Counting master integrals with PSLQ

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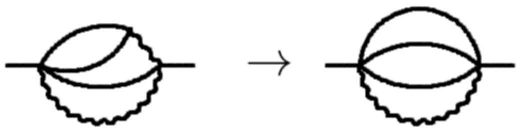
Counting the number of master integrals for a given sector of a Feynman diagram is a problem with a not trivial solution. The usual approaches are algebraic:

1. Analysys of the solution of the system of i.b.p. identities between Feynman integrals.
2. Application of the Lee-Pomeranski (2013) criterion to suitable integral representations of the Feynman integrals.

One needs simply to build a system of algebraic integration-by-parts identities for the given sector, solving them with the desired ordering and look for the integrals which are left unreduced.

- Pros: simple
- Cons: not known *a priori* how many identities are needed.
- Cons: some additional relations may come from simmetry relations, outside the scope of i.b.p. identities
- Cons: some additional relations may come from the generation of identities for higher sectors, i.e. sectors containg topologies more complicated. A famous (first?) example: master integrals for the 3-loop g-2 (S.L. and E.Remiddi 1996) (18 master integral $\rightarrow$ 17)



One needs a integral representation, with Symanzik or Baikov polynomials. One uses the criterion presented by Lee e Pomeransky in 2013.

Counting Master Integrals for a given sector

Is it possible to find some *alternative*, numerical, reliable, independent way to determine the number of master integrals?.

An useful tool for such a searches is the PSLQ algorithm (Ferguson and Bailey 1992). It is one (probably the most famous) practical algorithm able for determining if, given some (high-precision) real numbers $\{x_i\}$, there exist integers c_i such that $\sum_i c_i x_i = 0$, or, if unsuccesful, give a rigorous superior bound M such that $c_i > M$.

If $\{x_i\}$ are the valous for of some Feynman integrals of some sector, with different indices, it would be possible in line of principle to establish if there are linear relations among them, and the length of such relations. For example, given values $x_1 = \sqrt{2}$, $x_2 = \pi$, $x_3 = 5\sqrt{2} + 7\pi$, PSLQ with 5 digits discovers the relation $-5x_1 - 7x_2 + x_3 = 0$

In practice, calculating directly such Feynman integrals is a very difficult problem, so, as long as we are interested only to the number of master integrals, we can limit ourselves to cutted integrals.

$$I(\{n_j\}) = \int \frac{d^d k_i}{D_1^{n_1} \ldots D_M^{n_M}}$$

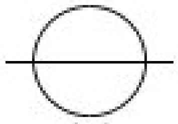
M denominators, $M+N$ irreducible scalar products We introduce the Baikov representation, where $z_i = D_i$

$$I(\{n_j\}) = K(d) \int_R z_1^{n_1} \ldots z_M^{n_M} P(\{z_j\}, j = 1, \ldots, M)^\delta)$$

P=0 on ∂R

On the cut $z_1 = z_2 = \ldots z_M = 0$

$$I(\{n_j\}) = K(d) \int z_{M+1}^{n_{M+1}} \ldots z_{M+N}^{n_{M+N}} P(\{z_j\}, j = M+1, \ldots, M+N)^\delta$$



$$I(n_4,n_5) = K(d)\int \frac{d^Dk_1 d^Dk_2}{D_1 D_2 D_3} D_4^{n_4} D_5^{n_5}$$

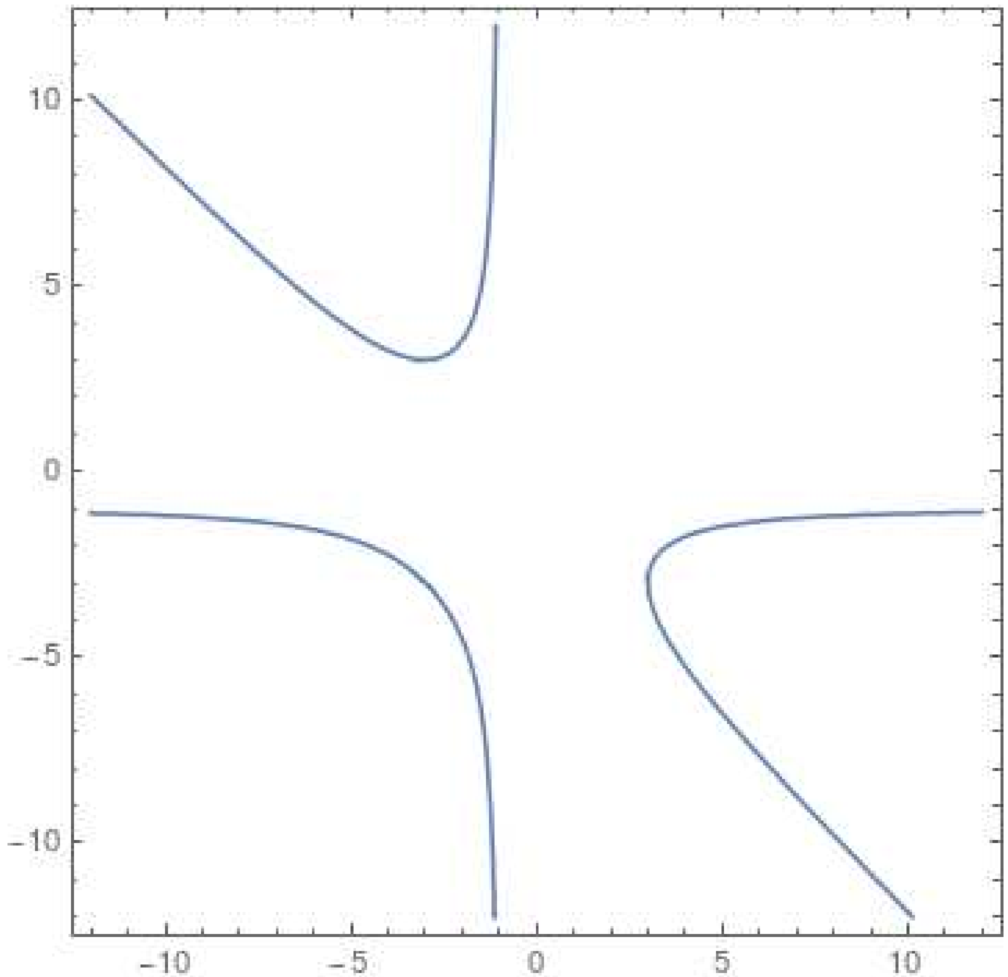
$D_1 = k_1^2 - 1,\ D_2 = k_2^2 - 1,\ D_3 = (p-k_1-k_2)^2 - 1,\ D_4 = (p-k_1)^2 - 1,$
$D_5 = (p-k_2)^2 - 1.$

Baikov representation on the maximal cut

$$I(z,w) = K(d)\int_R z^{n_4} w^{n_5} P(z,w,s)^\delta$$

$$P(z,w,s) = -S^2 - w^2 - w(1+w)z - (1+w)z^2 + S(3+w+(1+w)z)$$

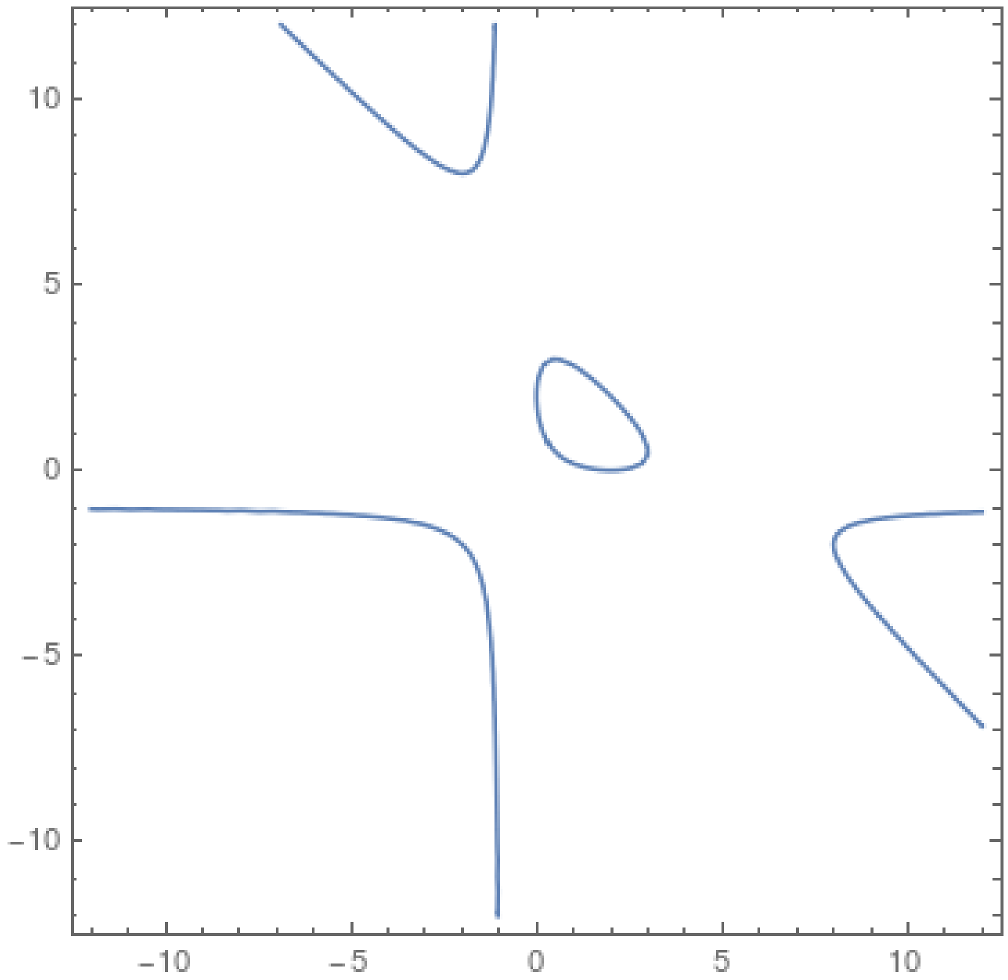
Plot of $P(z, w, s) = 0$, $s = -3$,



the lobes go to infinity

Critical points $s = 0, 1$ and 9.

For $s > 1$ a *closed* path appears

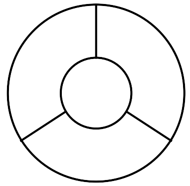


Plot of $P(z, w, s) = 0$, $s = 4$,

Defining an integral inside the closed path for $\delta = (d-4)/2 = -1/3$

$$G(n_4, n_5) = \int_0^3 dz \int_{w_1(z)}^{w_2(z)} z^{n_4} w^{n_5} P(z, w, s)^{-1/3}$$

with 10-digits precision PSLQ finds $424G(0,0) - 249G(2,0) + 36G(3,0) = 0$ which is the same identity which one finds by solving i.b.p. identities, selecting only the top sector integrals. The number of master integrals is 2.



$$I(n) = \int \frac{d^Dk_1 d^Dk_2 d^Dk_3 d^Dk_4}{D_1 D_2 D_3 D_4 D_5 D_6 D_7 D_8 D_9}$$

Baikov representation on the maximal cut

$$I(n) = \int_R z^n P(z)^\delta$$

$$P(z) = \frac{z}{2} - \frac{3}{16} z^2$$

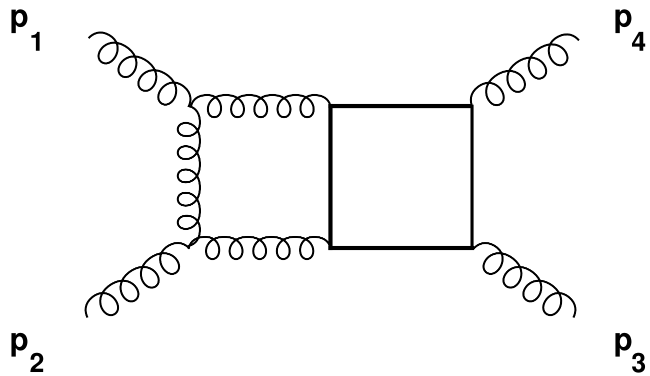
$P(z) = 0$ at the ends of the integration interval. We are interested to $n > 0$, so we consider only finite interval we choose as J: $0 < z < 8/3$

In this case one can integrating analytically

$$I(n) = \int_0^{\frac{8}{3}} z^n P(z)^\delta = \frac{2^{3+2d+3n}}{3^{n+d+1}} \frac{\Gamma(d+1)\Gamma(d+n+1)}{\Gamma(2d+n+2)} \tag{1}$$

$\frac{d}{dz}\ln(P(z)) = 0$ has 1 solution $\rightarrow$ 1 master integral

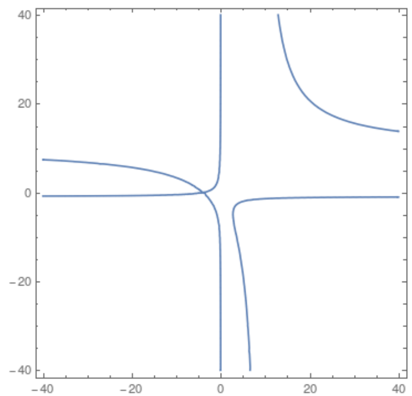
PSLQ finds easily two-terms relations among $\{I(n)\}$



Baikov representation on the maximal cut

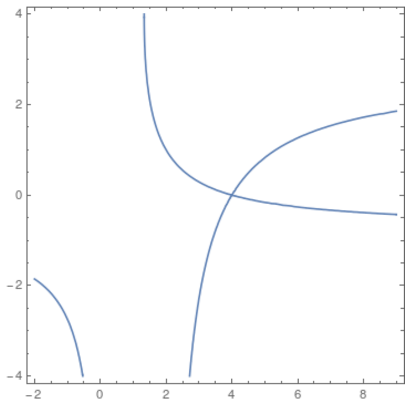
$$I(m,n) = \int_R z^m w^n P(z,w)^\delta$$

$$P(z,w) = w^2 z^2 + s(t^2 - t(2+w)z + z(z + w(w+z)))$$

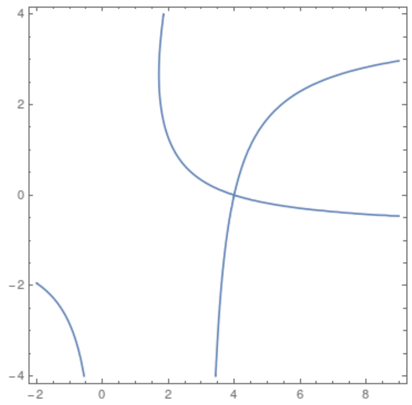


Problem: the curves $P(z,w) = 0$ goes to infinity

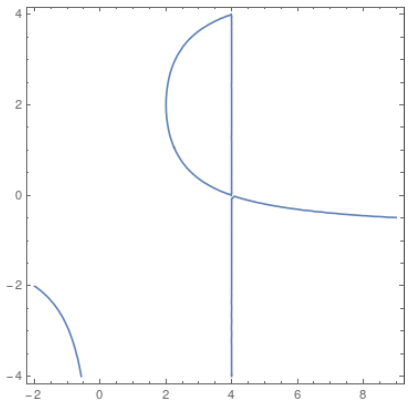
$\rightarrow$ divergent integrals. We look for *self-intersections* of the curves $P(z,w) = 0$



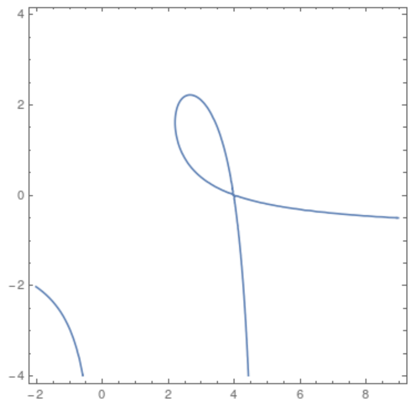
$s = -2, \; t = 4$



$s = -3, \; t = 4$



$s = -4, \; t = 4$



$s = -5, \; t = 4$

Two-loop box example 1

We chose $s = -$, $t = 4$, $\delta = 10/7$

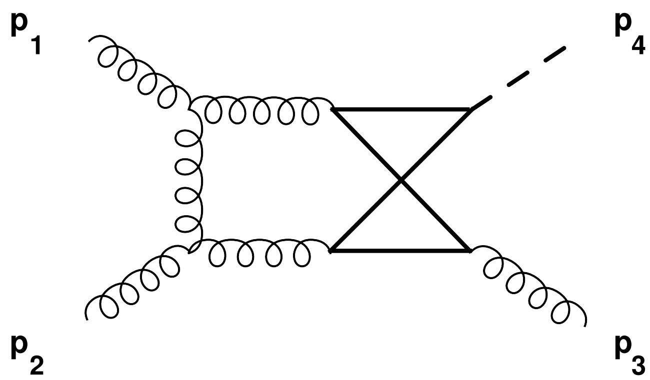
$$I(m,n) = \int_{\frac{36}{13}}^{4} dz \int_{w_1(z)}^{w_2(z)} dw z^m w^n P(z,w,s=-9,t=4)^{\delta}$$

$$I(0,0) = 0.288904389479758165\ldots$$
$$I(0,1) = 0.169477622359192948\ldots$$
$$I(0,2) = 0.108175540481366117\ldots$$
$$I(0,3) = 0.0735004777973063809\ldots$$
$$\ldots$$

the shortest relations that PSLQ finds have 4 terms, e.g.

$$48960I(0,0) - 89738I(0,1) + 9981I(0,2) - 216I(0,3) = 0$$

$\rightarrow$ 3 master integrals



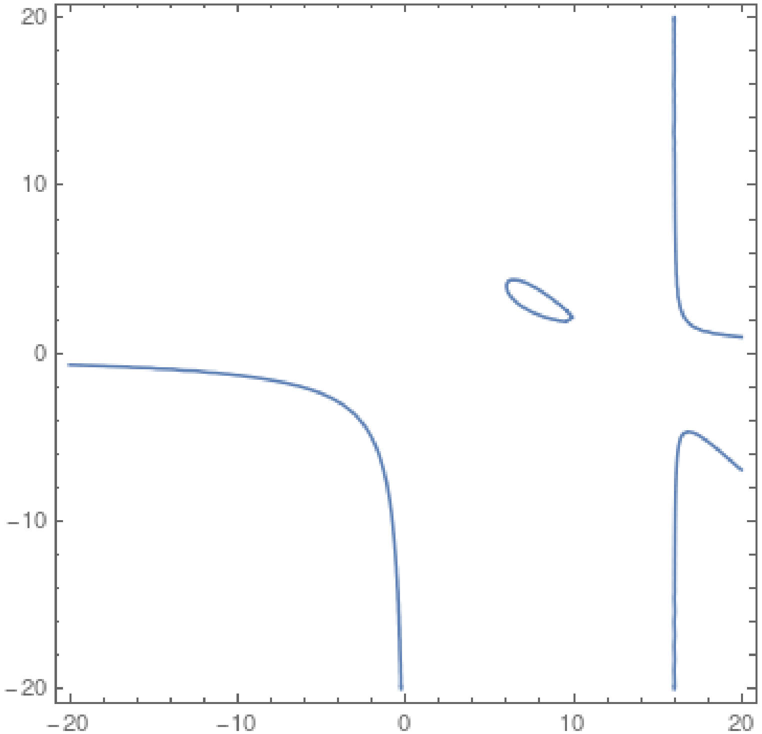
Baikov representation on the maximal cut

$$I(m,n) = K(d) \int\limits_{\gamma} z^m w^n P(z,w)^{\delta}$$

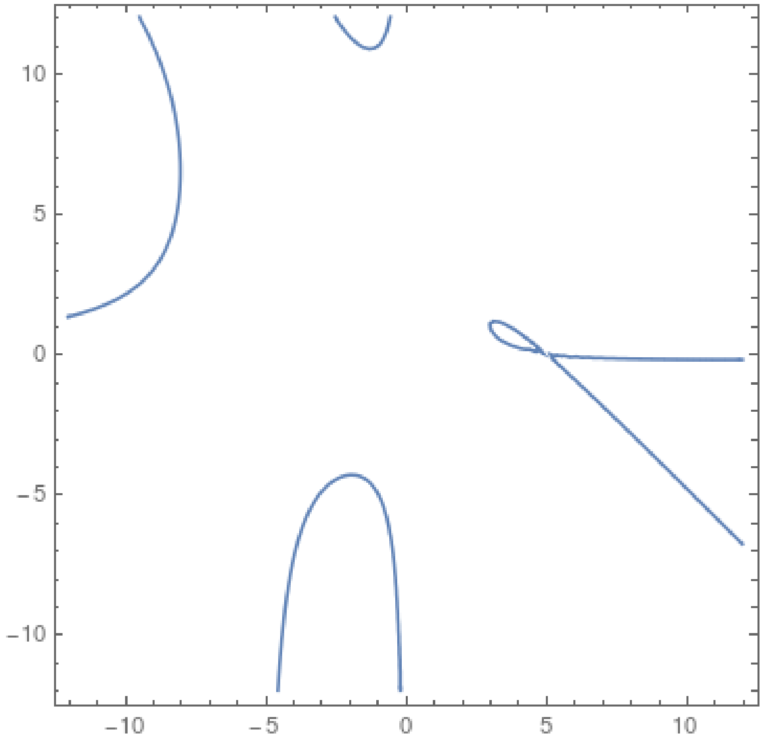
$$\begin{aligned} P(z,w) = &+w^2 z(-1+s+z) + wz(-1+s+z)(-1+s+t+z)s - 2s^2 + s^3 \\ &- 2st + 2s^2 t + st^2 - 2sz + 2s^2 z + 2stz + sz^2 \end{aligned}$$

Two-loop box example 2

Critical points/lines $s = 0, 1$, $s^2 + 14s + 1 = 0$, $s + t - 1 = 0$, $-4s - t + st + t^2 = 0$.

$s = -15, \ t = 2$ 

Here a closed curve appears.



If we choose $s = 6$, $t = -10$
A self-intersection appears.

$$I(m,n) = \int_3^5 dz \int_{w_1(z)}^{w_2(z)} dw\, z^m w^n P(z,w,s=6,t=-10)^{\frac{10}{7}}$$

We integrate over the small compact region. PSLQ, with 80 digits, finds 5-terms relations between $I(0\ldots 3, 0\ldots 3) \to 4$ master integrals.

- Baikov representation is usually considered not useful for numerical calculations, because of complications of the path and divergences.
- For some value of the kinematical invariants closed finite curves may appear; here the integrals are relatively easy to calculate, and high-precision values can be used to determine independently the number of irreducible integrals.

The End