquad \frac{d}{d\tau} \psi_p + \omega_p \, \psi_p - \phi_{i,p} = 0$$

- For $\varphi_L = \phi_1 = dz,\ \ \varphi_R = \phi_1 = dz$:

p	min	max	$\varphi_{L,p}$	ψ_p
0	1	-1	$d\tau$	$-$
1	1	-1	$d\tau$	$-$
∞	-1	1	$-d\tau/\tau^2$	$\sum_{i=-1}^{1} \alpha_i\, \tau^i$

$$\alpha_{-1} = \frac{1}{2\gamma+1}\ , \qquad \alpha_0 = -\frac{1}{2(2\gamma+1)}\ , \qquad \alpha_1 = -\frac{\gamma}{2(2\gamma-1)(2\gamma+1)}$$

Therefore $$\langle \phi_1 | \phi_1 \rangle = \mathrm{Res}_{z=\infty}(\psi_\infty \phi_1) = \frac{\gamma}{2(2\gamma-1)(2\gamma+1)}$$

- For $\varphi_L = \phi_2 = z\,dz,\ \ \varphi_R = \phi_1 = dz$:

p	min	max	$\varphi_{L,p}$	ψ_p
0	2	-1	$\tau\, d\tau$	$-$
1	1	-1	$d\tau$	$-$
∞	-2	1	$-d\tau/\tau^3$	$\sum_{i=-2}^{1} \alpha_i\, \tau^i$

$$\alpha_{-2} = \frac{1}{2(\gamma+1)}\ , \quad \alpha_{-1} = -\frac{\gamma}{2(\gamma+1)(2\gamma+1)}, \qquad \alpha_0 = -\frac{1}{4(2\gamma+1)}\ , \quad \alpha_1 = -\frac{\gamma}{4(2\gamma-1)(2\gamma+1)}$$

Therefore $$\langle \phi_2 | \phi_1 \rangle = \mathrm{Res}_{z=\infty}(\psi_\infty \phi_1) = \frac{\gamma}{4(2\gamma-1)(2\gamma+1)}$$

- **Master Decomposition Formula**

$$I_1 = c_1\, I_0\ ,$$

$$c_1 = \langle \phi_2 | \phi_1 \rangle \langle \phi_1 | \phi_1 \rangle^{-1} = \frac{1}{2}$$

dLog Basis

$$\varphi_1 = d\log\frac{z}{z-1} = \left(\frac{1}{z} - \frac{1}{z-1}\right) dz \ ,$$

and let us decompose both $\langle\phi_1|$ and $\langle\phi_2|$ in the basis of $\langle\varphi_1|$,

$$\langle\phi_1| = \langle\phi_1|\varphi_1\rangle\langle\varphi_1|\varphi_1\rangle^{-1}\langle\varphi_1|,$$

$$\langle\phi_2| = \langle\phi_2|\varphi_1\rangle\langle\varphi_1|\varphi_1\rangle^{-1}\langle\varphi_1|.$$

We need the intersection numbers,

$$\langle\varphi_1|\varphi_1\rangle = \frac{2}{\gamma}, \qquad \langle\phi_1|\varphi_1\rangle = \frac{1}{2\gamma+1}, \qquad \langle\phi_2|\varphi_1\rangle = \frac{1}{2(2\gamma+1)}$$

Therefore

$$\langle\phi_1| = \frac{\gamma}{2(2\gamma+1)}\langle\varphi_1|, \qquad \langle\phi_2| = \frac{\gamma}{4(2\gamma+1)}\langle\varphi_1|$$

from which $\quad \langle\phi_2| = 1/2\langle\phi_1|$

- **Observation:** $\mathbf{C}^{-1} = \langle\varphi_1|\varphi_1\rangle^{-1} = \gamma/2 \qquad \gamma$ factorizing out

dLog Basis

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We need the intersection numbers,

$$\langle\varphi_1|\varphi_1\rangle = \frac{2}{\gamma}, \qquad \langle\phi_1|\varphi_1\rangle = \frac{1}{2\gamma+1}, \qquad \langle\phi_2|\varphi_1\rangle = \frac{1}{2(2\gamma+1)}$$

Therefore

$$\langle\phi_1| = \frac{\gamma}{2(2\gamma+1)}\langle\varphi_1|, \qquad \langle\phi_2| = \frac{\gamma}{4(2\gamma+1)}\langle\varphi_1|$$

from which $\langle\phi_2| = 1/2\langle\phi_1|$

- **Observation:** $\mathbf{C}^{-1} = \langle\varphi_1|\varphi_1\rangle^{-1} = \gamma/2$ $\quad \gamma$ factorizing out

Shooting a cannonball on a fly?

Gauss ${}_2F_1$ Hypergeometric Functions

$$\beta(b, c-b)\ {}_2F_1(a, b, c; x) = \int_0^1 z^{b-1}(1-z)^{c-b-1}(1-xz)^{-a}\, dz$$

$$= \int_{\mathcal{C}} u\, \varphi = {}_{\omega}\langle \varphi | \mathcal{C}] \qquad u = z^{b-1}(1-xz)^{-a}(1-z)^{-b+c-1}\,, \qquad \varphi = dz$$

$$\omega = d \log u = \frac{xz^2(c-a-2) + z(ax-c+x+2) - bxz + b - 1}{(z-1)z(xz-1)}\, dz\,, \qquad \nu = 2\,, \qquad \mathcal{P} = \{0,\ 1,\ \tfrac{1}{x},\ \infty\}$$

- **Monomial Basis** $\quad \{\langle \phi_i|\}_{i=1,2} \qquad \phi_{n+1} \equiv z^n dz$

  - **Metric** $\qquad \mathbf{C} = \begin{pmatrix} \langle \phi_1|\phi_1\rangle & \langle \phi_1|\phi_2\rangle \\ \langle \phi_2|\phi_1\rangle & \langle \phi_1|\phi_2\rangle \end{pmatrix}$

$$\langle \phi_1|\phi_1\rangle = \Big(x^2(-(a-b+1))(b-c+1) - 2ax(-b+c-1) + a(c-2)\Big) \,/\, \Big(x^2(a-c+1)(a-c+2)(a-c+3)\Big),$$

$$\langle \phi_1|\phi_2\rangle = \Big(x^3(-(a-b+1))(a-b+2)(b-c+1) - ax^2(-b+c-1)(2a-3b+c+2) + ax(a+2c-5)(-b+c-1) - a(c-3)(c-2)\Big) \,/\, \Big(x^3(a-c+1)\,(a-c+2)(a-c+3)(a-c+4)\Big),$$

$$\langle \phi_2|\phi_1\rangle = \Big(x^3(-(a-b))(a-b+1)(b-c+1) - ax^2(-b+c-1)(2a-3b+c) + ax(a+2c-3)(-b+c-1) - a(c-2)(c-1)\Big) \,/\, \Big(x^3(a-c)(a-c+1)\,(a-c+2)(a-c+3)\Big),$$

$$\langle \phi_2|\phi_2\rangle = \Big(-ax^2(a^2b - a^2c + a^2 - 3ab^2 + 7abc - 8ab - 4ac^2 + 9ac - 5a - 3b^2c + 6b^2 + 4bc^2 - 10bc + 6b - c^3 + 2c^2 - c) + x^4(-(a^3 - 3a^2b + 3a^2 + 3ab^2 - 6ab + 2a - b^3 + 3b^2 - 2b))(b-c+1)$$
$$+ 2ax^3(a-b+1)(ab - ac + a - 2b^2 + 3bc - 2b - c^2 + c) + 2a(c-2)x(a+c-2)(b-c+1) + a(c^3 - 6c^2 + 11c - 6)\Big) \,/\, \Big(x^4(a-c)(a-c+1)(a-c+2)(a-c+3)(a-c+4)\Big).$$

- **Master Decomposition Formula** $$\langle \phi_n| = \sum_{i,j=1}^{2} \langle \phi_n | \phi_j \rangle \, \left(\mathbf{C}^{-1}\right)_{ji} \, \langle \phi_i |$$

- **Gauss' contiguity relation**

$$\langle \phi_3 | \mathcal{C}] \equiv \beta(b+2, c-b)\, {}_2F_1(a, b+2, c+2; x)$$

$$= \left(\frac{b}{x(a-c-1)} \right) \beta(b, c-b)\, {}_2F_1(a, b, c; x) + \left(\frac{(b-a+1)x + c}{x(c-a+1)} \right) \beta(b+1, c-b)\, {}_2F_1(a, b+1, c+1; x)$$

- **dLog Basis** $$\varphi_1 = \left(\frac{1}{z} - \frac{1}{z-1} \right) dz \qquad \varphi_2 = \left(\frac{1}{z-1} - \frac{x}{xz-1} \right) dz.$$

$$I_1 = \langle \varphi_1 | \mathcal{C}] = {}_2F_1(a, b-1, c-2; x)\,, \qquad I_2 = \langle \varphi_2 | \mathcal{C}] = \frac{(b-1)(x-1)}{c-2} {}_2F_1(a+1, b, c-1; x)$$

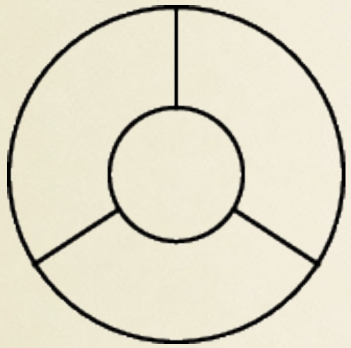
$$\mathbf{C}_{ij} = \langle \varphi_i | \varphi_j \rangle \qquad \mathbf{C} = \frac{1}{c-b-1} \begin{pmatrix} \frac{c-2}{b-1} & -1 \\ -1 & \frac{a+b-c+1}{a} \end{pmatrix}$$

- **Canonical system**
  $a = -\gamma,\ b = \gamma{+}1,\ c = 2(\gamma{+}1)$ $$\partial_x I_i = \mathbf{A}_{ij} I_j \qquad \mathbf{A} = \gamma \begin{pmatrix} 0 & \frac{-1}{x-1} \\ \frac{-1}{x} & \frac{2}{x-1} - \frac{2}{x} \end{pmatrix}$$

Feynman Integrals Decomposition
:: on the maximal cut :: 1-forms

- On the maximal cut :: simpler integrals
- 1-forms :: univariate integral representations
- Operation required :: Intersection Numbers for 1-forms

Four-Loop Vacuum Diagram



standard Baikov representation

$$u = \left(\frac{z}{2} - \frac{3z^2}{16}\right)^{\frac{d-5}{2}} \quad , \quad \omega = \frac{(d-5)(3z-4)}{z(3z-8)} dz \qquad \mathcal{P} = \{0, \tfrac{8}{3}, \infty\}.$$

$\omega = 0$ a has 1 solution, indicating 1 master integral $\boxed{\nu = 1}$

Monomial Basis. the decomposition of $I_{1,1,1,1,1,1,1,1,1;-2} = \langle \phi_3 | \mathcal{C}]$ in terms of $J_1 = I_{1,1,1,1,1,1,1,1,1;0} = \langle \phi_1 | \mathcal{C}]$

$$\mathbf{C} = \langle \phi_1 | \phi_1 \rangle = \frac{16(d-5)}{9(d-6)(d-4)} \qquad \langle \phi_n | = \langle \phi_n | \phi_1 \rangle \, \mathbf{C}^{-1}_{11} \, \langle \phi_1 |$$

$$\langle \phi_3 | \phi_1 \rangle = \frac{256(d-5)(d-1)}{81(d-6)(d-4)(d-2)} \qquad I_{1,1,1,1,1,1,1,1,1;-2} = \frac{16(d-1)}{9(d-2)} J_1,$$

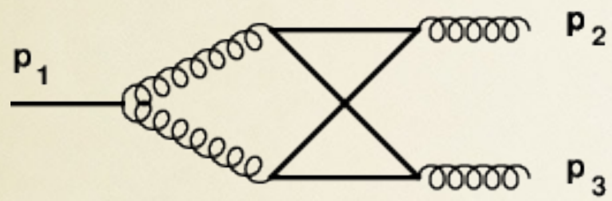
in agreement with SYS.

dlog-Basis. the decomposition of $\langle \phi_3 | \mathcal{C}]$ in terms of $\langle \varphi_1 | \mathcal{C}]$, $\quad \hat{\varphi}_1 = \dfrac{1}{z} - \dfrac{3}{3z-8}$

$$\mathbf{C} = \langle \varphi_1 | \varphi_1 \rangle = \frac{4}{d-5}$$

$$\langle \phi_3 | \varphi_1 \rangle = \frac{128(d-1)}{27(d-4)(d-2)} \qquad \langle \phi_3 | = \frac{32(d-5)(d-1)}{27(d-4)(d-2)} \langle \varphi_1 |$$

Two-Loop Non-Planar Triangle



$D_1 = k_1^2\,,\ D_2 = k_2^2 - m^2\,,\ D_3 = (p_1 - k_1)^2\,,\ D_4 = (p_3 - k_1 + k_2)^2 - m^2,$
$D_5 = (k_1 - k_2)^2 - m^2\,,\ D_6 = (p_2 - k_2)^2 - m^2\,.$

$$z = D_7 = 2(p_2 + k_1)^2 - p_1^2$$

$$u = B^\gamma\,, \qquad B = \left(z^2 - \tau_1^2\right)\left(z^2 - \tau_2^2\right)\,, \qquad \tau_1 = s\sqrt{1 + (4m)^2/s}\,, \qquad \tau_2 = s\,,$$

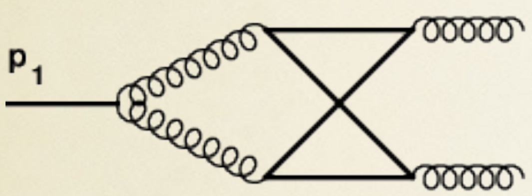
$$\gamma = \frac{d-5}{2} \qquad \omega = \frac{2\gamma z\left(2z^2 - \tau_1^2 - \tau_2^2\right)}{\left(z^2 - \tau_1^2\right)\left(z^2 - \tau_2^2\right)}dz\,, \qquad \boxed{\nu = 3\,,} \qquad \mathcal{P} = \{-\tau_1, -\tau_2, \tau_2, \tau_1, \infty\}$$

dlog-basis. $\varphi_1 = \left(\frac{1}{\tau_1 + z} - \frac{1}{\tau_2 + z}\right)dz\,, \qquad \varphi_2 = \left(\frac{1}{\tau_2 + z} - \frac{1}{z - \tau_2}\right)dz\,, \qquad \varphi_3 = \left(\frac{1}{z - \tau_2} - \frac{1}{z - \tau_1}\right)dz\,,$

$$\mathbf{C} = \begin{pmatrix} \langle\varphi_1|\varphi_1\rangle & \langle\varphi_1|\varphi_2\rangle & \langle\varphi_1|\varphi_3\rangle \\ \langle\varphi_2|\varphi_1\rangle & \langle\varphi_2|\varphi_2\rangle & \langle\varphi_2|\varphi_3\rangle \\ \langle\varphi_3|\varphi_1\rangle & \langle\varphi_3|\varphi_2\rangle & \langle\varphi_3|\varphi_3\rangle \end{pmatrix} = \frac{1}{\gamma}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \qquad \mathbf{C}^{-1} = \gamma\begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix}$$

the projection of $\phi_1 = dz$ is $\quad \langle\phi_1| = \frac{\gamma\tau_1}{4\gamma+1}\langle\varphi_1| + \frac{\gamma\left(\tau_1 + \tau_2\right)}{4\gamma+1}\langle\varphi_2| + \frac{\gamma\tau_1}{4\gamma+1}\langle\varphi_3| \quad$ verified with REDUZE.

Two-Loop Non-Planar Triangle



System of Differential Equations $\quad x \equiv \frac{\tau_1}{\tau_2} \qquad \sigma(x) = \partial_x \log\left(B(z,x)^\gamma\right) = -\frac{2\gamma\tau_2^2 x}{z^2 - \tau_2^2 x^2}.$

$$\langle \Phi_i(x)| \equiv \langle (\partial_x + \sigma(x))\varphi_i|$$

$$\langle\Phi_1(x)| = -\frac{\tau_2\left(2\gamma\tau_2^2x^2 - 2\gamma\tau_2^2 x + \tau_2^2 x + \tau_2 x z - z^2 - \tau_2 z\right)}{\left(\tau_2+z\right)\left(\tau_2 x - z\right)\left(\tau_2 x + z\right)^2} dz,$$

$$\langle\Phi_2(x)| = \frac{4\gamma\tau_2^3 x}{\left(\tau_2 - z\right)\left(\tau_2+z\right)\left(\tau_2 x - z\right)\left(\tau_2 x + z\right)} dz,$$

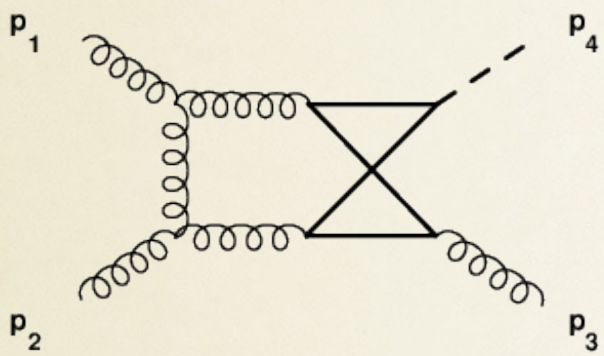
$$\langle\Phi_3(x)| = -\frac{\tau_2\left(2\gamma\tau_2^2x^2 - 2\gamma\tau_2^2 x + \tau_2^2 x - \tau_2 x z - z^2 + \tau_2 z\right)}{\left(\tau_2 - z\right)\left(\tau_2 x - z\right)^2\left(\tau_2 x + z\right)} dz.$$

$$\mathbf{F}_{ij} = \langle \Phi_i | \varphi_j \rangle \qquad \mathbf{F} = \begin{pmatrix} \frac{7x^2+2x-1}{(x-1)x(x+1)} & -\frac{2}{x-1} & -\frac{x-1}{x(x+1)} \\ -\frac{2}{x-1} & \frac{4x}{(x-1)(x+1)} & -\frac{2}{x-1} \\ -\frac{x-1}{x(x+1)} & -\frac{2}{x-1} & \frac{7x^2+2x-1}{(x-1)x(x+1)} \end{pmatrix}$$

$$\mathbf{\Omega} = \mathbf{F}\mathbf{C}^{-1} = \gamma \begin{pmatrix} \frac{4x^2+x-1}{(x-1)x(x+1)} & \frac{1}{x} & \frac{1}{x(x+1)} \\ -\frac{2}{(x-1)(x+1)} & \frac{2}{x+1} & -\frac{2}{(x-1)(x+1)} \\ \frac{1}{x(x+1)} & \frac{1}{x} & \frac{4x^2+x-1}{(x-1)x(x+1)} \end{pmatrix}$$

- Canonical

Non-Planar Contribution to $H+j$ Production



Loop-by-Loop form of the Baikov representation

$$D_1 = k_1^2\,, \quad D_2 = (k_1+p_1)^2\,, \quad D_3 = (k_1-p_3-p_4)^2\,,$$
$$D_4 = (k_2-p_3)^2 - m_t^2 \quad D_5 = k_2^2 - m_t^2\,, \quad D_6 = (k_1-k_2)^2 - m_t^2\,,$$
$$D_7 = (k_1-k_2-p_4)^2 - m_t^2\,.$$

$$z = D_8 = (k_1-p_3)^2$$

$$D_9 = (k_2+p_1)^2$$

$$u = \frac{\left(-m_H^2+s+t+z\right)^{d-5}\left(z\left(m_H^2-s-z\right)+4sm_t^2\right)^{\frac{d-5}{2}}}{\sqrt{z\left(-m_H^2+s+z\right)}}\,,$$

$$\omega = \frac{q_0+q_1\,z+q_2\,z^2+q_3\,z^3+q_4\,z^4}{2z\left(-m_H^2+s+z\right)\left(-m_H^2+s+t+z\right)\left(z\left(-m_H^2+s+z\right)-4sm_t^2\right)}\,dz\,, \qquad \boxed{\nu = 4\,,}$$

$$\mathcal{P} = \{0,\, m_H^2-s,\, \tfrac{1}{2}(m_H^2-s-\rho),\, \tfrac{1}{2}(m_H^2-s+\rho),\, m_H^2-s-t,\, \infty\}\,, \qquad \rho = \sqrt{m_H^4-2sm_H^2+16sm_t^2+s^2}\,.$$

Mixed Bases $J_1 = I_{1,1,1,1,1,1,1;0} = \langle e_1|\mathcal{C}]$, $J_2 = I_{1,2,1,1,1,1,1:0} = \langle e_2|\mathcal{C}]$, $J_3 = I_{1,1,1,2,1,1,1;0} = \langle e_3|\mathcal{C}]$ and $J_4 = I_{1,1,1,1,2,1,1;0} = \langle e_4|\mathcal{C}]$,

$$\begin{aligned}
\hat{e}_1 &= 1\,,\\
\hat{e}_2 &= \frac{(d-5)\left(m_H^4-m_H^2(2s+t+z)+s^2+s(t+z)+2tz\right)}{s(-m_H^2+s+t+z)^2}\,,\\
\hat{e}_3 &= \frac{(d-5)(s+z)}{z(m_H^2-s-z)+4sm_t^2}\,,\\
\hat{e}_4 &= \frac{(d-5)(m_H^2-z)}{z(m_H^2-s-z)+4sm_t^2}\,.
\end{aligned}$$

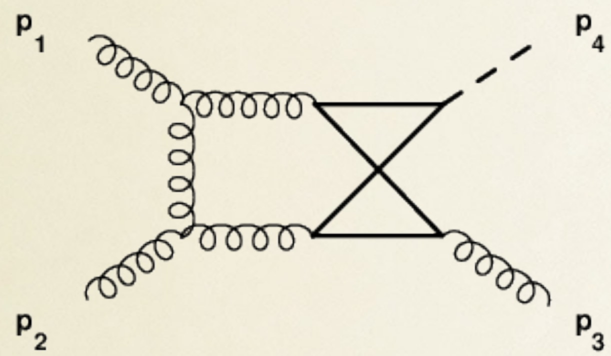
$$\begin{aligned}
\hat{\varphi}_1 &= \frac{1}{z} - \frac{1}{-m_H^2+s+z}\,,\\
\hat{\varphi}_2 &= \frac{1}{-m_H^2+s+z} - \frac{1}{\frac{1}{2}\left(-m_H^2+\rho+s\right)+z}\,,\\
\hat{\varphi}_3 &= \frac{1}{\frac{1}{2}\left(-m_H^2+\rho+s\right)+z} - \frac{1}{\frac{1}{2}\left(-m_H^2-\rho+s\right)+z}\,,\\
\hat{\varphi}_4 &= \frac{1}{\frac{1}{2}\left(-m_H^2-\rho+s\right)+z} - \frac{1}{-m_H^2+s+t+z}\,.
\end{aligned}$$

$$\mathbf{C}_{ij} = \langle e_i|\varphi_j\rangle\;, \quad 1 \le i,j \le 4,$$

$$\langle\varphi| = \sum_{i,j=1}^{\nu} \langle\varphi|h_j\rangle\left(\mathbf{C}^{-1}\right)_{ji}\langle e_i|$$

$$I_{1,1,1,1,1,1,1;-1} = c_1\,J_1 + c_2\,J_2 + c_3\,J_3 + c_4\,J_4$$

Non-Planar Contribution to $H+j$ Production



Loop-by-Loop form of the Baikov representation

$$D_1 = k_1^2\,,\quad D_2 = (k_1+p_1)^2\,,\quad D_3 = (k_1-p_3-p_4)^2\,,$$
$$D_4 = (k_2-p_3)^2 - m_t^2\quad D_5 = k_2^2 - m_t^2\,,\quad D_6 = (k_1-k_2)^2 - m_t^2\,,$$
$$D_7 = (k_1-k_2-p_4)^2 - m_t^2\,.$$

$$z = D_8 = (k_1-p_3)^2$$

$$D_9 = (k_2+p_1)^2$$

$$u = \frac{\left(-m_H^2+s+t+z\right)^{d-5}\left(z\left(m_H^2-s-z\right)+4sm_t^2\right)^{\frac{d-5}{2}}}{\sqrt{z\left(-m_H^2+s+z\right)}}\,,$$

$$\omega = \frac{q_0+q_1\,z+q_2\,z^2+q_3\,z^3+q_4\,z^4}{2z\left(-m_H^2+s+z\right)\left(-m_H^2+s+t+z\right)\left(z\left(-m_H^2+s+z\right)-4sm_t^2\right)}\,dz\,,\qquad \nu = 4\,,$$

$$\mathcal{P} = \{0,\, m_H^2-s,\, \tfrac{1}{2}(m_H^2-s-\rho),\, \tfrac{1}{2}(m_H^2-s+\rho),\, m_H^2-s-t,\, \infty\}\,,\qquad \rho = \sqrt{m_H^4-2sm_H^2+16sm_t^2+s^2}\,.$$

Mixed Bases $J_1 = I_{1,1,1,1,1,1,1;0} = \langle e_1|\mathcal{C}]$, $J_2 = I_{1,2,1,1,1,1,1:0} = \langle e_2|\mathcal{C}]$, $J_3 = I_{1,1,1,2,1,1,1;0} = \langle e_3|\mathcal{C}]$ and $J_4 = I_{1,1,1,1,2,1,1;0} = \langle e_4|\mathcal{C}]$,

Checks. KIRA, leaves us with 6 MIs, 2 more: $J_5 = I_{1,1,1,1,1,2,1;0,0}\,,\qquad J_6 = I_{1,1,2,1,1,1,1;0,0}\,.$

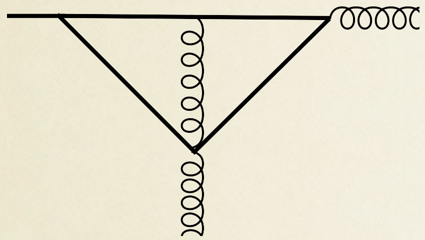
- **Higher sectors IBPs** $$J_6 = \frac{10-2d}{s}J_1 + \frac{(2m_t^2-m_H^2)s+m_H^4}{m_H^2 s}J_3 + \frac{2m_t^2}{s}J_4 + \frac{s(m_H^2-2m_t^2)+2m_H^2m_t^2}{m_H^2 s}J_5\,.$$ (on the cut)
- **Self similarity** $k_1 \to -k_1-p_1-p_2\,,\quad k_2 \to -k_2+p_3\,,\quad p_1 \leftrightarrows p_2\,,$ $$J_5 = \frac{s}{m_H^2+s}J_3 - \frac{m_H^2}{m_H^2+s}J_4$$ (on the cut)

after using these 2 extra relations KIRA is in perfect agreement

$\nu = 4$ verified with a numerical evaluation of the integrals on the maximal cut + PSLQ [80 digits]

Planar Double Triangle

[Lee; Zhang]



Loop-by-Loop form of Baikov representation

$$D_1 = k_1^2 - m^2, \qquad D_2 = k_2^2 - m^2, \qquad D_3 = (k_1 - k_2)^2,$$
$$D_4 = (k_2 - p_1)^2 - m^2, \qquad D_5 = (k_1 - p_1 - p_2)^2 - m^2.$$

$$z = D_6 = (k_1 - p_1)^2 - m^2.$$

$$u = \frac{(z(s+z) + sm^2)^{\frac{d-4}{2}}}{z}, \qquad \omega = \frac{z((d-6)s + 2(d-5)z) - 2m^2 s}{2z\,(z(s+z) + sm^2)}\, dz$$

$\nu = 2$

in agreement with the literature

- **warning:** z risen to an *integer* power
- **solution:** introducing a regulating exponent ρ $\quad z^{-1} \to z^{\rho-1}$

$$u = ((z - r_1)(z - r_2))^{\frac{d-4}{2}} z^{\rho-1}, \qquad \omega = \frac{2r_1 r_2(\rho - 1) - (r_1 + r_2)(d - 6 + 2\rho)z + 2(d - 5 + \rho)z^2}{2z(z - r_1)(z - r_2)}\, dz, \qquad \nu = 2,$$

$$r_1 = \frac{1}{2}\left(-\sqrt{s^2 - 4m^2 s} - s\right), \quad r_2 = \frac{1}{2}\left(\sqrt{s^2 - 4m^2 s} - s\right)$$

$$\mathcal{P} = \{0, r_1, r_2, \infty\}$$

Mixed Bases.

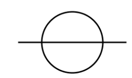
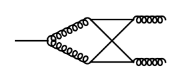
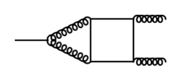
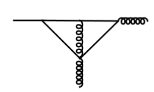
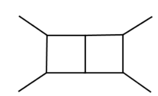
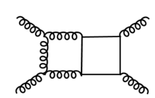
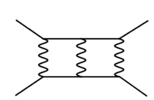
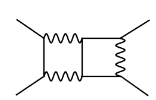
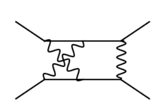
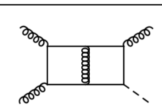
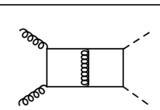
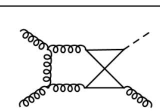
monomial basis $\quad \phi_1 = 1\, dz$ and $\phi_2 = z\, dz. \quad \Longleftrightarrow \quad J_1 = I_{1,1,1,1,1;0}, \quad J_2 = I_{1,1,1,1,1;-1}$

right basis $\quad \hat{\varphi}_1 = \frac{1}{z} - \frac{1}{z - r_1}, \qquad \hat{\varphi}_2 = \frac{1}{z - r_1} - \frac{1}{z - r_2}.$

$$\mathbf{C} = \langle \phi_i | \varphi_j \rangle$$

- **Master Decomposition Formula + limit** $\rho \to 0$ $\qquad$ agrees with the reduction obtained from LiteRed.

Other Applications :: *proof of concepts*

Integral family	Sec.	ν_{LBL}	ν_{std}
	7	1	1
	8	3	3
	9	1	1
	10	2	1
	11	2	2
	12	3	4
	13.1	2	2
	13.2	3	4
	13.3	3	4
	14.1	4	4
	14.1	4	4
	14.2	4	6

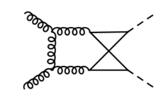
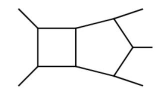
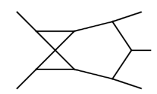
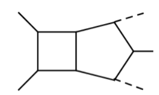
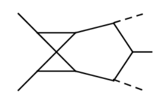
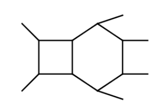
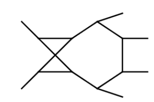
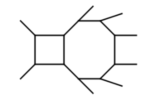
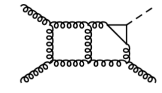
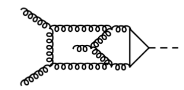
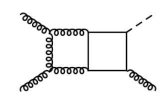
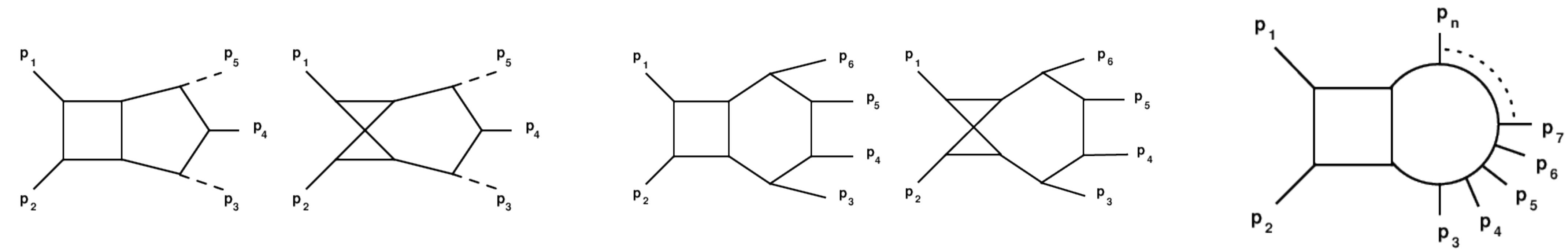
Integral family	Sec.	ν_{LBL}	ν_{std}
	14.3	4	6
	15.1	3	3
	15.2	3	3
	16	3	3
	16	3	3
	16	3	3
	16	3	3
	16.1	3	3
	17.1	2	2
	17.2	3	3
	17.3	3	4

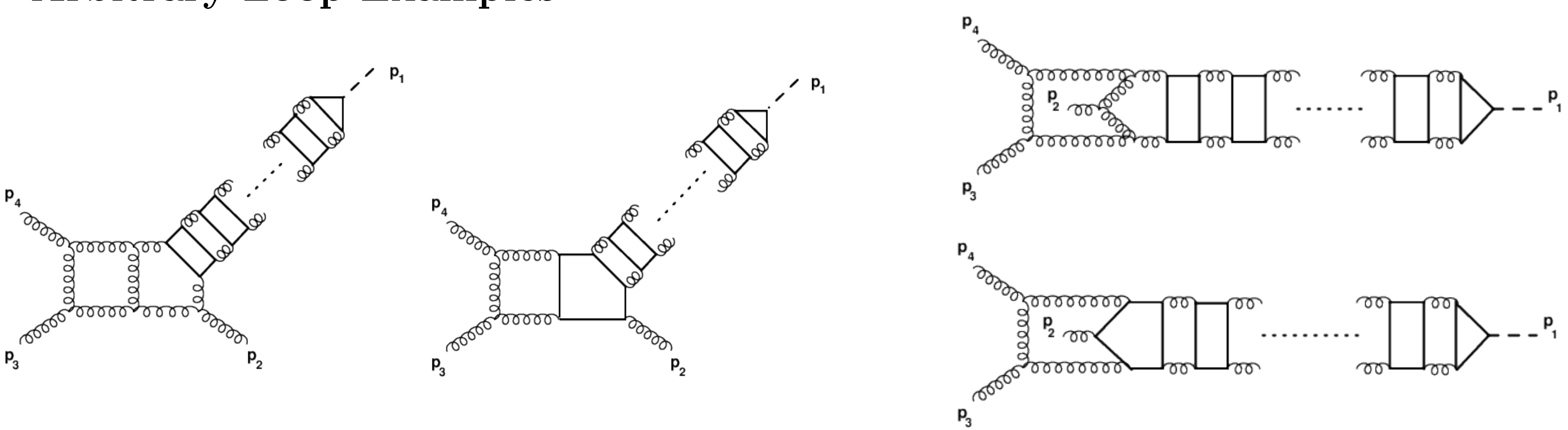
Table 1: Comparisons of the number of masters obtained by the LP criterion, from Loop-by-Loop (ν_{LBL}) and standard Baikov parametrization (ν_{std}).

Other Applications :: *proof of concepts*

Multileg and Massive Cases



Arbitrary Loop Examples



Feynman Integrals Decomposition
:: on the maximal cut :: 2-forms

- On the maximal cut :: simpler integrals
- 2-forms :: two-variable integral representations
- Operation required :: Intersection Numbers for 2-forms

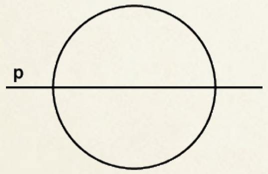
2-form Representations :: 2-form InterX

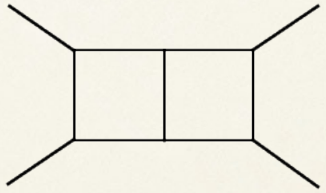
Matsumoto (1998)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

$$\int_{\mathcal{C}} u(x,y)\phi(x,y) \qquad u = B_1^{\gamma_1} B_2^{\gamma_2} \cdots B_m^{\gamma_m},$$

$$\omega = d\log u = \sum_{i=1}^{m} \gamma_i \left(\frac{\partial_x B_i}{B_i} dx + \frac{\partial_y B_i}{B_i} dy \right) \qquad \nabla_\omega \equiv d + \omega \wedge$$





Intersection Numbers for 2-forms

Poles of ω form hypersurfaces $\mathcal{H}_i$ $\qquad \mathcal{H}_i \equiv \{(x,y) \mid B_i(x,y) = 0\}.$ $\qquad dx \wedge dy = dz_i^{\perp} \wedge dz_i^{||}$ unit Jacobian.

hypersurfaces, in general, intersect at points $\mathcal{P}_{ij}$ $\qquad \mathcal{P}_{ij} \equiv \mathcal{H}_i \cap \mathcal{H}_j \quad$ for $\quad i \neq j.$

The algorithm for computing the intersection number $\langle \phi_L | \phi_R \rangle_\omega$ consists of three steps.

1. **Hypersurfaces.** construct the one-form ψ_i satisfying the equation: $\qquad \nabla_\omega \psi_i = \phi_L$ $\qquad$ locally near $\mathcal{H}_i$
2. **Intersections of Hypersurfaces.** construct the function ψ_{ij} satisfying the equation: $\qquad \nabla_\omega \psi_{ij} = \psi_i - \psi_j$ $\qquad$ locally near $\mathcal{P}_{ij}$.
3. **Intersection Numbers of Two-Forms.**

   double-residue formula: $$\langle \phi_L | \phi_R \rangle_\omega \equiv \sum_{\mathcal{P}_{ij}} \mathrm{Res}_{z_i^{\perp}=0} \, \mathrm{Res}_{z_j^{\perp}=0} \left(\psi_{ij} \, \phi_R \right).$$

2-form Representations :: Iterated 1-form InterX

$$I_{n,m} \equiv K \int_{\mathcal{C}_1}\int_{\mathcal{C}_2} u \; z_1^n \, z_2^m \, dz_1 \wedge dz_2$$

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

$$u = u(z_1, z_2) \; , \qquad \omega = \hat{\omega}_1 dz_1 + \hat{\omega}_2 dz_2 \; , \qquad \hat{\omega}_1 \equiv \partial_{z_1} \log u, \qquad \hat{\omega}_2 \equiv \partial_{z_2} \log u.$$

- **Intersection in z_1.** $\qquad u = u_{z_1} \; , \qquad \hat{\omega}_{z_1} \equiv \partial_{z_1} \log u_{z_1} = \partial_{z_1} \log u = \hat{\omega}_1 \, .$

$$I_{n,m} = \int_{\mathcal{C}_2} J_n \, z_2^m \, dz_2 \, , \qquad J_n = \int_{\mathcal{C}_1} u_{z_1} \, z_1^n \, dz_1 \equiv {}_{\omega_1}\langle \phi_{n+1} | \mathcal{C}_1] \, .$$

we *assume* that the J_n integral family admits $\nu_1 = 1$ master integral, say $\quad J_0 = \int_{\mathcal{C}_1} u_{z_1} \, dz_1 \equiv {}_{\omega_1}\langle \phi_1 | \mathcal{C}_1]$

- **Master Decomposition Formula**

$$J_n = c_n J_0 \qquad c_n = \langle \phi_{n+1} | \phi_1 \rangle_{\omega_1} \; \langle \phi_1 | \phi_1 \rangle_{\omega_1}^{-1} \; = c_n(z_2).$$

It depends on the other variable

2-form Representations :: Iterated 1-form InterX

$$I_{n,m} \equiv K \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} u \; z_1^n \, z_2^m \, dz_1 \wedge dz_2$$

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

$$u = u(z_1, z_2) \;, \qquad \omega = \hat{\omega}_1 dz_1 + \hat{\omega}_2 dz_2 \;, \qquad \hat{\omega}_1 \equiv \partial_{z_1} \log u, \qquad \hat{\omega}_2 \equiv \partial_{z_2} \log u.$$

- **Intersection in z_1.** $\quad u = u_{z_1} \;, \qquad \hat{\omega}_{z_1} \equiv \partial_{z_1} \log u_{z_1} = \partial_{z_1} \log u = \hat{\omega}_1 \,.$

$$I_{n,m} = \int_{\mathcal{C}_2} J_n \, z_2^m \, dz_2 \,, \qquad J_n = \int_{\mathcal{C}_1} u_{z_1} \, z_1^n \, dz_1 \equiv {}_{\omega_1}\langle \phi_{n+1} | \mathcal{C}_1] \,.$$

we *assume* that the J_n integral family admits $\nu_1 = 1$ master integral, say $\quad J_0 = \int_{\mathcal{C}_1} u_{z_1} \, dz_1 \equiv {}_{\omega_1}\langle \phi_1 | \mathcal{C}_1]$

- Master Deco Formula $\quad J_n = c_n J_0 \qquad c_n = \langle \phi_{n+1} | \phi_1 \rangle_{\omega_1} \; \langle \phi_1 | \phi_1 \rangle_{\omega_1}^{-1} \; = c_n(z_2).$

$$I_{n,m} = \int_{\mathcal{C}_2} c_n J_0 \, z_2^m \, dz_2 \; = \int_{\mathcal{C}_2} u_{z_2} \, \psi_{n,m} \; \equiv \; {}_{\omega_{z_2}}\langle \psi_{n,m} | \mathcal{C}_2] \;,$$

- **Intersection in z_2.** $\quad \psi_{n,m} \equiv c_n \, z_2^m \, dz_2 \,, \qquad u_{z_2} \equiv \; J_0 \,, \qquad \hat{\omega}_{z_2} = \partial_{z_2} \log u_{z_2} \,.$

- Master Deco Formula

monomial basis $\qquad {}_{\omega_{z_2}}\langle \phi_k | \equiv z_2^{k-1} \, dz_2$

$$I_{n,m} = \sum_{i=0}^{\nu_2 - 1} c_{n,m,i} \, I_{0,i} \;, \qquad c_{n,m,i} = \langle \psi_{n,m} | \phi_j \rangle_{\omega_{z_2}} (\mathbf{C}_{\omega_{z_2}}^{-1})_{ji}$$

What can we do with 1-form InterX Numbers?

- ***If*** the integral representation has 1 ISPs, ***Then*** we can get:
  - the coefficients of the **top sector** :: **maximal-cut**

- ***If*** the integral representation has 0 ISPs, ***Then*** we can get:
  - the coefficients of the **top sector** :: **maximal-cut**
  - the coefficients of the **level-1 subsectors** :: **next-to-maximal-cut**

What can we do with 2-form InterX Numbers?

- ***If*** the integral representation has 2 ISPs, ***Then*** we can get:
  - the coefficients of the **top sector** :: **maximal-cut**
- ***If*** the integral representation has 1 ISPs, ***Then*** we can get:
  - the coefficients of the **top sector** :: **maximal-cut**
  - the coefficients of the **level-1 subsectors** :: **next-to-maximal-cut**
- ***If*** the integral representation has 0 ISPs, ***Then*** we can get:
  - the coefficients of the **top sector** :: **maximal-cut**
  - the coefficients of the **level-1 subsectors** :: **next-to-maximal-cut**
  - the coefficients of the **level-2 subsectors** :: **next-to-next-to-maximal-cut**

Feynman Integrals Decomposition
:: n-forms ::

Frellesvig, Gasparotto, Laporta, Mandal,
Mattiazzi, Mizera **& P.M.**
(in progress)

- No cut
- n-forms :: n-variable integral representations
- Operation required :: Intersection Numbers for n-forms
- Intersection Numbers for n-forms :: n *steps* down in the decomposition

(other) **Parametric Representations:**

- **Schwinger Parameterization**
- **Lee-Pomeransky Parameterization**

Frellesvig, Gasparotto, Mandal, Mattiazzi,
Mizera, Ossola, Sameshima **& P.M.**
(in progress)

Gamma Function :: 1-variate InterX

$$\Gamma(s) = \int_{x=0}^{\infty} x^{s-1} e^{-x} dx.$$

$$u(x) := x^{s-1} e^{-x} \qquad C := [0, \infty]$$

$$\omega := d \ln u \; = \left(\frac{s-1}{x} - 1 \right) dx \qquad \nu = 1 \qquad P = \{0, \infty\}$$

$$I(n) := \int_C u \, \phi_n := \langle \phi_n | C] \;, \qquad \phi_n := x^n \, dx$$

$$\phi_0 = 1 dx \qquad I(0) := \langle \phi_0 | \, C] \qquad \langle \phi_0 | \, \phi_0 \rangle = s - 1$$

$$\phi_1 = x dx \qquad I(1) := \langle \phi_1 | \, C] \qquad \langle \phi_1 | \, \phi_0 \rangle = s(s-1).$$

- Master Decomposition Formula

$$\langle \phi_1 | = \langle \phi_1 | \, \phi_0 \rangle \, \langle \phi_0 | \, \phi_0 \rangle^{-1} \, \langle \phi_0 | \; = s \, \langle \phi_0 | \qquad \Longleftrightarrow \qquad I(1) = s I(0)$$

$$\Gamma(s+1) = s\Gamma(s).$$

Other parametric Representations:

- **Schwinger Parameterization**
- **Lee-Pomeransky Parameterization**

Frellesvig, Gasparotto, Mandal, Mattiazzi,
Mizera, Ossola, Sameshima **& P.M.**
(in progress)

Gamma Function :: 1-variate InterX

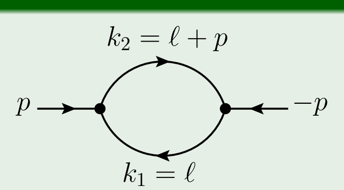
Eric's Example :: 2-variate InterX

An integral family is defined by a set of denominators $D_1, \ldots, D_N$ that are quadratic (or linear) forms in loop momenta $\ell_1, \ldots, \ell_L$:

$$\mathcal{I}(a_1, \ldots, a_N; d) = \left(\prod_{k=1}^{L} \int \frac{\mathrm{d}^d \ell_k}{\mathrm{i}\pi^{d/2}} \right) \frac{1}{D_1^{a_1} \cdots D_N^{a_N}}$$

Example

$$\mathcal{I}(a_1, a_2; d) = \int \frac{\mathrm{d}^d \ell}{\mathrm{i}\pi^{d/2}} \frac{1}{(\ell^2)^{a_1}((\ell+p)^2)^{a_2}}$$



A family is also described by a matrix Λ, vectors Q_i and a scalar J such that

$$\sum_{k=1}^{N} x_k D_k = -\sum_{i,j=1}^{L} \Lambda_{ij}(\ell_i \cdot \ell_j) + \sum_{i=1}^{L} 2(Q_i \cdot \ell_i) + J$$

Associated polynomials: $\mathcal{U} := \det \Lambda$, $\mathcal{F} := \mathcal{U}\,(Q^\intercal \Lambda^{-1} Q + J)$

In terms of $\omega := a_1 + \ldots + a_N - L\frac{d}{2}$ and $\mathcal{G} := \mathcal{U} + \mathcal{F}$ (Lee-Pomeransky),

$$\mathcal{I}(a) = \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d}{2} - \omega)} \left(\prod_{k=1}^{N} \int_0^\infty \frac{x_k^{a_k - 1} \mathrm{d}x_k}{\Gamma(a_k)} \right) \mathcal{G}^{-d/2}$$

Example

$$\mathcal{I}(a_1, a_2) = \frac{\Gamma(\frac{d}{2})}{\Gamma(d - a_1 - a_2)} \int_0^\infty \frac{x_1^{a_1-1} \mathrm{d}x_1}{\Gamma(a_1)} \int_0^\infty \frac{x_2^{a_2-1} \mathrm{d}x_2}{\Gamma(a_2)} \Big(\underbrace{x_1 + x_2}_{\mathcal{U}} \underbrace{- p^2 x_1 x_2}_{\mathcal{F}} \Big)^{-\frac{d}{2}}$$

The (twisted) Mellin transform of a function $f\colon \mathbb{R}_+^N \longrightarrow \mathbb{C}$ is

$$\mathcal{M}\{f\}(a) := \left(\prod_{k=1}^{N} \int_0^\infty \frac{x_k^{a_k - 1} \mathrm{d}x_k}{\Gamma(a_i)} \right) f(x_1, \ldots, x_N),$$

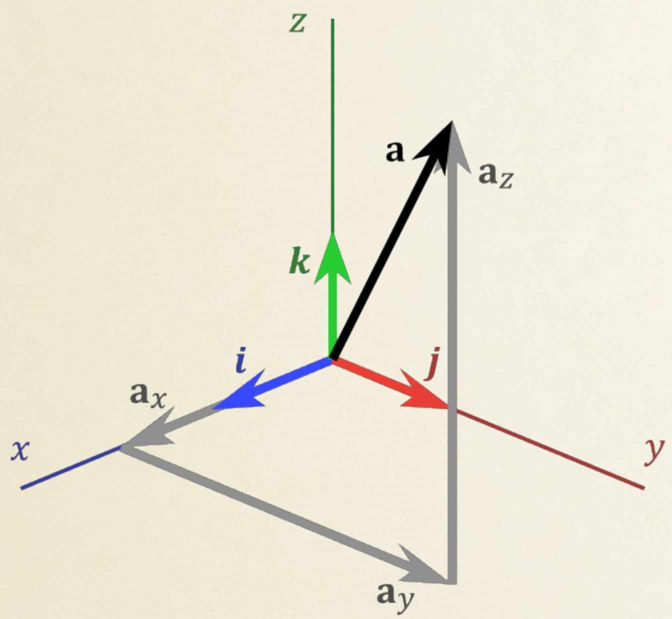
whenever this integral exists. The Feynman integral is a special case:

$$\mathcal{I}(a) = \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d}{2} - \omega)} \widetilde{\mathcal{I}}(a) \quad \text{for} \quad \widetilde{\mathcal{I}}(a) = \mathcal{M}\left\{ \mathcal{G}^{-d/2} \right\}(a).$$

To Conclude:

Amplitudes Decomposition:

the algebraic way



$\mathbf{a} = a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k}$

Basis: {**i j k**}

Scalar product/**Projection**:
to extract the components

$a_x = \mathbf{a}.\mathbf{i} \quad a_y = \mathbf{a}.\mathbf{j} \quad a_z = \mathbf{a}.\mathbf{k}$

Summary ::

- Novel Algebraic Property Unvealed
  - The algebra of Feynman Integrals is controlled by Intersection Numbers
  - Intersection Numbers ~ Scalar Product/Projection between Feynman Integrals
  - Exploiting the geometric properties of the integrands, dictated by graph polynomials
- (toward a) Novel Decomposition Method
  - Direct **decomposition** into a Integral Basis
  - Direct construction of system of **differential equations** for the Integral Basis
  - Direct construction of **finite difference equations** for the Integral Basis
  - NO intermediate relation required !
  - Analytic solution :: Integral representation (graph polynomials) + Algebra
- Interesting novel results also for math
  - Intersection numbers beyond dLog-forms

Outlook ::

- Complete Decomposition (involving subsectors)
    - Multivariate Intersection Numbers

- Novel Integral Representations

$$\int_{\mathcal{C}} u\,\varphi = \ {}_{\omega}\langle \varphi | \mathcal{C}]$$

- Compatibility between dimensional regularization and ***multivaluedness*** :: ***arbitrary*** denominator powers
- **Total Volume Integrals** (super-space) :: **Vacuum Graphs** :: :: *Schlaefli Differential Equations* :: *Bernstein-Sato Ideal*
- Volume of Symplex and Iterated Integral Representations Kohno

- Intersection Numbers for n-forms and D-module Theory
    - critical points of =?= Euler $\chi(\mathcal{G})$
    - Intersection Numbers in Mellin space ?

The unreasonable effectiveness of mathematics

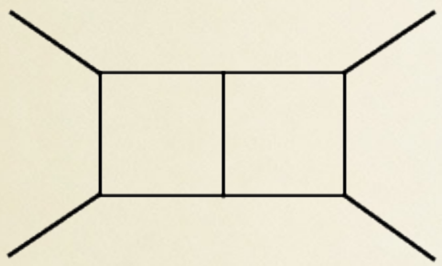
E. Wigner

Wigner was referring to the mysterious phenomenon in which areas of pure mathematics, originally constructed without regard to application, are suddenly discovered to be exactly what is required to describe the structure of the physical world.

M. Berry

Back up

Two-Loop Massless Double-Box Iterative Intersections



$$I_{n,m} \equiv \int_{\mathcal{C}_1}\int_{\mathcal{C}_2} u \;\; z_1^n \, z_2^m \, dz_1 \wedge dz_2 \;, \qquad \mathcal{C}_1, \mathcal{C}_2 = [0, \infty]$$

$$u = (z_1 z_2 (1 + a(z_1 + z_2) + b z_1 z_2))^\gamma \;, \qquad a = -1/t \text{ and } b = -1/(st)$$

$$\omega = \hat{\omega}_1 dz_1 + \hat{\omega}_2 dz_2 \qquad \hat{\omega}_1 = \frac{\gamma\,(a\,(2z_1 + z_2) + 2bz_1z_2 + 1)}{z_1\,(a\,(z_1+z_2) + bz_1z_2 + 1)}, \qquad \hat{\omega}_2 = \frac{\gamma\,(2z_2\,(a + bz_1) + az_1 + 1)}{z_2\,(a\,(z_1+z_2) + bz_1z_2 + 1)}.$$

$$I_{n,m} \equiv \int_{\mathcal{C}_2} dz_2 z_2^m \, J_n \;, \qquad J_n \equiv \int_{\mathcal{C}_1} dz_1 \; u_{z_1} \, z_1^n \;, \qquad u_{z_1} = (z_1 z_2 (1 + a(z_1 + z_2) + b z_1 z_2))^\gamma$$

Intersections in z_1.

$$\hat{\omega}_{z_1} = \partial_{z_1} \log\left(u_{z_1}\right) = \frac{\gamma\,(a\,(2z_1 + z_2) + 2bz_1z_2 + 1)}{z_1\,(a\,(z_1+z_2) + bz_1z_2 + 1)} = \hat{\omega}_1 \qquad \nu_1 = 1. \qquad {}_{\omega_1}\langle \phi_1 | = 1 dz_1$$

$$J_2 = c_2 \;J_0 \;, \qquad \Longleftrightarrow \qquad {}_{\omega_1}\langle \phi_3 | \mathcal{C}_1] \;= c_2 \;{}_{\omega_1}\langle \phi_1 | \mathcal{C}_1] \,, \qquad c_2 = \langle \phi_3 | \phi_1 \rangle_{\omega_1} \langle \phi_1 | \phi_1 \rangle_{\omega_1}^{-1} = \frac{(\gamma+2)\,(az_2+1)^{\,2}}{2(2\gamma+3)\,(a+bz_2)^{\,2}} \;.$$

Intersections in z_2.

$$\hat{\omega}_{z_2} = \partial_{z_2} \log(J_0) = \frac{a^2\,(3\gamma z_2 + z_2) + a\left(2b\gamma z_2^2 + \gamma\right) - bz_2}{z_2\,(az_2+1)\,(a+bz_2)} \;. \qquad \nu_2 = 2, \qquad {}_{\omega_{z_2}}\langle \phi_1 | = dz_2 \text{ and } {}_{\omega_{z_2}}\langle \phi_2 | = z_2\, dz_2$$

$$\psi_{2,0} = c_2 \, dz_2 \;, \qquad I_{2,0} = c_{2,0,0}\, I_{0,0} + c_{2,0,1}\, I_{0,1} \;,$$

$$c_{n,m,i} = \langle \psi_{n,m} | \phi_j \rangle_{\omega_{z_2}} (\mathbf{C}_{\omega_{z_2}}^{-1})_{ji} \qquad c_{2,0,0} = -\frac{\gamma+1}{b(2\gamma+3)} \;, \qquad c_{2,0,1} = -\frac{3a^2\gamma + 3a^2 + b}{ab(2\gamma+3)}$$

in agreement with Reduze.