Integration-by-parts reductions via unitarity cuts and syzygies

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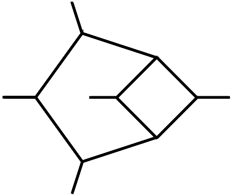
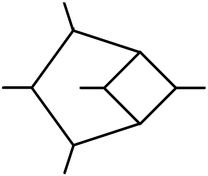


The Mathematics of Linear Relations between Feynman Integrals
Mainz Institute for Theoretical Physics
20th of March 2019

Based on PRD **93**(2016)041701, PRD **98**(2018)025023, JHEP **09**(2018)024
with J. Böhm, A. Georgoudis, H. Schönemann, M. Schulze, Y. Zhang

Overview

1. Motivation
2. IBP identities on unitarity cuts
3. Syzygy equations and their solution
4. Main example: 
5. AZURITE: a code to construct bases of integrals

Integration-by-parts reductions

IBP identities arise from the vanishing integration of total derivatives,

[Chertyrkin, Tchakov, Nucl. Phys. B **192**, 159 (1981)]

$$\int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\pi^{D/2}} \sum_{j=1}^{L} \frac{\partial}{\partial \ell_j^\mu} \frac{v_j^\mu \, P}{D_1^{a_1} \cdots D_k^{a_k}} \; = \; 0$$

where P and v_j^μ are polynomials in ℓ_i, p_j, and $a_i \in \mathbb{N}$.

Role in perturbative QFT calculations:

- **Reduction.** Reduce number of contributing loop integrals by factor of $\mathcal{O}(10^2) - \mathcal{O}(10^6)$ to basis.
- **Computing master integrals.** Enable setting up differential equations for basis integrals $\mathcal{I}_j$:

  [Gehrmann and Remiddi, Nucl. Phys. B **580**, 485 (2000)]

  [Henn, PRL **110** (2013) 251601]

$$\frac{\partial}{\partial x_m} \boldsymbol{\mathcal{I}}(\boldsymbol{x}, \epsilon) = A_m(\boldsymbol{x}, \epsilon) \boldsymbol{\mathcal{I}}(\boldsymbol{x}, \epsilon)$$

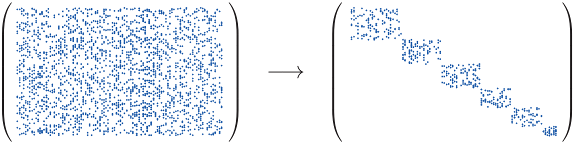
where x_m denotes a kinematical invariant.

IBP reductions on unitarity cuts

Standard approach: enumerate all linear relations and apply Gauss-Jordan elimination to *large* linear systems

[Laporta, Int.J.Mod.Phys. A **15** (2000) 5087-5159]

Idea here: use *unitarity cuts* to block-diagonalize system



We use the Baikov representation $\left(k = \frac{L(L+1)}{2} + LE\right)$,

$$I(N;a) \equiv \int \prod_{j=1}^{L} \frac{\mathrm{d}^D \ell_j}{i\pi^{D/2}} \frac{N}{D_1^{a_1} \cdots D_k^{a_k}} = \int \frac{\mathrm{d}z_1 \cdots \mathrm{d}z_k}{z_1^{a_1} \cdots z_k^{a_k}} \underset{(\widehat{\boldsymbol{p}},\boldsymbol{\ell})}{\mathrm{Gram}}(z)^{\frac{D-L-E-1}{2}} N$$

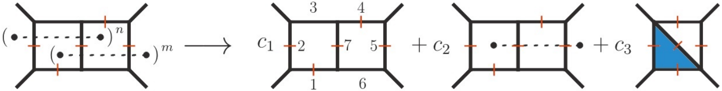
[Baikov, Phys.Lett. B **385** (1996) 404-410]

in which cuts are straightforward to apply,

$$\int \frac{\mathrm{d}z_i}{z_i^{a_i}} \xrightarrow{\text{cut}} \oint_{\Gamma_\epsilon(0)} \frac{\mathrm{d}z_i}{z_i^{a_i}} \qquad i \in \mathcal{S}_{\text{cut}}$$

Example: Zurich-flag cut

Let us construct IBP identities on the Zurich-flag cut



Define $S_{\text{cut}} = \{1, 2, 4, 5, 7\}$ and $G = \text{Gram}_{(\widehat{\boldsymbol{p}}, \boldsymbol{\ell})}$.

On S_{cut}, the double-box integral takes the form

$$I_{\text{cut}}^{\text{DB}}[P] = \prod_{i \in S_{\text{cut}}} \oint_{\Gamma_\epsilon(0)} \frac{d\widetilde{z}_i}{\widetilde{z}_i} \int \prod_{j \notin S_{\text{cut}}} d\widetilde{z}_j \frac{G(\widetilde{\boldsymbol{z}})^{\frac{D-6}{2}}}{\widetilde{z}_3\, \widetilde{z}_6} P(\widetilde{\boldsymbol{z}})$$

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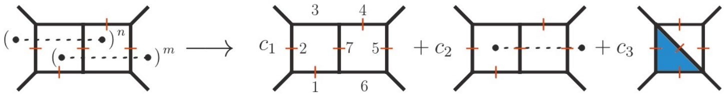

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Relabeling $z_{\{1,2,3,4\}} = \widetilde{z}_{\{3,6,8,9\}}$, this becomes

$$I_{\text{cut}}^{\text{DB}}[P] \; = \; \int \frac{\mathrm{d}z_1 \, \mathrm{d}z_2 \, \mathrm{d}z_3 \, \mathrm{d}z_4}{z_1 z_2} G(\boldsymbol{z})^{\frac{D-6}{2}} P(\boldsymbol{z})$$

Need to find IBP identities which involve

$$I_{\text{cut}}^{\text{DB}}[P] = \int \frac{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_4}{z_1 z_2} G(\boldsymbol{z})^{\frac{D-6}{2}} P(\boldsymbol{z})$$

Total derivatives $\longrightarrow$ IBP identities. Generic total derivative on cut:

$$\begin{aligned} 0 &= \int \left[\sum_{i=1}^{4} \frac{\partial}{\partial z_i} \left(\frac{a_i(\boldsymbol{z}) G(\boldsymbol{z})^{\frac{D-6}{2}}}{z_1 z_2} \right) \right] \mathrm{d}z_1 \cdots \mathrm{d}z_4 \\ &= \int \left[\sum_{i=1}^{4} \left(\frac{\partial a_i}{\partial z_i} + \frac{D-6}{2G} a_i \frac{\partial G}{\partial z_i} \right) - \sum_{j=1,2} \frac{a_j}{z_j} \right] \frac{G(\boldsymbol{z})^{\frac{D-6}{2}}}{z_1 z_2} \mathrm{d}z_1 \cdots \mathrm{d}z_4 \end{aligned}$$

The red term corresponds to an integral in $(D-2)$ dimensions, and the purple term in general produces doubled propagators.

To avoid dimension shifts and doubled propagators in

$$0 = \int \Big[\sum_{i=1}^{4} \Big(\frac{\partial a_i}{\partial z_i} + \frac{D-6}{2G} a_i \frac{\partial G}{\partial z_i} \Big) - \sum_{j=1,2} \frac{a_j}{z_j} \Big] \frac{G(\boldsymbol{z})^{\frac{D-6}{2}}}{z_1 z_2} \mathrm{d}z_1 \cdots \mathrm{d}z_4$$

we demand that each term is *polynomial,*

$$\sum_{i=1}^{4} a_i \frac{\partial G}{\partial z_i} + bG = 0$$

$$a_j + b_j z_j = 0$$

with a_i, b_i, b polynomials in z. Such eqs. are known as *syzygy equations*.
[Gluza, Kajda, Kosower, PRD**83**(2011)045012], [Schabinger, JHEP**01**(2012)077], [Ita, PRD**94**(2016)116015]

Obtain IBPs by plugging (a_i, b) into the top equation.
Note: (qa_i, qb) is also a solution, for polynomial q.

Strategy to solve syzygy equations

Solve syzygy equations with c cuts

$$a_j + b_j z_j = 0\,, \quad j = 1, \ldots, k-c \tag{1}$$

$$\sum_{j=1}^{m-c} a_j \frac{\partial G}{\partial z_k} + bG = 0 \tag{2}$$

as follows.

1) The generators of (1) are trivial:

$$\mathcal{M}_1 = \langle z_1 \mathbf{e}_1, \ldots, z_k \mathbf{e}_k, \mathbf{e}_{k+1}, \ldots, \mathbf{e}_m \rangle$$

2) Generators $\mathcal{M}_2 = \Big\langle (a_1, \ldots, a_m, b), \ \ldots \Big\rangle$ of (2) for the *off-shell case* $c = 0$ **can be explicitly found:**

$$(a_\alpha, b) = \left(\sum_{k=1}^{E+L} (1+\delta_{ik}) x_{jk} \frac{\partial z_\alpha}{\partial x_{ik}}, \ 2\delta_{ij} \right)$$

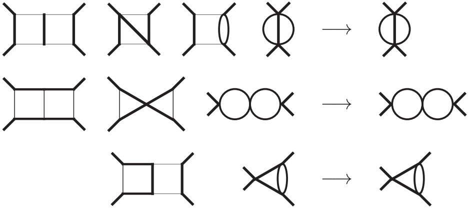
where $\quad x_{ij} = v_i \cdot v_j \quad$ with $\quad v_{i,j} \in \{p_1, \ldots, p_E, \ell_1, \ldots, \ell_L\}$.

[Böhm, Georgoudis, KJL, Schulze, Zhang, PRD **98**(2018)025023]

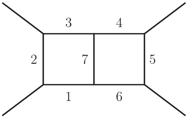
3) Take module intersection $\mathcal{M}_1\big|_{\text{cut}} \cap \mathcal{M}_2\big|_{\text{cut}}$

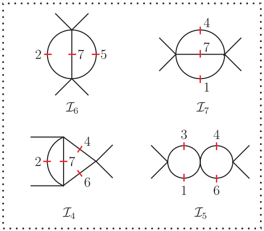
Spanning set of cuts for IBPs

To find the complete IBP reduction, we must consider the cuts associated with "uncollapsible" masters:



A bit more explicitly, the cuts we need to consider are

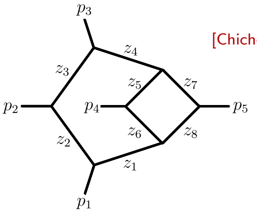




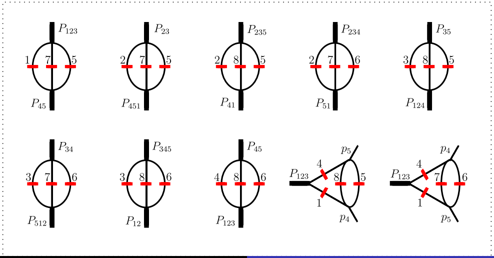
Main example: non-planar hexagon box

Task: IBP reduce non-planar hexagon box with numerator insertions of degree four in the z_i

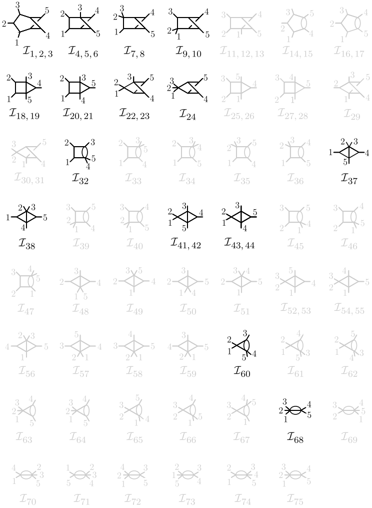
[Badger, Brønnum-Hansen, Hartanto, Peraro, PRL **120**(2018)092001]
[Chawdhry, Lim, Mitov, 1805.09182]
[Abreu, Page, Zeng, JHEP **01**(2019)006]
[Abreu, Dormans, Febres Cordero, Ita, Page, PRL **122**(2019)082002]
[Chicherin, Gehrmann, Henn, Lo Presti, Mitev, Wasser, JHEP **03**(2019)042]
[Abreu, Dixon, Herrmann, Page, Zeng, 1812.08941]
[Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia, 1812.11160, 1812.11057]



There are 10 cuts to consider:

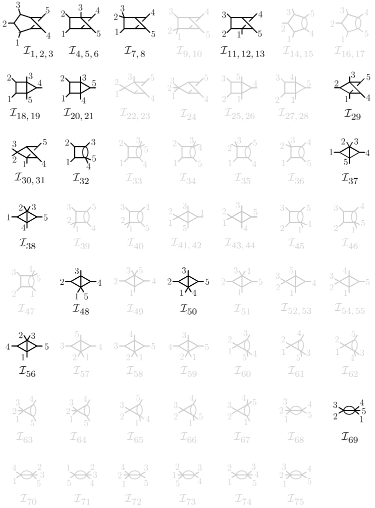


Construct and solve IBP identities on a spanning set of cuts.



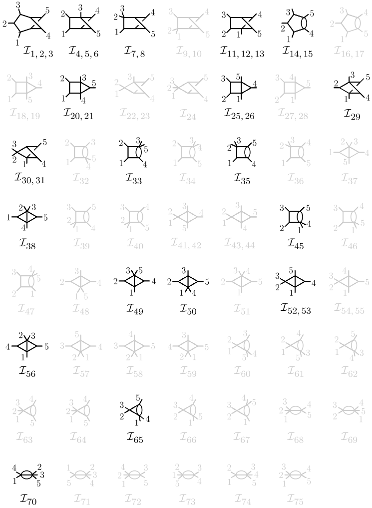
Cut $\{1,5,7\}$

Construct and solve IBP identities on a spanning set of cuts.



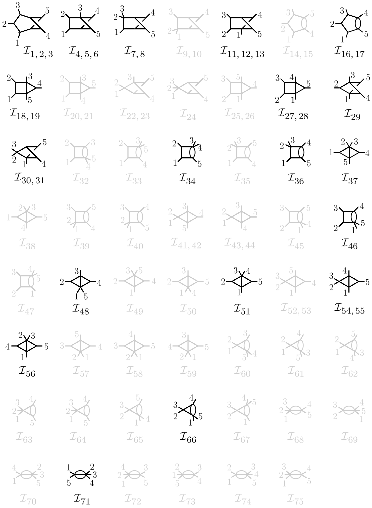
Cut $\{2, 5, 7\}$

Construct and solve IBP identities on a spanning set of cuts.



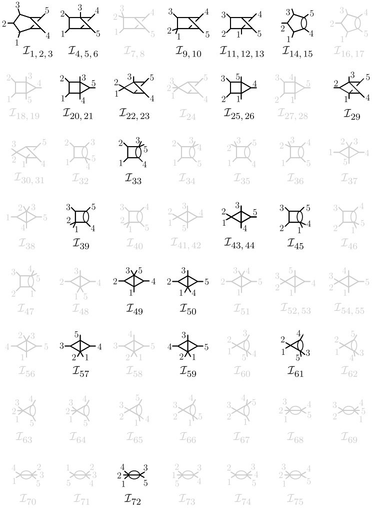
Cut $\{2, 5, 8\}$

Construct and solve IBP identities on a spanning set of cuts.



Cut $\{2, 6, 7\}$

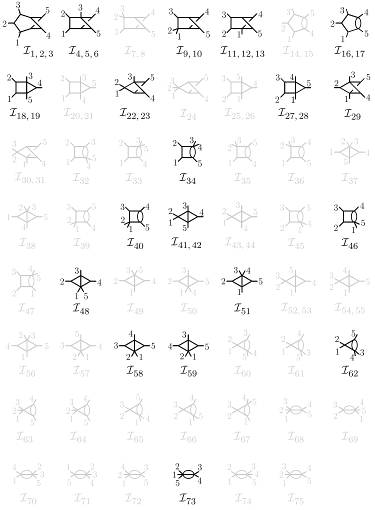
Construct and solve IBP identities on a spanning set of cuts.



Cut $\{3, 5, 8\}$

Non-planar hexagon box: spanning set of cuts

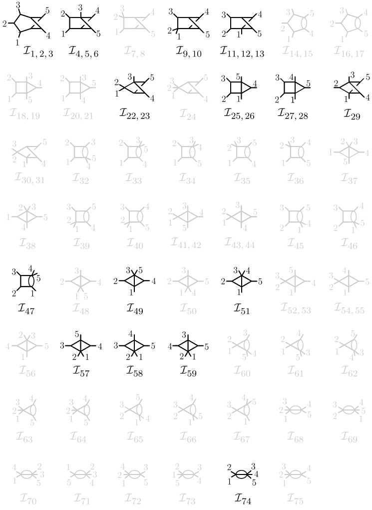
Construct and solve IBP identities on a spanning set of cuts.



Cut $\{3, 6, 7\}$

Non-planar hexagon box: spanning set of cuts

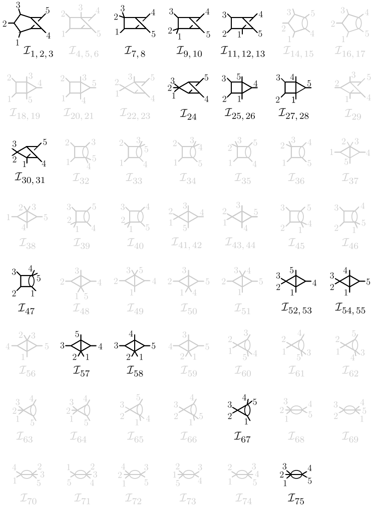
Construct and solve IBP identities on a spanning set of cuts.



Cut $\{3, 6, 8\}$

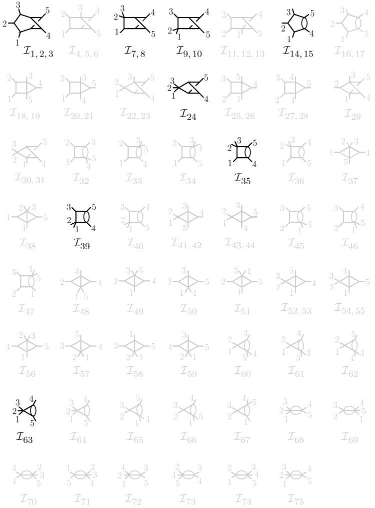
Non-planar hexagon box: spanning set of cuts

Construct and solve IBP identities on a spanning set of cuts.



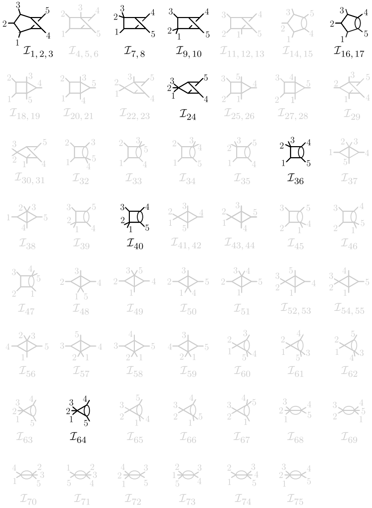
Cut $\{4, 6, 8\}$

Construct and solve IBP identities on a spanning set of cuts.



Cut $\{1, 4, 5, 8\}$

Construct and solve IBP identities on a spanning set of cuts.



Cut $\{1, 4, 6, 7\}$

Syzygies for the non-planar hexagon box

Syzygies for ensuring *D*-dimensionality:

$$
\begin{aligned}
M_1 = \langle &(z_1 - z_2, z_1 - z_2, -s_{12} + z_1 - z_2, -s_{12} - s_{13} + z_1 - z_2, s_{14} + z_1 - z_2 - z_8 + z_{10}, z_1 - z_2 - z_8 + z_{10}, 0, 0, -s_{12} - s_{13} - s_{14} + z_1 - z_2, 0, 0) \\
&(0, 0, 0, 0, s_{14} + z_1 - z_2 - z_8 + z_{10}, z_1 - z_2 - z_8 + z_{10}, s_{12} + s_{13} + s_{14} - z_8 + z_{10}, z_{10} - z_8, 0, z_{10} - z_8, s_{12} - z_8 + z_{10}) \\
&(s_{12} + z_2 - z_3, z_2 - z_3, z_2 - z_3, -s_{23} + z_2 - z_3, s_{12} + s_{24} + z_2 - z_3 - z_8 + z_{11}, s_{12} + z_2 - z_3 - z_8 + z_{11}, 0, 0, -s_{23} - s_{24} + z_2 - z_3, 0, 0) \\
&(0, 0, 0, 0, s_{12} + s_{24} + z_2 - z_3 - z_8 + z_{11}, s_{12} + z_2 - z_3 - z_8 + z_{11}, s_{12} + s_{23} + s_{24} - z_8 + z_{11}, z_{11} - z_8, 0, s_{12} - z_8 + z_{11}, z_{11} - z_8) \\
&(s_{13} + s_{23} + z_3 - z_4, s_{23} + z_3 - z_4, z_3 - z_4, z_3 - z_4, -2s_{12} - s_{13} - s_{14} - s_{23} - s_{24} + z_3 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, \\
&\quad -s_{12} + z_3 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, 0, 0, s_{12} + s_{13} + s_{14} + s_{23} + s_{24} + z_3 - z_4, 0, 0) \\
&(0, 0, 0, 0, -2s_{12} - s_{13} - s_{14} - s_{23} - s_{24} + z_3 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, -s_{12} + z_3 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, \\
&\quad -2s_{12} - s_{13} - s_{14} - s_{23} - s_{24} + z_4 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, -s_{12} - s_{13} - s_{23} + z_4 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, \\
&\quad 0, -s_{12} - s_{23} + z_4 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}, -s_{12} - s_{13} + z_4 - z_5 + z_6 + z_7 + z_8 - z_9 - z_{10} - z_{11}) \\
&(-s_{12} - s_{13} - s_{23} + z_4 - z_9, -s_{12} - s_{13} - s_{14} - s_{23} + z_4 - z_9, -s_{12} - s_{13} - s_{14} - s_{23} - s_{24} + z_4 - z_9, z_4 - z_9, z_5 - z_6, z_5 - z_6, 0, 0, z_4 - z_9, 0, 0) \\
&(0, 0, 0, 0, z_5 - z_6, z_5 - z_6, -z_4 + z_5 - z_6 + z_9, s_{12} + s_{13} + s_{23} - z_4 + z_5 - z_6 + z_9, \\
&\quad 0, s_{12} + s_{13} + s_{14} + s_{23} - z_4 + z_5 - z_6 + z_9, s_{12} + s_{13} + s_{23} + s_{24} - z_4 + z_5 - z_6 + z_9) \\
&(2z_1, z_1 + z_2, -s_{12} + z_1 + z_3, -s_{12} - s_{13} - s_{23} + z_1 + z_4, -s_{12} - s_{13} - s_{23} + z_1 + z_4 + z_6 - z_8 - z_9, z_1 + z_6 - z_8, 0, 0, z_1 + z_9, 0, 0) \\
&(0, 0, 0, 0, -s_{12} - s_{13} - s_{23} + z_1 + z_4 + z_6 - z_8 - z_9, z_1 + z_6 - z_8, z_6 - z_8 - z_9, -z_1 + z_6 - z_8, 0, -z_2 + z_6 - z_8, s_{12} - z_1 + z_2 - z_3 + z_6 - z_8) \\
&(-z_1 + z_6 - z_8, -z_1 + z_6 - z_{10}, -z_1 + z_6 + z_8 - z_{10} - z_{11}, s_{12} + s_{13} + s_{23} - z_1 - z_4 + z_5 - z_7 + z_9, \\
&\quad s_{12} + s_{13} + s_{23} - z_1 - z_4 + z_5 + z_8 + z_9, -z_1 + z_6 + z_8, 0, 0, -z_1 + z_6 - z_7, 0, 0) \\
&(0, 0, 0, 0, s_{12} + s_{13} + s_{23} - z_1 - z_4 + z_5 + z_8 + z_9, -z_1 + z_6 + z_8, z_7 + z_8, 2z_8, 0, z_8 + z_{10}, z_8 + z_{11}) \rangle \qquad (5.9)
\end{aligned}
$$

Syzygies for ensuring no doubled propagators:

$$
\begin{aligned}
M_2 = \langle &(z_1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), (0, z_2, 0, 0, 0, 0, 0, 0, 0, 0, 0) \\
&(0, 0, z_3, 0, 0, 0, 0, 0, 0, 0, 0), (0, 0, 0, z_4, 0, 0, 0, 0, 0, 0, 0) \\
&(0, 0, 0, 0, z_5, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, z_6, 0, 0, 0, 0, 0) \\
&(0, 0, 0, 0, 0, 0, z_7, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0, z_8, 0, 0, 0) \\
&(0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0) \\
&(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1) \rangle
\end{aligned}
$$

Compute intersection of $M_1\big|_{\text{cut}} \cap M_2\big|_{\text{cut}}$ on each of the 10 cuts.

Complexity of IBP systems

- Resources to compute $M_1\big|_{\text{cut}} \cap M_2\big|_{\text{cut}}$: **25-800 s** and **1-14 GB RAM**

  (on 24 cores, 3.40 GHz)

- Size of generating systems after trimming: **1.5-10 MB**

  Plug resulting generators into ansatz for total derivative:

$$0 = \int \left[\sum_{i=1}^{m-c} \left(\frac{\partial a_{r_i}}{\partial z_{r_i}} + \frac{D-L-E-1}{2G(z)} a_{r_i} \frac{\partial G}{\partial z_{r_i}} \right) - \sum_{i=1}^{k-c} \frac{a_{r_i}}{z_{r_i}} \right] \frac{G(z)^{\frac{D-L-E-1}{2}}}{z_{r_1} \cdots z_{r_{k-c}}} \mathrm{d}z_{r_1} \cdots \mathrm{d}z_{r_{m-c}}$$

- Resulting linear systems to solve:
  700-1200 equations, size **1 MB**, density **1.5%**

Gauss-Jordan elimination of IBP systems

To find the IBP reductions, Gauss-Jordan eliminate IBP systems.

Some remarks:

- To preserve sparsity, use a *total pivoting* strategy (i.e., allow column swaps)
- For cut $\{1, 4, 6, 7\}$, the RREF can be performed fully analytically, requiring 31 minutes on one core and 1.5 GB RAM.

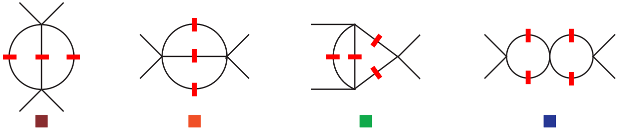
  $\vdots$

- For $\{3, 6, 7\}$, assigned numerical values to two s_{ij}.
  Ran 440 points on cluster (2.5 h and 1.8 GB RAM per job).
  Used interpolation code to get analytical results (23 min and 15 GB RAM on one core).

[von Manteuffel and Schabinger, PLB **744**(2015)101]
[Peraro, JHEP**12**(2016)030]

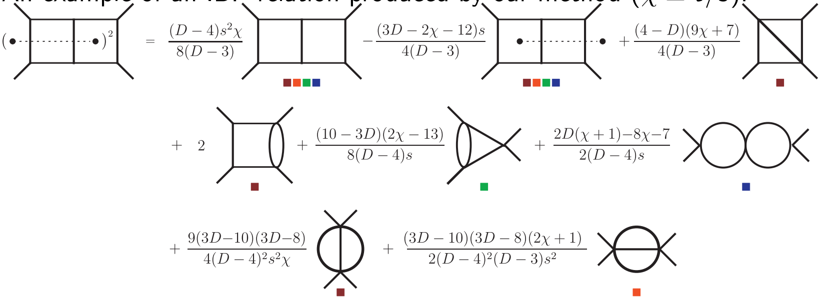
Merging on-shell IBP reductions

By solving the IBP identities on the following cuts



we reconstruct the *complete IBP reductions* by merging the partial results.

An example of an IBP relation produced by our method ($\chi \equiv t/s$):



Results for IBP reductions

- Fully analytic IBP reductions of the 32 hexagon boxes

$$
\begin{array}{lll}
\{I(1,1,1,1,1,1,1,1,0,0,-4), & I(1,1,1,1,1,1,1,1,0,-1,-3), & I(1,1,1,1,1,1,1,1,0,-2,-2) \\
I(1,1,1,1,1,1,1,1,0,-3,-1), & I(1,1,1,1,1,1,1,1,0,-4,0), & I(1,1,1,1,1,1,1,1,-1,0,-3) \\
I(1,1,1,1,1,1,1,1,-1,-1,-2), & I(1,1,1,1,1,1,1,1,-1,-2,-1), & I(1,1,1,1,1,1,1,1,-1,-3,0) \\
I(1,1,1,1,1,1,1,1,-2,0,-2), & I(1,1,1,1,1,1,1,1,-2,-1,-1), & I(1,1,1,1,1,1,1,1,-2,-2,0) \\
I(1,1,1,1,1,1,1,1,-3,0,-1), & I(1,1,1,1,1,1,1,1,-3,-1,0), & I(1,1,1,1,1,1,1,1,-4,0,0) \\
I(1,1,1,1,1,1,1,1,0,0,-3), & I(1,1,1,1,1,1,1,1,0,-1,-2), & I(1,1,1,1,1,1,1,1,0,-2,-1) \\
I(1,1,1,1,1,1,1,1,0,-3,0), & I(1,1,1,1,1,1,1,1,-1,0,-2), & I(1,1,1,1,1,1,1,1,-1,-1,-1) \\
I(1,1,1,1,1,1,1,1,-1,-2,0), & I(1,1,1,1,1,1,1,1,-2,0,-1), & I(1,1,1,1,1,1,1,1,-2,-1,0) \\
I(1,1,1,1,1,1,1,1,-3,0,0), & I(1,1,1,1,1,1,1,1,0,0,-2), & I(1,1,1,1,1,1,1,1,0,-1,-1) \\
I(1,1,1,1,1,1,1,1,0,-2,0), & I(1,1,1,1,1,1,1,1,-1,0,-1), & I(1,1,1,1,1,1,1,1,-1,-1,0) \\
I(1,1,1,1,1,1,1,1,0,0,-1), & I(1,1,1,1,1,1,1,1,0,-1,0)\} &
\end{array}
$$

  can be downloaded from (268 MB compressed / 790 MB uncompressed)

  https://github.com/yzhphy/hexagonbox_reduction/releases/download/1.0.0/hexagon_box_degree_4_Final.zip

- Our results agree with fully numerical results from FIRE5 C++ (6 hours per point).

  [A. Smirnov, CPC **189**(2015)182]

AZURITE: a code to construct bases of integrals

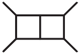
Given an input set of inverse propagators $D_1, \ldots, D_k$,
AZURITE determines a basis as follows.

[A. Georgoudis, KJL, Y. Zhang, CPC **221**(2017)203]

1. Find automorphism groups G of the graph Γ and its subgraphs.
2. Find a list $\mathcal{C}$ of cuts such that no two elements of $\mathcal{C}$ are related by a discrete symmetry of a (sub)graph of Γ.
3. For each cut $c \in \mathcal{C}$, construct IBP identities and symmetry relations on c (with $\mathbb{Z}_p$ values for kinematics and D).
4. Apply Gauss-Jordan elimination to the system of identities. The non-pivot columns correspond to basis integrals.

A priori $2^7 = 128$ cuts to consider for four gluons at two loops.
Mod out by discrete symmetries $\Longrightarrow$ only 34 cuts are needed.

128 cuts



4 symmetries

48 cuts

4 symmetries

42 cuts

4 symmetries

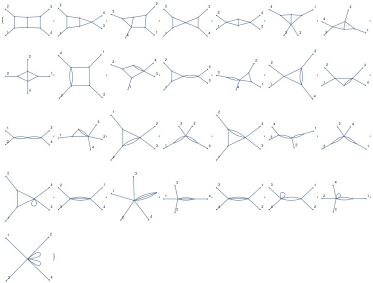
40 cuts

2 symmetries

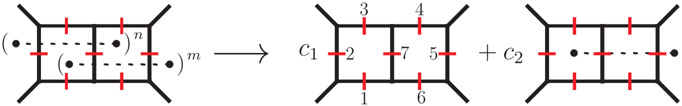
35 cuts

8 symmetries

34 cuts



Consider IBP identities on maximal cut {1,2,3,4,5,6,7}



With $I[m,n] \equiv I[(\ell_1+p_4)^{2m}(\ell_2+p_1)^{2n}]$ and $(s\equiv 1, t\equiv 3, D\equiv 8009 \pmod{9001})$:

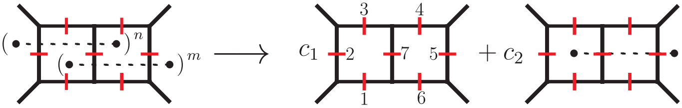
$$I[0,2] = -1075 I[0,0] + 3228 I[1,0]$$

Wrt. $\{I[0,3], I[1,2], I[2,1], I[3,0], I[0,2], I[1,1], I[2,0], I[0,1], I[1,0], I[0,0]\}$, record the linear relations as

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -3228 & 1075 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -449 & 3477 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -4499 & 4499 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1611 & -536 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -3228 & 1075 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1611 & -536 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -449 & 3477
\end{pmatrix}$$

Example: IBP relations on the maximal cut

Consider IBP identities on maximal cut $\{1,2,3,4,5,6,7\}$



With $I[m,n] \equiv I[(\ell_1+p_4)^{2m}(\ell_2+p_1)^{2n}]$ and $(s \equiv 1, t \equiv 3, D \equiv 8009 \pmod{9001})$:

$$I[0,2] = -1075 I[0,0] + 3228 I[1,0]$$

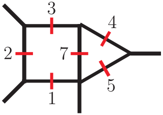
Wrt. $\{I[0,3], I[1,2], I[2,1], I[3,0], I[0,2], I[1,1], I[2,0], I[0,1], I[1,0], I[0,0]\}$, record the linear relations as

$$\begin{pmatrix} 0&0&0&0&0&0&0&1&-1&0\\ 0&0&0&0&1&0&0&0&-3228&1075\\ 1&0&0&0&0&0&0&0&-449&3477\\ 0&0&0&0&0&1&0&0&-4499&4499\\ 0&1&0&0&0&0&0&0&1611&-536\\ 0&0&0&0&0&0&1&0&-3228&1075\\ 0&0&1&0&0&0&0&0&1611&-536\\ 0&0&0&1&0&0&0&0&-449&3477 \end{pmatrix} \longrightarrow \begin{pmatrix} 1&0&0&0&0&0&0&0&-449&3477\\ 0&1&0&0&0&0&0&0&1611&-536\\ 0&0&1&0&0&0&0&0&1611&-536\\ 0&0&0&1&0&0&0&0&-449&3477\\ 0&0&0&0&1&0&0&0&-3228&1075\\ 0&0&0&0&0&1&0&0&-4499&4499\\ 0&0&0&0&0&0&1&0&-3228&1075\\ 0&0&0&0&0&0&0&1&-1&0 \end{pmatrix}$$

Add non-pivot elements $\{I[1,0], I[0,0]\}$ to basis.

Example: IBPs on less-than-maximal cuts

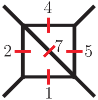
On the six-fold cut {1,2,3,4,5,7}:



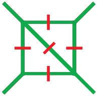
$$\left[(\ell_1+p_4)^{2n}(\ell_2+p_1)^{2p}(\ell_2+p_4)^{2q}\right] \longrightarrow 0$$

$\Longrightarrow$ no elements added to basis

On the five-fold cut {1,2,4,5,7}:



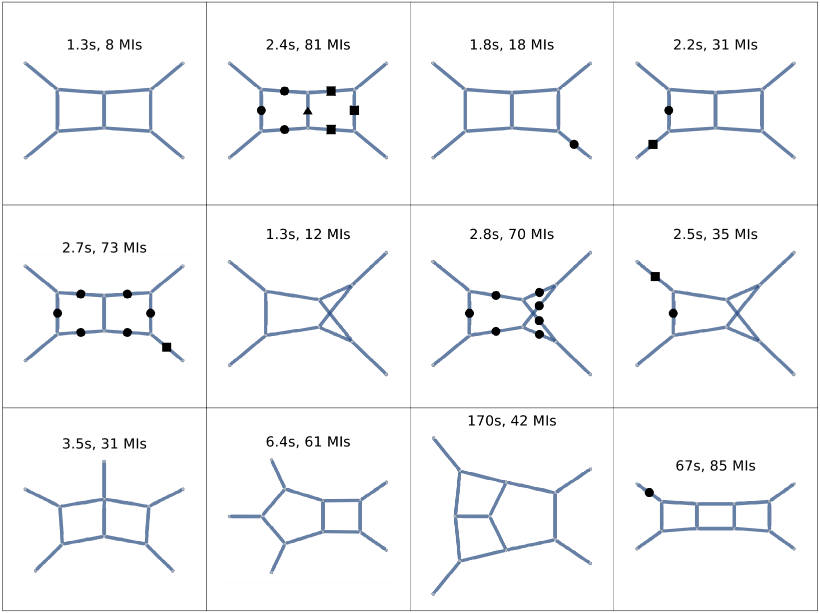
$$\left[(\ell_1+P_{12})^{2m}(\ell_1+p_4)^{2n}(\ell_2+p_1)^{2p}(\ell_2+p_4)^{2q}\right] \longrightarrow c_{mnpq}$$



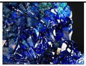
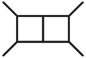
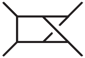
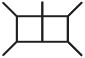
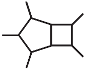
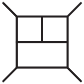
$[1]$

$\Longrightarrow$ $I[0,0,0,0]$ added to basis

AZURITE: sample results and timings

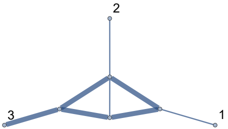
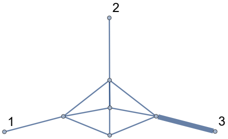


MINT: [Lee and Pomeransky, JHEP 1311 (2013) 165]

			
topology	MINT timing	AZURITE timing	ratio
	3.1 s	1.3 s	2.4
	3.2 s	1.3 s	2.5
	6.9 s	3.5 s	2.0
	29.5 s	6.4 s	4.6
	$> 2 \cdot 10^5$ s	170 s	> 1000

AZURITE: master integral count comparison

REDUZE2: [Manteuffel and Studerus, 1201.4330]

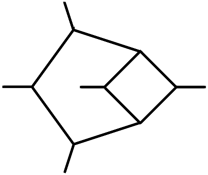
sector	MINT	REDUZE2	AZURITE
	1	2	2
	0	1	1

MINT undercounting: misses critical point(s) at infinity.

Conclusions

- New formalism for IBP reductions. Main ideas: cuts, IBP identities from syzygies, total pivoting, rational reconstruction

- Obtained the **fully analytic** IBP reductions of



  with numerator insertions up to degree 4 in the z_i.

- Powerful framework. IBP reductions for further $2 \to 3$ two-loop processes **seem well within reach**.